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Theoretical Aeroacoustics: Compiled Mathematical Derivations of Fereidoun ‘Feri’ Farassat

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April 2016

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Acknowledgments

The staff of the National Aeronautics and Space Administration (NASA) Langley Research Center Technical Library, Sarah Schwaner and Susan Miller, kindly provided access to Dr. Farassat's collection of notes. The family of Dr. Farassat, especially Linda Schneider and Nigel Schneider, are gratefully acknowledged for providing access to Dr. Farassat's books. Doug Boyd, Mark Dunn, Craig Hunter, Florence Hutcheson, Alexandra Loubeau, Doug Nark, and others provided substantial material. Discussions with Michael Myers and Mark Dunn provided insight into the life of Dr. Farassat. The author is grateful for the opportunity to organize this publication and for the time he spent within the same research group of Dr. Farassat.

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Abstract

Dr. Fereidoun 'Feri' Farassat was a theoretical aeroacoustician at the National Aeronautics and Space Administration (NASA) Langley Research Center. This document contains technical derivations, notes, and classes that Dr. Farassat produced during his professional career. The layout of the document has been carefully crafted so that foundational ideas through advanced theories, which altered the technical discipline of aeroacoustics, build upon one another. The document can be used to understand the theories of acoustics and learn one contemporary aeroacoustic prediction approach made popular by Dr. Farassat. Most importantly, this document gives the general reader insight into how one of NASA's best aeroacoustics theoreticians thought, constructed, and solved problems throughout his career.

Dedicated to our families.

The content identified below is believed to be third party content that was incorporated by Dr. Farassat into his course materials. Because Dr. Farassat is deceased, the author made a good faith effort to identify the source of the content but was unable to do so. Should a recipient of this NASA TM be aware of the source(s) of any of the identified content and notifies the author, NASA will use reasonable efforts to update this NASA TM to reflect the identified sources. The content is present on pages 34, 36, 37, 38, 39, 41, 48, 49, 51, 54-56, 57-60, 64-71, 72-76, 83, 113-115, 159, 164, 193-194, 195-196, 438, 439, 657, 658, 909, 913, 1139, and 1177-1181.

Contents

Figure 1. Portrait photograph of Dr. Fereidoun ‘Feri’ Farassat taken December 2006.

1 Introduction

Dr. Fereidoun 'Feri' Farassat was a researcher at the National Aeronautics and Space Administration (NASA), who not only was a leading theoretician in the field of aeroacoustics[1] but also had incredible influence on those around him. This document contains his comprehensive mathematical derivations, notes, and classes that represent his life's work. Dr. Farassat was an unwavering advocate for NASA research and always followed what he believed was the correct technical path. Beyond Dr. Farassat's notable theoretical contributions to the scientific field of aeroacoustics, he had an incredible influence on other researchers and those who surrounded him on a professional and personal level. Readers who examine this document, with or without a technical background, will gain an understanding of how one of NASA's best theoreticians thought, worked, and lived.

This document has a unique philosophy and purpose that emerged during its creation. Originally the document was intended as simply a comprehensive set of mathematical derivations from Dr. Farassat's career at NASA. Certainly this task has been accomplished, and this publication represents a comprehensive volume of the green books,[2] technical derivations, notes, and classes that Dr. Farassat produced during his professional career. However, once the document was created it was realized that it represents so much more than just a simple volume of research notes. The layout of the notebooks, classes, and derivations has been carefully chosen so that foundational ideas through theories, that altered the technical viewpoint of aeroacousticians, build upon one another. With this approach, the document could be used by a student with a basic technical undergraduate degree to understand the basics of acoustics and learn one class of very popular prediction approaches.

This is a historical document about the life of a NASA researcher and those he interacted with. It gives insight into the development of equations used internationally. Perhaps most importantly, this document gives the general reader insight into how one of NASA's best aeroacoustics theoreticians thought, constructed, and solved problems throughout his career. Dr. Farassat's career is evidence of the importance of fundamental theoretical research and how it can change the course of a technical discipline and the organization of NASA itself.

The content of this document is as follows: A brief biography of Dr. Farassat describes his life and impact; how the document was created and how the derivations, notes, and classes are organized are described; and finally, a comprehensive list of Dr. Farassat's publications is presented.

1.1 A Brief Biography of Fereidoun 'Feri' Farassat

Dr. Fereidoun 'Feri' Farassat, whose portrait is shown in Figure 1 on page 5, was born on October 20^{th}, 1944 in Ramhormoz, Iran. His father was Golamhassan Farassat, who was the head of the Gendarmerie[3] within the 10^{th} province of Esfahan, Iran, and his mother was Fatmeh Roozrokh Farassat. He had multiple sisters and brothers. The Farassat family moved throughout Iran during Feri's childhood, and they lived in the Iranian cities of Azarbaijan, Mazandaran, Yazd, South Khorasan, and Tehran. Dr. Farassat enjoyed an Iranian childhood and often claimed that he had received the highest quality education. Feri graduated from a public high school in Tehran, Iran. He received his undergraduate degree of Bachelor of Engineering with emphasis in Mechanics in 1967 from the American University of Beiruit. After completing his undergraduate degree he moved

[1] Aeroacoustics is the study of how sound is generated from moving bodies and turbulence, propagated and scattered through the fluid, and received by an observer.

[2] Many NASA researchers record their work within bounded books that have a green cover and are affectionately called 'green books.'

[3] A military organization whose task is aiding the police within a civilian population.

to Scotland and worked as an engineer.

Dr. Farassat started graduate studies at Syracuse University in the United States in 1968 and in 1970 earned his Master's degree.[4] His research[5] at Syracuse University was focused on experimental jet aeroacoustics. After this experimental investigation, he entirely focused on theoretical approaches.

Dr. Farassat started his doctoral studies at Cornell University in 1970, after graduating from Syracuse University. At Cornell University his doctoral advisor was Professor William Rees Sears.[6] Professor Sear's advisor was Professor Theodore von Karman, and Theodore von Karman's advisor was Professor Ludwig Prandtl. Dr. Farassat was undoubtedly proud of his academic lineage. While at Cornell University, Dr. Farassat worked on theoretical approaches to predict the noise from helicopter rotors based upon the research of Ffowcs Williams and Hawkings.[7] Perhaps the main focus of Dr. Farassat's Ph.D. dissertation is the use of generalized functions in conjunction with aeroacoustics and was the basis for his research throughout his life. Dr. Farassat writes, 'the period that I worked with Bill Sears at Cornell was one of the happiest periods of my life. I often visited Bill and his wife Mabel in Tucson, Arizona where Bill had retired.' He completed his Ph.D.[8] in 1973 and joined the faculty of the George Washington University as a senior scientist and eventually became an adjunct professor. He taught classes within the college of engineering and held this position concurrently with his position at NASA for approximately 25 years.

Dr. Farassat joined NASA in 1979. He became a naturalized citizen in 1981. While at NASA and fullfilling his research obligations, he continued to teach classes to both the researchers of NASA and students of the George Washington University. Some material from these classes is included within this publication. He remained at NASA throughout his professional career, where he quickly became a leading theoretician in aeroacoustics. However, his interests were not limited to theoretical aeroacoustics and included general acoustics, scattering, the Ffowcs Williams-Hawkings (FW-H) equation, Kirchhoff formulas, helicopter noise, ducted fans, noise from propellers, and subsonic and supersonic aerodynamics. He was also interested in more general mathematical areas such as non-standard analysis, differential geometry, topology, and generalized functions, as just a few select examples. These varied interests are readily apparent by examining the work within this document.

Dr. Farassat published approximately 150 papers, many in prestigious journals, and a great number of them are single author. He would often give away first authorship of many papers when working in conjunction with students or colleagues, and he was known to inspire them to work on very difficult problems. Undoubtedly, he was the main contributor of the majority of these joint research efforts. The majority of these publications are freely available to the public on the NASA Technical Reports Server.

Dr. Farassat, during his mid to late-career, was undoubtedly the theoretical backbone of the Aeroacoustics Branch at NASA Langley Research Center and perhaps across the entire organization. He had influenced the technical direction of many researchers within both the Aeroacoustics Branch and NASA as a whole and had a considerable influence throughout the community, all of which are still being felt today.

Dr. Farassat was highly recognized by international organizations and by NASA. He received

[4]Farassat, F., 'Noise from High Speed Coaxial Interacting Jets,' Masters Thesis, Syracuse University, 1970.

[5]Supported by NASA Grant NGL-33-022-082.

[6]Professor Sears wrote an excellent auto-biography that discusses some of his interactions with Theodore von Karman and Dr. Farassat. Sears, W. R., 'Stories from a Twentieth-Century Life,' Unknown Publisher, 1993.

[7]Ffowcs Williams, J. E. and Hawkings, D. L., 'Sound Generation by Turbulence and Surfaces in Arbitrary Motion,' Royal Society Philosophical Transactions A, Vol. 264, No. 1151, 1969, pp. 321-342. DOI: 10.1098/rsta.1969.0031

[8]Farassat, F., 'The Sound from Rigid Bodies in Arbitrary Motion,' Ph.D. Dissertation, Cornell University, 1973.

the NASA Exceptional Scientific Achievement Medal in 1987 and 1991. Dr. Farassat was a fellow of multiple prestigious technical societies including the American Institute of Aeronautics and Astronautics and the American Helicopter Society. The NASA H. J. E. Reid Award was given to Dr. Farassat in 1980.[9] He also received the American Institute of Aeronautics and Astronautics Aeroacoustics Award in 1996.

Dr. Farassat enjoyed life outside of his research. He loved his family, friends, colleagues, hobbies and he possessed an unwavering love for those around him. Dr. Farassat was very proud of his heritage and often shared stories about Iran at work, with friends, and family. He loved Iranian people, food, culture, history, and had an equal love for America. He was an avid cook, and he cooked for guests of NASA Langley and for his family. His wonderful culinary creations were shared at many NASA acoustics picnics and holiday parties. Dishes often consisted of traditional Iranian cooking or something more experimental. He had a great interest in gardening that likely grew out of his culinary skills and often spoke of focusing on gardening during potential retirement.

Dr. Farassat was a lover of books and possessed a considerable collection of volumes. His books were drawn from the NASA Technical Library and through traditional stores. In fact, at one point every book that NASA Langley purchased for the Technical Library was first reviewed by Dr. Farassat before it was shelved within the collection. Dr. Farassat had an incredible love for all mathematics and certainly explored almost all areas of mathematics to some degree. Perhaps beyond all other interests, Dr. Farassat enjoyed helping those who were less fortunate and was involved in volunteer efforts within the community. For example, he consistently volunteered at soup kitchens within Hampton Roads and particularly with the Salvation Army and with the St. Andrew Presbyterian Church.

Dr. Farassat, on July 9^{th}, 2011, passed away due to complications from cancer while surrounded by his family at his home in Hampton, Virginia. At the time of his death he was a Senior Technologist[10] at NASA. He was admired by his family, friends, colleagues, and many others. Dr. Farassat was an unwavering advocate within NASA for the importance of research and was extremely supportive of junior researchers. He helped those in need and always made time to listen. Certainly, he had an extremely positive impact on everyone who knew him.

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- Hardin, C., 'Some Elegant Derivations Employing Generalized Functions,' International Journal of Aeroacoustics, Vol. 14, No. 1-2, 2015, pp. 353-358. DOI: 10.1260/1475-472X.14.1-2.353

[9] H. J. E. Reid Award was given to Dr. Farassat for his paper, Farassat, F., 'Theory of Noise Generation for Moving Bodies with an Application to Helicopter Rotors,' NASA TR-R451, 1980.

[10] Senior Technologist and Senior Theoretical Aeroacoustician, which is the highest purely technical position achievable at NASA.

- Obituary published in Daily Press from July 12^{th} to July 13^{th}, 2011.
- Personal discussions with Michael Myers, Williamsburg, Virginia, November 2015.
- Personal discussions with Mark Dunn, Hampton, Virginia, November 2015.
- Personal discussions with Fereidoun Farassat, NASA Langley, 2008-2011.

1.2 Process of Preparation

This section discusses the process of preparing this document. A large number of belongings were left behind by Dr. Farassat that included books, journal articles, notes, derivations, and countless other items during his career at NASA. These belongings were placed by the members of the NASA Langley Aeroacoustics Branch and the NASA Langley Technical Library staff into sets of labeled boxes located at the Acoustics Research Lab and the NASA Langley Technical Library, respectively. Eighteen catalogued boxes reside at the NASA Acoustics Research Lab pictured in Figure 2, and twenty-one boxes reside at the NASA Langley Technical Library pictured in Figure 3. A number of important articles reside with the family of Dr. Farassat. Those of a technical nature were provided for the purpose of creating this document. In particular, Dr. Farassat's personal green books were provided to NASA for the purpose of publication and appear here. Dr. Farassat had a large number of international and local collaborators who possessed some of his writings, class notes, and derivations. These collaborators kindly made additional material available for publication. It is estimated that 45 boxes of research material were recovered, in addition to many loose notes and green books.

Once all the articles of Dr. Farassat were collected, they were meticulously searched, and items were digitized for publication. The originals were then returned to their sources, with the majority residing at the NASA Aeroacoustics Branch and NASA Langley Technical Library. The digitized documents were then organized by technical category or as a class. Particular digitized documents were then removed as they were identical, as some sources provided identical derivations or class notes. Also, some sources provided portions of a series of derivations or portions of a class, and these portions were combined to produce a complete derivation or class.

The importance of the green books cannot be understated, as they contain the majority of his original mathematical derivations. These green books are all represented within this publication. Great care has been taken to make them fully available to the reader. Many classes were taught by Dr. Farassat, and most of the corresponding notes and presentation slides are present. A number of unbound handwritten notes are also included.

Journal publications, conference proceedings, or presentations that are publicly available are not included in this document, as they are already available to the reader. A comprehensive list of these journal publications, conference proceedings, and presentations are included at the end of this document. Also, select notes of Dr. Farassat that were used to learn well-known fields of mathematics or physics are not included, as they have been established previously within technical communities. Finally, some notes containing highly personal comments were removed and an editorial comment written at their locations.

It is a challenge to present such a large and varied research material in a logical order. An obvious approach is to arrange the material in chronological order; however, given that many technical approaches and classes were developed intermittently, this would create a confusing publication. Instead, we adopt the approach of presenting material as one would in a technical book used for learning or teaching. The classes on the basics of acoustics, applied acoustics, aeroacoustics, etc. are presented first to give those less familiar with the subject the necessary background for its understanding. Then, the series of green books are presented in their order of importance, as they contain more intricate ideas based upon concepts taught in the classes. Finally, important unbound

notes are presented. With this approach, the arrangement of the material is similar to a graduate textbook on advanced mathematics, fluid dynamics, and acoustics.

Figure 2. Eighteen boxes of the collected articles of Dr. Farassat located at the NASA Langley Acoustics Research Lab.

Figure 3. Twenty-one boxes of the collected articles of Dr. Farassat located at the NASA Langley Technical Library. Access was courtesy of the library staff.

2 Summaries of Mathematical Derivations, Notes, and Courses

This section contains concise summaries of Dr. Farassat's various notebooks, unbound mathematical derivations, and courses. An effort was made to create summaries that are technically

descriptive and accessible to a wide audience. Each subsection contains an overall description of the corresponding content. Then, a description of the content on a page by page basis is given.

Dr. Farassat regularly wrote page numbers within his bound notebooks. We reference the page numbers of his notes within these summaries. For each summary section, page numbers are abbreviated by the acronym of the section title. Also, we reference page numbers that appear on the bottom of each page to aid the reader in finding the sections and corresponding pages labeled by Dr. Farassat. These two page numbering schemes are used concurrently. For example, for the section summarizing 'The Green's Function Short Course' (GFSC), we might write pages 329-330, which refers to the numbers at the bottom of each page of this document. Alternatively, we might write GFSC pages 3-4, which refers to pages of the GFSC shown on pages 329-330.

A Very Basic Course in Acoustics (Pages 31 - 76)

Dr. Farassat often presented lectures and courses to researchers of NASA and to students residing at universities. The most introductory course is titled, 'A Very Basic Course in Acoustics,' (AVBCA) and contains five main lectures. These slides and notes were originally developed as a short course for civil servants and contractors with no prerequisite knowledge of acoustics.

The purpose of the first lecture is to present basic definitions of linear acoustics with emphasis on acoustic amplitude, frequency, speed of sound, wave length, period, wavenumber, and types of signals. Dr. Farassat uses the word 'science' to describe the field and describes acoustics as the study of small perturbations of quantities in a material (e.g., air, water, plasma). Concepts are presented based on a planar wave of discrete frequency until page 38, where broadband waves are introduced through the summation of multiple sinusoidal waves. The concept of Fourier decomposition is examined without going into detail, as he would in more advanced courses. The first lecture ends with the definition of the decibel among other descriptive factors such as loudness.

AVBCA lecture two starts on page 44 and focuses on plane and spherical waves. Some amount of review material is presented from lecture one. An example is presented that shows how little energy is contained within acoustic waves. This is an important point as the energy within an acoustic wave is generally many orders of magnitude smaller than that contained in the prevailing flow-field. AVBCA lecture three (pages 54-61) introduces the field of psychoacoustics, discusses biology of the ear, critical octave and one-third octave bands, loudness, age related effects, subjective statistics of acoustics, and perceived noise level.

Lecture four, starting on page 62, discusses the basics of superposition and constructive interference, how waves stand in ducts, beating phenomenon, plane waves, reflections, spherical waves, diffraction, scattering, and refraction by flow gradients. The fifth and final lecture of AVBCA starts on page 71, which focuses on microphones and noise measurements.

Applied Acoustics (Pages 77 - 158)

Dr. Farassat presented another excellent course in acoustics called, 'Applied Acoustics,' (AA). The class is targeted towards practicing engineers at NASA and for students within the university classroom. Its purpose is to introduce acoustics at a more technical level than the introduction to acoustics class previously discussed. The prerequisite is likely a junior or senior level undergraduate engineering education, as the class is meant for those outside the field to quickly become familiar with acoustics. The course is divided into ten lectures.

The first lecture (pages 77-83) focuses on basic governing equations of acoustics. Fundamental concepts are introduced such as speed of sound within gases, wavelength, frequency, inviscid equations of motion, and derivation of the wave equation. Note that viscosity is not included in

the governing equations and that a perfect gas is assumed. Select canonical equations such as the Helmholtz equation and solutions are sought. Lecture one closes with the introduction of the decibel scale and sound power.

The second lecture (pages 84-92) explains the use of complex numbers in acoustics, steady state conditions, the advantages of the time versus frequency domain, and energy relations. Multiple types of waves (plane, spherical, evanescent) are used as examples to illustrate the developed material. AA lecture three, on pages 92-98, reviews plane waves that were presented in the second lecture and expands on previously developed results. Particle displacement as a function of wave intensity and the steady state assumption are explored for planar waves. The same relations are found for spherical waves, and the concept of the monopole is introduced.

The monopole is explored further in the fourth lecture (pages 98-101), and it is shown that the mathematical description of the monopole is also suitable for the solution of the wave equation with a monopole source. Integral solutions of the wave equations are obtained. Rayleigh's piston in a wall is examined, and integral solutions are derived that include the velocity field. Lecture four ends with the introduction of the dipole source. Lecture five (pages 101-105) explores the idea of compact and non-compact sources. An in-depth presentation of the concepts of wave kinematics is given. Of particular interest is the exploration of stationary and non-stationary compact sources, which are often a source of some confusion for many practicing acousticians. The lecture closes with an introduction to the compactness condition for a convecting monopole.

Lecture six (pages 105-116) is very technical and introduces the concept of noise from moving bodies through presentation of the Ffowcs-Williams Hawkings (FW-H) equation, which includes permeable data surfaces. The FW-H equation is presented in only two slides.[11] A number of other important concepts are introduced and related to the FW-H equation, including the Lowson formula, Curle formula, Rayleigh formula, and Green's function in an unbounded domain. These concepts are explained in the context of the theory of generalized functions. Finally, lecture six closes with the introduction of the theory of the Green's function.

Lecture seven, on pages 117-122, returns to more traditional acoustics topics and discusses the sound from a moving dipole using Lowson's formula. Gutin's result is then discussed, which focuses on the sound from steady rotating forces such as those from a propeller. The lecture closes with a discussion of Kirchoff's formula. Lecture eight (pages 123-130) continues with the development of the theory of noise generation from moving sources. The acoustic analogy of Lighthill is introduced. Some important points are made regarding Lighthill's approach in preparation for the derivation of the FW-H equation. The FW-H equation is derived, and its terms, consisting of thickness, loading, and quadrupole, are explained. Many of the intermediate steps are omitted given the prerequisite of the audience.

AA lecture nine (pages 130-141) focuses on the solution of the FW-H equation, which was developed within the previous lecture. The solution is shown for the loading and thickness source terms and placed within the context of Dr. Farassat's Formulation 1A. A number of the 'tricks of the trade' are presented with regard to evaluation of Formulation 1A. Based on the newly derived solution, Rayleigh's piston, Curle's formula, Kirchoff's formula, moving sources, Lowson's formula, and Succi's formula are revisited. The final lecture, starting on page 141, introduces perturbation theory and applies it to the governing equations within the context of duct acoustics. These newly developed equations are used to explain the phenomenon of acoustic waves traveling through engine nacelles. The course ends with a discussion on concepts of duct acoustics.

[11] One can imagine the large amount of discussion that occurred to explain this canonical result in aeroacoustics.

Aeroacoustics Lectures (Pages 159 - 257)

Dr. Farassat continually lectured throughout his career, and one excellent course he presented was on aeroacoustics. The Aeroacoustics Lectures (AL) notes are assembled from multiple sources into a coherent single class based on lectures conducted in 2000 and 2010. The AL notes are divided into nine lectures that were presented in 2000 and 2010 and four additional preliminary lectures that were presented in 2010. The latter lectures, as they are more introductory and meant for NASA Langley summer interns and relatively new employees within acoustics, are positioned after the first nine main lectures. The last four lectures contain select redundant material relative to the first nine lectures and are viewed as prerequisite material. In total, there are approximately 168 pages of material. These lectures are suitable for the basis of a graduate or senior level undergraduate course in aeroacoustics. The course generally assumes a working knowledge of differential equations and a portion of a technical undergraduate education. Lecture number nine combines all the material presented within the previous eight lectures into a single unified theory consisting of the solutions of the FW-H equation.

The first lecture, starting on page 159, introduces the students to the concepts of the speed, wavelength, and frequency of sound on AL pages 1/1-1/2. Common variables of aeroacoustics are defined. The governing equations are derived on AL pages 1/3-1/8 using these basic concepts. Resulting differential equations include simple relations for density with pressure, mass conservation, momentum conservation (without viscosity), the wave equation and operator, and the Helmholtz equation. A very basic result of the use of complex numbers in acoustics is presented using the developed equations on AL page 1/9. On AL page 1/10 the basics of the sound pressure level scale in decibels is introduced. Contours of loudness within a 'line form' are presented on AL page 1/11. The first lecture ends on AL page 1/12 with a table of the sound power range and a discussion of the scaling of subjective effects due to changes in sound power.

Lecture 2 begins on page 165 by discussing the linearity of acoustics and the advantages of treating problems in a linear fashion (AL pages 2/1-2/2). Rayleigh's complex amplitudes are introduced for the field variables of pressure and velocity. The concept of steady state is examined (introduced in AL lecture 1) on AL page 2/3 and extended with Fourier series representation of a signal. Differences between time and frequency domain solution approaches are discussed on AL page 2/4. Their relation is shown with use of the Fourier transform, and their advantages and disadvantages are discussed. The energy equation was not discussed within AL lecture 1, and here, on AL pages 2/7-2/9 it is introduced through modification of the momentum equation via the dot product of the velocity. The acoustic intensity is derived and related to the acoustic energy density through a differential equation. This equation is examined with the use of a volumetric (integral) approach and some energy relations (steady state) are derived on AL page 2/9. The second lecture ends by discussing various simplified models of waves (AL page 2/10). Plane waves are introduced first, and the acoustic energy density and intensity are derived within a long duct. Evanescent plane waves are described, and an example is presented for an evanescent plane wave originating from a vibrating plane (AL page 2/13). AL pages 2/15-2/16 introduce the phase velocity and trace velocity of plane waves. AL lecture 2 closes on page 2/17, where the wavelength and wavenumber that were introduced in AL lecture 1 are derived for the plane wave.

Lecture three, starting on page 173, continues developing the simplified theory of waves on AL pages 3/1-3/3. Plane waves defined in the time domain are introduced and derived from the momentum equation concepts of the time dependent intensity and particle displacement. The concept of acoustic pressure using the theory of steady state acoustics is shown on AL page 3/4 and introduces summation of acoustic pressure. The summation approach uses the concept of linearity introduced earlier. Some basic rules for finding mean square acoustic pressure are summarized on

AL page 3/5. Another model is introduced for simplified waves. Here, spherically symmetric waves are introduced on AL pages 3/6-3/12 with the use of the three-dimensional wave equation discussed in AL lecture 1. A number of characteristics are derived including intensity and phase angles between velocity and pressure. Important differences between plane and spherically spreading waves are noted. A boundary condition that is analogous to a pulsating sphere is introduced on AL page 3/10, which is used to explain the concept of the monopole and the pulsating monopole (AL pages 3/10-3/11). AL lecture 3 concludes by summarizing wavenumber relations for the steady monopole.

AL lecture 4 is relatively short (pages 179-182 and AL pages 4/1-4/6) and focuses on defining acoustic sources. The first acoustic source is the monopole that was introduced in AL lecture 3. A Rayleigh piston is defined that consists of a normal velocity distribution on an infinite wall on AL page 4/2. The velocity potential of Rayleigh's piston is derived on AL pages 4/3-4/4. The dipole is introduced through a physical argument, and the pressure from the dipole is derived (AL pages 4/5-4/6). The near-field and far-field pressure terms are shown to be independent terms within the solution.

The fifth lecture of AL starts on page 182 and discusses the idea of compact sources by using the compactness condition (AL page 5/1). Frequency and wavenumber relations for observers in motion and in wind tunnel coordinates are introduced for plane waves on AL pages 5/2-5/3. On the next two pages (AL pages 5/4-5/5) the same relations are introduced for the point source. The collapsing sphere approach is introduced for the solution of the wave equation on AL page 5/7. Some further notes on the compactness condition, but now in the context of a moving source, are shown on AL page 5/8. This compactness condition is placed within the context of the collapsing sphere solution and closes AL lecture 5.

The sixth lecture of the AL, starting on page 186, focuses on acoustic radiation from moving bodies and generalized function theory. It is noted on AL page 6/1 that generally two approaches are used for noise prediction for moving bodies: computational fluid dynamics[12] and acoustic analogies. The second approach is adopted, and the FW-H equation is introduced on AL page 6/2. Lowson's formula is introduced on AL page 6/3, and the Green's function is also introduced here. The focus of the lecture shifts towards the introduction of generalized functions on AL pages 6/4-6/16. An overview of the strengths of using generalized functions is presented, and then various functions are defined, such as the Dirac delta and Heaviside functions. One very important point is emphasized: that governing equations are valid if the derivatives are represented as generalized derivatives. The remaining portion of AL lecture 6 (AL pages 6/16-6/23) are handouts and examples related to the Green's function and wave equation.

AL lecture 7 starts on page 198 (AL pages 7/1-7/7) and focuses on noise generation from moving sources. Lowson's formula is reexamined in more detail on AL pages 7/1-7/5. Gutin's result is introduced on AL pages 7/6-7/7, which describes the noise from steady rotating forces (a propeller). A number of handouts are included with AL lecture 7 that focus on the Kirchhoff equation.

The eighth AL lecture (pages 204-211 and AL pages 8/1-8/12) completes the discussion from AL lecture 7 on the noise generation from moving sources. The method of descent is discussed for the solution of the wave equation in two-dimensions. The acoustic analogy is introduced on AL pages 8/3-8/6 within the context of Lighthill's acoustic analogy and uses a jet flow as an example source. The intensity of the noise from a jet is derived, and the spectral density is given as an integral involving the two-point cross-correlation of the turbulence (among other flow quantities). AL pages 8/7-8/10 discuss the FW-H equation through its derivation based upon the continuity and momentum equations with a generalized function source. The Kirchhoff equation is discussed

[12] Computational aeroacoustics.

within this context. A permeable surface formulation is found on AL page 8/10. A discussion of the implications and interpretation of the permeable FW-H equation is on AL pages 8/11-8/12. AL lecture 8 finishes with a number of handouts that focus on the FW-H equation, Green's functions for discontinuous solutions, and where the Dirac delta function appears in applications.

The ninth lecture is the last of the series, and it continues to discuss the FW-H equation and its solution (pages 212-222 and AL pages 9/1-9/22). It uses almost all the results from the previous eight lectures and places them on a sound theoretical basis within the FW-H equation. AL pages 9/1-9/7 discuss the thickness and loading terms of the FW-H equation, the solution of the FW-H equation in terms of thickness and loading noise sources, moving the partial time derivative within the integral of the solution, and a summary of the traditional solution. Dr. Farassat is particularly famous for his Formulation 1A, which is one solution of the FW-H equation that is presented on AL page 9/8. Implications of the solution of the FW-H equation and its interpretation, along with some 'tricks of the trade,' are discussed on AL pages 9/9-9/11. An alternative method of writing Formulation 1A is also included for the students. A number of classical solutions introduced in previous lectures are now discussed in the context of the FW-H equation and solution. Rayleigh's piston within the wall is revisited with the FW-H equation solution on AL page 9/12. Curle's formula for flow over a stationary surface is found using the solution of the FW-H equation on AL page 9/13. Kirchhoff's formula for moving surfaces is placed in the context of the FW-H equation on AL page 9/14. Moving observers are addressed on AL page 9/16. Lowson's formula is examined on AL pages 9/17-9/18 using the FW-H solution. The results of using Succi's thickness noise and Isom's thickness noise theories are also derived in the context of the FW-H solution on AL pages 9/19-9/20. The final portion of AL lecture 9 on AL pages 9/21-9/22 makes some conjecture regarding the volumetric term of Dr. Farassat's solution of the FW-H equation. These are addressed in Dr. Farassat's Formulation 4 (see Green Book 2).

During the summer of 2010 a number of introductory notes were presented for students supported by NASA internships and new researchers of NASA Langley. These lectures consisted of four parts, which were presented before the nine AL lectures. These begin on page 223. The first lecture on AL pages 1-8 (after the end of AL lecture 9) discusses the one-dimensional wave equation and its solution. The wave equation in two and three-dimensions is then introduced. Poisson's solution and Huygens' principle are discussed on AL page 5. A few minor corrections of previous derivations are shown, and the uniqueness theorem of the wave equation in one, two, and three-dimensions is presented to close the first lecture.

The second lecture (page 232 and AL prerequisite lecture 2 pages 1-6) discusses various examples for using the solution of the wave equation in aeroacoustics. An alternative solution of the wave equation is discussed on AL prerequisite lecture 2 pages 4-6. The third lecture (page 238 AL prerequisite lecture 3 pages 1-14) has more substance than the second. It begins by describing a strategy to solve the wave equation in two-dimensions. Differences between the solution of the wave equation in two-dimensions and three-dimensions are discussed. The concept of sources in motion is defined on AL prerequisite lecture 3 page 4. Lowson's formula and its solution are introduced. Compact sources are defined and discussed. The final AL lecture, on AL prerequisite lecture 4 pages 1-6, discusses Garrick's triangle, the zone of silence, and point sources and observers in rectilinear motion.

Lectures on the Aeroacoustics of Rotating Blades in Time Domain (Pages 258 - 327)

A number of lectures were presented on 'The Aeroacoustics of Rotating Blades in the Time Domain,' (LARBTD) by Dr. Farassat. The lectures were given in three parts and are approximately 71 pages

in length. Dr. Farassat's lectures often exceeded multiple sessions beyond the planned parts due to the large amount of developmental work performed on the chalk board. These lectures were originally delivered at the Department of Mechanics and Aeronautics at the University of Rome in 1989. Much like the lectures on aeroacoustics, the final lecture seamlessly combines large amounts of material that were developed throughout the course.

The first lecture discusses the history of propeller noise and its prediction on pages 259 through 262 (LARBTD pages 1/1 - 1/3). The historical focus is on major prediction developments and NASA's involvement. Different sources of noise from rotating blades are defined (LARBTD pages 1/4 - 1/7) in the context of a propeller or rotors of a helicopter. It is argued that prediction theories must account for the thickness noise, loading (steady, periodic, impulsive, random, etc.) noise, and quadrupole noise (shocks, vortices, turbulence, etc.). Due to the difficult nature of the problem, the focus of the lectures is on discrete frequency noise prediction, but it is acknowledged that the effects of the broadband levels, inclusion of the nacelle or fuselage, and propagation should be included. The first lecture closes by describing several prediction approaches, what the prediction approaches need to include, and the benefits of frequency or time domain formulations (LARBTD pages 1/9-1/10).

The second lecture introduces the FW-H equation (LARBTD pages 2/1-2/2), and particular care is taken to define the variables. Within the remainder of the second lecture, various mathematical concepts are introduced to the students in preparation for solving the FW-H equation in lecture 3. These mathematical concepts include the introduction of generalized functions (LARBTD page 2/3) and a number of examples, divergence theorem with generalized functions, changing the order of operations of limiting processes (LARBTD page 2/10), two applications of generalized derivatives (LARBTD pages 2/12), support of functions involving the Dirac delta function (LARBTD page 2/15), results of integration of surfaces involving the Dirac delta function (LARBTD page 2/17), a short discussion regarding the derivation of the FW-H equation (LARBTD page 2/18), and basic differential geometry results (LARBTD page 2/21).

The third and final lecture, starting on page 293, is focused on finding and interpreting solutions of the FW-H equation. The first part of the lecture (LARBTD pages 3/1-3/6) considers two approaches for finding solutions, as illustrated through the use of two canonical inhomogeneous wave equations. Two forms of the solution of an inhomogeneous wave equation are found on LARBTD pages 3/2 and 3/3. Concepts of the gamma (Γ) and sigma (Σ) surfaces are introduced on LARBTD page 3/4. Forms of solutions of the wave equation with right hand side '$Q\delta$' are shown on LARBTD pages 3/8, 3/12, and 3/14. Based on these basic forms of solutions of the wave equation with carefully selected sources, solutions of the FW-H equation are sought under certain conditions.

The first solution discussed within the third lecture is based upon the assumption of compact sources (LARBTD 3/15-3/17). Explicit solutions are derived for the acoustic pressure from the thickness and loading source terms. The solution is discussed in the context of Succi's formula (LARBTD page 3/17) and Lowson's formula (LARBTD page 3/18) is recovered.

A second solution is derived for the situation where a solid body is rotating subsonically with a shock-wave attached (LARBTD pages 3/19-3/23). The approach begins by examining the quadrupole term and placing it in a form that captures the noise from the shock wave. The modified FW-H equation is presented on LARBTD page 3/20, and each term is explained. A closed-form integral solution is derived on LARBTD page 3/23, and all of the arguments are analytical. Evaluation for cases when terms are singular requires special care.

The final portion of lecture three (LARBTD pages 3/24-3/35) discusses the solution of the FW-H equation for the case of rotating bodies of motion that have attached shock waves. A number of complex mathematical operations are performed to obtain the FW-H equation in an alternative

form shown on LARBTD page 3/28, where the terms are interpreted in detail. The right hand side of the FW-H equation now contains source terms that were examined previously and also some new source terms. Two forms of the FW-H equation are identified for solution that were not examined within the previous parts of LARBTD. Three solutions result as shown on LARBTD pages 3/33-3/35, and together with previously developed forms, the full solution of the FW-H equation for the noise from supersonic rotating bodies with attached shock waves can be constructed.

The Green's Function (Course) (Pages 328 - 411)

A short course on Green's functions (Green's Function Short Course, GFSC) was prepared by Dr. Farassat. The course uses the traditional note format as opposed to presentation slides. The length of the course is approximately 90 handwritten pages. These notes could be used as a general introduction to the mathematical field of Green's functions. As a prerequisite, one should have knowledge of partial differential equations and some knowledge of the canonical differential equations of physics. The course highlights include the definition and development of the Green's function, linear operators, adjoint Green's functions, the wave equation, the Helmholtz equation, the Laplace equation, and the uniqueness theorem. The closing of the course contains an important discussion of the application of Green's function theory to the field of aeroacoustics.

The Green's function notes (page 329 and GFSC pages 3-6)[13] start with an examination of the linear ordinary differential equation operator and define the Green's function. A basic example of using the Green's function is shown for a second-order ordinary differential equation. The adjoint operator is introduced with the use of examples on GFSC pages 7-10. The definitions of the adjoint operator and self-adjoint operator are presented. Existence of the Green's function is investigated. A note on self-adjoint operators of the Green's function is presented on GFSC page 11. It is illustrated through the use of the complex conjugate of the operator and corresponding Green's function. A method is presented on GFSC pages 12-13 to quickly check if the Green's function is self-adjoint for homogeneous second-order ordinary differential equations. A few simple examples are presented, and the question is posed regarding how to conduct the same analysis if the ordinary differential equation boundary conditions are inhomogeneous. GFSC pages 13-17 answer this question by showing two different methods.

The general second-order linear ordinary differential equation is again considered, but now generalized derivatives are used (GFSC pages 18-26).[14] Adjoint linear differential operators are examined on GFSC pages 27-30. An important characteristic of functions defined in three-dimensional space used for the solutions of the FW-H equation is presented. A few other examples and informal proofs of properties of Green's functions are given.

Dr. Farassat next presents an example using the previously developed theory of Green's functions. The Green's function of the Laplace equation is sought on GFSC pages 31-34. Details of the Dirac delta function are presented, and the spherical coordinate system is defined. The Dirac delta function is written in the spherical coordinate system, and the solution of Laplace's equation is derived on GFSC page 34. Using the previously developed Green's function for the Laplace equation in unbounded space, the solution of the Laplace equation is derived (GFSC pages 35-36). The general solution is shown to satisfy the Laplace equation. Before finding additional Green's functions, the partial differential equation invariant under translation of space and time (about the origins) is presented on GFSC pages 37-38. Invariants are useful to understand in the proceeding derivations of Green's functions that correspond to various partial differential equations.

[13] GFSC pages 1 and 2 are blank.

[14] Page 24 is blank and some pages contain redundant numbers.

The Green's function of the Helmholtz equation within an unbounded three-dimensional space (GFSC pages 39-41) is derived. This is a logical partial differential equation to examine because of its similarity to the Laplace equation, which was previously examined. The Green's function is derived by assuming a solution that is similar to the Green's function of the Laplace equation. It is curious that a complex solution is derived, and this is addressed.

The Green's function of the wave equation in three-dimensional space is derived on GFSC pages 42-43. Here, the Green's function is written with arguments containing both the observer and source, spacial and temporal positions, which are characteristically used in the field of acoustics. The Green's function is derived by making use of the Fourier transform and using the Green's function derived previously for the Helmholtz equation on GFSC pages 39-41. The retarded time solution of the wave equation is then derived in the time-domain on GFSC page 43.

The heat equation is another canonical partial differential equation, and the Green's function of the heat equation operator is derived on GFSC pages 44-47. The heat kernel, that is the Green's function of the heat equation, is derived in one-dimension through three-dimensions. Solutions using the heat kernel are presented.

Focus is returned to the Laplace equation (GFSC page 48), and the Green's function of the Laplace equation is examined in higher dimensions than three within an unbounded domain. The result is shown without derivation. The Green's function of the Laplace equation in a bounded domain is derived on GFSC pages 49-51. GFSC pages 52-59 show the solution of the Laplace equation in a bounded domain in two-dimensions, and some pages are omitted as they are simple plots of the solution. GFSC pages 60-65 explore the uniqueness theorem for the Laplacian operator within bounded domains. Proofs are presented for the uniqueness theorem for the Direchlet and Neumann boundary conditions. Pages 66-67 of GFSC summarize Green's theorem.

The Green's function of the heat equation in an unbounded domain was found on GFSC pages 44-47, and here on GFSC pages 68-72, the Green's function is derived within a bounded domain. Page 73 of GFSC contains a note on the Green's function of the wave equation. Here, an alternative approach is presented for deriving the Green's function relative to that presented on GFSC pages 42-43.

At this point, the basics of Green's functions have been presented, and on GFSC pages 74-79 some notes are presented on their application. In particular, on GFSC pages 78-79 their application to the Kirchoff formula is discussed. The final entry on GFSC pages 80-85 discusses the uniqueness theorem of the Green's function for the wave equation in a bounded domain.

An appendix is included on GFSC pages A1-A5 that shows a general method for finding adjoint boundary conditions for second-order linear ordinary differential equations. When working with adjoint equations, corresponding adjoint boundary conditions must also be used. The appendix closes with three recommended references for further study.

The Mathematics of Near Field Acoustical Holography (Pages 412 - 442)

Dr. Farassat taught a class at NASA Langley on 'The Mathematics of Near Field Acoustical Holography' (NAF) at the end of the summer of 2000. Acoustic holography is an inverse problem that attempts to characterize or map the source of acoustic radiation. NAF consists of seven main lectures plus an appendix that summarizes some analytical results. In total, there are sixty pages.

The class starts on NAF page 1/1 by defining it as an inverse problem. NAF involves the combination of mathematics, physics of acoustics, and experimental measurement. The first lecture continues on NAF pages 1/2-1/5 with the introduction of Fourier transforms. The Dirac delta function and corresponding Fourier transform are defined on NAF page 1/6. Within the second lecture, more definitions and development work are presented. The acoustic intensity, steady state

acoustics, wave equation, and decomposition are shown on NAF pages 2/1-2/7. Evanescent waves are reviewed on NAF pages 2/8-2/10.

The third lecture, starting on NAF page 3/1, introduces standing waves on an infinite wall or vibrating plane. These waves are interpreted in wavenumber space on NAF page 3/2. Velocities of particles within the fluid field are found on NAF page 3/3. Additional discussion of evanescent waves is given on NAF page 3/4. The so-called angular spectrum Fourier transform for acoustics is introduced on NAF pages 3/5-3/7.

The fourth lecture, on NAF pages 4/1-4/8, introduces some basic acoustic concepts such as the monopole and dipole. Rayleigh's first and second integrals are derived on NAF pages 4/3-4/8. The concept of the propagator, that is like a Green's function, is introduced in the fifth lecture on NAF page 5/1. Ewald's sphere construction is introduced on NAF pages 5/3-5/6. This approach allows for a method of visualizing the directivity of planar sources. Planar near field construction is examined on NAF pages 5/6-6/2. The remaining part of the sixth lecture discusses strategies to handle ill-posed problems.

The seventh NAF lecture examines traveling waves on square plates (NAF pages 7/1-7/2). Other plate radiation problems are examined, such as the radiation from a simply supported structure. Bouwkamp's result (NAF page 7/7) is found. Edge and corner modes are discussed on NAF page 7/8. The remaining part of the seventh lecture shows various figures and diagrams. The final seven pages show some analytical results for the prediction of trailing edge noise.

The Workshop on Kirchhoff Formulas (Pages 443 - 472)

Dr. Farassat hosted a workshop at NASA Langley Research Center on Kirchhoff formulas. The Workshop on Kirchhoff Formulas (WKF) was hosted in February of 1995. The workshop sought to convey to the participants the derivation of two Kirchhoff formulas for the noise radiation from subsonic and supersonic surfaces, respectively. The advantage over traditional approaches is a more direct and simple derivation which is subsequently easier to interpret. Also, the participants gained some understanding of generalized functions, partial differential equations, and differential geometry. The workshop contains 57 pages of carefully prepared handwritten slides that were also distributed to participants.

Introductory material is presented first on page 443. The workshop (WKF page 1) opens by discussing three possible methods for noise prediction within the field of aeroacoustics. They are acoustic analogies, Kirchhoff methods, and computational fluid dynamic-based methods (computational aeroacoustics). After discussing these three approaches, the classical Kirchhoff formula is introduced on WKF pages 2-3. Here, the form of the equation is compared with the Laplace equation. Dr. Farassat attempts to convince the workshop participants that the Kirchoff approach has potential benefits over the other approaches on WKF page 4. WKF page 5 states the purpose of the workshop, as previously mentioned, as the derivation of two Kirchhoff formulas for subsonic and supersonic surfaces. As a secondary goal, tools are developed for generalized functions, partial differential equations, and differential geometry.

The methodology for deriving the Kirchhoff formulas (WKF page 6) is discussed through reduction of solutions of the wave equation with select generalized function sources. WKF pages 7-8 discuss traditional and generalized functions through simple examples. A motivational slide (WKF page 9) for generalized functions with emphasis on the so-called sifting property is presented based upon the Dirac delta function. Generalized functions are defined on WKF pages 10-14. In particular, the concepts and examples of ordinary, continuous, regular, singular, and symbolic functions are defined. Particular operations on generalized functions are presented on WKF pages 15-16. Differentiation of generalized functions is discussed on WKF pages 17-18. Here, Dr. Farassat de-

scribes the general properties of derivatives applied to generalized functions using examples. Some important results of generalized function theory in the context of the WKF are presented on WKF page 19.

Our attention turns temporarily from generalized functions to the Green's function (WKF pages 20-21) of a second-order linear ordinary differential equation. Here, the definition of the Green's function is introduced through example and connected to the Dirac delta function. It is noted that the boundary conditions must also be posed correctly for the Green's function to be found. Focus is returned to generalized functions within multiple dimensions (WKF page 22). As the Dirac delta function and generalized functions can be perceived as very abstract to those not familiar with the subject, Dr. Farassat on WKF pages 23-24 shows some examples of how they appear in the system of equations for the Green's function and acoustics of shock waves.

The mathematical theory of differential geometry plays a key role in the work of Dr. Farassat. On WKF pages 25-32 differential geometry is introduced and many important results are shown that are used later in the workshop. Major results are presented without a great deal of mathematical derivation.

The integration and differentiation of the Dirac delta function are presented on WKF page 33. Integration of a function that is a product of the Dirac delta function is presented on WKF page 34. These concepts are important for evaluating Kirchhoff formulas. Some illustrations explain generalized function manipulation (WKF page 35-36).

Focus is returned to the theory of Green's functions on WKF pages 37-38. In particular, the Green's function of the wave equation is examined in unbounded space. Source-to-observer vectors are defined. The domain of dependence is illustrated. WKF page 39 gives an example for the Green's function of the Laplace equation and its relation to discontinuous solutions. Two forms of the solution of the wave equation in unbounded space with volumetric sources are presented on WKF page 40.

The governing wave equation for deriving Kirchhoff formulas is presented on WKF page 41. Here, the wave equation is chosen to have a very specific right hand side. The entire WKF up to this point has been carefully devised to present this equation and its set of solutions. The first solution sought is the classical Kirchhoff formula (WKF page 42). Perhaps the main point is the proposed governing wave equation can recover the classical result. The first major result of WKF on pages WKF 43-46 is the derivation of the so-called subsonic Kirchhoff formula. Here, a deformable moving surface that is restricted to subsonic motion creates acoustic radiation. The final form of the subsonic Kirchhoff formula is shown on WKF page 46. Dr. Farassat introduces a trick to simplify the formulation of the solution of the governing wave equation of WKF page 41 on pages WKF 47-48. Finally, the derivation and solution of the governing wave equation is presented on WKF pages 49-53 using the theories developed throughout the workshop. A note on the evaluation of the supersonic Kirchhoff formulation is made on WKF page 54, where it is emphasized that singularities might occur. Nonetheless, the result remains integratable.

The WKF closes with a number of references that were used for development of the workshop. Finally, Dr. Farassat acknowledges his collaboration with M. K. Myers of George Washington University.

Summary of Notebook One (Pages 473 - 655)

As mentioned previously, Dr. Farassat took excellent research notes, and many of these appear within his green books. The first entry appears in green book (GB1) on April of 1978 and the last entry appears on February of 1982. GB1 contains, as summarized below, an extremely varied research portfolio. Here, we gain insight into how an early NASA researcher explores mathematical

concepts that are eventually integrated into a coherent research methodology. It is difficult to concisely classify or summarize the material within GB1 due to its wide variety. With this difficulty in mind, GB1 could be generally summarized to contain introductory material, derivations of the solutions of the wave equation, integrals involving generalized functions, notes on nonlinear ordinary differential equations, derivatives of integrals and singular functions, and the thickness noise from wings. We now concisely describe the contents of GB1.

The first entry of GB1 on April of 1978 (GB1 pages 1-4) is a review of boundary layer turbulence pressure fluctuations based upon order of magnitude analysis. The same month a short initial review of propeller noise (GB1 pages 5-6) is conducted in the context of the theory of Gutin. A short overview of the statistical theory of isotropic turbulence is presented (GB1 pages 7-9), which discusses the result of Kolmogorov ($\kappa^{-5/3}$ energy spectrum in the inertial range where κ is wavenumber). Most basic results of the kinematic theory of gases is recorded on GB1 pages 10-11. Nonlinear acoustics of plane waves are explored using the approach of Riemann (GB1 pages 12-15), which are governed by the Euler equations. Select inequalities and other algebraic results[15] are summarized (GB1 pages 16-22) in a proof theorem format. Important theorems of partial differential equations (GB1 pages 23-26) are reviewed based on the work of Weinberger.[16] Of particular interest to Dr. Farassat is the maximum principle, which was often used within some of his lectures.

Dr. Farassat, who was undoubtedly an international expert on the theory of generalized functions, gives some concise proofs in GB1 pages 27-29. A large section is devoted to a review of difference (or differential) equations and finite differences (GB1 pages 30-51) in April of 1979. Most of the review represents the basis of differential equations until GB1 page 43, where a number of solution techniques are summarized and special classifications are noted. Mikusinski calculus, which is also known as operational calculus or operational analysis, is a mathematical approach to transform differential equations into an algebraic form involving a polynomial equation. Mikusinski's approach is reviewed by Dr. Farassat on GB1 pages 52-64 based upon Fenyo and Frey.[17] On GB1 page 59, a general polynomial is found (using the method of Feyno) and on GB1 page 62 an example is presented for the solution of a linear differential equation with constant coefficients. Dr. Farassat writes on GB1 page 64 that he, 'finds Mikusinski operational calculus as one of the most beautiful parts of modern mathematics.' Some brief notes are presented on number theory on GB1 pages 65-68, including a note on Fermat primes and coordinate changes on GB1 pages 69-71. Isoperimetric inequalities are summarized (GB1 pages 72-76), where the inequality involves the square of the perimeter and area within a plane. On GB1 pages 77-78, he proposes a proof to a problem given in The American Mathematical Monthly, 1978, page 496. Additional exploration of proofs of various inequalities appear on GB1 pages 79-83.

The first entry that contains a more direct application to acoustics appears on GB1 pages 84-86, where an identity is derived that has application for an acoustic source on a surface in motion. Dr. Farassat writes an interesting note at the bottom of GB1 page 86 regarding the inclusion (during numerical evaluation) of the blade tip, which he thought had a non-negligible contribution to the resulting acoustic pressure. In May of 1980, results from Dr. Farassat's study on conversion of spacial derivatives to temporal derivatives within solutions of the wave equation with application to sources on moving surfaces was completed (GB1 pages 87-91). He considers solutions of the wave equation with a sigma (Σ) surface. These notes were corrected at the later date of February 1982

[15] These are adapted from Crystal, G., 'Algebra: An Elementary Text-Book,' Adam and Charles Black, 1904.

[16] Weinberger, H. F., 'A First Course in Partial Differential Equations: with Complex Variables and Transform Methods,' Blaisdell Publishing Company, 1965.

[17] Fenyo, S. and Frey, T., 'Modern Mathematical Methods in Technology,' North-Holland Publishing Company, 1975.

on GB1 page 161.

The focus returns to more pure mathematical explorations of inequalities on GB1 page 92 and total differential equations on GB1 pages 93-99. Boundary conditions and their derivatives within a one-dimensional second-order inhomogeneous differential equation are explored (GB1 pages 100-108). It is argued that, under certain circumstances, the two boundary conditions (involving zeroth and first-order derivatives) can be related to one another.

From January through March of 1981, GB1 focuses on integrals and generalized functions. Two integrals involving the Dirac delta function and their evaluation (one from Dr. Farassat's Ph.D. thesis) are discussed on GB1 pages 109-114. The evaluation is much more compact relative to his Ph.D. thesis. He returned to this entry approximately ten years later in 1991 and illustrated an arguably better approach. Divergent integrals can appear upon taking a derivative within a convergent integral. Regularization of divergent integrals involves the use of the Heaviside function. GB1 pages 115-118 address divergent integrals with corrections noted on GB1 pages 119-120. A notable definition of the generalized derivative resides on the middle of GB1 page 120. There are some divergences from investigations of integrals on GB1 pages 112-114 on Mersenne primes[18] and notational discussions of partial derivatives.[19]

Solutions for first-order non-linear partial differential equations are examined (GB1 pages 121-124) using Charpit's method. The generalized Fourier transform of the natural logarithm of the absolute value of an argument is examined (GB1 pages 125-127). This approach draws on the definition of the regularization of the divergent integral developed previously on GB1 pages 115-118, and the result is shown on the bottom of GB1 page 126. In April of 1981, a philosophical entry on generalized functions appears on GB1 pages 128-139 that has application to discontinuous integrals. These represent more exploratory thoughts and mathematics and, relative to earlier entries, contains less review. Dr. Farassat presents multiple proofs (GB1 pages 140-141) of a problem proposed by Lass,[20] which involves the relation between two curves intersecting and their curvature. A number of examples are devised by Dr. Farassat (GB1 pages 142-151) to illustrate the use of variable transforms for the solution of partial differential equations. Geometric acoustics or ray theory and its assumptions are reviewed on GB1 pages 152-153 based upon the wave equation and the assumption that the wavelength is small. Technical readers might note the similarities of the results with those of classical optics. A number of theorems and proofs involving complex variables are presented on GB1 pages 154-157 that are based on the notes of MacRobert.[21]

Pages 158-160 of GB1 present an example of 'regularizing' an integral of the solution for the Laplace equation defined in the space above the $x - y$ plane. This basic example leads to the more complicated regularization of the divergent surface integral that represents the acoustic pressure from the aerodynamic pressure on the surface of a rotating blade (GB1 pages 161-170). These pages heavily use methods developed previously in GB1, such as overcoming the divergence of integrals when the time derivative is applied within the integrand. Note that some of the development and explanation reside in GB2.

The final set of technical notes within GB1 are on GB1 pages 171-177 and discuss what he calls the thickness problem for wings. Here, the intersection between aerodynamics and acoustics is not entirely physically intuitive. This is because the velocity potential is governed by an acoustic equation, specifically the wave equation. It is shown that the classical results found in aerodynamic theory, for subsonic and supersonic flow, correspond to those found with this purely acoustic theory. These results continue in GB2.

[18] Dr. Farassat's proof is incorrect for Mersenne primes as counter examples exist.
[19] A form adopted that involves the partial derivatives represented as a matrix.
[20] Lass, H., 'Vector and Tensor Analysis,' McGraw Hill, 1950.
[21] MacRobert, T. M., 'Functions of a Complex Variable,' MacMillan, 1954.

The final five pages (GB1 pages 5-1)[22] contain equations of motion and 'things to remember.' Throughout the other green books and unbounded notes these constants and units are considered universal unless stated otherwise.

Summary of Notebook Two (Pages 656 - 843)

The second notebook, green book two (GB2), is a direct continuation of the first (green book one, GB1). Entries within GB2 span from February 8^{th}, 1982 to May 8^{th}, 1998. Unlike GB1, Dr. Farassat's focus narrows within GB2 towards more specific research goals. Major sections consist of the examination of thickness noise, equations describing the noise and aerodynamics of moving bodies, a very large section on generalized function theory, development of some of his formulations, and much more. A large number of notes within the notebook focus on predicting noise from propellers moving supersonically. As one can imagine, this is a very difficult problem due to the shock waves, which are radiated from the supersonic moving body. The preface of GB2 begins with a drawing by Dr. Farassat's daughter, Daria. We now concisely describe the contents of GB2.

Examination of the thickness of wings and its implication on acoustics is continued from GB1 pages 171-177. Here, the potential flow is derived from an acoustics solution and compared with the results of Ashley and Landahl.[23] Dr. Farassat's solution is greatly simplified compared to the traditional approach. A new formulation of the solution of the FW-H equation is proposed for the prediction of noise from a propeller on GB2 pages 7-11. The thickness and loading noise terms are treated separately, the time derivative remains outside the surface integrals, and a frame of reference is used that rotates with the propeller. Sources on edges, that appear within solutions of the wave equation and involve Dirac delta functions, are examined on GB2 pages 12-15. The integrand involving multiple Dirac delta functions is examined in three and four dimensions and is written in more simplified forms on GB2 page 15. Missing terms in the integral equation for the acoustics from moving bodies are examined on GB2 pages 16-25, and this work is a continuation of GB1 pages 161-170. Here, the surface integral is evaluated at a small distance above the surface of the moving body, which is not the usual approach of Dr. Farassat, who normally evaluated integrals upon the moving surface. It is noted that his student, Lyle N. Long,[24] found an incompatibility with the previous approach that is corrected here.

One of the largest sections, GB2 pages 26-91, discusses generalized function theory and its application to a new prediction approach involving supersonic propeller noise. Integration of non-closed surfaces is discussed on GB2 pages 26-33. On GB2 pages 34-36, Dr. Farassat notes his efforts to overcome assumptions of his previously developed models (specification of pressure, steady surface pressures, and thin airfoil theory). His simplified approach for the prediction of supersonic propeller noise and associated aerodynamics is outlined in a number of steps starting on GB2 page 37. The basis is the FW-H equation, and within the first step the right hand side is written as a decomposition of two vectors: the first in the normal direction and the second in the tangential direction. On step 2 (GB2 page 40) the solution is solved directly with the Green's function. Step 3 (GB2 page 41) eliminates the derivative of the Dirac delta function. The near-field term is isolated (involving the inverse square of the propagation distance) on step 4 (GB2 pages 42-43). Now, the far-field term involves two main components that are isolated and simplified on step 5 (GB2 pages 43-45). Further simplification is performed on step 6, and some new notation is

[22] Note that the last five pages are in reverse order and labeled from 5, 4, 3, 2, and 1. This is likely due to the fact that the number of pages in the end of the book are unknown as they are created.

[23] Ashley, H. and Landahl, M., 'Aerodynamics of Wings and Bodies,' Dover Publications, 1985.

[24] Professor Long is now a professor at The Pennsylvania State University Department of Aerospace Engineering.

introduced (GB2 pages 45-53). The discontinuity of the trailing edge is addressed on step 7 (GB2 pages 54-55). The near-field and far-field terms are combined, and the delta functions within the integrands are examined on step 8 (GB2 pages 56-60). Step 9 revisits focus on evaluation of the terms resulting from the previous steps (GB2 pages 61-82). A number of notes (represented as step 10), simplifications, and methods for evaluation of the prediction equation are presented on GB2 pages 82-89. Finally, on GB2 pages 90-91, a summary of the final equation suited for numerical evaluation is presented, which includes the corrections noted within the previous steps.[25]

GB2 returns to more exploratory work on GB2 pages 92-99 with generalized functions. Comments are written on the theory of partial differential equations, their difficulty, and Dr. Farassat's view of the subject. Dr. Farassat reexamines the appearance of line integrals within his supersonic propeller prediction theory (GB2 pages 100-106) and confirms that they should be present. The spatial integration involving Delta prime is investigated on GB2 pages 107-109. Recall that the mathematical term Delta prime appeared within step 3 of GB2 page 41. The supersonic propeller noise theory previously developed on GB2 pages 26-91 is revisited on GB2 pages 110-119 after it was programmed by Sharon Padula. Dr. Farassat notes a missing term that involves a line integral along the tip of the propeller blade. On the bottom of GB2 page 91, it is noted that the equations were correctly derived, but this missing line integral greatly improved the agreement between prediction and measurement.

A general entry is made approximately eight years after the previous entry within GB2 as Dr. Farassat was working on loose paper and not making regular entries into his research notebooks. On GB2 pages 120-122 he summarizes his achievements over these years that include: creating the Advanced Subsonic and Supersonic Propeller Induced Noise computer program, lectures involving the FW-H equation, the application of the acoustic equations to traditional aerodynamic problems, examining the Kirchhoff theory for moving surfaces, examining the quadrupole term within the FW-H equation, and examining singularities within the solutions of acoustics from rotating blades.

Entries are continued on GB2 pages 123-129, and involve a discussion of the FW-H equation and its evaluation. Emphasis is placed on the thickness and loading terms. Examination of the solution for the wave equation with source terms that consist of generalized functions is explored on GB2 pages 130-140. Some additional related miscellaneous investigations are shown on GB2 pages 141-149. A second general entry appears in November of 1993 on GB2 page 150 and discusses the publication of a paper on the 80^{th} birthday of Professor William Sears, who was Dr. Farassat's advisor at Cornell.

Technical entries continue on GB2 pages 151-152 with a small note on normal vectors. A general entry is made on GB2 page 153 that discusses reviews of non-standard analysis, algebra, topology, and generalized functions. Some of these review notes are shown in other sections of this document. A few non-technical notes are written regarding personnel assignments with respect to evaluating developed theory. Thoughts on the development of Dr. Farassat's Formulation 3 are shown on GB2 pages 154-157. This approach eventually appeared in print in 1983. A large entry is created on the mean curvature of the sigma surface that is rigid (GB2 pages 158-173) and deformable (GB2 pages 174-180). This theory is required because the curvature of a surface is an argument in the theory of the noise from supersonic propellers. On GB2 page 181, Dr. Farassat notes that another researcher confirms his deformable surface theorem on GB2 pages 174-180 and that he has been recording some of his research in other notebooks.

The final entries of GB2, on GB2 pages 184-185, show some minor corrections to previous results on solutions of the wave equation containing a derivative Dirac delta function source. The

[25]It is noteworthy (bottom of GB2 page 91) that the final equation was written in February of 1983, and it took until June of 1991 for a computer program without error to be developed and validated to evaluate the final equation.

notebook ends on his 1998 vacation in Duck, North Carolina.

Summary of Notebook Three (Pages 844 - 876)

The third and final general notebook, green book 3 (GB3), follows the second but has many fewer entires. GB3 begins in June of 1998, and the last entry is in October of 1998. It contains 32 pages and is focused on four topics.

GB3 begins with further examination of the symbol of Christoffel, which represents arrays of real numbers and has applications in topology. Christoffel symbols are used in Farassat et al.[26] The Christoffel symbol actually appeared within GB2 in many discussions for the prediction of noise from supersonic propeller blades. Two major corrections of one of Dr. Farassat's AIAA papers are discussed on GB3 pages 4-8. Examinations of a classical approach and another approach using Formulation 4 for solving the wave equation are shown on GB3 pages 9-28. Computer aided symbolic mathematical software[27] is used, and the results are printed and copied to GB3. Further examination of edge length elements appears on GB3 page 28, which was developed in GB2. In the context of Formulation 4, the final three pages of GB3 (GB3 pages 29-31) discuss a singularity of $\sin\theta$ in the denominator, and the ability to handle higher order singularities is explored. It is shown that for a second power involving $\sin\theta$ that Formulation 4 remains integratable. The notebook closes on GB3 pages 31-32 by presenting an example solution for dipoles on a sphere or circle using Dr. Farassat's Formulation 4.

Ducted Fans (Pages 877 - 933)

Dr. Farassat's notebook on ducted fans (DF) is a comprehensive treatment of the fundamentals of predicting the noise from a rotor residing within a duct. The methods have application to aircraft engines where the fan creates very high intensity tones that are scattered or suppressed by the duct. Highlights of the DF notebook include the development of perturbation equations, acoustics within cylindrical ducts with a mean flow, graphical solutions, and rotating sources in a moving frame. The DF notebook consists of 56 pages.

The first part of the DF notebook on DF pages 1-4 discusses perturbation equations for the acoustic waves within a fluid mean flow. Each of the field variables of the governing equations is written as a mean and perturbation quantity. These are substituted into the governing equations of mass, momentum, and energy. Irrotational flow and the equation of state are also considered. Equations for the velocity potential are derived on DF pages 5-6.

The focus of DF pages 7-20 turns toward the acoustics of a cylindrical duct with uniform flow. The wavenumber of the acoustic wave is an important quantity within this analysis. Modal solutions of the system of equations are found, and the roots are analyzed. Roots are real or complex valued. The cut-off ratio is defined on DF page 9. Wavenumber is defined to be dependent on the circumferential mode on DF page 10. The pressure within the flow-field is derived and is in the form of a double summation on DF page 11. Attention is turned toward the analysis of circumferential modes on DF pages 13-16. The question of which modes are excited is then addressed through DF page 20.

DF pages 21-25 (starting on page 898) examine a graphical solution approach for the propagating modes in a duct with a uniform flow. The approach is used in two-dimensions and then extended

[26] Farassat, F., Brentner, K. S., and Dunn, M. H., 'A Study of Supersonic Surface Sources - The Ffowcs Williams-Hawkings Equation and the Kirchhoff Formula,' AIAA Paper 1998-2375, 1998.

[27] Dr. Farassat, an early proponent of using symbolic mathematical software for research, makes use of the software program Mathematica.

to cylindrical ducts. An example illustrates the two and three-dimensional approaches. A few miscellaneous notes are written on DF page 26 that include fan properties of several large turbofan engines for aircraft and a correction to DF pages 18-19. Some explanation is written by Dr. Farassat on DF pages 27-28 on how a rotating microphone[28] can be used to discern the acoustic mode within a duct. Additional data on engines with large ducted fans[29] are shown on DF page 29.

Rotating sources in a moving frame are addressed on DF pages 30-36 for the purpose of modeling rotating blades within a duct. The problem is modeled with the inhomogeneous wave equation with a periodic forcing function. DF pages 32-36 contain a number of pages copied from Kinsler et al.,[30] which discusses power radiation from pipes. Acoustic energy and intensity relations within moving media (DF pages 37-40) are reviewed based on the journal article of Myers.[31] A differential equation for the conservation of energy is derived. Analysis based on the review of Myers is continued through DF page 43, and a number of similar analyses are explored on DF pages 43-47. A note on data analysis from a circular microphone measurement is made on DF page 48. This theory is reviewed on DF pages 48-54. The final entry occurs on DF pages 55-56 and is a summary of the derivation of the energy equation for uniform flow with a perturbation velocity. It is noted that the mean velocity is divergence free and the result should apply to non-uniform mean flows.

Sound Propagation in a Duct and Interaction Tones (Pages 934 - 956)

Based upon the ducted fan notebook previously discussed, Dr. Farassat developed a class called Sound Propagation in a Duct and Interaction Tones (SPDIT). The class contains 43 slides that are directly based upon the notebook. The presentation of the material is very graphical compared to the notebook.

Lecture slides of SPDIT 1 through 34 follow the DF notebook very closely. Like the DF notebook, the slides examine the governing equations and perturbation theory, time independent uniform mean flow (SPDIT page 5), acoustics of cylindrical ducts with uniform flow (SPDIT pages 6-10), mode cut-off (SPDIT pages 11-14), mode cut-on (SPDIT page 15), graphical approaches (SPDIT pages 16-18), phase and group velocity (SPDIT pages 19-20), cut-off frequency (SPDIT pages 21-22), vector wavenumber (SPDIT page 23), annular ducts (SPDIT page 24), directivity of peak radiation given a mode (SPDIT page 25), modes in a duct with rotors and vanes (SPDIT pages 26-28), interaction modes (SPDIT pages 29-33), and how rotating microphones separate modes (SPDIT page 34). All these topics reside in detail in the DF notebook. An additional nine pages (SPDIT pages labeled 1-9 starting after SPDIT page 34) discuss spinning acoustic modes within annular ducts with rigid non-porous walls.

Nonstandard Analysis Notebook (Pages 957 - 1006)

A notebook on non-standard analysis (NSA) examines the fluid dynamic properties of a shock wave using the relatively newly developed field in mathematics of non-standard analysis. Insall and Weisstein[32] define non-standard analysis as 'a branch of mathematical logic, which introduces hyperreal numbers to allow for the existence of "genuine infinitesimals," which are numbers that are less than 1/2, 1/3, 1/4, 1/5, ..., but greater than 0.' Non-standard analysis, among some other

[28]The microphone rotates about a fixed point in space and not about its axis.

[29]Aircraft engines with a high bypass ratio, which is the ratio of the airflow through the fan to the airflow through the core.

[30]Kinsler, L. E., Frey, A. R., Coppens, A. B., and Sanders, J. V., 'Fundamentals of Acoustics,' Wiley, 1982.

[31]Myers, M. K., 'An Exact Energy Corollary for Homentropic Flow,' Journal of Sound and Vibration, Vol. 109, No. 2, 1986, pp. 277-284. DOI: 10.1016/S0022-460X(86)80008-6

[32]Insall, M. and Weisstein, E. W. 'Nonstandard Analysis,' MathWorld - A Wolfram Web Resource, 2015.

mathematical methods, is used to estimate the shock jump conditions using the conservation laws of fluid dynamics. In particular, the entropy generation, shape, and thickness of a shock wave under varying conditions is examined. It is argued that the shock wave thickness decreases as Mach number increases. The notebook has 49 pages.

The NSA Greenbook begins (NSA pages 1-2) by discussing applications of nonstandard analysis for finding shock jump conditions from conservation laws in the non-conservative form. Here, a shock jump condition is interpreted as a discontinuity of a function. The product of the function with a Dirac delta function can be explored if the concepts of NA are used. Shock jump conditions (NSA pages 3-6) are reviewed based upon Salas and Iollo.[33] Through an entropy conservation law, presented on NSA pages 7-9, the Heaviside function is explored through two approaches. NSA page 9 provides some interpretation of these two approaches. Variation of the conditions with increasing Mach number (in front of the shock wave) is explored on NSA pages 9-11.

Almost a year after NSA pages 9-11 were written, Dr. Farassat's interest returned to the subject on NSA page 12, where he hoped to find the structure of one of the Heaviside functions within the context of the shock problem. The governing equations within a single dimension are examined on NSA pages 12-15 and written with a jump condition. It is concluded that the resulting system of equations can be solved numerically, but here Dr. Farassat chooses to continue using NSA (NSA pages 16-23). An improved approach is discussed on NSA pages 24-26, where a closed-form solution of a second-order nonlinear ordinary differential equation is found. Much of these analyses are performed by Dr. Farassat through the use of Mathematica. Some of the output of Mathematica of the preceding analyses are shown on NSA pages 27-31. A note on the bottom of NSA page 31 lays claim by Dr. Farassat of the asymptotic (analytical) result of the width of a shock wave in the limit of Mach number. NSA page 32 notes journal articles used in the development of the preceding results. NSA pages 32-39 examine a second case for the shock structure of a viscous and heat-conducting fluid. Mathematica notebooks for its evaluation are shown on NSA pages 39-48. Some general conclusions are drawn regarding the shock thickness as a function of Mach number on NSA page 49. It is concluded that shock thickness decreases as Mach number increases and that the form of the Heaviside functions for entropy and temperature have a peculiar behavior near the shock.

Notes on Differential Geometry (Pages 1007 - 1067)

Dr. Farassat was interested in differential geometry (DG) as it has applications within the formulations he developed in the field of aeroacoustics. Differential geometry involves the use of calculus to solve geometry problems. These notes are suitable for a detailed review of DG and are not meant as an introduction to this advanced topic in mathematics. The DG notebook is approximately 60 pages in length.

The DG notes begin with some basic theory. The theory of curves is introduced along with the fundamental theory of the curve (DG pages 1-2). Surface theory is examined following the development of curve theory on DG pages 3-5. Change of variables on a surface is introduced on DG pages 6-8. A contravariant basis vector is introduced on DG pages 8-10. The developed contravariant basis vector dot product with a surface normal is defined to be zero (DG page 10). Classifications of points on surfaces are placed within three categories on DG pages 11-12.

Spherical and Gaussian mapping are introduced on DG pages 12-14. The Gauss and Weingarten formulas are defined on DG page 15. Christoffel symbols are defined on DG pages 16-17 and were also introduced in the beginning of green book three. Formulas of Gauss and Codazzi are defined on

[33] Salas, M. D. and Iollo, A., 'Entropy Jump Across an Inviscid Shock Wave,' Theoretical and Computational Fluid Mechanics, Vol. 8., 1996, pp. 365-375. DOI: 10.1007/BF00456376

DG pages 18-19, and the fundamental theorem of surfaces is defined on DG pages 19-20. Attention is turned toward the geometry of surfaces and the curvature of a curve on the surface on DG pages 21-29. In particular, the differentiation of vectors is addressed. The parallel transport of vectors on a curve, which preserves the scalar product of two vectors, is discussed on DG pages 29-31.

The Gauss-Bonnet theorem is discussed on DG pages 32-38. A few short notes on extrinsic geometry of surfaces are shown on DG page 39. Meusnier's theorem is introduced. DG pages 40-45 examine the spherical image of tangent vectors. Useful results of differential geometry are summarized on DG pages 45-51. The notebook closes with an introduction to manifolds on DG pages 52-60.

Aerodynamics, Aeroacoustics, and Propfan Notebook (Pages 1068 - 1147)

Dr. Farassat, in this early green book called Aerodynamics, Aeroacoustics, and Propfan Notebook (AAPN), sets out to develop a simplified computer program to predict the noise from propeller fan blade geometry. Major topics include the design of early computer programs to evaluate the noise from propellers and to plot their results. The AAPN is not characteristic of his other notebooks in that it is more of a working document and contains a number of loose scratch pages. The AAPN was started in December of 1977 and contains 82 pages.

The problem to be solved and blade geometry are defined on AAPN pages 1-5. It is concluded that a simpler approach than those that exist is required for numerical evaluation. A computer program for this purpose is devised on AAPN pages 6-10 with a flow-chart on AAPN page 7. Input and output of the computer program are defined on AAPN pages 11-15. Part of the code is shown on AAPN page 16, and the programming language is FORTRAN 77. Additional subroutines and example output are shown on AAPN pages 17-23. A number of common block definitions and subroutine descriptions are shown on AAPN pages 24-41. Dr. Farassat describes a discretization scheme along the chord of the blade on AAPN pages 42-44. A numerical check of the computer code for the Fourier transform is written on AAPN pages 45-50. Example outputs from various test functions are shown. Analysis of an actuator disk (AAPN page 51) is performed. Examination of a deformed biconvex parabolic curve is examined for application to the propeller noise problem on AAPN pages 52-53. AAPN page 54 discusses the inclusion of friction and wave drag in the acoustic calculation. The final numbered pages of AAPN, AAPN pages 55-60, are used to check the developed code for errors. The remaining portion of AAPN, dated February to May of 1982, is used to describe various aspects of the computer program PROPFAN.[34] In particular, a numerical root finding algorithm is developed. A number of potential research directions are noted on page 1140 in May of 1982 within AAPN. The final portion of the notebook describes miscellaneous subroutines and Fourier transform calculations. The last pages of the notebook contain graph paper used to plot functions and are omitted here.

Ray Acoustics (Pages 1148 - 1161)

A very short notebook with the title 'Acoustics' focuses on reviewing ray acoustics. The ray acoustics (RA) notebook[35] is focused on examining some canonical results of acoustic ray theory. Ray theory draws on classical optics and assumes that the wavelength of the radiating waves is much shorter than other length scales within the domain. The RA notebook has 13 pages.

[34] Martin, R. M. and Farassat, F., 'User's Manual for a Computer Program to Calculate Discrete Frequency Noise of Conventional and Advanced Propellers,' NASA TM-83135, 1981.

[35] This notebook has a blue cover while all others have a green cover.

This notebook starts on RA pages 1-2 with a discussion on stationary medium, Huygens' principle, and the differential equation for the slowness vector. Acoustic rays contained within a moving medium are discussed on RA pages 3-4. The amplitude variation along the ray is examined on RA pages 5-6, which is based on Pierce.[36] Conservation of energy along a ray within a stationary medium is reviewed on RA pages 7-8. The short RA notebook is closed on RA pages 9-13 after examining energy conservation of acoustic rays within a mean flow. Conservation equations of mass, momentum, entropy, energy, and the constitutive relation are examined. The Blokhintzev invariant is also found.

Computational Fluid Dynamics Notebook (Pages 1162 - 1170)

The computational fluid dynamics (CFD) notebook examines particular basics for numerical solutions of partial differential equations that govern fluid motion. The year that the notebook was started (1979) corresponded to the early era of using computers to solve fluid dynamics problems. The CFD notebook contains eight pages.

The CFD notebook starts with some entries on the numerical solution of partial differential equations on CFD page 1. An overview of the steps required to find a numerical solution are summarized. Finite differences are introduced through the simple time-dependent heat equation. The heat equation was programmed by Dr. Farassat in FORTRAN, and the code is reproduced on CFD page 2. The solution is graphed on CFD pages 3-4. The solution is found by marching through time, and an alternative approach is proposed on CFD page 5 that involves substituting an assumed form of the solution. The last two pages (CFD pages 7-8) discuss the use of analytic solutions to quantify the rate of convergence.

Airframe Noise Notebook (Pages 1171 - 1183)

The Airframe Noise (AN) notebook focuses on the lesser known Formulation 1B of Dr. Farassat. The AN notebook is another short notebook that contains 12 pages. Notebook AN pages 1-5 summarize the derivation of Formulation 1B, which is used to predict the noise from unsteady surface pressures. The derivation is based upon the movement of a large flat surface through a fluid medium with a fixed frame observer. The final derivation appears on AN page 5 with a note regarding the Kutta condition. AN pages 6-10 are extracted from Farassat and Casper.[37] Using Formulation 1B, a statistical approach for the auto-correlation of acoustic pressure is formed due to pressure fluctuations on an airframe. Finally, AN pages 10-12 discuss Formulation 1B versus Formulation 1A for airframe calculations. It is concluded for subsonic flow that Formulation 1A remains simpler; however, formulations are not applicable for supersonic propellers.

Abstract Algebra Notebook (Pages 1184 - 1194)

Dr. Farassat, likely for intellectual reasons and curiosity, decided to study the field of abstract algebra of mathematics. Generally, abstract algebra is the study of fields, groups, modules, lattices, rings, and vector spaces. This short Abstract Algebra (AA) notebook has ten pages. Dr. Farassat discusses his motivation for learning AA on AA page 1. Euclid's algorithm is described on AA pages 2-6, and some proofs are reproduced. AA page 7 discusses correspondence with professors at

[36] Pierce, A. D., 'Acoustics: An Introduction to its Physical Principles and Applications,' Acoustical Society of American, 1989.

[37] Farassat, F. and Casper, J., 'Broadband Noise Prediction when Turbulence Simulation is Available - Derivation of Formulation 2B and its Statistical Analysis,' Journal of Sound and Vibration, Vol. 331, No. 10, 2012, pp. 2203-2208. DOI: 10.1016/j.jsv.2011.07.044

various universities regarding the countability of real numbers. The last section of the AA notebook on AA pages 8-10 is based on Littlewood.[38]

Notes on Ffowcs Williams 1963 Journal Article (Pages 1195 - 1204)

Dr. Farassat studied many journal articles written by his contemporaries. One of importance to technical readers in aeroacoustics is the article of Ffowcs Williams.[39] Dr. Farassat's analysis of the article is shown in his 'Notes on Ffowcs Williams 1963 Journal Article' (NFW1963) and is 10 pages long. Some notational changes from the original paper are discussed on NFW1963 page 1. Relations between the fixed-frame and moving frame are discussed. A number of relations are derived from the Garrick triangle. On NFW1963 page 8 the FW-H equation is derived. The final two remaining pages (NFW1963 pages 8-10) show the derivation of the auto-correlation of the acoustic pressure.

Final Derivations (Pages 1205 - 1210)

Near the end of Dr. Farassat's life, he continued to produce excellent prediction theories for aeroacoustic problems. Here, some of these final derivations (FD) are shown and six select[40] pages are presented. They represent a formulation to find the aerodynamic velocity potential. These last pages were produced by Dr. Farassat just two days before his passing. They are certainly a testament to Dr. Farassat's unwavering dedication to research.

[38]Littlewood, D. E., 'The Skeleton Key of Mathematics - A simple Account of Complex Algebraic Theories,' Dover Publications, 1949.

[39]Ffowcs Williams, 'The Noise from Turbulence Convected at High Speed,' Philosophical Transactions of the Royal Society, Vol. A255, 1963, pp. 496-503.

[40]These notes were kindly provided by Mark Dunn.

3 A Very Basic Course in Acoustics

What is Acoustics?

The science of acoustics is the study of all phenomena associated with propagation of small perturbations (e.g.,pressure, velocity, displacement) in air, water or solids.

We will discuss wave propagation in air only. This subject is usually known as general acoustics. Wave propagation in water is studied in underwater acoustics and in solids is studied in ultrasonics and physical acoustics. Wave propagation in solids is much more complicated and of more varieties than in air and water.

Some Areas of Acoustics (From Acoustical Society of America)

Acoustics is now a very broad subject and is actively studied. Here are some of the areas of acoustics: General Linear Acoustics, Nonlinear Acoustics, Atmospheric Acoustics, Aeroacoustics, Underwater Sound, Ultrasonics and Physical Acoustics, Transduction, Acoustical Measurements, Instrumentation, Applied Acoustics, Structural Acoustics and Vibration, Acoustic Signal Processing, Physiological Acoustics, Psychological Acoustics, Speech Production, Speech Perception, Musical Acoustics, Bioacoustics, Computational Acoustics

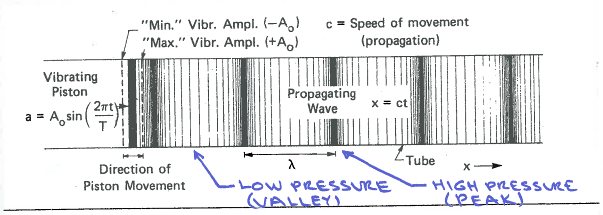
Lecture 1

–Acoustic waves in the air

Amplitude, frequency, speed of sound, acoustic wave, wave length, period, audible frequency range, wave number, phase, useful definitions for acoustic signals, types of common signals, the decibel scale, loudness scale, finding rms pressure for some common signals

ACOUSTIC WAVES IN AIR

SOUND IS DEFINED AS ANY PRESSURE VARIATION IN THE AIR SENSED BY THE EAR. IN AIR SOUND TRAVELS AT THE SPEED OF ABOUT 343 m/s AT 20°C. A VIBRATING PISTON AT THE END OF A TUBE SENDS ACOUSTIC WAVES INTO THE TUBE. LET THE PISTON MOVE AT THE FREQUENCY f (NO. OF OSCILLATIONS/SEC) THE UNIT OF FREQUENCY IS HERTZ (HZ). A FIXED MICROPHONE IN THE TUBE SENSES



The transformation of vibrations into waves

A PRESSURE VARIATION AT THE SAME FREQUENCY f. AT A FIXED TIME t, THE PRESSURE VARIATION OF HIGH AND LOW VALUES THAT ARE REGULARLY SPACED ALONG THE TUBE. WAVE LENGTH IS THE DISTANCE BETWEEN TWO SUCCESSIVE PEAKS OR VALLEYS IN PRESSURE (SHOWN AS λ IN THE FIGURE ABOVE). WE WILL CONSIDER SINUSOIDAL PRESSURE "WAVES" OF THE TYPE $p' = P \sin(2\pi f t) = P \sin\left(\frac{2\pi t}{T}\right)$.

ACOUSTIC WAVES IN AIR (CONT'D)

THE PRESSURE PATTERN IN THE FIGURE ON PREVIOUS PAGE TRAVELS TO THE RIGHT AT CONSTANT SPEED C KNOWN AS SPEED OF SOUND.

IMPORTANT RELATION : FREQUENCY X WAVELENGTH = SPEED OF SOUND

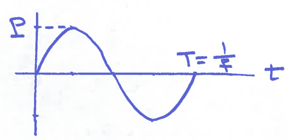
OR $\boxed{f\lambda = c}$

f IN HZ, λ IN m AND c IN m/s

PERIOD OF OSCILLATION $T = \dfrac{1}{f}$ SECONDS

FOR THE SINUSOIDAL "WAVE" $p' = P \sin(2\pi f t)$,

THE AMPLITUDE P IS DEFINED AS THE MAXIMUM OF PRESSURE AS SHOWN ON THE RIGHT. NOTE THAT THIS IS THE PRESSURE SENSED BY A MICROPHONE FIXED AT A POINT.



ACOUSTIC WAVES IN AIR (CONT'D)

MORE ON ACOUSTIC PRESSURE WAVE PROPAGATING IN DIRECTION x AT SPEED OF SOUND c

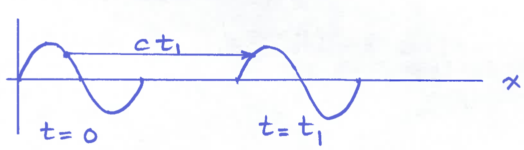
$$p' = P \sin(\omega t - kx)$$

$\omega = 2\pi f$ ANGULAR FREQUENCY RADIAN/S

$k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi f}{f\lambda} = \dfrac{\omega}{c}$ WAVE NUMBER m^{-1}

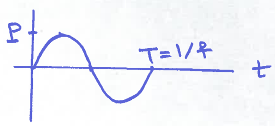
WHY IS THIS A WAVE PROPAGATING IN THE DIRECTION x AT SPEED c ?

ANSWER : LET US FOLLOW THE POINT IN SPACE WITH CONSTANT PRESSURE, i.e. WITH $\omega t - kx = \theta =$ CONST. THEN THIS POINT TRAVELS AT THE SPEED FOUND FROM THE RELATION $\dfrac{d\theta}{dt} = 0 = \omega - k\dot{x}$ OR $\dot{x} = \dfrac{dx}{dt} = \dfrac{\omega}{k} = c$

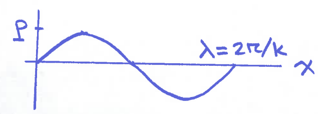


ACOUSTIC WAVES IN AIR (CONT'D)

THE ABOVE RESULT SHOWS THAT A SINUSOIDAL PRESSURE PATTERN AT TIME t_0 PROPAGATES UNCHANGED ALONG x DIRECTION AT SPEED OF SOUND c. ALSO FOR FIXED $x = x_1$, WE HAVE $p' = P \sin(\omega t - k x_0)$, i.e. WE HAVE SINUSOIDAL VARIATION IN TIME. SIMILARLY FOR A FIXED TIME t_0, WE HAVE SINUSOIDAL VARIATION IN SPACE $p' = P \sin(\omega t_0 - k x\cancel{x})$. THE SINUSOIDAL VARIATION IN TIME HAS FREQUENCY $f = \omega / 2\pi$ AND THE SINUSOIDAL VARIATION IN SPACE HAS THE WAVE LENGTH $\lambda = \frac{2\pi}{k}$.



PRESSURE VARIATION AS MEASURED BY A FIXED MICROPHONE IN SPACE



PRESSURE VARIATION IN SPACE FOR A FIXED TIME AS MEASURED BY MANY MICROPHONES POSITIONED AT VARIOUS DISTANCES ALONG X

ACOUSTIC WAVES IN AIR (CONT'D)

WAVE NUMBER k = THE NUMBER OF WAVELENGTHS IN 6.28 m OF LENGTH OF SPACE

EXAMPLES: IF $\lambda = 1$ m , $k = 6.28$ m^{-1}
IF $\lambda = 0.1$ m , $k = 62.8$ m^{-1}

THEREFORE IF $\lambda \uparrow$, WE HAVE $k \downarrow$ AND
IF $\lambda \downarrow$, WE HAVE $k \uparrow$

AND BECAUSE $f\lambda = c = $ CONST.
IF $f \uparrow$, WE HAVE $k \uparrow$
IF $f \downarrow$, WE HAVE $k \downarrow$

i.e. λ AND k VARY INVERSELY
f AND k VARY DIRECTLY

NOTE THAT PERIOD $T = 1/f$ AND λ VARY DIRECTLY : AS $\lambda \downarrow$ (i.e. $f \uparrow$) WE HAVE $T \downarrow$.

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ACOUSTIC WAVES IN AIR (CONT'D)

MORE ON SPEED OF SOUND C IN THE AIR

SPEED OF SOUND DEPENDS ON TEMPERATURE OF THE AIR

$$C = \sqrt{\gamma R T} \quad m/s$$

$$\gamma = 1.4 \;, \quad R = \frac{8314}{29} = 286.7 \quad \left\{ \begin{array}{l} 8314 \text{ GAS CONSTANT} \\ 29 \text{ MOL. WEIGHT OF AIR} \end{array} \right.$$

$T = 273 + T\,°C$, WHERE $T\,°C$ IS AIR TEMPERATURE IN DEGREES CELSIUS

$$C = 20.03\sqrt{273 + T°C}$$

$T°C = (T°F - 32)/1.8$, $T°F$ IS AIR TEMP. IN DEGREES FARENHEIT

EXAMPLE : AT $20°C$, $C = 20.03\sqrt{293} = 343$ m/s

$20°C = 20 \times 1.8 + 32 = 68°F$

NOTE : IN ACOUSTICS WE ALWAYS USE SI UNITS, i.e. KILOGRAMS (FOR MASS), METERS (FOR LENGTH), ETC.

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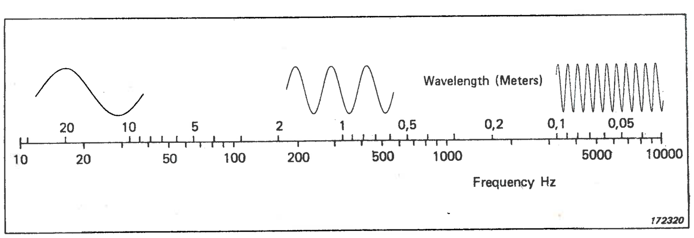
ACOUSTIC WAVES IN AIR (CONT'D)

THE AUDIBLE RANGE OF FREQUENCY

20 HZ TO 20 kHZ

CORRESPONDING TO WAVELENGTH RANGE OF

17 m TO 1.7 cm (0.017 m)



Wavelength in air versus frequency under normal conditions

NOTE THAT HUMAN EAR IS MOST SENSITIVE TO FREQUENCY RANGE 1 kHz TO 5 kHz (WAVELENGTH 34 cm TO 7 cm)

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ACOUSTIC WAVES IN AIR (CONT'D)

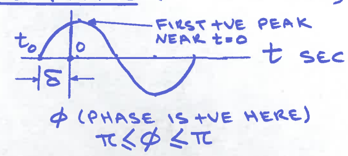
ACOUSTIC SIGNAL CHARACTERISTICS

FOR A SINUSOIDAL SIGNAL $p' = P \sin(2\pi f t + \phi)$

P IS THE AMPLITUDE AND ϕ IS THE PHASE (IN RADIANS).

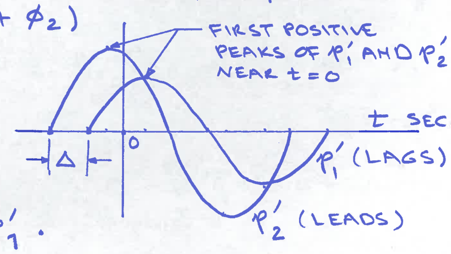
$\phi = 2\pi f \delta$ RADIANS, $\boxed{-\pi \leq \phi \leq \pi}$

NOTE THAT $\delta > 0$, i.e. $\delta = -t_0$ IN THE FIGURE.



ϕ (PHASE IS +VE HERE)
$\pi \leq \phi \leq \pi$

FOR TWO SIGNALS OF THE SAME FREQUENCY, WE OFTEN NEED RELATIVE PHASE. CONSIDER $p'_1 = P_1 \sin(2\pi f t + \phi_1)$ AND $p'_2 = P_2 \sin(2\pi f t + \phi_2)$



RELATIVE PHASE $= 2\pi f \Delta$

$\Delta > 0$, $\Delta = |\phi_1 - \phi_2|$

WE SAY p'_2 LEADS p'_1 BECAUSE IT STARTS RISING NEAR $t=0$ EARLIER THAN p'_1.

SIMILARLY, WE SAY p'_1 LAGS p'_2 BECAUSE IT STARTS RISING NEAR $t=0$ AFTER p'_2.

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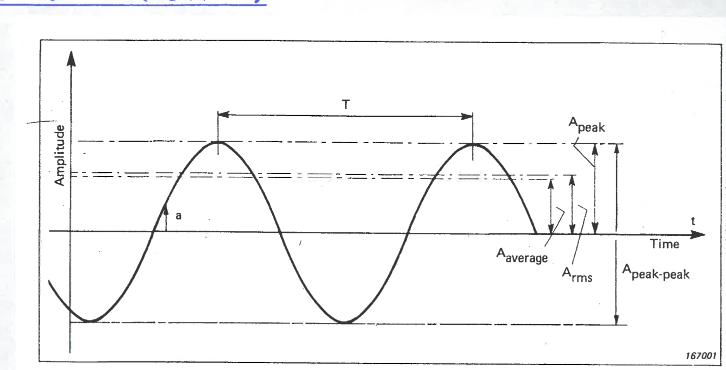
ACOUSTIC WAVES IN AIR (CONT'D)

SOME USEFUL DEFINITIONS

FOR ANY ACOUSTIC SIGNAL $p' = F(t)$, WE DEFINE THE FOLLOWING QUANTITIES

$$p'_{MEAN} = \frac{1}{T}\int_0^T F(t)\,dt$$

T LARGE



Sinusoidal signal showing various measures of signal amplitude

$$p'_{AVERAGE} = \frac{1}{T}\int_0^T |F(t)|\,dt, \quad p'_{rms} = \sqrt{\frac{1}{T}\int_0^T F^2(t)\,dt}$$

rms : ROOT MEAN SQUARE

FOR A SINUSOIDAL SIGNAL $p' = P \sin(\omega t + \phi)$, WE HAVE

$$p'_{MEAN} = 0, \quad p'_{AVERAGE} = \frac{2}{\pi} P \approx 0.67 P, \quad p'_{rms} = \frac{1}{\sqrt{2}} P \approx 0.71 P$$

FOR ANY SIGNAL, WE DEFINE CREST FACTOR $F_c = \dfrac{p'_{PEAK}}{p'_{rms}}$

AND FORM FACTOR $F_F = \dfrac{p'_{rms}}{p'_{AVERAGE}}$

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ACOUSTIC WAVES IN AIR (CONT'D)

SOME IMPORTANT ACOUSTIC SIGNALS

PURE TONE (SINUSOIDAL)

$p' = P \sin(\omega t + \phi)$

PERIODIC SIGNAL : $\omega = 2\pi f$

$p' = P_0 + P_1 \sin(\omega t + \phi_1)$ FUNDAMENTAL

$+ P_2 \sin(2\omega t + \phi_2)$ 1ST HARMONIC

$+ P_3 \sin(3\omega t + \phi_3) + \cdots$

THE SUM MAY OR MAY NOT BE INFINITE!

PERIOD $= 1/f = T$, f: FUNDAMENTAL FREQ.

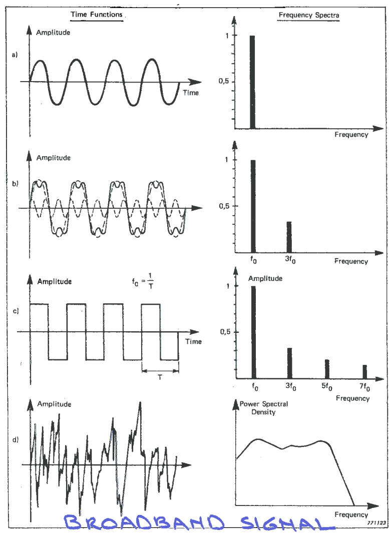
APERIODIC SIGNAL : f_1/f_2 IRRATIONAL

$p' = P_1 \sin(2\pi f_1 t + \phi_1)$

$+ P_2 \sin(2\pi f_2 t + \phi_2)$

NOTE :- IN PERIODIC SIGNALS, P_0, $P_1, \phi_1, P_2, \phi_2, \ldots$ ARE ALL CONSTANTS.

- WE CAN DEFINE p'_{MEAN}, $p'_{AVERAGE}$, p'_{PEAK}, p'_{rms}, F_C AND F_F FOR A BROADBAND SIGNAL



Sound signals and their spectra
a) pure sinusoid (simple and periodic) c) square wave (complex but periodic)
b) combination of two sinusoids d) random noise (complex and non-periodic)

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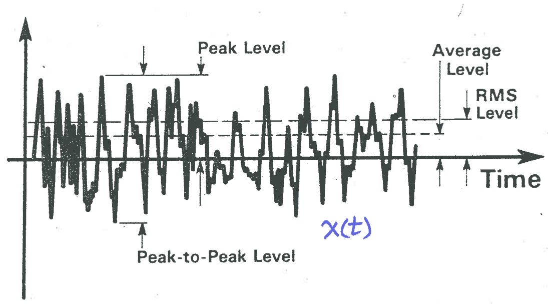
ACOUSTIC SIGNALS IN AIR (CONT'D)

COMMON ACOUSTIC INSTRUMENTS GIVE p'_{MEAN}, $p'_{AVERAGE}$, p'_{PEAK}, p'_{rms}

$$\text{RMS Level} = \sqrt{\frac{1}{T}\int_0^T x^2(t)\,dt}$$

$$\text{Average Level} = \frac{1}{T}\int_0^T |x|\,dt$$

FOR PERIODIC AND BROADBAND SIGNALS. MOST SIGNALS OF INTEREST ARE OF THE TYPE THAT THESE QUANTITIES ARE TIME INDEPENDENT. HOWEVER, WE ALSO WORK WITH SIGNALS THAT THESE QUANTITIES ARE FUNCTIONS OF TIME. THEY ARE CALLED TRANSIENT SIGNALS.



ONE SHOULD KNOW THE NATURE OF THE SIGNAL BEFORE ANALYZING IT.

ACOUSTIC SIGNALS IN AIR (CONT'D)

THE DECIBEL SCALE

— THE ACOUSTIC PRESSURE IN SI UNITS IS IN PASCALS (Pa).

$\boxed{1 \text{ PASCAL} = 1 \text{ N/m}^2}$, N : NEWTON, UNIT OF FORCE IN SI UNITS.
THIS IS A SMALL FORCE IN TERMS OF EVERYDAY EXPERIENCE.
FOR EXAMPLE 1 POUND FORCE = 4.44 N. SINCE $1 \text{ m}^2 = 10.75 \text{ ft}^2$, 1 PASCALL APPEARS TO BE A VERY SMALL UNIT OF PRESSURE.

REMEMBER 1 ATMOSPHERE = 14.7 PSI ≈ 1,00000 Pa !
BUT IN ACOUSTICS 1 Pa IS A VERY LARGE PRESSURE BECAUSE THE EAR IS A VERY SENSITIVE INSTRUMENT! WHEN WE TALK ABOUT ACOUSTIC PRESSURE, WE USUALLY MEAN p'_{rms}.

— RANGE OF ACOUSTIC PRESSURE EAR CAN SENSE :

p'_{rms} FROM $20\,\mu\text{Pa}$ TO 20 Pa $\quad \mu$: MICRO $= 10^{-6}$

THIS IS A RANGE WITH A FACTOR OF 1,000,000 !
THEREFORE, PASCAL IS NOT A GOOD UNIT FOR ACOUSTIC PRESSURE.

— THE EAR DOES NOT RESPOND TO ACOUSTIC PRESSURE **LINEARLY**. WE NEED TO COMPRESS THIS RANGE TO A MUCH MORE MANAGABLE UNIT — IT IS DECIBELS !

ACOUSTIC SIGNALS IN AIR (CONT'D)

THE DECIBEL SCALE

DECI A PREFIX MEANING ONE TENTH

BEL TAKEN FROM THE NAME OF THE GREAT SCOTTISH-AMERICAN SCIENTIST AND INVENTOR ALEXANDER GRAHAM BELL

SOUND PRESSURE LEVEL $L_p = 10 \log_{10} \left(\frac{p'_{rms}}{p'_0}\right)^2$

$$= 20 \log_{10} \left(\frac{p'_{rms}}{p'_0}\right) \text{ dB (ref: } 20\,\mu\text{Pa)}$$

$p'_0 = 20\,\mu\text{Pa}$: REFERENCE PRESSURE

THIS IS THE SMALLEST ACOUSTIC PRESSURE A PERSON WITH VERY GOOD EARS CAN HEAR AT f = 1000 TO 5000 Hz

$\log_{10}$: LOGARITHM TO THE BASE 10, i.e. $x = 10^{\log_{10} x}$

WHICH MEANS $\log_{10} x$ IS THE NUMBER THAT WE MUST RAISE 10 TO THIS POWER TO GET x.

EXAMPLES: $\log_{10} 1 = 0$ BECAUSE $10^0 = 1$

$\log_{10} 2 = 0.301$ BECAUSE $10^{0.301} = 2$

ACOUSTIC SIGNALS IN AIR (CONT'D)

EXAMPLES (CONT'D)

$\log_{10} 10 = 1$ BECAUSE $10^1 = 10$

RULES YOU MUST KNOW ABOUT LOGARITHMS IN ANY BASE:

$$\begin{cases} \log(a\,b) = \log a + \log b \\ \log(a^n) = n \log a \\ \log(a/b) = \log a - \log b \end{cases}$$

HERE $a > 0$ AND $b > 0$. WE DO NOT DEFINE LOGARITHMS FOR NEGATIVE NUMBERS. NOTE THAT $p_{rms} > 0$ ALWAYS!

- RANGE OF DECIBEL SCALE (IN AUDIBLE RANGE)

  TRESHOLD OF HEARING : 0 DECIBELS
  TRESHOLD OF PAIN : 120-130 DECIBELS

- THE DECIBEL SCALE IS USED FOR SOUND INTENSITY AND POWER WITH THEIR OWN REFERENCE QUANTITIES. WE WILL DISCUSS THEM LATER.

ACOUSTIC SIGNALS IN AIR (CONT'D)

THE DECIBEL SCALE (CONT'D)

- IF WE DOUBLE THE rms PRESSURE FROM p'_{rms} TO $2p'_{rms}$ THE CHANGE (INCREASE) IN DECIBEL LEVEL IS

$$\Delta dB = 20 \log_{10}\left(\frac{2p'_{rms}}{p_0}\right) - 20 \log_{10}\left(\frac{p'_{rms}}{p_0}\right)$$
$$= 20 \log_{10} 2 = 6 \quad \text{DECIBELS INCREASE}$$

SIMILARLY IF WE INCREASE p'_{rms} TO $10\,p'_{rms}$

$$\Delta dB = 20 \log_{10}\left(\frac{10\,p'_{rms}}{p_0}\right) - 20 \log_{10}\left(\frac{p'_{rms}}{p_0}\right)$$
$$= 20 \log_{10} 10 = 20 \quad \text{DECIBELS INCREASE}$$

- THE DECIBEL SCALE GIVES ROUGHLY A MEASURE OF LOUDNESS. FOR A PERSON A CHANGE OF

  3 dB IS JUST PERCEPTIBLE
  5 dB IS CLEARLY PERCEPTIBLE
  10 dB IS "TWICE" AS LOUD!

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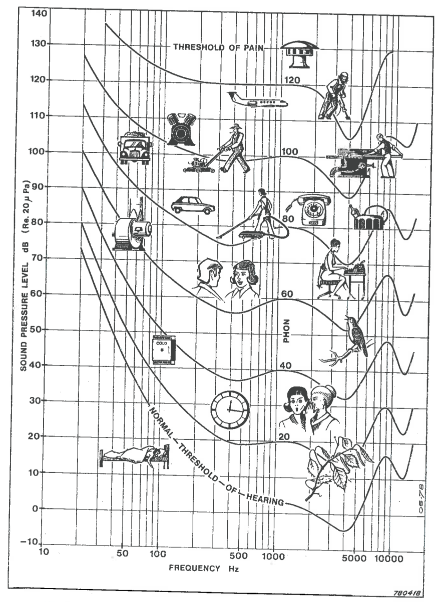
ACOUSTIC SIGNALS IN AIR (CONT'D)

THE DECIBEL SCALE (CONT'D)

- THIS FIGURE GIVES YOU AN IDEA OF THE DECIBEL LEVELS OF COMMON NOISE SOURCES.
- NOTE THAT THE LOUDNESS OF SOUND IS SUBJECTIVE AND IS FREQUENCY DEPENDENT. THE UNIT OF LOUDNESS IS PHON. IT IS THE DECIBEL LEVEL OF THE SOUND WHEN THE EQUAL LOUDNESS CURVE IS AT 1000 Hz.
- THE EAR IS MOST SENSITIVE FROM 1000 TO 5000 HZ, LEAST SENSITIVE AT LOW AND HIGH FREQUENCIES.



Typical sound pressure levels of common noise sources

EQUAL LOUDNESS CURVES ARE SHOWN.

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ACOUSTIC SIGNALS IN AIR (CONT'D)

FINDING p'_{rms} FOR SOME COMMON SIGNALS

1. PERIODIC SIGNALS

$$p' = P_o + P_1 \sin(\omega t + \phi_1) + P_2 \sin(2\omega t + \phi_2) + \cdots$$

$$p'_{rms} = \sqrt{\frac{1}{T}\int_0^T p'^2\, dt}$$

WE CAN SHOW THAT

$$p'^2_{rms} = P_o^2 + \frac{1}{2}(P_1^2 + P_2^2 + \cdots)$$

MOST MICROPHONES ARE DESIGNED SO THAT P_o IS NOT MEASURED. THIS QUANTITY P_o IS CALLED THE DC SHIFT. THEREFORE, WHEN $P_o = 0$, WE HAVE

$$p'^2_{rms} = \frac{1}{2}(P_1^2 + P_2^2 + \cdots) = (P_{1,rms})^2 + (P_{2,rms})^2 + \cdots$$

NOTE THAT THE PHASE ϕ DOES NOT PLAY ANY ROLE IN EVALUATION OF p'_{rms}. REMEMBER THE RULE:

$$p'^2_{rms} = \text{SUM OF } (p'_{rms})^2 \text{ OF ALL HARMONICS}$$

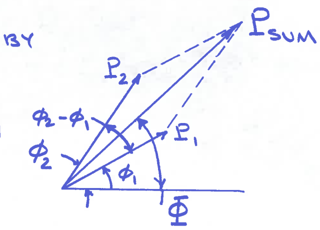
ACOUSTIC SIGNALS IN AIR (CONT'D)

FINDING p'_{rms} FOR SOME COMMON SIGNALS (CONT'D)

2. TWO SINUSOIDAL SIGNALS OF THE SAME FREQUENCY

$$p'_1 = P_1 \sin(\omega t + \phi_1), \quad p'_2 = P_2 \sin(\omega t + \phi_2)$$

p'^2_{rms} FOR $p' = p'_1 + p'_2$ IS GIVEN BY



$$p'^2_{rms} = \frac{1}{2}\left[P_1^2 + P_2^2 + 2 P_1 P_2 \cos(\phi_2 - \phi_1)\right]$$
$$= \frac{1}{2} P_{SUM}^2$$

P_{SUM} IS THE AMPLITUDE AND Φ IS THE PHASE OF THE $p' = p'_1 + p'_2 = P_{SUM} \sin(\omega t + \Phi)$

– IF $\cos(\phi_2 - \phi_1) = 1$ i.e. $\phi_1 = \phi_1 \Rightarrow p'^2_{rms} = \frac{1}{2}(P_1 + P_2)^2$

IF $\cos(\phi_2 - \phi_1) = -1$ i.e. $\phi_2 - \phi_1 = \pm\pi \Rightarrow p'^2_{rms} = \frac{1}{2}(P_1 - P_2)^2$

IN THIS CASE IF $P_1 = P_2 \Rightarrow p'_{rms} = 0$, i.e. THE TWO SIGNALS COMPLETELY CANCEL EACH OTHER. THIS IS THE IDEA BEHIND ANTINOISE.

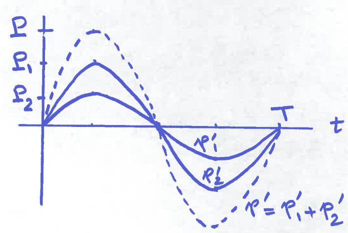
ACOUSTIC SIGNALS IN AIR (CONT'D)

FINDING p'_{rms} FOR SOME COMMON SIGNALS (CONT'D)

WE HAVE SHOWN THAT FOR TWO SIGNALS OF COMMON FREQUENCY f:

p'_{rms} IS MAXIMUM IF THE TWO SIGNALS ARE IN PHASE

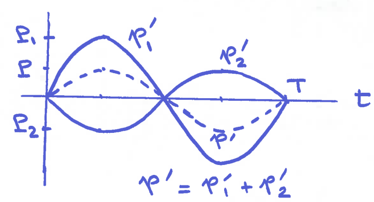
p'_{rms} IS MINIMUM IF THE TWO SIGNALS ARE 180° OUT OF PHASE.



$P = P_1 + P_2$

IN PHASE

$$p'_{rms} = \frac{1}{\sqrt{2}}(P_1 + P_2)$$



$P = |P_1 - P_2|$

180° OUT OF PHASE

$$p'_{rms} = \frac{1}{\sqrt{2}}|P_1 - P_2|$$

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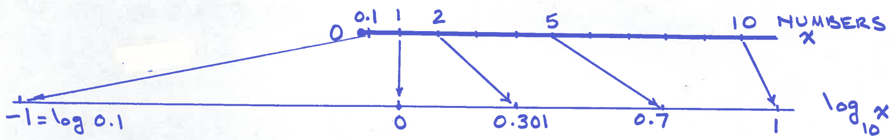
LOGARITHM AND THE DECIBEL SCALE

LOGARITHM IS DEFINED FOR POSITIVE NUMBERS.

- FOR $0 < x < 1$, $\log_{10} x < 0$
- FOR $x > 1$, $\log_{10} x > 0$
- $\log_{10} 1 = 0$, $\log_{10} 10 = 1$, $\log_{10} 1000 = 3$, $\log_{10} 10^n = n$



- A DOUBLING OF p'_{rms} INCREASES THE SOUND LEVEL BY 6 dB
- INCREASING p'_{rms} TO $5\,p'_{rms}$, INCREASES THE SOUND LEVEL BY 14 dB
- INCREASING p'_{rms} TO $10\,p'_{rms}$, INCREASES THE SOUND LEVEL BY 20 dB
- INCREASING p'_{rms} TO $100\,p'_{rms}$, INCREASES THE SOUND LEVEL BY 40 dB

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HOW DO WE FIND p'_{rms} FROM DECIBLE LEVEL?

WE HAVE $L_p = 20 \log_{10}\left(\frac{p'_{rms}}{p_o}\right)$ DECIBELS re: p_o

FROM THIS WE FIND THAT

$$p'_{rms} = p_o \times 10^{L_p/20}, \quad p_o = 20 \times 10^{-6}\ \text{Pa}$$

$$= 20 \times 10^{-6} \times 10^{L_p/20}$$

$$= 20 \times 10^{(L_p/20)-6}\ \text{Pa}$$

EXAMPLES - IF THE SOUND LEVEL IS 60 DECIBELS

$$p'_{rms} = 20 \times 10^{(60/20)-6} = 20 \times 10^{-3} = 0.02\ \text{Pa}$$

FOR $L_p = 80$, $p'_{rms} = 20 \times 10^{(80/20)-6} = 20 \times 10^{-2} = 0.2$ Pa

FOR $L_p = 100$, $p'_{rms} = 20 \times 10^{(100/20)-6} = 20 \times 10^{-1} = 2$ Pa

FOR $L_p = 130$, $p'_{rms} = 20 \times 10^{(130/20)-6} = 20 \times 10^{0.5} = 20 \times 3.16$
$= 63.2$ Pa

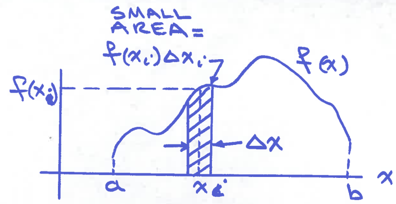
SOME MATHEMATICAL TERMS

THE MEANING OF AN INTEGRAL

$$I = \int_a^b f(x)\,dx$$

$$\approx \text{SUM}\left[f(x_i)\,\Delta x\right]$$

$$= \text{AREA UNDER THE CURVE}$$

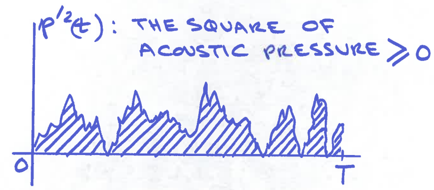


$$\Delta x = \frac{b-a}{n}$$

n SOME LARGE POSITIVE NUMBER

FOR EXAMPLE $\quad p'^2_{rms} = \frac{1}{T}\int_0^T p'^2(t)\,dt$

WE HAVE SHOWN THE INTEGRAL ON THE RIGHT.



THE CROSSED AREA IS: $\int_0^T p'^2(t)\,dt$

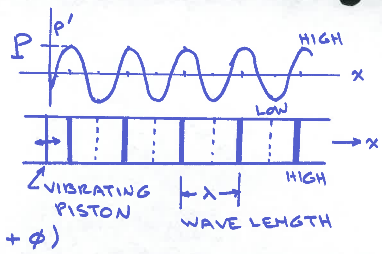
Lecture 2

– Plane and Spherical Waves

Plane sinusoidal waves, acoustic velocity and displacement amplitudes, acoustic intensity and power, acoustic power level, acoustic energy density, phasor diagram, spherical acoustic waves, amplitudes and radiation patterns of monopoles and dipoles, moving sources- the Doppler effect

PLANE ACOUSTIC WAVES

A VIBRATING PISTON AT THE END OF A TUBE PRODUCES PLANE WAVES IN THE TUBE OF THE FORM:



ACOUSTIC PRESSURE $\left.\right\}$ $p'(x,t) = P \sin(\omega t - kx + \phi)$

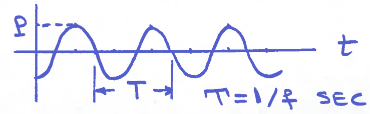
WHERE P IS THE AMPLITUDE IN PASCALS

ω IS $2\pi f$ RADIAN/S , f IS IN HERTZ (Hz)

$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$ WAVE NUMBER IN m^{-1}, c SPEED OF SOUND m/s

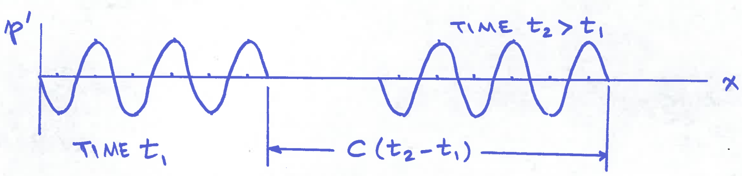
ϕ RADIANS IS A CONSTANT, KNOWN AS THE PHASE.

- THIS IS A PURE TONE. THE ABOVE PICTURE OF SINUSOID IS FOUND BY USING MANY MICROPHONES ON THE WALL OF THE TUBE AND MEASURING p' AT A FIXED TIME t.
- FOR A FIXED MICROPHONE, A SINUSOIDAL SIGNAL OF FREQUENCY $\omega/2\pi$ IS MEASURED WITH THE SAME AMPLITUDE P

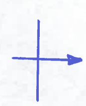


PLANE ACOUSTIC WAVES (CONT'D)

- THE PATTERN OF PRESSURE FOR A FIXED TIME t_1 PROPAGATES TO THE RIGHT AT SPEED OF SOUND c UNCHANGED



- THE AIR MOVES BACK AND FORTH ALONG THE DIRECTION OF MOTION OF THE WAVE:



VELOCITY $v' = V \sin(\omega t - kx + \phi)$ — AIR PARTICLE

DISPLACEMENT $d' = D \cos(\omega t - kx + \phi)$ — AIR PARTICLE

PLANE WAVE $p' = P \sin(\omega t - kx + \phi)$ MOVING TO THE RIGHT

$V = \frac{P}{\rho_o c}$ AMPLITUDE OF VELOCITY FLUCTUATIONS m/s

AT 20°C $\left\{ \begin{array}{l} \rho_o = 1.2 \text{ kg/m}^3 \text{ DENSITY OF THE AIR} \\ \rho_o c = 410 \text{ kg/m}^2\text{s OR RAYLS (FROM LORD RAYLEIGH)} \end{array} \right.$

PLANE ACOUSTIC WAVES (CONT'D)

THE AMPLITUDE OF DISPLACEMENT OF PARTICLES OF AIR IS

$$D = \frac{V}{\omega} = \frac{P}{\rho_0 c \omega} = \frac{P}{415\,\omega} \quad m$$

- LET US GET SOME IDEA OF THE MAGNITUDES OF THESE AMPLITUDES AT 70 dB AND 120 dB, re: 20 μPa, WHICH ARE THE LEVELS OF NORMAL CONVERSATION AND TRESHOLD OF PAIN, RESPECTIVELY. WE TAKE $f = 1000$ Hz, $\omega = 6280$ rad/s

70 dB re: 20 μPa

$$p'_{rms} = 20 \times 10^{(70/20)-6} = 0.063 \text{ Pa}$$

$$P = \sqrt{2}\, p'_{rms} = 0.089 \text{ Pa}$$

$$V = \frac{0.089}{410} = 0.00022 \text{ m/s}$$

$$D = \frac{V}{\omega} = \frac{0.00022}{6280} = 3.35 \times 10^{-8} \text{ m} = 3.35 \times 10^{-5} \text{ mm}$$

120 dB re: 20 μPa

$$p'_{rms} = 20 \times 10^{(120/20)-6} = 20 \text{ Pa}$$

$$P = \sqrt{2}\, p'_{rms} = 28.28 \text{ Pa}$$

$$V = \frac{28.28}{410} = 0.069 \text{ m/s}$$

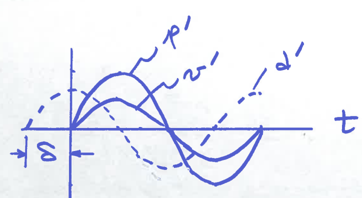
$$D = \frac{0.069}{6280} = 1.10 \times 10^{-5} \text{ m} = 1.10 \times 10^{-2} \text{ mm}$$

PLANE ACOUSTIC WAVES (CONT'D)

WE SEE THAT EVEN AT THE TRESHOLD OF PAIN, THE AMPLITUDES OF VELOCITY AND DISPLACEMENT ARE EXTREMELY SMALL ! SIZE OF A MOLECULE (OXYGEN OR NITROGEN) $\approx 0.3 \times 10^{-8}$ m !

- PHASE INFORMATION

ACOUSTIC PRESSURE AND ACOUSTIC VELOCITY ARE IN PHASE. THE DISPLACEMENT LEADS p' AND v' BY $\pi/2$ OR $90°$



$$\text{PHASE} = \frac{2\pi\delta}{T} = \frac{\pi}{T}$$

BECAUSE $\delta = \frac{T}{4}$

- ACOUSTIC WAVES TRANSFER ENERGY FROM THE SOURCE TO THE SPACE AROUND THE SOURCE. POWER = ENERGY/s

UNIT OF POWER IN SI UNITS : WATTS (W)

1 WATT = 1 N·m/s

1 HORSE POWER = 746 WATTS

ACOUSTIC ENERGY IS GENERALLY VERY SMALL.

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PLANE ACOUSTIC WAVES (CONT'D)

ACOUSTIC ENERGY QUANTITIES

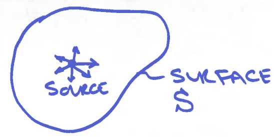
I: ACOUSTIC INTENSITY = ACOUSTIC POWER CROSSING $1\ m^2$ AREA

$$I = \frac{p_{rms}^2}{\rho_0 c} \quad W/m^2 \quad \text{(FOR PLANE WAVES)}$$

$\rho_0 c$ IS CALLED CHARACTERISTIC ACOUSTIC IMPEDANCE OF THE MEDIUM WHICH IS AIR HERE. (410 RAYLS AT 20°C)

ACOUSTIC POWER FROM A SOURCE = W

$$W = \int_S I\, dS \quad \text{WATTS}$$



WHERE S IS FAR ENOUGH FROM THE SOURCE SUCH THAT THE ACOUSTIC WAVES LOOK LIKE PLANE WAVES.

w: ENERGY DENSITY = ACOUSTIC ENERGY / UNIT VOLUME OF SPACE

$$= \frac{p_{rms}^2}{\rho_0 c^2} = \frac{I}{c} \quad N\text{-}m/m^3$$

FOR PLANE WAVES (THIS HAS THE UNITS OF PRESSURE)

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PLANE ACOUSTIC WAVES (CONT'D):

- THE DECIBLE SCALE: FOR ACOUSTIC POWER

$$L_W = 10 \log_{10}\left(\frac{W}{W_0}\right)$$

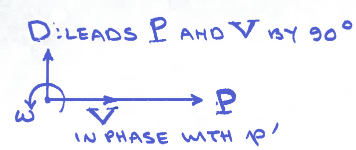
W : ACOUSTIC POWER $= \int_S I\, dS$

W_0 : REFERENCE POWER $= 10^{-12}$ WATT

W_0 IS THE POWER THAT A PLANE WAVE WITH $p'_{rms} = 20\ \mu Pa$ TRANSFERS ACROSS A SURFACE WITH AREA $1\ m^2$.

- THE USE OF PHASORS (ROTATING VECTORS) TO SHOW THE RELATION BETWEEN p', v' AND d' FOR PLANE ACOUSTIC WAVES

IF WE LET THE PHASORS SHOWN ON THE RIGHT TO ROTATE WITH ANGULAR VELOCITY $\omega = 2\pi f$ AND PROJECT THESE ON A VERTICAL OR HORIZONTAL LINE, WE GET THE SINUSOIDAL CURVES FOR p', v' AND d' ON SLIDE 2/4.



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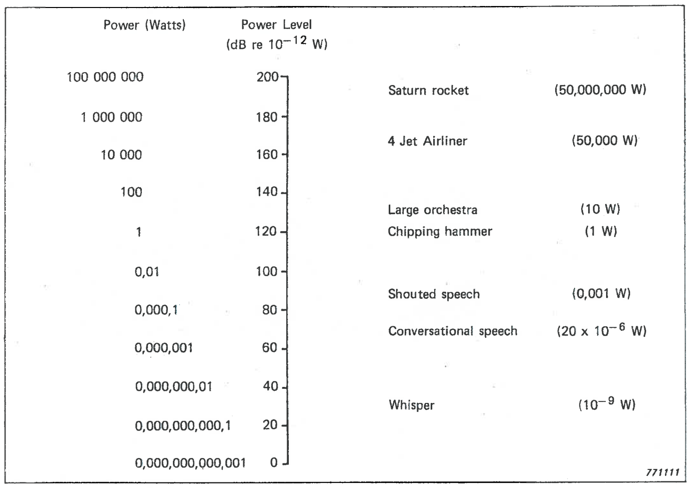
PLANE ACOUSTIC WAVES (CONT'D)



Sound Power output of some typical noise sources

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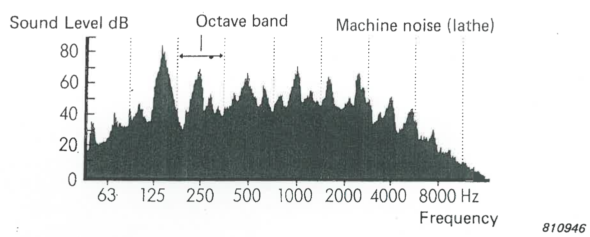
PLANE ACOUSTIC WAVES (CONT'D)



Noise is an irregular combination of tones at all frequencies. The octave band centre frequencies are shown on the scale (WE WILL DISCUSS OCTAVE BAND LATER)

NEAR A SOURCE OF SOUND, LIKE A NOISY MACHINERY, SOUND WAVES ARE VERY COMPLEX AND UNLIKE A PLANE WAVE. FAR FROM THE SOURCE AND IN THE ABSENCE OF SOLID REFLECTING SURFACES, THE SOUND WAVE IS PLANE. WHEN WE HAVE REFLECTING SURFACES, THE SOUND FIELD IS MORE COMPLICATED. THIS SITUATION WILL BE DISCUSSED LATER.

- TWO TERMS : INFRASOUND : SOUND WITH FREQ. LESS THAN 20 HZ
  ULTRASOUND : SOUND WITH FREQ. OVER 20,000 HZ

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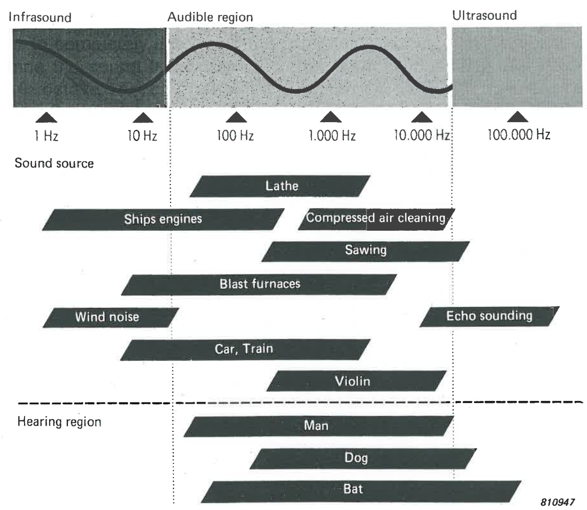
PLANE ACOUSTIC WAVES (CONT'D)



Approximate limits for the audible ranges of different mammals and the frequency ranges of different sound sources.

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PLANE ACOUSTIC WAVES (CONT'D)

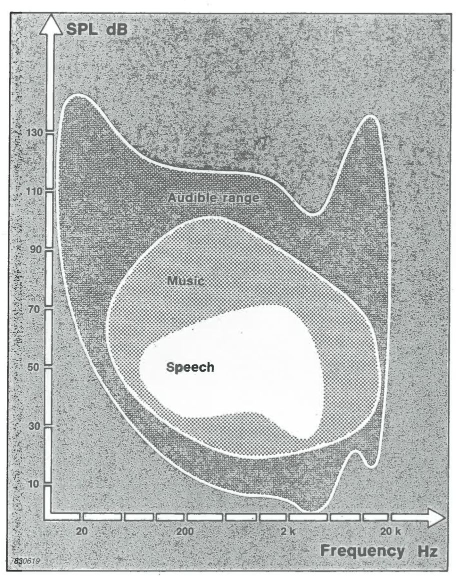
- THE FIGURE ON THE RIGHT SHOWS THE RANGE OF FREQUENCY AND ACOUSTIC LEVEL OF INTEREST TO HUMANS. THE AUDIBLE RANGE IS PRODUCED BY THE RESPONSE OF HUMAN EARS.

- FOR PLANE ACOUSTIC WAVES WHICH ARE COMBINATION OF MANY TONES, WE HAVE

  ACOUSTIC INTENSITY = SUM OF ACOUSTIC INTENSITIES OF THE TONES (IN W/m^2)

  ACOUSTIC POWER = SUM OF ACOUSTIC POWERS OF THE TONES (IN WATTS)

  THEN USE $L_W = 10 \log_{10}\left(\frac{W}{W_0}\right)$



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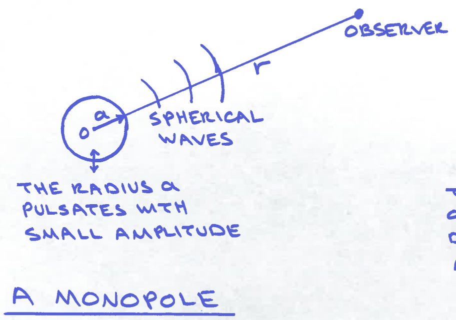
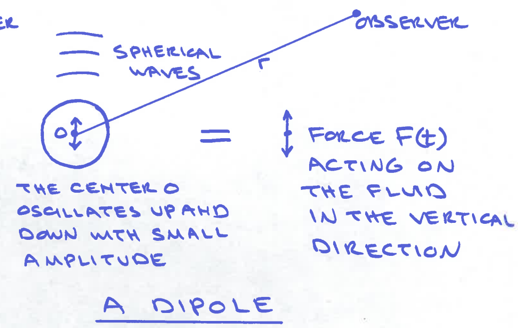
SPHERICAL ACOUSTIC WAVES

MANY ACOUSTIC SOURCES ARE SPHERICAL. WE WILL STUDY TWO MODELS KNOWN AS MONOPOLES (PULSATING SPHERE) AND DIPOLES (OSCILLATING SPHERE OR FLUCTUATING FORCES)





WE ARE GOING TO ASSUME THAT THE MEASUREMENT DISTANCE r FROM THE OBSERVER TO THE CENTER OF THE SPHERE IS MUCH LARGER THAN THE RADIUS OF THE SPHERE a.

SPHERICAL ACOUSTIC WAVES (CONT'D)

MONOPOLE SOURCE - THE ACOUSTIC PRESSURE IS SYMMETRICAL IN ALL DIRECTIONS, i.e. $p' = p'(r,t)$, $\omega = 2\pi f$, $k = \frac{\omega}{c}$

$$p'(r,t) = \frac{P \sin(\omega t - kr)}{r}$$

r ← SPHERICAL ATTENUATION

WHERE P IS A CONSTANT. YOU MAY THINK OF P/r AS AMPLITUDE. IT CAN BE RELATED TO THE AMPLITUDE OF PULSATION OF THE RADIUS OF SPHERE. WE WILL NOT NEED TO DO IT HERE. WE NOTE THAT

$$p'(2r, t) = \frac{P \sin(\omega t - 2kr)}{2r}, \text{ AMPLITUDE} = \frac{P}{2r}$$

i.e. DOUBLING THE DISTANCE HAS REDUCED THE AMPLITUDE BY HALF.

ALSO $r p' = P \sin(\omega t - kr)$ THIS MEANS THAT THE QUANTITY rp' IS LIKE A PLANE WAVE AND TRAVELS WITHOUT CHANGING ITS SHAPE.

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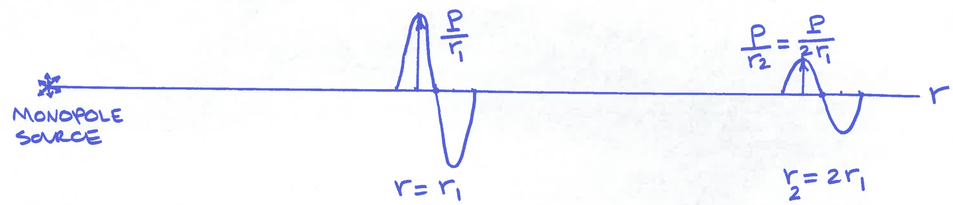
SPHERICAL ACOUSTIC WAVES (CONT'D)

MONOPOLE SOURCE (CONT'D)



ACOUSTIC VELOCITY $v'(r,t) = \dfrac{p'(r,t)}{\rho_0 c} = \dfrac{p'(r,t)}{410}$

ACOUSTIC VELOCITY BEHAVES LIKE ACOUSTIC PRESSURE

ACOUSTIC VELOCITY AMPLITUDE $= \dfrac{P}{\rho_0 c r}$

ACOUSTIC INTENSITY $= \dfrac{P^2}{2\rho_0 c r^2}$ GOES AS $1/r^2$

ACOUSTIC POWER $= 4\pi r^2 \times \dfrac{P^2}{2\rho_0 c r^2}$

$$= \frac{2\pi P^2}{\rho_0 c}$$

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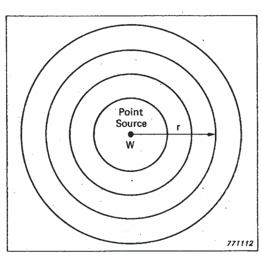
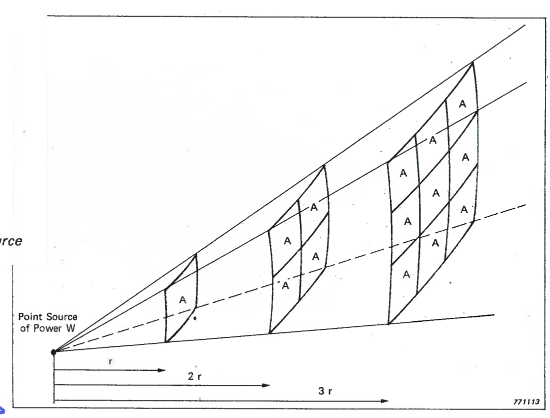
SPHERICAL ACOUSTIC WAVES (CONT'D)

MONOPOLE SOURCE (CONT'D)



The propagation of spherical wavefronts from a point source



The dispersion of sound from a point source

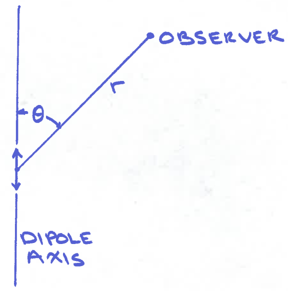
THE ACOUSTIC INTENSITY VARIES AS $1/r^2$ BECAUSE DOUBLING THE DISTANCE FROM THE SOURCE INCREASES THE AREA THAT ENERGY MUST SPREAD BY A FACTOR OF FOUR. THEREFORE, SINCE ACOUSTIC POWER = AC. INTENSITY X AREA, THE AC. INTENSITY MUST DECREAS BY A FACTOR OF 4.

SPHERICAL ACOUSTIC WAVES (CONT'D)

DIPOLE SOURCE

$$p'(r,\theta,t) = \frac{P\cos\theta \sin(\omega t - kr)}{r}$$

$$v'(r,\theta,t) = \frac{P\cos\theta \sin(\omega t - kr)}{\rho_0 c r}$$

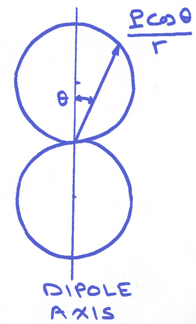


THIS TIME THE RADIATION PATTERN DEPENDS ON ANGLE θ

AC. PRESSURE AMPLITUDE $= \dfrac{P\cos\theta}{r}$

AC. VELOCITY AMPLITUDE $= \dfrac{P\cos\theta}{\rho_0 c r}$

THE AMPLITUDE VARIATION ON A SPHERE OF RADIUS r AND IN A PLANE CONTAINING THE DIPOLE AXIS IS SHOWN ON THE RIGHT. TO GET THE FULL 3D PATTERN, ROTATE THIS FIGURE AROUND THE DIPOLE AXIS TO GET TWO SPHERES.



SPHERICAL ACOUSTIC WAVES (CONT'D)

DIPOLE SOURCE (CONT'D)

- NOTE THAT THE MAXIMUM AMPLITUDE (NOSE) IS ON THE DIPOLE AXIS AND IS EQUAL TO $\frac{P}{r}$.

  THE AMPLITUDE IN THE PLANE $\theta = 90°$, i.e. THE PLANE NORMAL TO THE DIPOLE AXIS IS ZERO.

- ACOUSTIC INTENSITY $= \dfrac{P^2\cos^2\theta}{2\rho_0 c r^2}$ $\quad W/m^2$

  FOR FIXED θ, THIS FALLS AS $1/r^2$. MAX. AC. INTENSITY IS AGAIN ON THE DIPOLE AXIS

- ACOUSTIC POWER $= \dfrac{\pi P^2}{\rho_0 c} \displaystyle\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta$

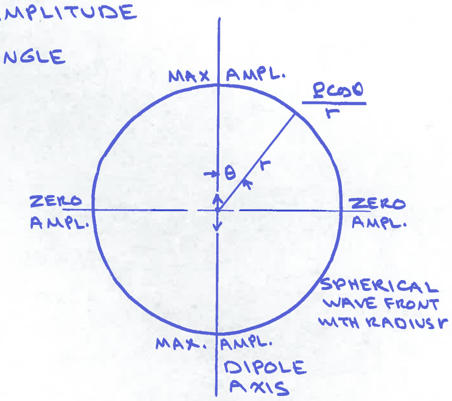
$$= \frac{2\pi P^2}{3\rho_0 c}$$

THEREFORE, IF A MONOPOLE AND DIPOLE HAVE THE SAME ACOUSTIC PRESSURE AMPLITUDE ON THE DIPOLE AXIS, THEN THE MONOPOLE RADIATES ACOUSTIC ENERGY THREE TIMES THAT OF A DIPOLE. A MONOPOLE IS SAID TO BE A MORE EFFICIENT SOUND RADIATOR.

SPHERICAL ACOUSTIC WAVES (CONT'D)

DIPOLE SOURCE (CONT'D)

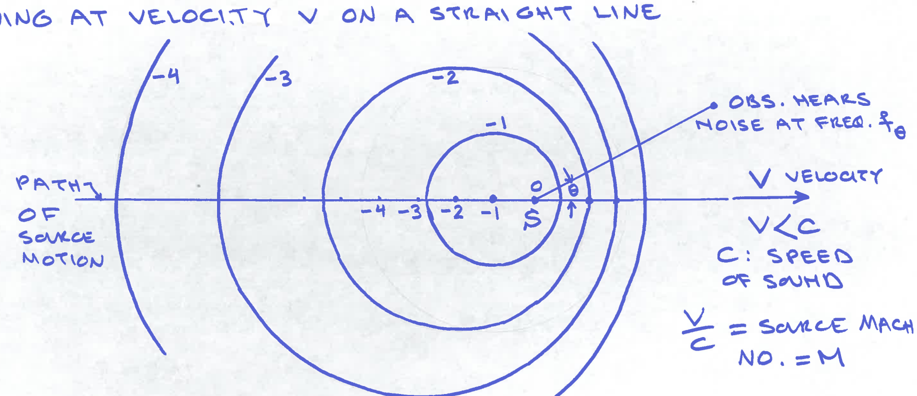
- NOTE THAT A DIPOLE, LIKE A MONOPOLE, STILL SENDS OUT SPHERICAL WAVES. BUT THE AMPLITUDE VARIES AS THE COSINE OF THE ANGLE θ FROM THE DIPOLE AXIS



- DIPOLES ARE MODELS OF FLUCTUATING FORCES ON THE AIR. THESE FORCES ARE PRODUCED BY PROPELLER AND HELICOPTER ROTORS. MONOPOLES ARE THE MODEL OF THE THICKNESS OF AIRFOILS AS THEY MOVE IN THE AIR. THE NOISE PRODUCED BY THICKNESS IS CALCULATED USING A MONOPOLE MODEL.

MOVING SOURCES – THE DOPPLER EFFECT

SOURCE GENERATING PURE TONE AT FREQUENCY f AND MOVING AT VELOCITY V ON A STRAIGHT LINE



THIS IS A PICTURE OF THE WAVEFRONTS TAKEN AT TIME $t=0$ WHEN S IS AT POSITION 0. S: SOURCE. AT TIMES $t=-1,-2,\ldots$ THE SOURCE WAS AT POSITIONS $-1,-2,\ldots$. THE CORRESPONDING WAVEFRONTS AT $t=0$ ARE MARKED BY $-1,-2,\ldots$ THE FREQUENCY OF SOUND HEARD BY AN OBSERVER AT ANGLE θ IS

$$f_\theta = \frac{f}{1-(V/c)\cos\theta} = \frac{f}{1-M\cos\theta},$$

$f_\theta > f$ SOURCE APPROACHING

$f_\theta < f$ SOURCE RECEDING

<u>THIS IS THE DOPPLER EFFECT</u>

Lecture 3

– The Ear and Subjective Effects of Noise

The structure of ear, how ear hears noise, masking and critical bands, octave and 1/3 octave band filters, loudness and its determination, the phon and sone scales, age related and noise related hearing loss, subjective rating using acoustic measurements, the weighting network, equivalent continuous sound level, perceived noise level

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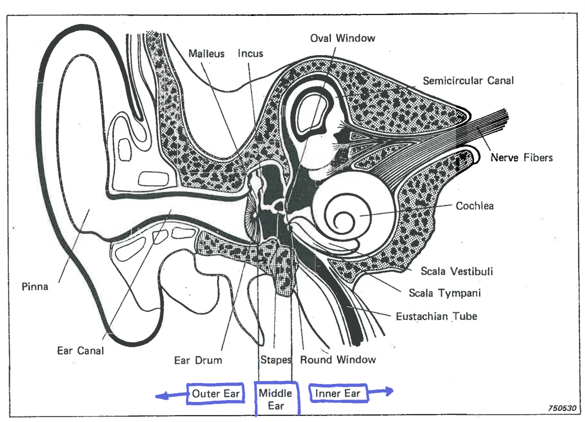
THE EAR AND SUBJECTIVE EFFECTS OF NOISE

THE STRUCTURE OF EAR

- HUMAN EAR IS A VERY SENSITIVE INSTRUMENT. OUR HEARING IS USED FOR
  - COMMUNICATION (SPEECH)
  - ENJOYMENT (MUSIC)
  - WARNING OF DANGER
- THE OUTER EAR IS FOR DIRECTING THE SOUND TO EAR DRUM
- THE MIDDLE EAR IS FOR TRANSFERING THE EAR DRUM VIBRATION MECHANICALLY (IMPEDANCE MATCHING)
- THE INNER EAR IS FOR SENSING THE SOUND AND CONVERTING IT TO A SIGNAL TO SEND TO THE BRAIN (ELECTRONICS!)



The main parts of the ear

- SEMICIRCULAR CANAL FOR SENSING BALANCE

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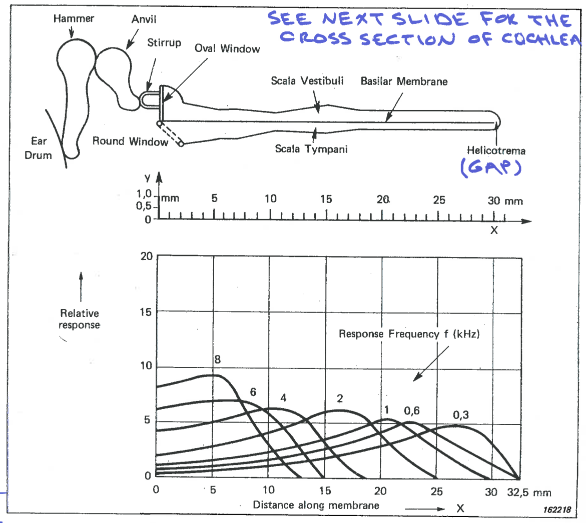
THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

THE STRUCTURE OF EAR (CONT'D)

THE EAR DRUM VIBRATION IS TRANSMITTED TO THE OVAL WINDOW BY THE STIRRUP (OR STAPES). THE COCHLEA IS LIQUID FILLED (APPROX. 1500 m/s SOUND SPEED). A WAVE TRAVELS IN THE UPPER CANAL (SCALA VESTIBULI), PAST HELICOTEREMA INTO LOWER CANAL (SCALA TYMPANI). THE ROUND WINDOW SERVES TO REDUCE REFLECTIONS. AS THE WAVE MOVES THROUGH THE CANALS, THE BASILAR MEMBRANE DEFLECTS WITH LOCATION OF MAXIMUM DEFLECTION DEPENDING ON FREQUENCY: HIGH FREQ. NEAR OVAL WINDOW, LOW FREQ. NEAR HELICOTREMA.



Longitudinal section of the cochlea showing the positions of response maxima (THE COCHLEA IS SHOWN UNFURLED)

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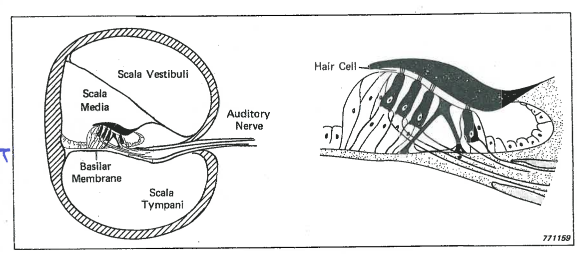
THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

THE STRUCTURE OF EAR (CONT'D)

THE DEFLECTION OF BASILAR MEMBRANE IS SENSED BY A MULTITUDE OF EXTREMELY SENSITIVE HAIR CELLS WHICH CONVERT THE ANALOG SIGNAL INTO NERVE IMPULSES. THESE ARE TRANSMITTED TO THE BRAIN BY AUDITORY NERVES.



Section across the cochlea

THE NERVE IMPULSES HAVE FREQUENCY AND LOUDNESS INFORMATION THAT THE LISTENER FEELS.

THE HEALTHY HUMAN EAR CAN HEAR FROM 20 Hz TO 20 kHz (A RATIO OF 1000) AND THE INTENSITY RATIO OF THE LOUDEST TO THE QUIETEST NOISE IS 10^{12}!

LOUDNESS OF NOISE IS A SUBJECTIVE EFFECT THAT IS NOT DIRECTLY RELATED TO THE ACOUSTIC PRESSURE LEVEL.

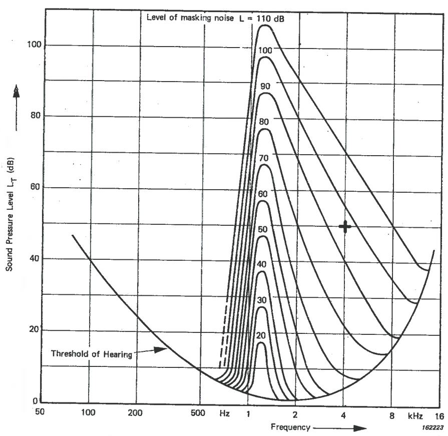
THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

MASKING AND THE CRITICAL BANDS

THE BACKGROUND NOISE CAN PREVENT THE INTELLIGIBILITY OF SPEECH. WE SAY THAT THE SPEECH IS MASKED BY THE BACKGROUND NOISE (MASKING NOISE). INSTEAD OF SPEECH, WE MAY HAVE A WARNING SIGNAL IN AN AIRCRAFT COCKPIT WHICH IS MASKED.

SINCE THE BACKGROUND NOISE CAN BE OF MANY VARIETIES, MANY PARAMETERS INFLUENCE MASKING. FOR NARROW BAND NOISE, WE HAVE THESE THREE RULES:

1. A narrow band of noise causes more masking than does a pure tone of the same intensity centred at the same frequency.
2. At low levels, masking is confined to a fairly narrow band around the masking noise's centre frequency. As the level of the masking noise increases so does the frequency range over which it has an effect.
3. The masking effect is not symmetrical about the centre frequency of the masking noise. Frequencies above the centre frequency are more easily masked than those below.



Masking effect of a narrow band noise centred at 1200 Hz at various levels (after Zwicker). A 50 dB 4 kHz tone (marked +) can be heard if the masking noise level is 90 dB, but is masked if its level rises to 100 dB

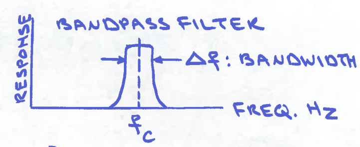
MASKING EFFECT IS SOMETIMES DESIRED IN QUIET OFFICES WHERE PEOPLE CAN HEAR CONVERSATION NEARBY AND GET DISTRACTED.

THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

MASKING AND THE CRITICAL BANDS (CONT'D)

THE EAR ACTS AS A SET OF OVERLAPPING CONSTANT PERCENTAGE BANDWIDTH BANDPASS FILTERS.

- A FILTER IS A DEVICE WHICH ALLOWS A RANGE OF FREQUENCY (BANDWIDTH) OF SIGNAL (NOISE, CURRENT, ETC.) TO PASS THROUGH THE DEVICE.



f_c : CENTER FREQ.

FOR A CONSTANT % BANDWIDTH BANDPASS FILTER, $\Delta f / f_c$ = CONST.

- INCREASING BANDWIDTH OF A NOISE BEYOND A CRITICAL VALUE DOES NOT INCREASE ITS MASKING EFFECT OF A PURE TONE AT ITS CENTER FREQUENCY. THIS IS THE CRITICAL BANDWIDTH.
- THE CRITICAL BANDWIDTH OF THE EAR IS APPROXIMATELY 23% OF THE CENTER FREQ., i.e. 1/3 OF OCTAVE. THIS JUSTIFIES THE USE OF 1/3 OCTAVE BAND ANALYSIS IN NOISE MEASUREMENTS
- CRITICAL BANDS ARE DESIGNATED AS "BARK".

Critical Band (Bark)	1	2	3	4	5	6	7	8
Centre Frequency (Hz)	50	150	250	350	450	570	700	840
Bandwidth f (Hz)	100	100	100	100	110	120	140	150
Critical Band (Bark)	9	10	11	12	13	14	15	16
Centre Frequency (Hz)	1000	1170	1370	1600	1850	2150	2500	2900
Bandwidth f (Hz)	160	190	210	240	280	320	380	450
Critical Band (Bark)	17	18	19	20	21	22	23	24
Centre Frequency (Hz)	3400	4000	4800	5800	7000	8500	10500	13500
Bandwidth f (Hz)	550	700	900	1100	1300	1800	2500	3500

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Table of critical bands

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THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

DIGRESSION: OCTAVE AND 1/3 OCTAVE BAND FILTERS

- OCTAVE BAND FILTERS : THE AUDIBLE FREQ. RANGE IS DIVIDED INTO TEN CONTIGUOUS (i.e. NO GAPS!) BANDS OF FREQUENCIES CALLED OCTAVES SUCH THAT

$$\frac{f_u}{f_L} = \frac{\text{UPPER FREQ.}}{\text{LOWER FREQ.}} = 2 \quad , \text{CENTER FREQ.} = f_c = \sqrt{f_L \cdot f_u} \quad \text{BY DEFINITION}$$

WE CAN SHOW THAT $f_L = \dfrac{f_c}{\sqrt{2}}$, $f_u = \sqrt{2}\, f_c$

OCTAVE BANDS ARE IDENTIFIED BY CENTER FREQ.

EXAMPLE : THE $f_c = 500$ Hz OCTAVE BAND HAS $f_L = 500/\sqrt{2}$ $= 354$ Hz , $f_u = \sqrt{2} \times 500 = 707$ Hz

$$\Delta f = \text{BANDWIDTH} = f_u - f_L = \left(\sqrt{2} - \frac{1}{\sqrt{2}}\right) f_c = \frac{f_c}{\sqrt{2}}$$

THEREFORE $\dfrac{\Delta f}{f_c} = \dfrac{1}{\sqrt{2}} = 0.71$ OR 71 % , i.e. AN OCTAVE BAND FILTER IS A CONST. % FILTER.

- 1/3 OCTAVE BAND FILTERS : $\Delta f / f_c = 0.23$

THREE ADJACENT BANDS COVER ONE OCTAVE. THE CENTER FREQUENCIES ARE SET BY INTERNATIONAL AGREEMENT.

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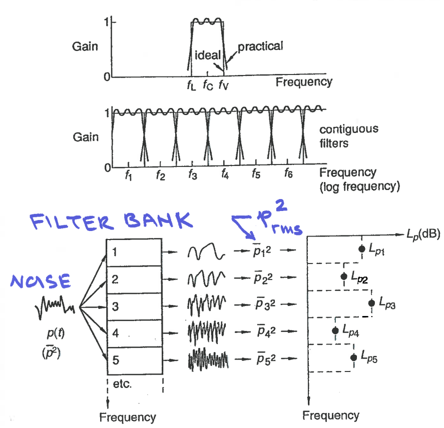
THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

DIGRESSION: OCTAVE AND 1/3 OCTAVE FILTERS (CONT'D)



$$\begin{cases} \bar{p}^2 = \sum_n \bar{p}_n{}^2 \\ L_{p0A} = 10 \log_{10}\left[\dfrac{\bar{p}_{0A}{}^2}{p_0{}^2}\right] \neq \sum_n L_{pn} \end{cases}$$ NOTE !

NOTE HOW WE GET THE SOUND PRESSURE LEVEL OF THE NOISE.

Table A2.1 Standard frequency bands (Hz) and A-weighting

Band number	Octave band centre frequency	One-third octave band centre frequency	Band limits Lower	Band limits Upper	A-weighting (dB)
14		25	22	28	−44.7
15	31.5	31.5	28	35	−39.4
16		40	35	44	−34.6
17		50	44	57	−30.2
18	63	63	57	71	−26.2
19		80	71	88	−22.5
20		100	88	113	−19.1
21	125	125	113	141	−16.1
22		160	141	176	−13.4
23		200	176	225	−10.9
24	250	250	225	283	−8.6
25		315	283	353	−6.6
26		400	353	440	−4.2
27	500	500	440	565	−3.2
28		630	565	707	−1.9
29		800	707	880	−0.8
30	1 000	1 000	880	1 130	0.0
31		1 250	1 130	1 414	+0.6
32		1 600	1 414	1 760	+1.0
33	2 000	2 000	1 760	2 250	+1.2
34		2 500	2 250	2 825	+1.3
35		3 150	2 825	3 530	+1.2
36	4 000	4 000	3 530	4 400	+1.0
37		5 000	4 400	5 650	+0.5
38		6 300	5 650	7 070	−0.1
39	8 000	8 000	7 070	8 800	−1.1
40		10 000	8 800	11 300	−2.5
41		12 500	11 300	14 140	−4.3
42	16 000	16 000	14 140	17 600	−6.6
43		20 000	17 600	22 500	−9.3

Reproduced in part from *Fundamentals of Noise and Vibration* (Fahy and Walker, 1998) – see Bibliography.

FROM : FRANK FAHY " FOUNDATIONS OF ENGINEERING ACOUSTICS ", 2001

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THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

DIGRESSION : OCTAVE AND 1/3 OCTAVE FILTERS (CONT'D)

IN READING AN OUTPUT OF AN OCTAVE OR 1/3 OCTAVE FILTER, ONE SHOULD BE AWARE THAT THE SOUND PRESSURE LEVEL AT THE CENTER FREQUENCY OF A FILTER BAND IS DRAWN ACROSS THE BAND AT THAT LEVEL. THIS WILL GIVE THE WRONG IMPRESSION THAT THERE IS LESS ENERGY IN THE 1/3 OCTAVE SPECTRUM THAN THE OCTAVE SPECTRUM FOR THE SAME ACOUSTIC SIGNAL.

NOTE THAT IN AN OCTAVE BAND WE HAVE THREE 1/3 OCTAVE BANDS SO THAT

$$(p^2_{rms})_{OCT.} = (p^2_{rms})_{1/3\,OCT.1} + (p^2_{rms})_{1/3\,OCT.2} + (p^2_{rms})_{1/3\,OCT.3}$$

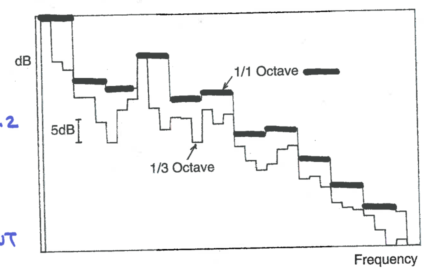

Fig. A2.5 1/1 and 1/3 octave spectra of the same signal.

(FROM FRANK FAHY)

AND IF THE LEVELS IN THE THREE BANDS ARE APPROXIMATELY CONSTANT $(p^2_{rms})_{1/3\,OCT}$, SAY. THEN

$(p^2_{rms})_{OCT} \approx 3(p^2_{rms})_{1/3\,OC}$ OR

$(L_p)_{OCT} = (L_p)_{1/3\,OCT} + 5$ dB re: $20\mu Pa$, $5 \approx 10 \log_{10} 3$.

(END OF DIGRESSION)

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THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

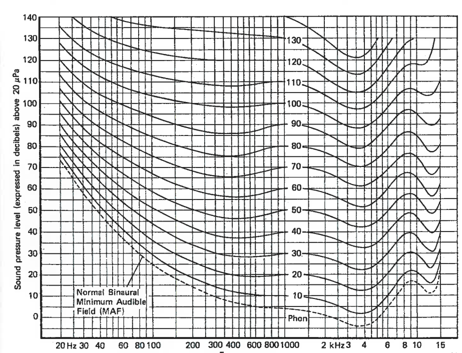
LOUDNESS AND ITS DETERMINATION

THE PHON AND SONE SCALES

EQUAL LOUDNESS CONTOURS FOR PURE TONES IS SHOWN ON THE RIGHT (FOUND EXPERIMENTALLY)

THE VERTICAL SCALE IS SOUND PRESSURE LEVEL dB re: $20\mu Pa$.

WHEN EQUAL LOUDNESS CURVES ARE AT 1000 Hz, THEIR DECIBEL LEVELS DEFINE THE LOUDNESS IN PHONS.



Normal Equal Loudness Contours for pure tones

SUBJECTIVELY, A 10 dB INCREASE IN THE LEVEL OF THE SAME NOISE IS PERCEIVED AS DOUBLING OF LOUDNESS. A SCALE WHICH INCREASES LINEARLY WITH PERCEIVED LOUDNESS CAN BE DEFINED BY

$$S = 2^{\frac{P-40}{10}} \quad \text{SONES}$$

WHERE P IS LOUDNESS IN PHONS. WE SEE THAT FOR $P = 40$ PHONS, $S = 1$, FOR $P = 50$ PHONS, $S = 2$. FOR ANY P, S INCREASES BY A FACTOR OF 2 FOR $P + 10$ PHONS.

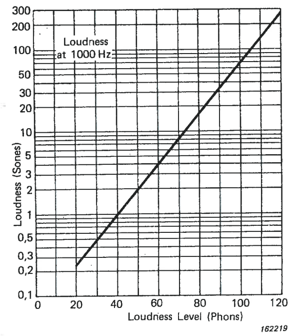
SEE THE FIGURE IN THE NEXT SLIDE.

THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

LOUDNESS AND ITS DETERMINATION (CONT'D)

THE PHON AND SONE SCALES (CONT'D)

THIS METHOD OF CALCULATING LOUDNESS IS GOOD FOR PURE TONES. FOR MORE COMPLEX NOISE INVOLVING BROADBANDNOISE MIXED WITH PURE TONE AND NARROW BAND COMPONENTS THERE ARE TWO METHODS BY ZWICKER AND STEVENS THAT ARE WIDELY USED. BOTH THESE METHODS ARE ACCEPTED BY ISO. THEY USE OCTAVE & 1/3 OCTAVE SPECTRA OF THE NOISE. THE STEVENS METHOD IS SIMPLER BUT MORE RESTRICTED THAN ZWICKER METHOD. THESE METHODS ARE DESCRIBED IN ANY ACOUSTICS HANDBOOK.



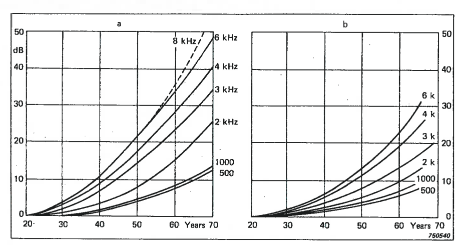
The relationship between loudness in Sones and loudness level in Phons

- FOR SHORT DURATION SOUNDS OF LESS THAN ONE SECOND DURATION, THE LOUDNESS DEPENDS ON THE ENERGY IN THE PULSE. THE SHORTER THE PULSE, THE LESS THE LOUDNESS.

THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

AGE RELATED AND NOISE RELATED HEARING LOSS

AS WE AGE, OUR HEARING LOSS INCREASES AT ALL FREQUENCIES. WE LOSE MORE HEARING AT HIGH THAN LOW FREQUENCIES. THIS PHENOMENON IS KNOWN AS PRESBYCUSIS.

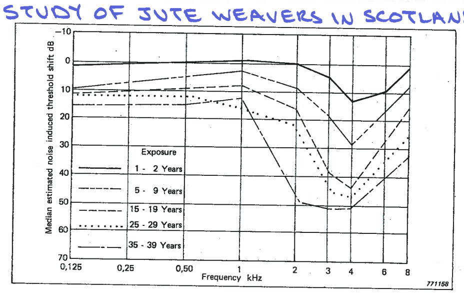


Average normal age-related hearing loss
a) According to Spoor *b) According to Hinchcliffe*

NOISE EXPOSURE CAN CAUSE BOTH TEMPORARY AND PERMANENT HEARING LOSSES. PERMANENT HEARING LOSS CAN OCCUR IF ONE IS EXPOSED TO LOUD NOISE OVER A LONG PERIOD OF TIME.

THE NOISE RELATED HEARING LOSS ALWAYS APPEAR AT ABOUT 4 KHZ. NOTE THAT THE MOST SEVERE HEARING LOSS IS IN THE RANGE OF UNDERSTANDING OF SPEECH

STUDY OF JUTE WEAVERS IN SCOTLAND



The development of noise induced hearing loss

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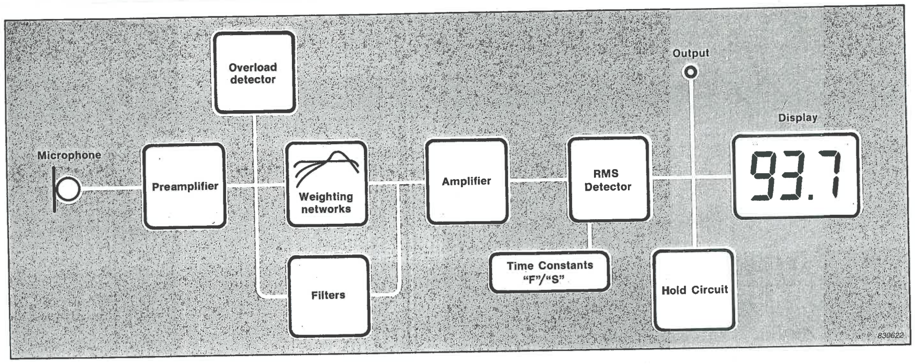
THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

SUBJECTIVE RATING USING ACOUSTIC MEASUREMENTS

MANY PARAMETERS ARE INVOLVED IN THE LOUDNESS EVALUATION OF NOISE. WE CAN USE COMMON ACOUSTIC MEASUREMENT INSTRUMENTS TO GET A GOOD IDEA OF THE LOUDNESS LEVEL. LET US LOOK AT A BASIC SOUND LEVEL METER SHOWN BELOW.



THE SOUND PRESSURE LEVEL WE HAVE DEFINED SO FAR IS SAID TO USE LINEAR SCALE. THIS HAS PRACTICALLY NO RELATION TO LOUDNESS. TO GET A SUBJECTIVE EVALUATION OF NOISE, WE USE THE WEIGHTING NETWORKS.

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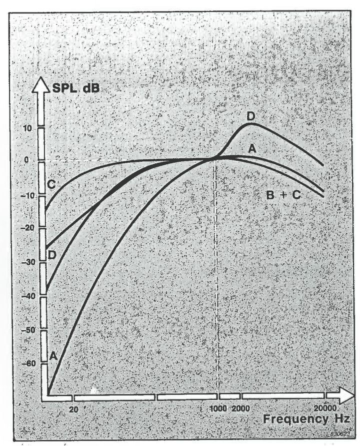
THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

THE WEIGHTING NETWORKS

WE KNOW THAT AT LOW AND AT HIGH FREQUENCIES, THE CONSTANT LOUDNESS CURVES (SLIDE 3/9) CURVE UP SO THAT ON THESE CURVES, THE SPL IS HIGHER THAN THE PHON LEVELS. THEREFORE, TO GET AN IDEA OF LOUDNESS AT THESE FREQUENCIES WE SHOULD GIVE LESS WEIGHT TO THEIR SPL'S. THIS IS DONE BY THE WEIGHTING NETWORKS A, B, C AND D. NOTE THAT A CURVE IS LIKE AN INVERTED EQUAL LOUDNESS CURVE AND IT IS THE MOST WIDELY USED WEIGHTING NETWORK. NOTE ALSO THAT THE A CURVE IS LIKE AN INVERTED EQUAL LOUDNESS CURVE AT LOW SPL,



B IS LIKE AN INVERTED EQUAL LOUDNESS CURVE AT MEDIUM SPL AND C IS LIKE AN INVERTED EQUAL LOUDNESS CURVE AT HIGH SPL. THE D CURVE WAS DEVELOPED FOR AIRCRAFT NOISE MEASUREMENTS. WE INDICATE THESE LEVELS AS dB(A), dB(B), ETC.

THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

SOME OTHER IMPORTANT DEFINITIONS

L_{eq} : EQUIVALENT CONTINUOUS SOUND LEVEL

IF THE A-WEIGHTED LEVEL OF NOISE IS TIME DEPENDENT, THEN $p_A(t)$ THE A-WEIGHTED INSTANTANEOUS ACOUSTIC PRESSURE IS TIME DEPENDENT. THIS IS DEFINED AS $20 \times 10^{-6} \times 10^{dB(A)/20}$ Pa. THE EQUIVALENT CONTINUOUS SOUND LEVEL L_{eq} IS DEFINED AS

$$L_{eq} = 10 \log_{10}\left[\frac{1}{T}\int_0^T \left(\frac{p_A(t)}{p_0}\right)^2 dT\right] \quad dB$$

T : MEASUREMENT TIME, $p_0 = 20\ \mu Pa$

THIS IS USED FOR DEFINING COMMUNITY NOISE STANDARDS SUCH AS L_{DN} : THE DAY-NIGHT AVERAGE SOUND LEVEL, AND

L_{NP} : THE NOISE POLLUTION LEVEL.

IN L_{DN} MORE WEIGHT IS GIVEN TO THE NOISE BETWEEN THE HOURS 22:00 TO 7:00. IN L_{NP}, THE EFFECT OF FLUCTUATIONS IN THE LEVEL OF SOUND IS ADDED TO L_{eq}.

THE EAR AND SUBJECTIVE EFFECTS OF NOISE (CONT'D)

PERCEIVED NOISE LEVEL : PNL

PNL WAS DEVELOPED FOR A SINGLE AIRCRAFT FLYOVER NOISINESS EVALUATION. IT REQUIRES EXTENSIVE CALCULATION PROCEDURE USING 1/3 OCTAVE BAND MEASUREMENT EVERY 1/2 SECONDS OR LESS. THESE MEASUREMENTS ARE WEIGHED AND SUMMED TO GET PERCEIVED NOISINESS IN NOYS AT EACH TIME INTERVAL. THIS VALUE IS THEN CONVERTED TO PERCEIVED NOISE LEVEL IN PNdB. AFTER CORRECTIONS FOR DURATION OF FLYOVER AND TONAL CONTENTS (FROM FANS, FOR EXAMPLE), THE EFFECTIVE PERCEIVED NOISE LEVEL IS OBTAINED. THIS IS TONE-CORRECTED EPNLdB (OR EPNdB).

SEE "ACOUSTIC NOISE MEASUREMENTS" BY BRÜEL & KJAER

Lecture 4

– Some Acoustic Phenomena

Superposition and interference, standing waves in a tube, resonance, the beat phenomenon, plane waves traveling at right angle, the Huygens principle, reflection of plane and spherical waves, diffraction, scattering, refraction by wind and temperature gradients

SOME ACOUSTIC PHENOMENA

SUPERPOSITION AND INTERFERENCE

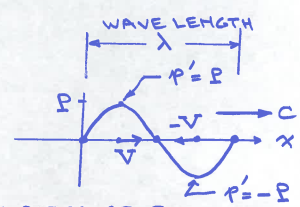
ACOUSTIC WAVES FROM DIFFERENT SOURCES ADD LINEARLY AT EACH POINT OF SPACE AND AT ANY TIME t. THIS IS CALLED THE SUPERPOSITION PRINCIPLE. THIS CAN PRODUCE A DISTRIBUTION OF PRESSURE IN SPACE KNOWN AS INTERFERENCE PATTERN. INTERFERENCE PATTERNS ARE MOST INTERESTING FOR PERIODIC PLANE OR SPHERICAL WAVES.

PLANE WAVES

RIGHT AND LEFT MOVING WAVES

RIGHT MOVING WAVE
$$\begin{cases} p'_+ = P_+ \sin(\omega t - kx) \\ v_+ = \dfrac{P_+}{\rho_0 c} \sin(\omega t - kx) = V_+ \sin(\omega t - kx) \end{cases}$$

p' AND v' ARE IN PHASE



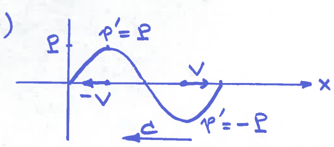
AC. PRESSURE FOR A FIXED t
RIGHT MOVING WAVE

$$\begin{cases} p'_- = P_- \sin(\omega t + kx) \\ v_- = -\dfrac{P_-}{\rho_0 c} \sin(\omega t + kx) = -V_- \sin(\omega t + kx) \end{cases}$$

p' AND v' ARE 180° OUT OF PHASE

$\omega = 2\pi f$ rad/s, $f\lambda = c$ m/s

$k = \dfrac{\omega}{c} = \dfrac{2\pi}{\lambda}$ m^{-1}



AC. PRESSURE FOR A FIXED t
LEFT MOVING WAVE

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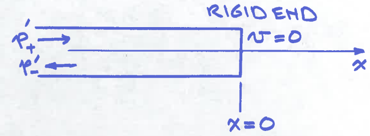
SOME ACOUSTIC PHENOMENA (CONT'D)
SUPERPOSITION AND INTERFERANCE (CONT'D)

STANDING WAVES IN A TUBE

CLOSED TUBE WITH RIGID END WALL



$p'_+ = P_+ \sin(\omega t - kx)$ RIGHT MOVING

$p'_- = P_- \sin(\omega t + kx)$ LEFT MOVING

$$v(0,t) = [p'_+(0,t)/\rho_0 c] - [p'_-(0,t)/\rho_0 c]$$

$$= \frac{1}{\rho_0 c}(P_+ - P_-)\sin\omega t = 0 \Rightarrow P_+ = P_-$$

$$p'(x,t) = p'_+(x,t) + p'_-(x,t) = P_+[\sin(\omega t - kx) + \sin(\omega t + kx)]$$

$$= 2P_+ \cos kx \sin\omega t \qquad \boxed{\text{AMP.} = 2P_+|\cos kx|}$$

$$v(x,t) = \frac{P_+}{\rho_0 c}[\sin(\omega t - kx) - \sin(\omega t + kx)]$$

$$= \frac{-2P_+}{\rho_0 c}\sin kx \cos\omega t = \frac{2P_+}{\rho_0 c}\sin kx \sin(\omega t + \tfrac{\pi}{2})$$

$$= 2V_+ \sin kx \sin(\omega t + \tfrac{\pi}{2}) \qquad \boxed{\text{AMP.} = 2V_+|\sin kx|}$$

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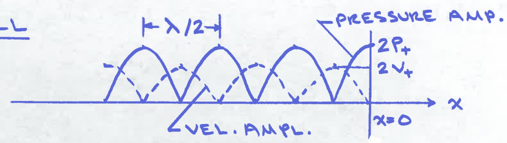
SOME ACOUSTIC PHENOMENA (CONT'D)
SUPERPOSITION AND INTERFERENCE (CONT'D)
STANDING WAVES IN A TUBE (CONT'D)

PRESSURE AND VELOCITY AMPLITUDE DISTRIBUTION IN A TUBE WITH RIGID END WALL



NOTE : AT EACH x, THE ACOUSTIC PRESSURE AND VELOCITY ARE PERIODIC WITH FREQUENCY $\omega/2\pi$. THE AMPLITUDE OF OSCILLATION VARIES ALONG THE TUBE AS A FUNCTION OF x. THE ACOUSTIC VELOCITY LEADS THE ACOUSTIC PRESSURE BY 90° AT ANY POINT ALONG THE TUBE. THE MAXIMUM OF PRESSURE AMPLITUDE OCCURS AT ZERO AC. VELOCITY AMPLITUDE. A PISTON VIBRATING AT A DISTANCE OF A MULTIPLE OF $\lambda/2$ FROM THE END WALL (i.e. AT POINTS WHERE VEL. AMP. = 0) EASILY EXCITES ACOUSTIC WAVES IN THE TUBE. THIS IS THE RESONANCE PHENOMENON.

– WE CAN DO A SIMILAR ANALYSIS FOR OPEN END TUBE WHERE
$p'(0,t) = p'_+(0,t) + p'_-(0,t) = 0 \Rightarrow P_+ = -P_-$, ETC.

SOME ACOUSTIC PHENOMENA (CONT'D)
SUPERPOSITION AND INTERFERENCE (CONT'D)

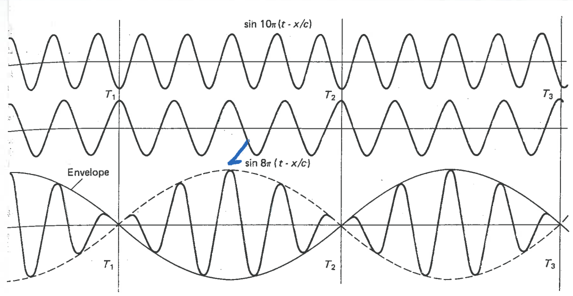
THE BEAT PHENOMENON

LET US TAKE TWO RIGHT MOVING WAVES OF AMPLITUDE UNITY WITH ANG. FREQ. ω_1 AND ω_2 SUCH THAT $|\omega_1 - \omega_2| = \Delta\omega$ SMALL. THE SUPERPOSITION OF THESE WAVES IS

$$p'(x,t) = p_1'(x,t) + p_2'(x,t) = \sin(\omega_1 t - kx) + \sin(\omega_2 t - kx)$$

$$= 2\cos\frac{\Delta\omega}{2}t \; \sin(\omega_{AV} t - kx)$$

WHERE $\omega_{AV} = \dfrac{\omega_1 + \omega_2}{2}$



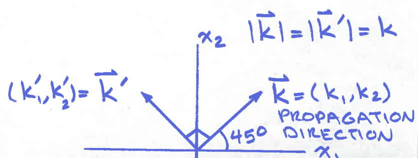
sin $10\pi(t-x/c) + \sin 8\pi(t-x/c) = 2\sin 9\pi(t-x/c)\cos \pi t$

The formation of beats. (WAVE PATTERN SHOWN FOR A FIXED t IN SPACE)

FOR A FIXED x, WE HEAR A PERIODIC SOUND OF FREQUENCY $\omega_{AV}/2\pi$ WHOSE AMPLITUDE VARIES WITH FREQ. $\Delta\omega/4\pi$. THE EAR SENSES THE HIGH FREQ. $\omega_{AV}/2\pi$ WHICH BEATS AT FREQ. $\Delta\omega/4\pi$. BOTH FREQUENCIES ARE SENSED IF THEY ARE IN AUDIBLE RANGE.

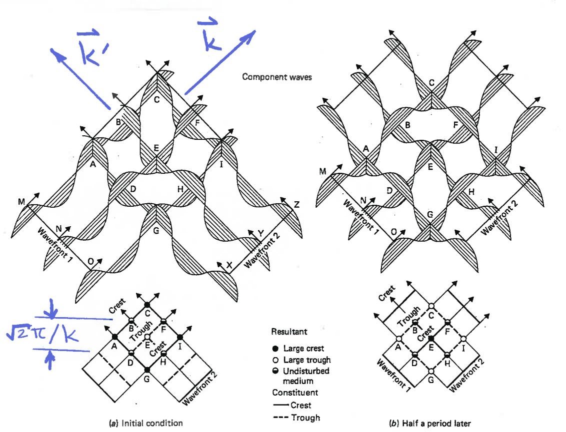
SOME ACOUSTIC PHENOMENA (CONT'D)
SUPERPOSITION AND INTERFERENCE (CONT'D)

INTERFERENCE PATTERNS OF TWO PLANE WAVES TRAVELING AT RIGHT ANGLES TO EACH OTHER



$$\vec{k}\cdot\vec{x} = k_1 x_1 + k_2 x_2$$
$$\vec{k}'\cdot\vec{x} = k_1' x_1 + k_2' x_2$$

$$p_1' = P\sin(\omega t - \vec{k}\cdot\vec{x})$$
$$p_2' = P\sin(\omega t - \vec{k}'\cdot\vec{x})$$



Two identical wave motions crossing at right angles.

WE HAVE A GRID PATTERN OF LINES PARALLEL TO x_1 AND x_2 AXES AT DISTANCES OF $\lambda/2 = \sqrt{2}\pi/k$ WHERE THE PRESSURE AMPLITUDE IS ZERO. THE PRESSURE IS PERIODIC WITH FREQ. $\omega/2\pi$. THE MAXIMUM AMPLITUDE OCCURS AT THE CENTER OF THE SQUARES FORMED BY ZERO PRESSURE LINES.

SOME ACOUSTIC PHENOMENA (CONT'D)

SUPERPOSITION AND INTERFERENCE (CONT'D)

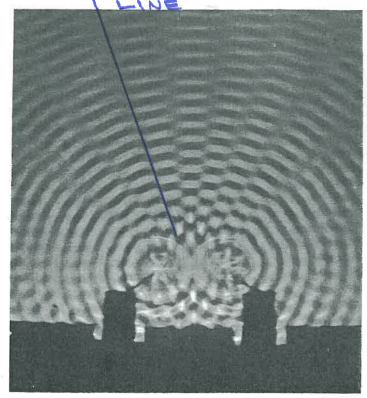
SPHERICAL SOURCES

TWO SPHERICAL WAVES $p_1'(\vec{x},t)$ AND $p_2'(\vec{x},t)$ WILL SUPERPOSE IN SPACE GIVING THE WAVE $p'(\vec{x},t) = p_1'(\vec{x},t) + p_2'(\vec{x},t)$.

THIS EFFECT IS SIMULATED IN A RIPPLE TANK ON THE RIGHT BY TWO OSCILLATING SOURCES AT A FINITE DISTANCE APART.

NOTE THE CONSTRUCTIVE AND DESTRUCTIVE INTERFERENCE PATTERNS.

DESTRUCTIVE INTERFERENCE LINE



A two-source interference pattern in a ripple tank.

SOURCES MOVING IN PHASE

- RIPPLE TANK CAN BE USED TO STUDY MANY WAVE PROPAGATION PHENOMENA.

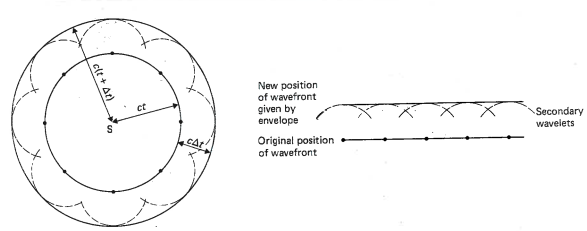
SOME ACOUSTIC PHENOMENA (CONT'D)

THE HUYGENS PRINCIPLE

- THIS PRINCIPLE STATES: i) EACH POINT ON THE WAVEFRONT ACTS AS A SOURCE OF NEW SPHERICAL WAVES, ii) THE NEW WAVEFRONT IS THE ENVELOPE OF ALL THE SPHERICAL WAVES AT A GIVEN TIME.

  THE ENVELOPE IS A SURFACE TANGENT TO THE SPHERICAL WAVES RADIATING OUT FROM A WAVEFRONT.

- THIS PRINCIPLE IS A VERY USEFUL TOOL FOR DISCOVERING MANY ACOUSTIC PHENOMENA (AS WELL AS OTHER WAVE PROPAGATION PHENOMENA)



(a) A circular wavefront (b) A plane wavefront

Fig. 3.3

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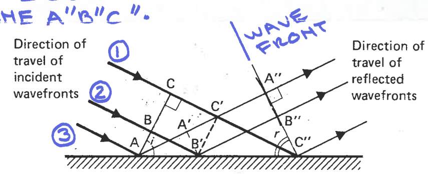
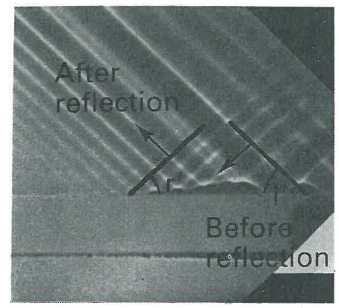
SOME ACOUSTIC PHENOMENON (CONT'D)
THE HUYGENS PRINCIPLE (CONT'D)
REFLECTION PHENOMENON

THE FIGURE ON THE LEFT SHOWS A PLANE WAVE REFLECTING FROM A RIGID SURFACE. THE RAY 1 TO 3 ARE NORMAL TO THE PLANE WAVE. WE WANT TO FIND THE WAVEFROM AT THE MOMENT RAY 3 REACHES THE SURFACE. WE DO THIS BY FINDING WHERE THE SURFACE OF THE SPHERE RADIATING OUT FROM POINT B' IS WHEN RAY 2 REACHED THE SURFACE. THE RADIUS OF THE SPHERE HAS TO BE EQUAL TO $C'C''$, i.e. $B'B'' = C'C''$.

SIMILARLY THE WAVE RADIATED FROM A, WHEN RAY 3 REACHES THE SURFACE FORM A SPHERE OF RADIUS $AA'' = CC''$. THE ENVELOPE IS THE PLANE $A''B''C''$.



Huygens' principle applied to reflection.



A plane wave reflected in a ripple tank.

BY THE ABOVE CONSTRUCTION WE CAN SHOW ANGLE OF INCIDENCE i = ANGLE OF REFLECTION r.

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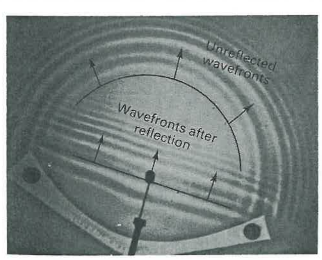
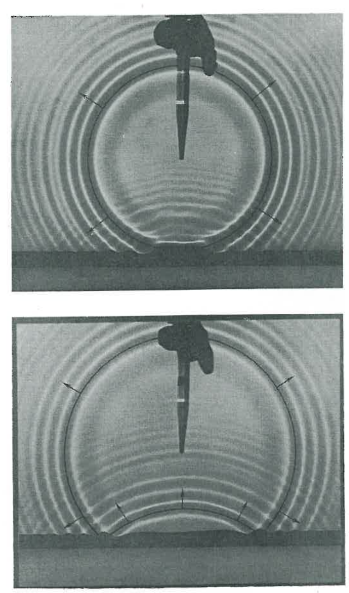
SOME ACOUSTIC PHENOMENA (CONT'D)
THE HUYGENS PRINCIPLE (CONT'D)
REFLECTION PHENOMENON (CONT'D)
SPHERICAL WAVES

HUYGENS PRINCIPLE CAN BE USED TO STUDY THE STRUCTURE OF THE WAVEFRONT PRODUCED BY THE REFLECTION OF A SPHERICAL WAVE FROM A RIGID PLANE OR PARABOLIC SURFACE.



(*b*) The reflection of circular waves from a parabolic reflector; the incident waves are formed at the focus of the parabola and the reflected wavefronts are plane.



(*a*) The reflection of circular waves from a straight barrier. In the lower photograph part of each wavefront has been reflected.

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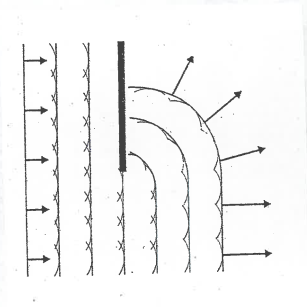
SOME ACOUSTIC PHENOMENA (CONT'D)

THE HUYGENS PRINCIPLE (CONT'D)

DIFFRACTION OF WAVES AT A BREAKWATER AND ITS ANALYSIS BY HUYGENS PRINCIPLE



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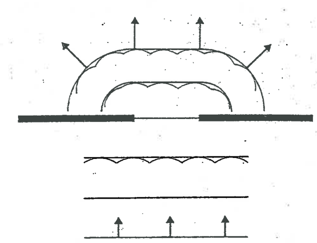
SOME ACOUSTIC PHENOMENA (CONT'D)

DIFFRACTION OF A PLANE WAVE BY AN APERTURE

NOTE THAT FOR LOW FREQUENCY ($\lambda > d$), d THE WIDTH OF THE APERTURE, THE DIFFRACTED WAVE COVERS A WIDER ANGLE. FOR HIGH FREQUENCY ($\lambda < d$), WE HAVE ALMOST A SHARP BEAM. THE HUYGENS PRINCIPLE EXPLAINS SOME OF THE FEATURES OF THE DIFFRACTED FIELD.



HUYGENS PRINCIPLE

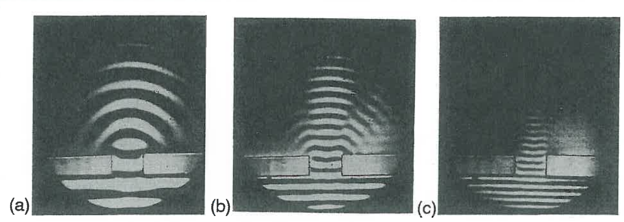


Fig. 12.8 Diffraction of a plane wave by an aperture in a screen: (a) low frequency; (b) medium frequency; (c) high frequency (source unknown).

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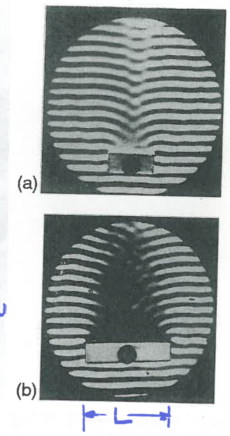
SOME ACOUSTIC PHENOMENON (CONT'D)

SCATTERING BY A RIGID OBJECT

AN OBJECT'S LENGTH SCALE L DETERMINES HOW MUCH IT AFFECTS SCATTERING OF SOUND. IF $L < \lambda$, λ THE WAVELENGTH OF THE SOUND, THEN THE WAVES GO AROUND THE OBJECT AND WE HAVE LITTLE SCATTERING. IF $L > \lambda$, THEN THE SCATTERING EFFECT IS MORE PROMINENT AND WE HAVE A SHADOW ZONE.



a – SMALL OBJECT $L \approx d$

b – LARGE OBJECT $L > d$

NOTE THAT THE BOUNDARY BETWEEN THE SHADOW AND RADIATED REGION IS VERY COMPLICATED AND IS NOT SHARP. IT IS STUDIED BY DIFFRACTION THEORY.

THIS EXPLAINS WHY THE MICROPHONE CAP INFLUENCES THE MEASUREMENTS AT HIGH FREQUENCIES.

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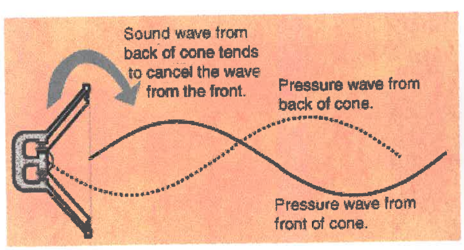
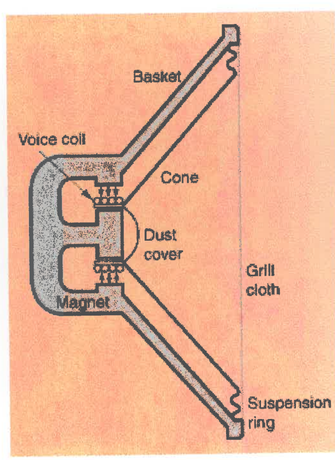
SOME ACOUSTIC PHENOMENA (CONT'D)

SOUND PRODUCTION BY A SPEAKER WITHOUT AN ENCLOSURE AT LOW FREQUENCY





AT LOW FREQUENCY (LONG WAVELENTH COMPARABLE TO SPEAKER DIAMETER) THE WAVES FROM BACK AND FRONT OF THE SPEAKER WHICH ARE ALWAYS 90° OUT OF PHASE ALMOST CANCEL EACH OTHER. THE LOUDSPEAKE IS, THEREFORE, VERY INEFFICIENT AT LOW FREQUENCIES IF IT IS NOT IN AN EN-CLOSURE. ALSO, ALWAYS MAKE SURE THAT YOUR WOOFERS ARE WIRED CORRECTLY TO OPERATE IN PHASE.

SOME ACOUSTIC PHENOMENA (CONT'D)

FRESNEL AND FRAUNHOFER ZONES

THE ACOUSTIC FIELD IN THE VICINITY OF OSCILLATING SURFACES, LIKE LOUDSPEAKERS CAN BE QUITE COMPLICATED. IN THE FARFIELD THE FIELD GENERALLY IS MUCH SIMPLER. THE FIGURE BELOW SHOWS THE RATIO OF ACOUSTIC AMPLITUDE $|\hat{p}|$ TO $\rho c |\hat{v}_n|$, WHERE $|\hat{v}_n|$ IS VELOCITY AMPLITUDE OF AN OSCILLATING PISTON, AS A FUNCTION OF NONDIMENSIONAL DISTANCE z/a ON THE PISTON AXIS. HERE a IS THE PISTON RADIUS. NOTE THAT WE HAVE OSCILLATORY BEHAVIOR NEAR THE PISTON AND THEN WE HAVE THE TYPICAL FAR FIELD BEHAVIOR (1 / DISTANCE). THE FRESNEL AND FRAUNHOFER ZONES ARE DEFINED AS SHOWN IN THE FIGURE.

NOTE THAT IN TERMS OF WAVE LENGTH λ, THE FRESNEL ZONE IS DEFINED BY

$$\frac{z}{\lambda} = \frac{z}{a} \cdot \frac{a}{\lambda} \leq 6 \times 5.5 = 33.$$

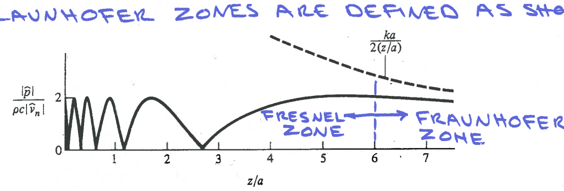


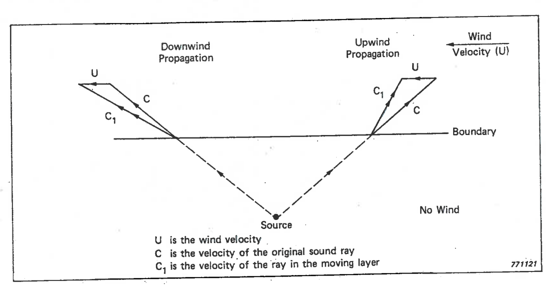
Figure 5-10 Variation along symmetry axis of acoustic-pressure amplitude $|\hat{p}|$ with distance z (units of a) from center of oscillating circular piston of radius a. Plot of $|\hat{p}|/|\rho c \hat{v}_n|$ versus z/a is for $ka/2\pi = 5.5. = \frac{a}{\lambda}$

SOME ACOUSTIC PHENOMENA (CONT'D)

REFRACTION EFFECTS BY TEMPERATURE AND WIND GRADIENTS

THE WIND EFFECT

THE SOUND WAVE RIDES OVER A PARTICLE IN MOTION. THE RAY DIRECTION CHANGES BY THE EFFECT OF THE WIND (REFRACTION). THIS CHANGE OF DIRECTION CAN BE FOUND FROM THE RELATION



Sound propagation across a boundary between layers with different velocities

$$\vec{C}_1 = \vec{U} + c\vec{n}$$

$\vec{C}_1$ EFFECTIVE PROPAGATION SPEED, c SPEED OF SOUND (LOCAL)

$\vec{U}$ WIND VELOCITY, $\vec{n}$ ORIGINAL RAY DIRECTION (UNIT VECTOR)

$\vec{n}_1 = \frac{\vec{C}_1}{|\vec{C}_1|}$ IS THE NEW RAY DIRECTION

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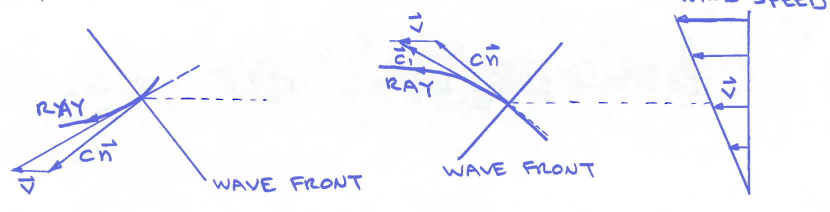
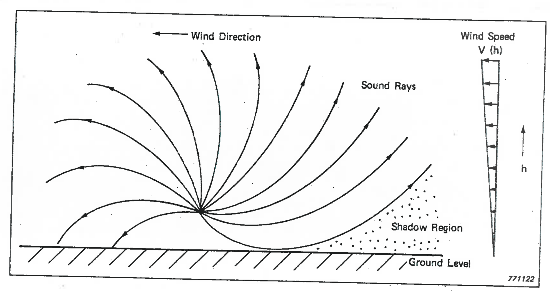
SOME ACOUSTIC PHENOMENA (CONT'D)

THE WIND EFFECT (CONT'D)

HOW CAN WE FIGURE THE RAY BENDING





Sound refraction in a boundary layer

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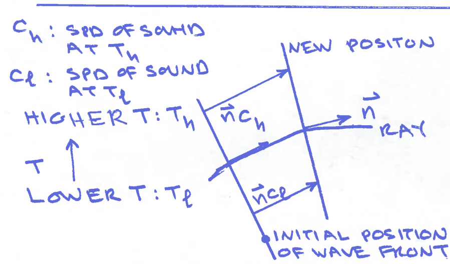
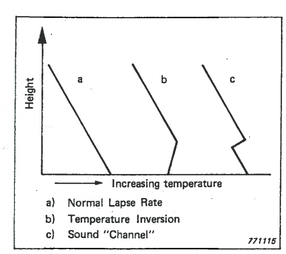
SOME ACOUSTIC PHENOMENA (CONT'D)

THE TEMPERATURE GRADIENT EFFECT ON REFRACTION





Typical atmospheric temperature gradients

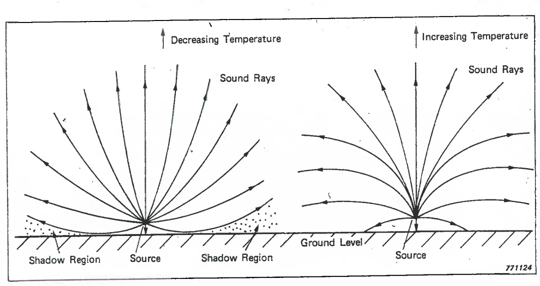


Fig. 2.12. Refraction of sound in an atmosphere with
a) a normal lapse rate
b) an inverted lapse rate

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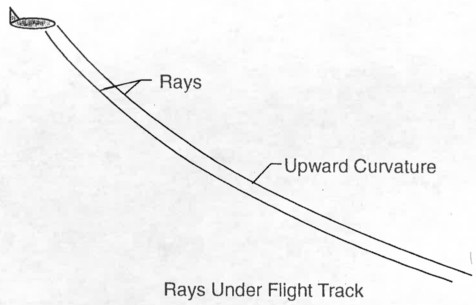
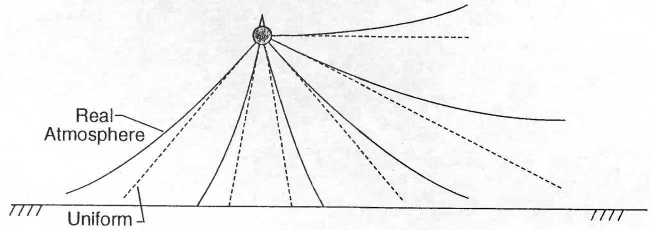
CURVATURE OF SONIC BOOM RAYS IN THE ATMOSPHERE



Rays Under Flight Track



Rays Across Flight Track

Lecture 5

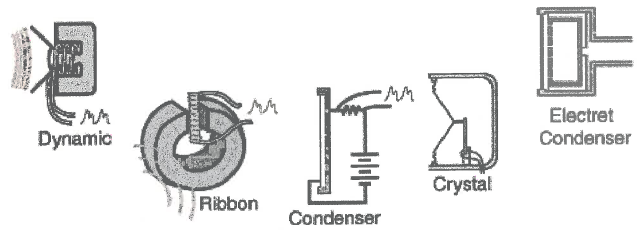
– Microphones and Noise Measurements

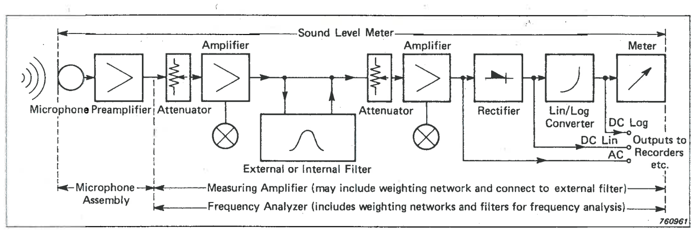
– Principles of Noise Control

MICROPHONES AND NOISE MEASUREMENT

Types of Microphones

CONDENSER MICROPHONES ARE COMMONLY USED FOR NOISE MEASUREMENTS. ELECTRET MICROPHONES ARE CHEAP, EASY TO MANUFACTURE AND VERY LIGHT. IT IS BECOMING MORE POPULAR FOR NOISE MEASUREMENTS.





Block diagram of a noise measuring system

MICROPHONES AND NOISE MEASUREMENT

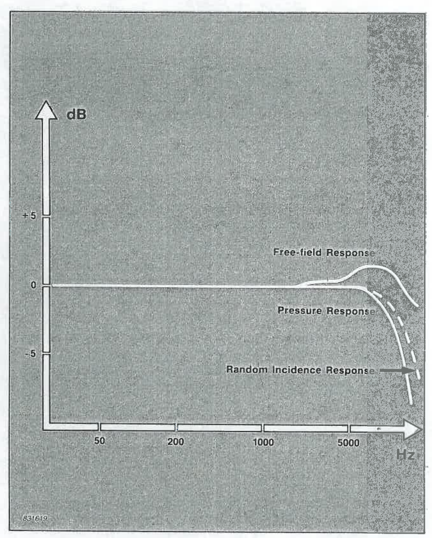
The Microphone in the Sound Field

The type of microphone and its orientation in the sound field also influence the accuracy of measurements. A measurement microphone should have a uniform frequency response, that is the microphone must be equally sensitive throughout the frequency range.

A microphone is normally characterized by one of three types of frequency response characteristics — **free-field** (usually at 0° incidence), **pressure**, and **random-incidence**, and is named after the response that is the most linear. Thus, the response curves shown in the diagram are for a random incidence microphone.

It is important to note that any microphone will disturb a sound field, but the free-field microphone compensates for the disturbance it causes in the sound field. The pressure microphone however, responds uniformly to the actual SPL, including the pressure disturbance caused by the microphone itself. The random incidence microphone is designed to respond uniformly to sounds arriving simultaneously from all angles, as is the case in highly reverberent or diffuse sound fields. (For most microphones the pressure and random incidence responses are very similar so a pressure microphone may also be used for random incidence measurements).

In general, when making free-field measurements (most outdoor measurements are essentially free-field), use a free-field microphone. In a diffuse-field, the microphone should be as omnidirectional as possible.



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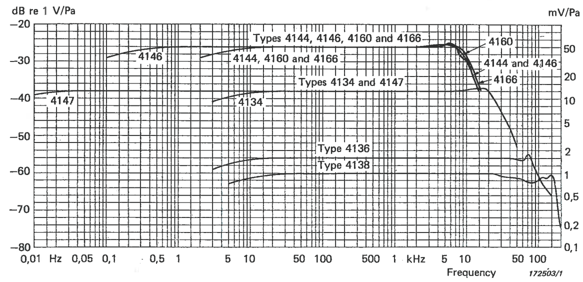
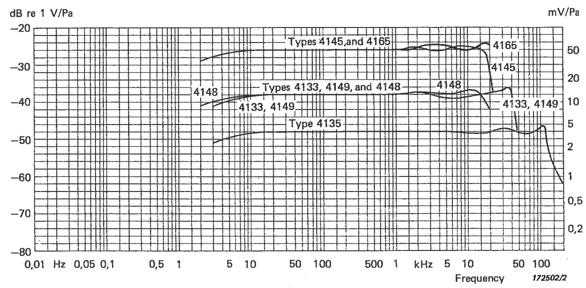
MICROPHONES AND NOISE MEASUREMENT

TYPICAL B & K MICROPHONE RESPONSES

SMALLER MICROPHONES HAVE LESS SENSITIVITY BUT HIGHER FREQUENCY RESPONSE.



Typical frequency responses of different pressure microphones



Typical 0° incidence frequency responses of the different free-field microphones

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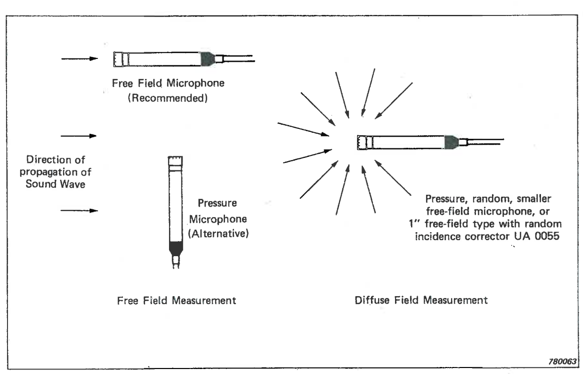
MICROPHONES AND NOISE MEASUREMENT

BEFORE SELECTING A MICROPHONE, ONE MUST HAVE A IDEA OF THE TYPE OF THE SOUND FIELD : DIFFUSE OR FREE FIELD.



Orientation of microphones in the sound field

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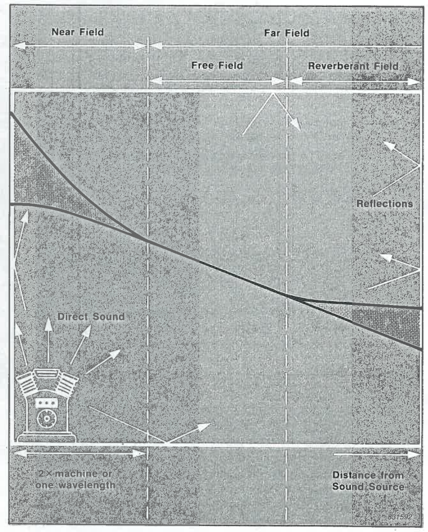
MICROPHONES AND NOISE MEASUREMENT

The Practical Room

In practice the majority of sound measurements are made in rooms that are neither anechoic nor reverberant — but somewhere in-between. This makes it difficult to find the correct measuring positions when the noise emission from a given source must be measured.

When determining emission from a single source, several errors are possible. If measurements are made too close to the machine, the SPL may vary significantly with a small change in sound level meter position. This will occur at a distance less than the wavelength of the lowest frequency emitted from the machine, or at less than twice the greatest dimension of the machine, whichever distance is the greater. This area is termed the **near-field** of the machine, and measurements in this region should be avoided if possible.

Other errors may arise if you measure too far away from the machine. Here, reflection from walls and other objects may be just as strong as the direct sound from the machine and correct measurements will not be possible. This region is termed the **reverberant-field.** Between the reverberant and near-field is the free-field which can be found by noting that the level drops 6 dB for each doubling in distance from the source. SPL measurements should be made in this region. However, it is quite possible, that the conditions are so reverberant or the room is so small that no free-field exists. In such cases some standards (such as ISO 3746) suggest an environmental correction to account for the effect of reflected sound.



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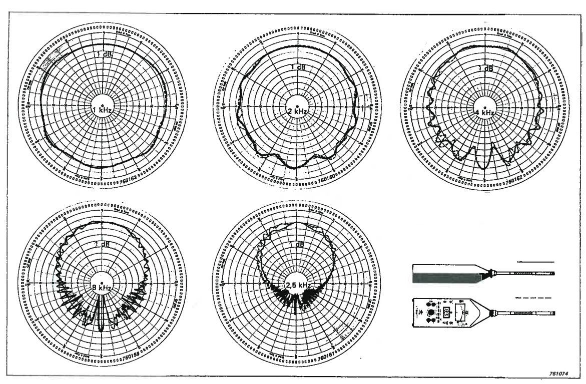
MICROPHONES AND NOISE MEASUREMENT

AT HIGH FREQUENCY (WAVELENGTH ≲ SIZE OF MICROPHONE OR HOLDER), THE DIRECTIONAL CHARACTERISTIC OF A MICROPHONE CAN CHANGE DRASTICALLY.



Directional characteristics of complete sound level meter in a free field

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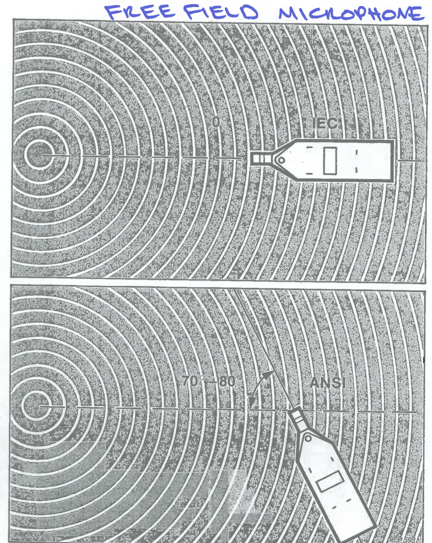
MICROPHONES AND NOISE MEASUREMENT

Selection of the most appropriate microphone may also be influenced by applicable national or international standards. For example the International Electrotechnical Commission **(IEC)** specifies sound level meters with a free-field response, whilst the American National Standard **(ANSI)** calls for a random incidence response to be used.

If a random-incidence microphone is used in a free-field environment, the most accurate measurement is obtained when it is oriented at an angle of between 70° and 80° to the sound source. If pointed directly at the source, the resulting measurement will be too high. Conversely, a free-field microphone used in a diffuse field under-estimates the true SPL.

The response of a free-field microphone can be changed by fitting an acoustic resonator. The resonator increases the pressure in front of the microphone at high frequencies, modifying the response to one that is closer to a flat random response. Such a resonator cap should be used when measuring indoors (i.e. in a predominantly diffuse field). In other cases and when measuring outdoors, remove the resonator cap and point the microphone towards the source. Some sound level meters can change from a correct free-field response to a very accurate random-incidence response by simply switching-in specially incorporated circuits.

FREE FIELD MICROPHONE



RANDOM INCIDENCE MICROPHONE

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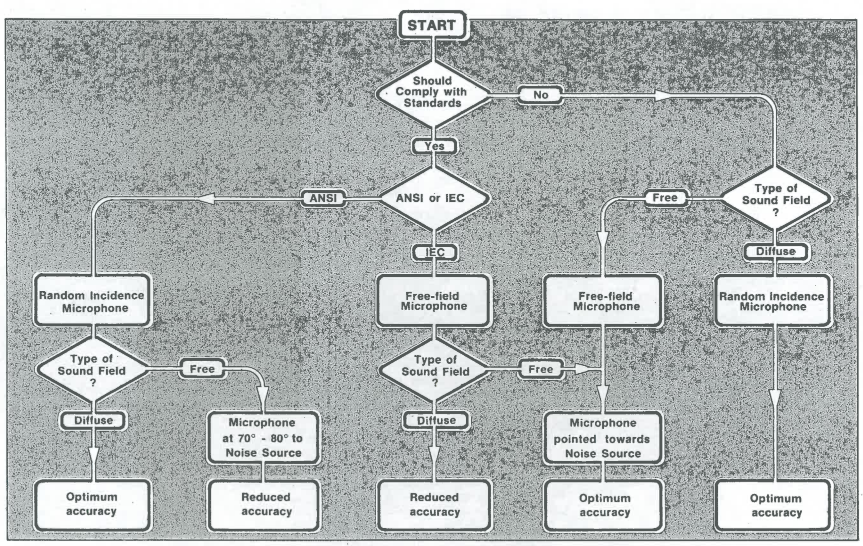
MICROPHONES AND NOISE MEASUREMENT

The Microphone in the Sound Field



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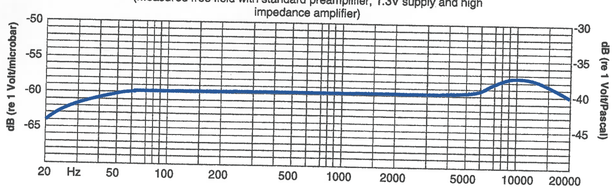
MICROPHONES AND NOISE MEASUREMENT

ELECTRET MICROPHONES ARE USED IN ENTERTAINMENT SYSTEMS AND CAN HAVE GOOD CHARACTERISTICS. THE FIGURE BELOW IS FOR LECTROSONICS M150 MICROPHONE.

M150 Electret Microphone

The M150 is a high performance microphone with heavily suppressed mechanical noise for critical applications. A wide, flat frequency response is augmented by a 2 or 3 dB bump in the upper octaves to add a crisper sound. The mic capsule should be positioned in the center of the talker's chest, not to either side, as high up as is practical. In windy conditions, it can be placed underneath a collar or thin clothing to reduce wind noise, although this will dull the sound somewhat.

FREQUENCY RESPONSE
(Measured free field with standard preamplifier, 1.3V supply and high impedance amplifier)



4 Applied Acoustics (Course)

Applied Aeroacoustics

MAE 692 – University of Virginia

Slides of Lectures

F. Farassat

NASA Langley Research Center

Summer Term 2004

F. Farassat- ME - 692, Appl. AA
NASA Langley Research Center
1 of 2

LEC. 1/1

SPEED OF SOUND IN A GAS

$c = \sqrt{\gamma R T}$

γ = RATIO OF SPECIFIC HEATS
= 1.4 FOR AIR (OR ANY DIATOMIC GAS)

$R = \dfrac{8314}{M}$ GAS CONSTANT

M = MOLECULAR MASS (= 29 FOR AIR)

T = TEMPERATURE °K
= 273 + T°C

c = 344 m/s AT 20°C

DENSITY OF AIR AT SEA LEVEL AND 20 °C = 1.17 kg/m³

IN ACOUSTICS WE USE SI UNITS ALWAYS.

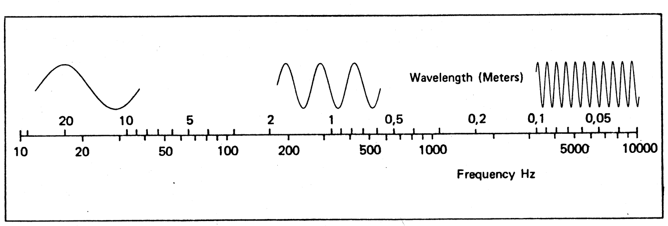
LEC. 1/2

WAVELENGTH VS FREQUENCY

AUDIO FREQUENCY RANGE: 20 Hz TO 20 kHz

FROM $\lambda = \dfrac{c}{f}$, WE GET

WAVELENGTH RANGE (FOR AUDIO FREQ.) : 17 m TO 1.7 cm



Wavelength in air versus frequency under normal conditions

- HUMAN EAR IS MOST SENSITIVE IN 1 kHz TO 5 kHz RANGE (34 cm TO 7 cm WAVELENGTH RANGE

THE GOVERNING EQUATIONS

WE NEED SOME SIMPLE RELATIONS TO GET THE GOV. EQS.

PLANE WAVE

PRESSURE WAVE AT A FIXED TIME

$\boxed{\lambda f = c}$ SPEED OF SOUND m/s

WAVELENGTH m

FREQUENCY HZ (CYCLES/S)

THIS MEANS THAT THE WAVES IN A CYLINDER OF LENGTH c m WILL PASS THROUGH A FIXED POINT PRODUCING $f = \frac{c}{\lambda}$ CYCLES IN ONE SECOND !

PERIOD $\boxed{T = \frac{1}{f}}$ SECONDS $\boxed{\lambda = cT}$

$$\boxed{\frac{\partial p}{\partial t} \approx \frac{P}{T} = \frac{cP}{\lambda}} \quad , \quad \boxed{\frac{\partial p}{\partial x} \approx \frac{P}{\lambda}}$$

WE USE SIMILAR APPROXIMATIONS FOR ρ, $\vec{u}$, ETC.

P : AMPLITUDE

THE GOVERNING EQUATIONS (CONT'D)

MASS CONTINUITY $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$, $\boxed{\vec{u}_0 = 0 \text{, i.e. MEDIUM IS AT REST}}$

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\sim \frac{\tilde{\rho}'}{T} = \frac{\tilde{\rho}' c}{\lambda}} + \underbrace{\rho \nabla \cdot \vec{u}}_{\sim \frac{\rho_0 U}{\lambda}} + \underbrace{\vec{u} \cdot \nabla \rho}_{\sim \frac{\tilde{\rho}' U}{\lambda}} = 0$$

$\tilde{\rho}'$ AMPLITUDE OF DENSITY PERTURBATION, U AMP. OF $|\vec{u}|$

ASSUME $\boxed{\tilde{\rho}' \ll \rho_0}$ $(\approx 1.2 \text{ kg/m}^2)$

$\Rightarrow \frac{\rho_0 U}{\lambda} \gg \frac{\tilde{\rho}' U}{\lambda}$ $\therefore$ IGNORE $\vec{u} \cdot \nabla \rho$ COMPARED TO $\rho \nabla \cdot \vec{u}$. APPROXIMATE ρ AS ρ_0 .

$$\boxed{\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0} \qquad \boxed{\rho' = \rho - \rho_0}$$

MASS CONTINUITY

– NOTE THAT BALANCING $\partial \rho / \partial t$ WITH $\rho_0 \nabla \cdot \vec{u}$ WILL GIVE

$$\boxed{\tilde{\rho}' \sim \rho_0 \frac{U}{c}}$$

THE GOVERNING EQUATIONS (CONT'D)

MOMENTUM EQUATION $\quad \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{\nabla p}{\rho}$

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{\sim \frac{cU}{\lambda}} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{\sim \frac{U^2}{\lambda}} = -\underbrace{\frac{\nabla p}{\rho}}_{\frac{P}{\rho_0 \lambda}} \qquad (P \text{ AMP. OF } p)$$

IF $U \ll c$ (SP. OF SOUND), WE CAN NEGLECT $\vec{u}\cdot\nabla\vec{u}$ COMPARED TO $\partial\vec{u}/\partial t$.

$$\boxed{\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p' = 0} \quad \text{MOMENTUM} \quad \left\{ \begin{array}{l} \text{CONDITION OF VALIDITY} \\ \boxed{\rho' \ll \rho_0 \,,\, |\vec{u}| \ll c} \end{array} \right.$$

$$\boxed{p' = p - p_0}$$

– NOTE THAT BALANCING $\rho_0 \partial\vec{u}/\partial t$ WITH ∇p GIVES

$$\boxed{P \sim \rho_0 c\, U} \qquad \boxed{\rho_0 c \approx 407 \text{ RAYLS}}$$

– THE RELATION BETWEEN $p' = p - p_0$ & $\rho' = \rho - \rho_0$ $\quad \boxed{p' = c^2 \rho'}$

THE GOVERNING EQUATIONS (CONT'D)

OTHER RELATIONS

$$\begin{array}{l} \frac{\partial}{\partial t} \\ \nabla\cdot \end{array} \left\{ \begin{array}{l} \frac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot\vec{u} = 0 \\ \rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla(c^2\rho') = 0 \end{array} \right. \Rightarrow \underbrace{\left\{ \begin{array}{l} \frac{\partial^2 \rho'}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\nabla\cdot\vec{u} = 0 \\ \rho_0 \frac{\partial}{\partial t}\nabla\cdot\vec{u} + c^2\nabla^2\rho' = 0 \end{array} \right.}_{\text{SUBTRACT, DIVIDE BY } c^2}$$

$$\Rightarrow \boxed{\frac{1}{c^2}\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 \rho' = \Box^2 \rho' = 0}$$

$\Box^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$: D'ALEMBERTIAN, WAVE OPERATOR

MULTIPLY ABOVE RESULT BY c^2, USE $p' = c^2\rho'$ TO GET

$$\boxed{\Box^2 p' = 0}$$

– LET $\vec{\xi} = \nabla\times\vec{u}$ VORTICITY, TAKE CURLE OF THE MOMENTUM EQ : $\rho_0 \frac{\partial}{\partial t}(\nabla\times\vec{u}) + \underbrace{\nabla\times\nabla p'}_{=0} = \rho_0 \frac{\partial \vec{\xi}}{\partial t} = 0$

i.e. $\partial\vec{\xi}/\partial t = 0$. SINCE $\vec{\xi}(\vec{x},0) = 0 \Rightarrow \boxed{\vec{\xi} = 0}$ FOR ALL TIME!

THE GOVERNING EQUATIONS (CONT'D)

THE CONDITION $\vec{\xi} = 0$ IMPLIES THAT THERE IS A VELOCITY POTENTIAL $\phi(\vec{x}, t)$ SUCH THAT $\vec{u} = \nabla\phi$. THE MOMENTUM EQ. GIVES $\nabla\left[\rho_0 \frac{\partial \phi}{\partial t} + p'\right] = 0 \Rightarrow \rho_0 \frac{\partial\phi}{\partial t} + p' = f(t)$

WE CAN TAKE $f(t) = 0$ BECAUSE $\phi(\vec{x}, t)$ IS ALWAYS DETERMINED UPTO A FUNCTION OF TIME : IF ϕ IS A VELOCITY POTENTIAL $\Rightarrow$ $\phi_1(\vec{x}, t) = \phi(\vec{x}, t) + F(t)$ IS ALSO A VELOCITY POTENTIAL.

$\Rightarrow \boxed{p' = -\rho_0 \frac{\partial \phi}{\partial t}}$, $\Box^2 p' = 0$ GIVES $\boxed{\Box^2 \phi = 0}$ (*)

TAKE THE GRADIENT OF THIS TO GET $\boxed{\Box^2 \vec{u} = 0}$

$\therefore$ $\boxed{p', \rho', \vec{u} \text{ SATISFY THE WAVE EQUATION}}$

— (*) $\Box^2 \phi = 0$ PUTS A RESTRICTION ON $F(t)$, i.e. $F(t)$ CAN ONLY BE A LINEAR FUNCTION $at + b$ $\Rightarrow p' = -\rho_0 \partial\phi/\partial t$ CAN BE FOUND UPTO A CONSTANT $-\rho_0 a$ WHICH IS SET TO ZERO SINCE $\langle p' \rangle = 0$, IN GENERAL.

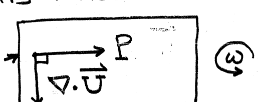
THE GOVERNING EQUATIONS (CONT'D)

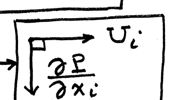
STEADY STATE CASE

OFTEN THE TIME DEPENDENCE IS PERIODIC. WE TAKE THE TIME DEPENDENCE AS $e^{i\omega t}$, $\omega = 2\pi f$ (ANGULAR FREQ.) rad/s . THEN $\partial/\partial t = i\omega$ AND MASS CONTINUITY AND MOMENTUM EQS. BECOME :

$$\boxed{\begin{aligned} i\omega P + \rho_0 c^2 \nabla\cdot\vec{U} &= 0 \\ i\omega \rho_0 \vec{U} + \nabla P &= 0 \end{aligned}}$$

PHASOR DIAGRAMS





$i = 1$ TO 3

$$p' = P(\vec{x})\, e^{i\omega t} \quad , \quad \vec{u} = \vec{U}(\vec{x})\, e^{i\omega t}$$

$\underbrace{\Box^2 p' = 0}_{\text{HYPERBOLIC}}$ BECOMES $\underbrace{\boxed{\nabla^2 P + \frac{\omega^2}{c^2} P = 0}}_{\text{ELLIPTIC}}$

HELMHOLTZ EQUATION

— NOTE THAT P AND U ARE COMPLEX QUANTITIES CALLED PHASORS. THE QUANTITIES OF INTEREST ARE $\mathrm{Re}[P e^{i\omega t}]$ AND $\mathrm{Re}[\vec{U} e^{i\omega t}]$.

SOME SIMPLE RESULTS

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

LET $a(t) = A e^{i\omega t}$, $b(t) = B e^{i\omega t}$, A AND B COMPLEX, $T = \frac{1}{f}$, $\omega = 2\pi f$, THEN

$$\boxed{\frac{1}{T}\int_0^T \mathrm{Re}\, a(t)\cdot \mathrm{Re}\, b(t)\, dt = \frac{1}{2}\mathrm{Re}\,(A B^*)}$$

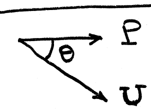
B^* = COMPLEX CONJUGATE OF B

EXAMPLE : $I = \frac{1}{T}\int_0^T \mathrm{Re}\, p\ \mathrm{Re}\, u\, dt$

$$= \frac{1}{2}\mathrm{Re}\,(P U^*) = \frac{1}{2}|P||U|\cos\theta$$

$p = P e^{i\omega t}$, $u = U e^{i\omega t}$

$|P|$ ABS. VALUE OF P



PHASOR DIAGRAM

THE DECIBEL SCALE FOR NOISE

THE RANGE OF THE PRESSURE VARIATION IN ACOUSTICS IS VERY WIDE. WE, THEREFORE, DEFINE A LOGARITHMIC SCALE CALLED THE DECIBEL SCALE :

$$\boxed{L_p = \text{SOUND PRESSURE LEVEL} = 20 \log_{10}\frac{P_{rms}}{P_{ref}}} \quad \text{DECIBELS}$$

$$p_{rms}^2 = \frac{1}{T}\int_0^T p^2(t)\, dt \quad , \quad p_{rms} = \frac{P}{\sqrt{2}} \quad \begin{cases} \text{SINUSOIDAL WAVE} \\ P = \text{AMP. OF WAVE} \end{cases}$$

$\boxed{P_{ref} = 20\ \mu Pa}$, Pa (PASCAL) = 1 N/m^2

ATMOSPHERIC PRESSURE (SEA LEVEL) $\approx 10^5$ Pa

— WE HAVE $p_{rms} = 20 \times 10^{-6} \times 10^{L_p/20}$ Pa

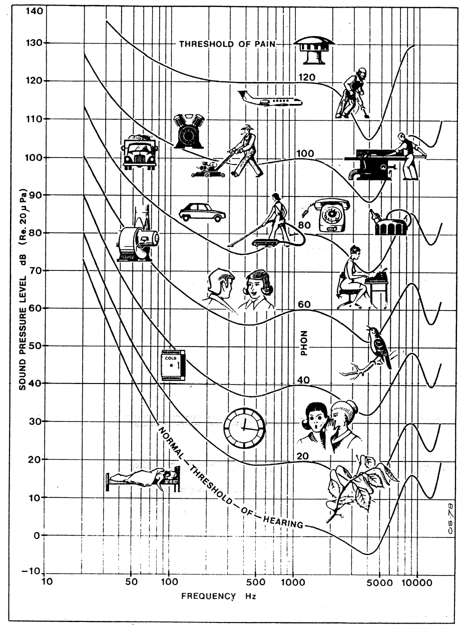
$$\boxed{L_W = \text{SOUND POWER LEVEL} = 10 \log_{10}\frac{W}{W_{ref}}} \quad \text{DECIBELS}$$

$W = \int_S p\, \vec{u}\cdot\vec{n}\, dS$ (FAR FIELD) , $\boxed{W_{ref} = 10^{-12} \text{ WATT}}$

W IS THE SOUND POWER IN WATTS

CONSTANT LOUDNESS CONTOURS

LEC. 1/11



Typical sound pressure levels of common noise sources

SOUND POWER RANGE

LEC. 1/12

Power (Watts)	Power Level (dB re 10^{-12} W)
100 000 000	200
1 000 000	180
10 000	160
100	140
1	120
0,01	100
0,000,1	80
0,000,001	60
0,000,000,01	40
0,000,000,000,1	20
0,000,000,000,001	0

Saturn rocket	(50,000,000 W)
4 Jet Airliner	(50,000 W)
Large orchestra	(10 W)
Chipping hammer	(1 W)
Shouted speech	(0,001 W)
Conversational speech	(20 x 10^{-6} W)
Whisper	(10^{-9} W)

Sound Power output of some typical noise sources

CHANGE IN LEVEL dB	SUBJECTIVE EFFECT
3	just perceptible
5	clearly perceptible
10	twice as loud

LEC. 2/1

2/1

THE USE OF COMPLEX NUMBERS IN ACOUSTICS

THE GOVERNING EQS. OF ACOUSTICS ARE LINEAR. LET US USE THE SUBSCRIPTS c, r AND i FOR COMPLEX, REAL AND IMAGINARY: $p_c = p_r + i\,p_i$, ETC. NOW IF p_i AND $\vec{u}_i$ SATISFY THE GOVERNING EQS. OF ACOUSTICS, LIKE p_r AND $\vec{u}_r$ $\Rightarrow$

$$\frac{\partial \rho_c}{\partial t} + \rho_0 \nabla\cdot\vec{u}_c = 0 \,, \quad \rho_0 \frac{\partial \vec{u}_c}{\partial t} + \nabla p_c = 0$$

NOTE: WE DROP PRIME FROM ρ'_c AND p'_c!

IT IS AT TIMES EASIER TO SOLVE THIS PROBLEM FOR p_c AND $\vec{u}_c$ AND TAKE THE REAL PARTS p_r AND $\vec{u}_r$ AS THE SOLUTION TO THE ORIGINAL PROBLEM. THIS METHOD IS CALLED COMPLEXIFICATION OF THE PROBLEM. THE MOST USEFUL FORM OF COMPLEXIFICATION IS BY TAKING $p_c = P\, e^{i\omega t}$, $\vec{u}_c = \vec{U}\, e^{i\omega t}$ WHERE P AND $\vec{U}$ ARE COMPLEX AMPLITUDES. THIS WAS INTRODUCED BY RAYLEIGH (THEORY OF SOUND). P AND $\vec{U}$ SATISFY

$$\boxed{\begin{aligned} i\omega P + \rho_0 c^2\, \nabla\cdot\vec{U} &= 0 \\ i\rho_0\omega \vec{U} + \nabla P &= 0 \end{aligned}} \qquad \begin{aligned} P &= P(\vec{x}) \\ \vec{U} &= \vec{U}(\vec{x}) \end{aligned}$$

STEADY STATE CONDITION

LEC. 2/2

2/2

SOME USEFUL RESULTS

$$P = |P|\, e^{i\phi} \,, \quad p_c(\vec{x},t) = P\, e^{i\omega t} = |P|\, e^{i(\omega t+\phi)}$$

$$p \equiv p_r = \mathrm{Re}\, p_c = |P| \cos(\omega t + \phi) \qquad (\text{FROM: } e^{i\theta} = \cos\theta + i\sin\theta)$$

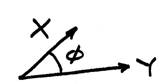
$|P|$: AMPLITUDE OF p , ϕ : PHASE OF p

$$p_{rms}^2 = \frac{1}{T}\int_0^T p^2\, dt = \frac{1}{2}|P|^2 \,, \quad \boxed{p_{rms} = \frac{|P|}{\sqrt{2}}}$$

LET $x = X\, e^{i\omega t}$, $y = Y\, e^{i\omega t}$, THE TIME AVERAGE OF $x_r y_r$ IS:

$$\underbrace{\langle x_r, y_r \rangle}_{\text{ALSO WRITTEN } \overline{x_r y_r}} = \frac{1}{T}\int_0^T x_r\, y_r\, dt = \frac{1}{2}\mathrm{Re}(X Y^*)$$

$$= \frac{1}{2}|X|\,|Y| \cos\phi$$



PHASOR DIAGRAM

$\langle x_r, y_r \rangle = 0$ IF $\phi = 90^\circ$

$\langle x_r, y_r \rangle$ IS MAXIMUM FOR $\phi = 0$.

$$\boxed{\langle x_r, y_r \rangle = \frac{1}{2}\mathrm{Re}(X Y^*)}$$

STEADY STATE CONDITION

WE HAVE S.S. CONDITION IF

i) THE BOUNDARY CONDITIONS ARE PERIODIC IN TIME

ii) WE ALLOW ALL TRANSIENTS TO DIE, $i.e.$ $t \to \infty$

— A PERIODIC FUNCTION $f(t)$ CAN BE WRITTEN AS

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-in\omega t} dt$$

— $f(t)$, IN GENERAL IS REAL $\Rightarrow c_{-n} = c_n^*$ AND

$Re(c_{-n} e^{-in\omega t}) = Re(c_n e^{in\omega t}) \Rightarrow$ WE CAN COMPLEXIFY THE PROBLEM FOR $n > 0$ BUT USE BC $2c_n e^{in\omega t}$.

— BY SUPERPOSITION PRINCIPLE, WE CAN SUM THE EFFECTS OF ALL HARMONICS OVER $n = 1, 2, 3, \ldots$. IN GENERAL, $\langle f(t) \rangle = 0$ SO THAT $c_0 = 0$.

TIME DOMAIN VS FREQUENCY DOMAIN

FOR THE STEADY STATE CASE, BY USING $p = P e^{i\omega t}$, WE GET FROM $\Box^2 p = 0$, $\mathcal{H}P = \nabla^2 P + \frac{\omega^2}{c^2} P = 0$, $P = P(\vec{x})$.

IN GENERAL, USING FOURIER TRANSFORM IN TIME, $t \to \omega$

$$\hat{f}(\vec{x}, \omega) = \int_{-\infty}^{\infty} f(\vec{x}, t) e^{-i\omega t} dt$$

WE TRANSFORM $\Box^2 p = Q(\vec{x}, t)$ TO

HELMHOLTZ OPERATOR $\quad \mathcal{H} \hat{p}(\vec{x}, \omega) = \nabla^2 \hat{p} + \frac{\omega^2}{c^2} \hat{p} = -\hat{Q}(\vec{x}, \omega)$

THE BC'S ARE ALSO TRANSFORMED TO FUNCTIONS OF $(\vec{x}, \omega)$. WE THEN SOLVE FOR $\hat{p}(\vec{x}, \omega)$ FOR ALL $|\omega| < \infty$. FROM $\hat{p}$ WE FIND

$$p(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(\vec{x}, \omega) e^{i\omega t} d\omega$$

THIS IS FREQUENCY DOMAIN METHOD. IF WE SOLVE DIRECTLY FOR $p(\vec{x}, t)$ FROM $\Box^2 p = Q(\vec{x}, t)$ + BC'S, WE ARE WORKING IN TIME DOMAIN. FREQUENCY DOMAIN METHOD IS MOSTLY APPLIED TO STEADY STATE PROBLEMS.

TIME VS FREQUENCY DOMAIN (CONT'D)

TIME DOMAIN AND FREQUENCY DOMAIN APPROACHES ARE COMPLEMENTARY. NEITHER METHOD IS SUPERIOR TO THE OTHER. WE LEARN SOMETHING NEW FROM EACH METHOD. OFTEN THE METHOD ONE USES DEPENDS ON THE EXPERIENCE AND THE TRADITION IN THE FIELD.

THESE ARE SOME CHARACTERISTICS OF F.D. AND T.D. METHODS:

- MANY PROBLEMS OF ENGINEERING ARE STEADY STATE SO THAT F.D. METHOD IS AN OPTION,
- MANY SIMPLE SOURCE AND PROPAGATION MODELS ARE AVAILABLE USING F.D. METHOD,
- IN F.D. METHOD, BY USING $t \to \omega$, WE HAVE ESSENTIALLY A 3 DIMENSIONAL PROBLEM. SOME PEOPLE FEEL MORE COMFORTABLE TO WORK IN 3D THAN FOUR DIMENSIONAL T.D. METHOD. HOWEVER, BECAUSE OF THE PARTICULARLY SIMPLE GREEN'S FUNCTION OF THE WAVE OPERATOR, ONE CAN ALSO LEARN TO WORK JUST AS EASILY IN T.D.

TIME DOMAIN VS FREQUENCY DOMAIN (CONT'D)

- IT APPEARS THAT MORE ANALYTIC (CLOSED FORM) SOLUTIONS ARE AVAILABLE USING F.D. APPROACH. MOST OFTEN, THIS HAS BEEN ACHIEVED BY SOME APPROXIMATIONS. APPROXIMATIONS IN GEOMETRY, OBSERVER DISTANCE AND SOURCE MOTION MAY NOT BE ACCEPTABLE IN SOME AEROACOUSTIC PROBLEMS.
- FOR SOME PROBLEMS OF AEROACOUSTICS, e.g. HIGH SPEED HELICOPTER ROTOR NOISE AND PROPELLER NOISE PREDICTION, BOTH T.D. AND F.D. METHODS CAN BE TIME CONSUMING ON A COMPUTER. HOWEVER, EXECUTION TIME ON A COMPUTER IS VERY MUCH DEPENDENT ON THE SKILL AND EXPERIENCE OF CODE DEVELOPER. SO AFTER YOU SELECT A METHOD, SPEND A LOT OF TIME THINKING ABOUT ALGORITHMS YOU USE IN YOUR CODE. THE DIFFERENCE BETWEEN A GOOD AND A BAD ALGORITHM COULD BE A FACTOR OF 100 IN EXECUTION TIME ON A COMPUTER!
- T.D. ANALYSIS IS MORE RECENT IN AEROACOUSTICS. EXPERIENCE HERE IS LIMITED!

LEC. 2/7
2/7

ENERGY RELATIONS (STATIONARY MEDIUM)

STARTING WITH MOMENTUM EQ. $\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p = 0$, TAKE DOT PRODUCT OF THIS WITH $\vec{u}$:

$$\rho_0 \vec{u} \cdot \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla p = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 u^2\right) + \nabla\cdot(p\vec{u}) - p\nabla\cdot\vec{u} = 0 \qquad (1)$$

WHERE $u = |\vec{u}(\vec{x},t)|$ VEC. NORM !

MASS CONT. EQ : $\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_0 \nabla\cdot\vec{u} = 0 \Rightarrow p\nabla\cdot\vec{u} = -\frac{1}{\rho_0 c^2}\, p\frac{\partial p}{\partial t}$

$$= -\frac{1}{2\rho_0 c^2}\frac{\partial p^2}{\partial t}$$

SUBSTITUTE IN (1) TO GET

$$\frac{\partial}{\partial t}\underbrace{\left[\frac{1}{2}\rho_0 u^2 + \frac{1}{2\rho_0 c^2}p^2\right]}_{e\ :\ \text{ACOUSTIC ENERGY DENSITY}} + \nabla\cdot\underbrace{(p\vec{u})}_{\vec{I}\ :\ \text{ACOUSTIC INTENSITY VECTOR}} = 0$$

$$\boxed{\frac{\partial e}{\partial t} + \nabla\cdot\vec{I} = 0}$$

UNITS OF e JOULES

UNITS OF $\vec{I}$ W/m^2

NOTE : $\vec{I}$ IS ALWAYS A REAL QUANTITY IN THIS COURSE !

LEC. 2/8
2/8

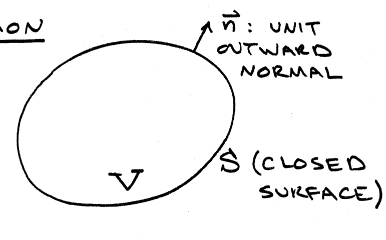
ENERGY RELATIONS (CONT'D)

$\frac{1}{2}\rho_0 u^2$ IN e IS THE INSTANTANEOUS KINETIC ENERGY/UNIT VOL.

$\frac{p^2}{2\rho_0 c^2}$ IN e IS THE INSTANTANEOUS POTENTIAL ENERGY STORED BY COMPRESSION/UNIT VOL.

ANOTHER VIEW OF ENERGY RELATION

$$\underbrace{\frac{\partial}{\partial t}\int_V e\, dV}_{\text{RATE OF INCREASE OF ENERGY IN } V} = -\underbrace{\int_S \vec{I}\cdot\vec{n}\, dS}_{\text{RATE OF ENERGY LEAVING } S}$$



$$\langle e\rangle = \frac{1}{4}\rho_0|\vec{U}|^2 + \frac{1}{4\rho_0 c^2}|P|^2 \qquad \text{FOR HARMONIC WAVES}$$

WHERE P AND $\vec{U}$ ARE COMPLEX AMPLITUDES OF p & $\vec{u}$

$$\langle\vec{I}\rangle = \frac{1}{2}\,\mathrm{Re}(P\vec{U}^*)$$

NOTE : $|\vec{U}|^2 = \underbrace{\vec{U}\cdot\vec{U}^*}_{\text{VEC. DOT PRODUCT}} = |U_1|^2 + |U_2|^2 + |U_3|^2$, U_1, U_2, U_3 COMPLEX.

ENERGY RELATIONS (CONT'D)

CONSIDER AGAIN STEADY STATE CASE : $\frac{\partial e}{\partial t} + \nabla \cdot \vec{I} = 0$

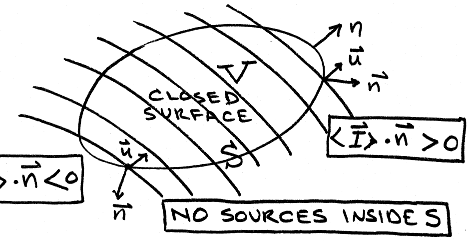
$$\boxed{\left\langle \frac{\partial e}{\partial t} \right\rangle = \frac{1}{T}\int_0^T \frac{\partial e}{\partial t}\, dt = \frac{e(T) - e(0)}{T} = 0 \; !}$$

i.e. IN STEADY STATE CASE NO ACCUMULATION OF ACOUSTIC ENERGY ON AVERAGE IS POSSIBLE ! $\Rightarrow \langle \nabla \cdot \vec{I} \rangle = 0$

$$\Rightarrow \boxed{\left\langle \int_S \vec{I} \cdot \vec{n}\, dS \right\rangle = \int_S \langle \vec{I} \rangle \cdot \vec{n}\, dS = 0} \quad (*)$$

i.e. ON AVERAGE, THE ENERGY ENTERING A CLOSED SURFACE IS EQUAL TO ENERGY LEAVING THE SURFACE

$\therefore$ NO ACCUMULATION OF ENERGY IN VOLUME V WITHIN THE SURFACE !



(*) NOTE THAT, IN GENERAL,

$$\boxed{\langle \vec{I} \rangle \neq 0}$$

SIMPLE MODELS OF WAVES

i) PLANE WAVES

$$p(\vec{x}, t) = A\, e^{i(\omega t - \vec{k} \cdot \vec{x})} = P(\vec{x})\, e^{i\omega t}$$

$P(\vec{x}) = A\, e^{-i \vec{k} \cdot \vec{x}}$. ASSUME $\vec{k} = (k_1, k_2, k_3)$ REAL .

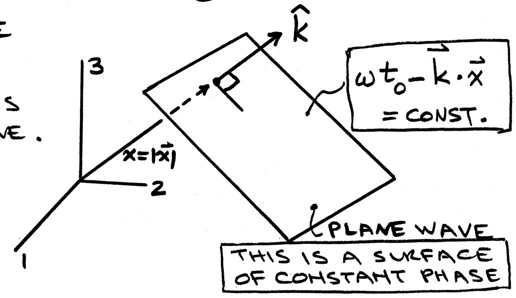
TO SATISFY $\nabla^2 P + \frac{\omega^2}{c^2} P = 0$, WE MUST HAVE

$\frac{\omega^2}{c^2} = k_1^2 + k_2^2 + k_3^2 = |\vec{k}|^2$ OR $\boxed{k = \frac{\omega}{c}}$. THE DIRECTION OF $\vec{k}$ IS ARBITRARY . LET $\hat{\vec{k}} = \frac{\vec{k}}{k}$ AND $\vec{x} = \hat{\vec{k}}\, x$

$$\Rightarrow p(\vec{x}, t) = A\, e^{i(\omega t - kx)} = A\, e^{-ik(x - ct)}$$

i.e. WE HAVE A PLANE WAVE TRAVELLING WITH SPEED c NORMAL TO THE PLANE . $\hat{\vec{k}}$ IS THE UNIT NORMAL TO THE PLANE .

THE PLANE WAVE WITH $\vec{k}$ REAL IS A PROPAGATING WAVE WITHOUT DECAY .



SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

FROM $i\rho_0 \omega \vec{U} + \nabla P = 0$, WE GET

$$\boxed{\vec{U} = \frac{P}{\rho_0 c}\hat{\vec{k}}} \Rightarrow \boxed{\langle \vec{I} \rangle = \frac{1}{2\rho_0 c}|P|^2 \hat{\vec{k}} = \frac{p_{rms}^2}{\rho_0 c}\hat{\vec{k}}}$$

THIS MEANS THAT $\vec{U} \parallel \hat{\vec{k}}$ AND COMPONENTS OF $\vec{U}$ VIEWED AS PHASORS ARE IN PHASE WITH P $\boxed{\vec{U_i} \longrightarrow P}$

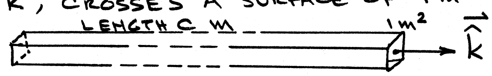
ACOUSTIC ENERGY DENSITY

$$\langle e \rangle = \underbrace{\frac{1}{4}\rho_0 |\vec{U}|^2}_{= \frac{1}{4\rho_0 c}|P|^2} + \frac{1}{4\rho_0 c^2}|P|^2 = \frac{|P|^2}{2\rho_0 c^2} = \frac{p_{rms}^2}{\rho_0 c^2} = \frac{|\langle \vec{I} \rangle|}{c}$$

$$= \frac{1}{4\rho_0 c}|P|^2 \quad \left(\text{USING } \vec{U} = \frac{P}{\rho_0 c}\hat{\vec{k}}\right)$$

WE HAVE <u>EQUIPARTITION</u> OF KINETIC AND POTENTIAL ENERGY DENSITIES.

$\boxed{|\langle \vec{I} \rangle| = c\langle e \rangle}$ MEANS THAT THE ACOUSTIC ENERGY IN A CYLINDER OF UNIT AREA ($1\ m^2$) AND LENGTH c, WITH AXIS PARALLEL TO $\hat{\vec{k}}$, CROSSES A SURFACE OF $1\ m^2$ AND $\perp$ TO $\hat{\vec{k}}$!



SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

ii) <u>EVANESCENT WAVES</u> $\quad p = A e^{i(\omega t - \vec{k}\cdot\vec{x})}$

THESE ARE ALSO PLANE WAVES BUT $\vec{k} = \vec{k}_R + i\vec{k}_i$ WHERE $\vec{k}_R$ AND $\vec{k}_i$ ARE VECTORS WITH <u>REAL</u> COMPONENTS. TO SATISFY HELMHOLTZ EQ., WE MUST HAVE

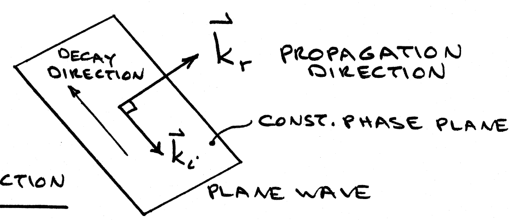
$$k_1^2 + k_2^2 + k_3^2 = \underbrace{|\vec{k}_r|^2 - |\vec{k}_i|^2}_{\text{REAL}} + \underbrace{2i\vec{k}_r\cdot\vec{k}_i}_{\text{IMAGINARY}} = \underbrace{\frac{\omega^2}{c^2}}_{\text{REAL}}$$

$$\Rightarrow \vec{k}_r \cdot \vec{k}_i = 0 \quad i.e.\ \vec{k}_r \perp \vec{k}_i$$

$$|\vec{k}_r|^2 - |\vec{k}_i|^2 = \frac{\omega^2}{c^2} > 0, \quad i.e.\ \boxed{|\vec{k}_r|^2 = \frac{\omega^2}{c^2} + |\vec{k}_i|^2}$$

$$P = A e^{-i\vec{k}\cdot\vec{x}}$$
$$= A e^{\vec{k}_i\cdot\vec{x}} e^{-i\vec{k}_r\cdot\vec{x}}$$
$$|P| = |A| e^{\vec{k}_i\cdot\vec{x}}$$

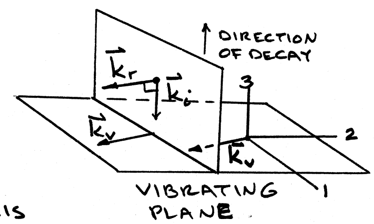
<u>DECAYS IN $-\vec{k}_i$ DIRECTION</u>



SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

EXAMPLE OF EVANESCENT WAVES

WAVES IN THE VICINITY OF A VIBRATING PLANE FOR WHICH WE HAVE $k_1^2 + k_2^2 > \frac{\omega^2}{c^2}$. HERE $(k_1, k_2, 0) = \vec{k}_v = \vec{k}_r$ WHERE $\vec{k}_v$ THE WAVE NUMBER VECTOR OF THE VIBRATION. $\vec{k}_i$ IS PARALLEL TO x_3-AXIS



– PROPAGATION SPEED OF EVANESCENT WAVE $= \frac{\omega}{|\vec{k}_r|} < c$
i.e. EVAN. WAVES TRAVEL AT SUBSONIC SPEED IN THE DIRECTION OF $\vec{k}_r$.

– $\vec{U} = \frac{P}{\rho_0 \omega} \vec{k} = \frac{P}{\rho_0 \omega}(\vec{k}_r + i\vec{k}_i)$, P AND $\vec{U}$ ARE NO LONGER IN PHASE , $\vec{U}$ ALSO DECAYS IN THE DIREC-TION $-\vec{k}_i$ (SEE NEXT SLIDE)

– $\langle \vec{I} \rangle = \frac{1}{2} \mathrm{Re}(P \vec{U}^*) = \frac{|P|^2}{2\rho_0 \omega} \vec{k}_r$

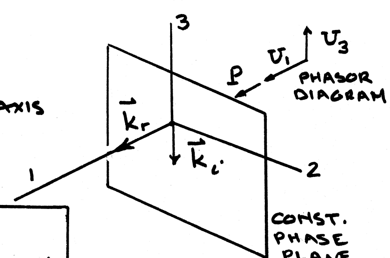
ENERGY FLOWS IN THE DIRECTION OF $\vec{k}_r$ ONLY !

BECAUSE OF THE DECAY, ENERGY DOES NOT PROPAGATE TO INFINITY BUT STAYS IN THE VICINITY OF THE VIBRATING PLANE.

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

EVANESCENT WAVES (CONT'D)

NOW LET US DEFINE x_1-AXIS ALONG $\vec{k}_r$, x_3-AXIS ALONG $-\vec{k}_i$ AND x_2-AXIS IN SUCH A WAY THAT THE $\vec{x}$-FRAME IS RIGHT HANDED. THEN



$\boxed{\vec{k}_r \,\&\, \vec{k}_i \text{ REAL VECTORS}}$

$\boxed{U_1 = \frac{P}{\rho_0 \omega} |\vec{k}_r| \qquad U_1 \longrightarrow P \text{ PHASOR DIAGRAM}}$

$\boxed{\langle I_1 \rangle = \frac{|P|^2}{2\rho_0 \omega} |\vec{k}_r| \quad , \langle I_2 \rangle = \langle I_3 \rangle = 0}$

$\boxed{U_3 = \frac{iP}{\rho_0 \omega} |\vec{k}_i| \qquad U_3 \uparrow \longrightarrow P \text{ PHASOR DIAGRAM}}$ $\Rightarrow \langle I_3 \rangle = \frac{1}{2}\mathrm{Re}(P U_3^*) = 0$

$\boxed{U_2 = 0 \text{ NO VELOCITY COMPONENT IN } x_2 \text{ DIRECTION}}$ $\Rightarrow \langle I_2 \rangle = 0$

ENERGY FLOWS IN x_1 DIRECTION ONLY. $\boxed{|P| = |A|\, e^{-|\vec{k}_i| x_3}}$

i.e. THE AMPLITUDE $|P|$ DECAYS IN x_3 DIRECTION.

– THE ENERGY DENSITY DECAYS IN x_3 DIRECTION ALSO.

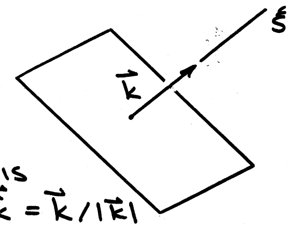
SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

PHASE VELOCITY

$\theta = \vec{k}\cdot\vec{x} - \omega t$ IS CALLED

THE PHASE OF THE PLANE WAVE

$p(\vec{x},t) = A\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$



FOR THIS PLANE WAVE, WE DEFINE AXIS ξ ALONG $\vec{k}$, TAKING $\vec{\xi} = \xi\hat{k}$, $\hat{k} = \vec{k}/|\vec{k}|$

$\theta = \vec{k}\cdot\hat{k}\,\xi - \omega t = |\vec{k}|\xi - \omega t$. NOW WE CAN STUDY THE VELOCITY OF THE SURFACE OF CONSTANT PHASE $\theta = \text{CONST.}$ ALONG ξ-AXIS : $\dot{\theta} = 0 = |\vec{k}|\,\dot{\xi} - \omega$, $\dot{\xi} = \omega/|\vec{k}|$. FOR A PROPAGATING PLANE WAVE, WE HAVE FROM HELMHOLTZ EQ. $|\vec{k}|^2 = \omega^2/c^2$ OR $\boxed{\dot{\xi} = \omega/|\vec{k}| = c}$ SPEED OF SOUND

FOR AN EVANESCENT WAVE : $p(\vec{x},t) = A\, e^{\vec{k}_i\cdot\vec{x}} \cdot e^{i(\vec{k}_r\cdot\vec{x} - \omega t)}$

$\theta = \vec{k}_r\cdot\vec{x} - \omega t = \vec{k}_r\cdot\hat{k}_r\,\xi - \omega t = |\vec{k}_r|\xi - \omega t$

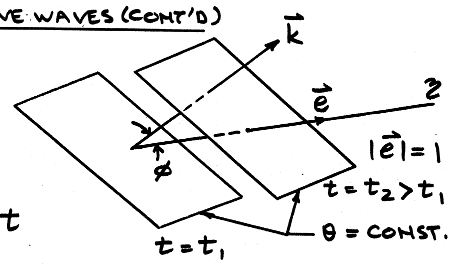
$\dot{\theta} = 0 = |\vec{k}_r|\,\dot{\xi} - \omega$, $\boxed{\dot{\xi} = \dfrac{\omega}{|\vec{k}_r|}}$, $|\vec{k}_r|^2 = \dfrac{\omega^2}{c^2} + |\vec{k}_i|^2 > \dfrac{\omega^2}{c^2}$

$\Rightarrow \boxed{\dot{\xi} < c}$ i.e. THE PHASE VELOCITY OF AN EVANESCENT WAVE IS SUBSONIC

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

TRACE VELOCITY

NOW DEFINE A FIXED AXIS η BY SPECIFYING THE UNIT VECTOR $\vec{e}$ ALONG IT. WE HAVE



$\theta = \vec{k}\cdot\vec{x} - \omega t = \vec{k}\cdot\vec{e}\,\eta - \omega t$

$= (|\vec{k}|\cos\phi)\eta - \omega t$

$\dot{\theta} = 0 = (|\vec{k}|\cos\phi)\,\dot{\eta} - \omega$, $\boxed{\dot{\eta} = \dfrac{\omega}{|\vec{k}|\cos\phi} > \dfrac{\omega}{|\vec{k}|}}$

THIS IS THE TRACE VELOCITY OF THE PLANE WAVE $p = A e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ ALONG η-AXIS. $\boxed{\dot{\eta} = \dfrac{c}{\cos\phi}}$. NOTE THAT WE ARE FOLLOWING THE SAME PLANE WAVE INTERSECTING η-AXIS AND $\dot{\eta}$ GIVES USE THE SPEED OF THE POINT OF INTERSECTION FOR AN OBSERVER FIXED TO THE UNDISTURBED MEDIUM. NOTE ALSO THAT $\cos\phi$ IS THE DIRECTION COSINE OF $\vec{k}$ WRT η-AXIS

TRACE SPEED OF $p = A e^{i(\vec{k}\cdot\vec{x} - \omega t)}$ ALONG x_1, x_2 AND x_3-AXES

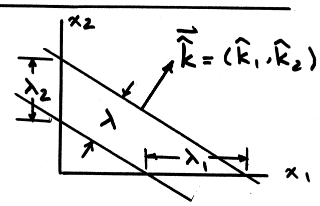
$\boxed{\dot{x}_i = \dfrac{c}{\hat{k}_i}}$, $\hat{k} = \vec{k}/|\vec{k}|$ UNIT VEC. ALONG $\vec{k}$

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

RELATIONS BETWEEN WAVELENGTH AND WAVE NUMBER

$f\lambda = c$ SPEED OF SOUND

$k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi f}{\lambda f} = \dfrac{\omega}{c}$ WAVE NO.



f = THE NO. OF PEAKS/SEC. PASSING OVER A POINT $\Rightarrow$ f DOES NOT CHANGE ANYWHERE FOR AN OBSERVER FIXED TO THE UNDISTURBED MEDIUM. FROM THE FIGURE

$t = t_1$ FIXED

$$\lambda_1 = \frac{\lambda}{\hat{k}_1}, \quad \frac{2\pi}{\lambda_1} = k_1 = \frac{2\pi f}{\lambda_1 f} = \frac{\omega}{(\lambda f)/\hat{k}_1} = \frac{\omega}{c/\hat{k}_1} = \frac{\omega}{\dot{x}_1}$$

WHERE $\dot{x}_1$ IS TRACE VELOCITY ALONG x_1-AXIS. WE HAVE

$$\boxed{\lambda_i = \frac{\lambda}{\hat{k}_i}} \qquad \boxed{k_i = \frac{2\pi}{\lambda_i} = \frac{\omega}{\dot{x}_i}} \qquad \vec{k} = (k_1, k_2, k_3)$$

THESE RELATIONS ARE FOR AN OBSERVER FIXED TO THE MEDIUM.

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

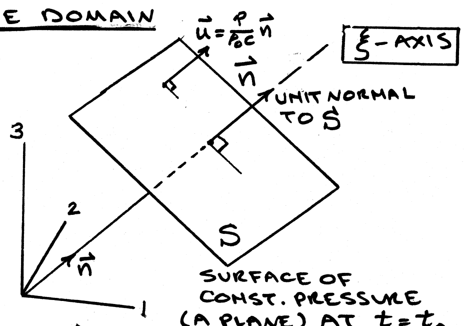
PROPAGATING PLANE WAVES IN TIME DOMAIN

$p = f(\vec{n}\cdot\vec{x} - ct)$

$\vec{n}$ PROPAGATION DIRECTION

$|\vec{n}| = 1$, f ARBITRARY FUNCTION

LET $\xi = \vec{n}\cdot\vec{x} \Rightarrow \xi - ct =$ $\vec{n}\cdot\vec{x} - ct$ = CONST. IS A PLANE SURFACE



– FROM MOM. EQ.

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p = -f'(\vec{n}\cdot\vec{x} - ct)\,\vec{n}$$

$$\vec{u} = \frac{\vec{n}}{\rho_0 c} f(\vec{n}\cdot\vec{x} - ct) + g(\vec{x}),$$ g ARBITRARY. SINCE

$\vec{u} = 0$ WHEN $p = 0 \Rightarrow g(\vec{x}) = 0$.

$$\boxed{\begin{aligned}\vec{u} &= \frac{\vec{n}}{\rho_0 c} f(\vec{n}\cdot\vec{x} - ct) \\ &= \frac{p}{\rho_0 c}\,\vec{n}\end{aligned}}$$

– NOTE THAT BOTH p AND u ARE FUNCTIONS OF $\xi - ct$, $\xi = \vec{x}\cdot\vec{n}$ (SEE ABOVE FIG.)

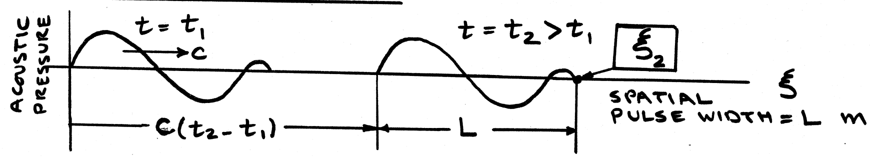
SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

PROPAGATING PLANE WAVES IN TIME DOMAIN (CONT'D)

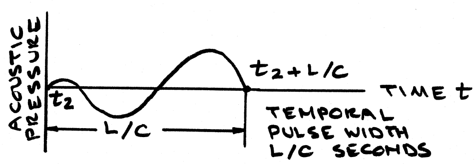
THE INSTANTANEOUS ACOUSTIC INTENSITY $\vec{I}(\vec{x},t)$ IS

$$\boxed{\vec{I} = \frac{p^2(\vec{x},t)}{\rho_0 c}\,\vec{n}} \Rightarrow \boxed{\langle\vec{I}\rangle = \frac{p_{rms}^2}{\rho_0 c}\,\vec{n}}$$

– SOMETHING TO REMEMBER



THE SPATIAL DISTRIBUTION OF A PRESSURE PULSE AT TWO TIMES t_1 AND $t_2 > t_1$



ACOUSTIC PRESSURE SIGNAL MEASURED BY A MICROPHONE AT THE POINT ξ_2

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

PARTICLE DISPLACEMENT $\vec{d}$: $\vec{u} = \frac{\partial \vec{d}}{\partial t}$, $\vec{d} = \vec{D}(\vec{x})\, e^{i\omega t}$

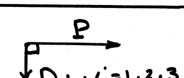
FROM MOMENTUM EQ., WE GET FOR STEADY STATE

$$\boxed{\vec{D}(\vec{x}) = \frac{1}{\rho_0 \omega^2}\nabla P}$$

FOR PLANE WAVES $P = A\, e^{-i\vec{k}\cdot\vec{x}}$, $\nabla P = -i P \vec{k}$, $|\vec{k}| = \frac{\omega}{c}$

$$\vec{D}(\vec{x}) = -\frac{iP}{\rho_0 c\omega}\hat{k}$$

$$|\vec{D}| = \frac{|P|}{\rho_0 c \omega} = \frac{|\vec{U}|}{\omega}$$



PHASOR DIAGRAM

$$\hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

ORDER OF MAGNITUDE OF ACOUSTIC QUANTITIES (PLANE WAVES)

FREQUENCY	LEVEL dB re: 20 μPa	|P| Pa	$\|\vec{U}\|$ m/s	$\|\vec{D}\|$ m
1 kHz	60	0.028	6.88×10^{-5}	1.1×10^{-8}
1 kHz	120 TRESHOLD OF PAIN	28.2	6.88×10^{-2}	1.1×10^{-5}
10 kHz	60	0.028	6.88×10^{-5}	1.1×10^{-9}
10 kHz	120	28.2	6.88×10^{-2}	1.1×10^{-6}

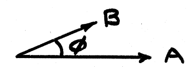
NOTE VERY SMALL VALUES OF $|\vec{D}|$!

COMBINATION OF ACOUSTIC PRESSURES IN STEADY STATE

WE NEED TO EVALUATE p^2_{rms} TO FIND THE DECIBEL LEVEL OF SOUND.

i) TWO WAVES OF THE SAME FREQUENCY

$$p_c = A e^{i\omega t} + B e^{i\omega t} = (A+B)e^{i\omega t}$$



PHASOR DIAGRAM

$$\boxed{\langle p_r^2\rangle = \frac{1}{2}|P|^2 = \frac{1}{2}|A+B|^2 = \frac{1}{2}\left[|A|^2+|B|^2+2|A||B|\cos\phi\right]}$$

$|P|$ IS <u>MAXIMUM</u> IF $\phi = 0$, $i.e.$ $B = \alpha A$, $\alpha > 0$, α REAL

$|P|$ IS <u>MINIMUM</u> IF $\phi = \pi$, $i.e.$ $B = \alpha A$, $\alpha < 0$, α REAL

$|P| = 0$ IF $B = -A$, $i.e.$ $|A| = |B|$ AND $\phi = \pi$

THIS IS THE IDEA BEHIND <u>ANTI-NOISE</u>.

ii) $p_c = A_1 e^{i\omega t} + A_2 e^{2i\omega t} + \cdots + A_n e^{ni\omega t} = p_r(\vec{x},t) + i\, p_i(\vec{x},t)$

$\boxed{\langle p_r^2\rangle = \frac{1}{2}\sum_{i=1}^{n}|A_i|^2}$. THE REASON IS AS FOLLOWS:

$$\langle p_r^2\rangle = \frac{1}{4}\langle (P + p_c^*)^2\rangle = \frac{1}{4}\left[\langle p_c^2\rangle + \langle p_c^{*2}\rangle + 2\langle p_c p_c^*\rangle\right]$$

$$= \frac{1}{2}\langle |p_c|^2\rangle = \frac{1}{2}\sum_{i=1}^{n}|A_i|^2 + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{n}\underbrace{\langle A_j A_k^* e^{i(j-k)\omega t}\rangle}_{=0}$$

$\left\{\begin{array}{l}\text{WE CAN SHOW } \langle p_c^2\rangle = \langle p_c^{*2}\rangle = 0. \\ \text{THIS FOLLOWS FROM } \langle e^{im\omega t}\rangle = 0,\ m \neq 0.\end{array}\right.$

COMBINATION OF ACOUSTIC PRESSURES IN STEADY STATE (CONT'D)

WE SUMMARIZE THE ABOVE RESULTS, GIVING A NEW RESULT(ii):

i) FOR TWO WAVES OF <u>IDENTICAL FREQUENCY</u>, $\langle p_r^2\rangle$ DEPENDS ON <u>THE PHASOR ADDITION</u> OF COMPLEX AMPLITUDES WHEN THE PHASE BETWEEN THE TWO WAVES IS KNOWN.

ii) FROM TIME SERIES ANALYSIS, IT CAN BE SHOWN THAT WHEN THE PHASE BETWEEN TWO WAVES OF <u>IDENTICAL FREQUENCY</u> IS <u>RANDOM</u>, THEN $\langle p_r^2\rangle = \frac{1}{2}(|A_1|^2 + |A_2|^2)$, $i.e.$ OF THE TWO WAVES ADD UP.

iii) FOR WAVES OF <u>DIFFERENT FREQUENCIES</u>, THE MEAN SQ. OF THE RESULTING PRESSURE IS THE SUM OF THE MEAN SQ. OF THE COMPONENTS:

$$p^2_{rms} = \langle p_r^2\rangle = \frac{1}{2}\sum_{i=1}^{n}|A_i|^2 = \sum_{i=1}^{n}(p^2_{rms})_i.$$

$$\Rightarrow \langle \vec{I}\rangle = \frac{\langle p_r^2\rangle}{\rho_0 c}\vec{n} = \sum_{i=1}^{n}\langle \vec{I}_i\rangle$$ PLANE WAVE PROPAGATING IN THE DIRECTION $\vec{n}$.

SIMPLE MODELS OF WAVES - SPHERICAL WAVES

TIME DOMAIN : p SPHERICALLY SYMMETRIC IF $p(\vec{x},t) = p(r,t)$

r DISTANCE FROM ORIGIN. WE CAN SHOW THAT IN THIS CASE

$$\Box^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp) = 0 \Rightarrow$$

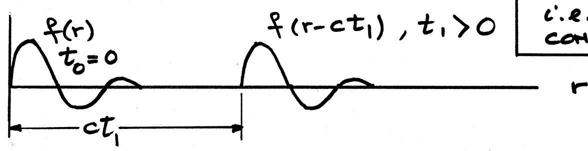
$$\boxed{\frac{1}{c^2}\frac{\partial^2 (rp)}{\partial t^2} - \frac{\partial^2 (rp)}{\partial r^2} = 0} \Rightarrow \boxed{p(r,t) = \underbrace{\frac{f(r-ct)}{r}}_{\text{OUTGOING WAVE}} + \underbrace{\frac{g(r+ct)}{r}}_{\text{INCOMING WAVE}}}$$

f AND g ARBITRARY FUNCTIONS

— WE WILL CONCENTRATE ON OUTGOING WAVES.

— INTERPRETATION OF $f(r-ct)$

NOTE : FOR r = CONST. WE HAVE p = CONST. i.e. SURFACES OF CONST. p ARE SPHERES

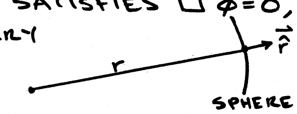


— THE r IN THE DENOMINATOR OF p IS THE SPHERICAL ATTENUATION. A FINITE PRESSURE ATTENUATES TO ZERO AS $r \to \infty$.

SIMPLE MODELS OF WAVES - SPHERICAL WAVES (CONT'D)

TIME DOMAIN (CONT'D)

WE NOTE THAT SINCE VEL. POTENTIAL ALSO SATISFIES $\Box^2 \phi = 0$,
WE HAVE $\phi(r,t) = \frac{f(r-ct)}{r}$, f ARBITRARY



$$p(r,t) = -\rho_0 \frac{\partial \phi}{\partial t} = \frac{\rho_0 c}{r} f'(r-ct)$$

$$\vec{u}(r,t) = \nabla\phi = \underbrace{\frac{\hat{\vec{r}}}{r} f'(r-ct)}_{\text{FAR FIELD}} - \underbrace{\frac{\hat{\vec{r}}}{r^2} f(r-ct)}_{\text{NEAR FIELD}}$$

$$\hat{\vec{r}} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{r} \ (*)$$

WE NOTE THAT IN THE FAR FIELD — $\boxed{\vec{u} = \frac{p}{\rho_0 c}\hat{\vec{r}}}$ WE HAVE ONLY RADIAL COMPONENT

THIS RELATION IS LOCALLY LIKE PLANE WAVE PROPAGATING IN THE DIRECTION $\hat{\vec{r}}$. BUT NOTE THAT IN THE FAR FIELD BOTH p AND $\vec{u}$ FALL OFF AS $1/r$, i.e. BOTH HAVE SPHERICAL ATTENUATION.

— WE CAN LEARN MORE FROM FREQUENCY ANALYSIS. IN PARTICULAR THE ENERGY DENSITY AND ACOUSTIC INTENSITY RELATIONS CAN BE STUDIED MORE EASILY IN FREQ. DOMAIN.

(*) — NOTE $|\hat{\vec{r}}| = 1$, i.e. $\hat{\vec{r}}$ IS UNIT OUTWARD NORMAL TO SPHERE r = CONST.

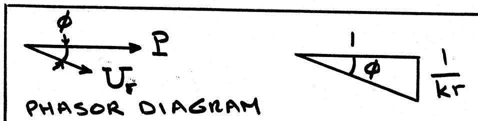
SIMPLE MODELS OF WAVES – SPHERICAL WAVES (CONT'D)

FREQUENCY DOMAIN : $p(r,t) = \frac{A}{r} e^{i(\omega t - kr)}$

$\underline{p}(r) = \frac{A}{r} e^{-ikr}$, $k = \frac{\omega}{c}$ WAVE NUMBER , $k = \frac{2\pi}{\lambda} =$

λ = WAVE LENGTH m , FROM MOMENTUM EQ. WE GET

$$\boxed{\begin{aligned} \vec{U} &= \frac{\underline{p}}{\rho_0 c}\left(1 - \frac{i}{kr}\right)\hat{r} \\ &= U_r \hat{r} \\ U_r &= \frac{\underline{p}}{\rho_0 c}\left(1 - \frac{i}{kr}\right) \end{aligned}}$$



PHASOR DIAGRAM

TO GET THIS RESULT, REMEMBER THAT

$$\nabla f(r) = f'(r)\nabla r \ , \ \nabla r = \frac{\vec{x}}{|\vec{x}|} = \hat{r}$$

FOR OUTGOING WAVES , U_r ALWAYS <u>LAGS</u> $\underline{p}$ BY $\phi = \tan^{-1}(1/kr)$.

$$\boxed{\phi \to \frac{\pi}{2} \text{ IF } kr \to 0 \ , \ \phi \to 0 \text{ IF } kr \to \infty}$$

$kr \to 0$ IF $k \to 0$ (i.e. $\omega \to 0$) FOR A FIXED r

$kr \to 0$ IF $r \to 0$ FOR A FIXED k (OR ω)

$kr \to \infty$ IF $k \to \infty$ (i.e. $\omega \to \infty$ HIGH FREQ.) FOR A FIXED r

$kr \to \infty$ IF $r \to \infty$ FOR ALL FINITE k

SIMPLE MODELS OF WAVES – SPHERICAL WAVES (CONT'D)

AVERAGE ACOUSTIC INTENSITY

$$\boxed{\langle \vec{I} \rangle = \frac{1}{2} \mathrm{Re}\,(\underline{p}\,\vec{U}^*) = \frac{|A|^2 \hat{r}}{2\rho_0 c\, r^2}}$$ VARIES AS r^{-2}

ON A SPHERE OF RADIUS r : $\int_S \langle \vec{I} \rangle \cdot \hat{r}\, d\vec{S} = \frac{2\pi |A|^2}{\rho_0 c} = \text{CONST.}$

AS EXPECTED.

ACOUSTIC ENERGY DENSITY (AVE.) $\langle e \rangle = \underbrace{\langle \frac{1}{2}\rho_0 u_r^2 \rangle}_{\text{K.E. DENS.}} + \underbrace{\langle \frac{p_r^2}{2\rho_0 c^2} \rangle}_{\text{P.E. DENS.}}$

$$\boxed{\langle \tfrac{1}{2}\rho_0 u_r^2 \rangle = \frac{|A|^2}{4\rho_0 c^2 r^2}\left(1 + \frac{1}{k^2 r^2}\right)}$$

$$\boxed{\langle \frac{p_r^2}{2\rho_0 c^2} \rangle = \frac{|A|^2}{4\rho_0 c^2 r^2}} \Rightarrow \boxed{\langle \tfrac{1}{2}\rho_0 u_r^2 \rangle = \left(1 + \frac{1}{k^2 r^2}\right)\langle \frac{p_r^2}{2\rho_0 c^2} \rangle}$$

– AVE. POT. ENERGY DENSITY VARIES AS r^{-2}

– AVE. K.E. DENSITY = AVE P.E. DENSITY AS $kr \to \infty$

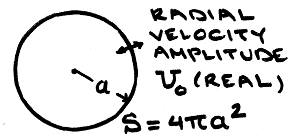
– IF $kr \ll 1 \Rightarrow$ AVE. K.E. DENSITY $\gg$ AVE. P.E. DENSITY
THIS MAKES SENSE BECAUSE $\phi \to \pi/2$ AND LARGE VELOCITY $|U_r|$ IS NEEDED TO SEND OUT ANY ACOUSTIC ENERGY.

WE WILL LEARN MORE FROM THE FOLLOWING PROBLEM.

SIMPLE MODELS OF WAVES – SPHERICAL WAVES (CONT'D)

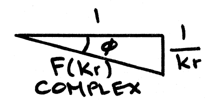
A PULSATING SPHERE: RADIUS a

LET THE RADIAL VELOCITY AMPLITUDE BE U_0 (REAL) SUCH THAT THE DISPLACEMENT AMPLITUDE $|D_0| = \frac{U_0}{\omega} \ll a$.



LET $\boxed{F(kr) = 1 - \frac{i}{kr}}$, THEN THE RESULTS OF THE PREVIOUS SLIDE CAN BE USED TO SHOW THAT

$$\boxed{\begin{aligned} P(r) &= \frac{\rho_0 c\, a\, U_0}{r\, F(ka)} e^{-ik(r-a)} \\ \vec{U}(r) &= \frac{a\, F(kr)}{r\, F(ka)} U_0\, e^{-ik(r-a)}\, \vec{\hat{r}} \end{aligned}}$$



$$\boxed{\langle \vec{I} \rangle = \frac{\rho_0 c\, a^2 U_0^2}{2 r^2 |F(ka)|^2}\, \vec{\hat{r}}} \Rightarrow \boxed{\langle W \rangle = \frac{\rho_0 c\, S\, U_0^2}{2|F(ka)|^2}}$$ MEAN RADIATED ACOUSTIC POWER

$$\boxed{\begin{aligned} \langle W \rangle &\approx \tfrac{1}{2} \rho_0 c (ka)^2 U_0^2 S \quad &\text{IF } ka \ll 1 \\ \langle W \rangle &\approx \tfrac{1}{2} \rho_0 c\, U_0^2 S \quad &\text{IF } ka \gg 1 \end{aligned}}$$ $\leftarrow \phi \approx \pi/2$; $\leftarrow \phi \approx 0$

THUS $\langle W \rangle$ THE TOTAL AVE. ACOUSTIC POWER RADIATED IS A FUNCTION OF ka. FOR SMALL $ka (\ll 1)$, THE PULSATING SPHERE IS A VERY INEFFICIENT RADIATOR. $\langle W \rangle_{MAX} = \frac{1}{2} \rho_0 c\, U_0^2 S$

SIMPLE MODELS OF WAVES – SPHERICAL WAVES (CONT'D)

STATIONARY MONOPOLE: ASSUME $ka \ll 1$

$$P(r) \approx \frac{i \rho_0 c k a^2}{r} U_0\, e^{-ik(r-a)}$$
$$= \frac{i \rho_0 \omega}{4\pi r} (4\pi a^2 U_0)\, e^{-ik(r-a)} = \frac{i \rho_0 \omega S U_0}{4\pi r} e^{-ik(r-a)}$$

IF $q(t) = \rho_0 S U_0 e^{i\omega t}$ IS THE RATE OF MASS INJECTION, THEN $\dot{q}(t) = i\rho_0 \omega S U_0 e^{i\omega t}$, i.e. $i\rho_0 \omega S U_0 = \dot{Q} = i\omega Q$ IS THE COMPLEX AMPLITUDE OF $\dot{q}(t)$. NOW LET $a \to 0$ KEEPING Q FINITE, THEN

$$\boxed{P(r) = \frac{\dot{Q}}{4\pi r} e^{-ikr} = \frac{i\omega Q}{4\pi r} e^{-ikr}} \quad \boxed{\vec{U}(r) = \frac{i\omega Q F(kr)}{4\pi \rho_0 c\, r} e^{-ikr}\, \vec{\hat{r}}}$$

IS THE SOLUTION OF THE HELMHOLTZ EQUATION

IDEAL OR POINT MONOPOLE $\boxed{\nabla^2 P + \frac{\omega^2}{c^2} P = -i\omega Q\, \delta(\vec{x}) = -\dot{Q}\, \delta(\vec{x})}$

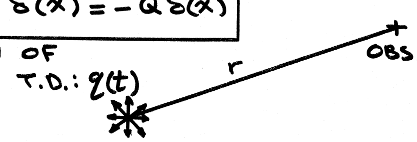
IN TIME DOMAIN THE SOLUTION OF

$$\boxed{\Box^2 p = \dot{q}(t)\, \delta(\vec{x})}$$

IS

$$\boxed{p(r,t) = \frac{\dot{q}(t - r/c)}{4\pi r}}$$

RAYLEIGH'S SOLUTION



Q: RATE OF MASS INJECTION AMPLITUDE } F.D.

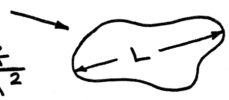
SIMPLE MODELS OF WAVES — SPHERICAL WAVES(CONT'D)

STATIONARY MONOPOLE (CONT'D)

i) $|p(r)| =$ CONST. , $|\vec{u}(r)| =$ CONST. ON SPHERE $r =$ CONST.

i.e. A POINT MONOPOLE HAS SPHERICALLY SYMMETRIC DIRECTIVITY

ii) FOR $ka \gg 1$, i.e. $a \gg \lambda$, $\langle W \rangle \approx \frac{1}{2}\rho_0 c U_0^2 S$ IMPLIES THAT EACH PART OF THE SPHERE OF RADIUS a ACTS LIKE A PLANAR SURFACE. WE CAN SHOW THAT THIS RESULT IS TRUE FOR ANY PULSATING SHAPE IF $L \gg \lambda$ WHERE L IS THE LENGTH SCALE OF THE BODY.



iii) FOR $ka \ll 1$, $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{p}{r}\right) \sim \frac{\omega^2 p}{rc^2} = \frac{k^2 p}{r} \sim \frac{p}{r\lambda^2}$

$\nabla^2\left(\frac{p}{r}\right) \sim \frac{\partial^2}{\partial r^2}\left(\frac{p}{r}\right) \sim \frac{p}{r^3}$. IF $a \lesssim r \ll \lambda \Rightarrow \frac{p}{r\lambda^2} \ll \frac{p}{r^3}$

THIS MEANS THAT FOR $a \lesssim r \ll \lambda$, $ka \ll 1$, p AND $u(\vec{x},t)$ SATISFY LAPLACIAN EQS. $\nabla^2 p = 0$, $\nabla^2 \vec{u} = 0$, i.e. INCOMPRESSIBLE FLOW EQS. THIS MEANS THAT WE CAN TREAT THE PROBLEM AS QUASI-STEADY WITH t AS A PARAMETER OF THE PROBLEM.

ACOUSTIC SOURCES

MONOPOLE SOURCE (IDEAL). BY CONVENTION, THIS IS DEFINED AS:

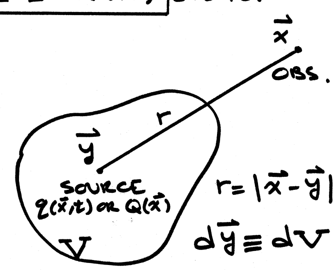
$$\boxed{\begin{array}{c}\Box^2 p = q(t)\,\delta(\vec{x}) \\ 4\pi p(\vec{x},t) = \frac{q(t-r/c)}{r}\end{array}} \qquad \boxed{\begin{array}{c}H P = Q\,\delta(\vec{x}) \\ 4\pi P(\vec{x}) = -\frac{Q\,e^{-ikr}}{r}\end{array}}$$

STEADY STATE
$H = \nabla^2 + k^2$
$k = \omega/c$

THESE ARE (IDEAL) POINT MONOPOLE SOURCES. FROM THESE WE FIND THE SOLUTION TO THE FOLLOWING TWO IMPORTANT PDE'S : $\boxed{\Box^2 p = q(\vec{x},t)}$, $\boxed{H P = Q(\vec{x})}$ STEADY STATE

$$\boxed{4\pi p(\vec{x},t) = \int_V \frac{q(t-r/c)}{r}\,d\vec{y}}$$

$$\boxed{4\pi P(\vec{x}) = -\int_V \frac{Q(\vec{y})\,e^{-ikr}}{r}\,d\vec{y} \quad \text{STEADY STATE}}$$



$r = |\vec{x} - \vec{y}|$

$d\vec{y} \equiv dV$

MONOPOLE DISTRIBUTION

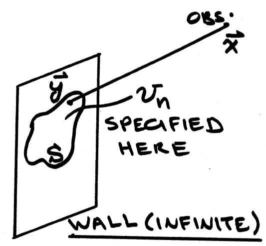
THESE ARE VERY IMPORTANT RESULTS.

WE GAVE THE MODEL OF AN IDEAL POINT MONOPOLE IN THE LAST LECTURE AS A PULSATING SPHERE (NOTE CHANGE OF NOTATION HERE!).

ACOUSTIC SOURCES (CONT'D)

RAYLEIGH PISTON IN THE WALL FORMULA

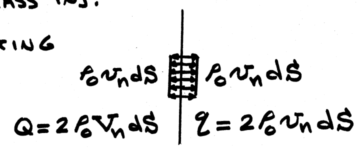
WE HAVE A NORMAL VELOCITY DISTRIBUTION ON AN INFINITE WALL. RAYLEIGH REASONED THAT q AND Q FOR AN ELEMENT OF THE SURFACE AREA dS ARE



$q = 2\rho_0 v_n(\vec{x},t)\,dS$ RATE OF MASS INJECTION

$Q = 2\rho_0 V_n(\vec{x})\,dS$ AMP. OF RATE OF MASS INJ.

* (WE ARE USING THE NOTATION FOR PULSATING SPHERE AGAIN!)



LOOKING EDGEWISE AT THE WALL

$$\boxed{p(\vec{x},t) = \frac{\rho_0}{2\pi}\int_S \frac{\dot{v}_n(\vec{y},t-r/c)}{r}\,dS}$$

$$\boxed{P(\vec{x}) = \frac{i\omega\rho_0}{2\pi}\int_S \frac{V_n(\vec{y})e^{-ikr}}{r}\,dS} \quad \text{STEADY STATE}$$

THESE ARE ALSO VERY IMPORTANT RESULTS.

N

ACOUSTIC SOURCES (CONT'D)

VELOCITY POTENTIAL FOR RAYLEIGH'S PISTON IN THE WALL FORMULA

$\phi(\vec{x},t)$ VEL. POT. $\Rightarrow p = -\rho_0 \frac{\partial\phi}{\partial t}$

$\Phi(\vec{x})$ VEL. POT. (STEADY STATE) $\Rightarrow P = -i\omega\rho_0\Phi$

FROM FORMULAS ON PREVIOUS PAGE

$$\boxed{\phi(\vec{x},t) = -\frac{1}{2\pi}\int_S \frac{v_n(\vec{y},t-r/c)}{r}\,dS}$$

$$\boxed{\Phi(\vec{x}) = -\frac{1}{2\pi}\int_S \frac{V_n(\vec{y})e^{-ikr}}{r}\,dS} \quad \text{STEADY STATE}$$

FROM $\nabla_x r = \hat{r} = \vec{r}/r$, $\vec{r} = \vec{x}-\vec{y}$, WE GET

$$\boxed{\vec{u}(\vec{x},t) = \nabla_x\phi = \frac{1}{2\pi}\int_S \left\{\frac{[\dot{v}_n]_{ret}}{cr} + \frac{[v_n]_{ret}}{r^2}\right\}\hat{r}\,dS}$$

$$\vec{U}(\vec{x}) = \nabla_x\Phi = \frac{1}{2\pi}\int_S \left\{\frac{ik}{r} + \frac{1}{r^2}\right\} V_n(\vec{y})\hat{r}e^{-ikr}\,dS$$

STEADY STATE

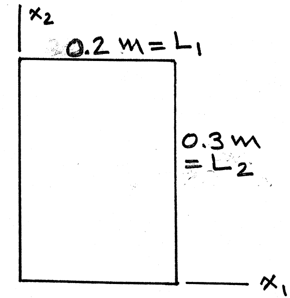
$$\boxed{\vec{U}(\vec{x}) = \frac{ik}{2\pi}\int_S \frac{F(kr)}{r}V_n(\vec{y})\hat{r}e^{-ikr}\,dS}$$

$F(kr) = 1 - i/kr$

ACOUSTIC SOURCES (CONT'D)

RAYLEIGH PISTON IN THE WALL FORMULA (CONT'D)

LET US CONSIDER A PISTON IN AN INFINITE WALL, 0.2 X 0.3 m AS SHOWN ON THE RIGHT.



LET $f = 3000$ HZ AND

$$V_n(x_1, x_2) = \sin\left(\frac{j\pi x_1}{L_1}\right)\sin\left(\frac{\ell\pi x_2}{L_2}\right)$$

$$j = 1, 2, \dots \quad , \ell = 1, 2, \dots$$

FIND $\underline{p}(\vec{x})$ NUMERICALLY. TAKE $c = 340$ m/s. PLOT $|\underline{p}|$ IN PLANES $x_3 =$ CONST. FROM SMALL TO LARGE x_3.

PLOT $|\underline{p}|$ IN THE PLANES $x_1 = 0, L_1/2, L_1$ AND PLANES $x_2 = 0, L_2/2$ AND L_2. PLOT VECTORS $\vec{\underline{U}}(\vec{x})$ IN THESE PLANES ALSO. STUDY ACOUSTIC ENERGY DENSITY NEAR THE PISTON.

WHAT WE ARE LOOKING FOR: FRESNEL & FRAUNHOFER ZONES, DIRECTIVITY PATTERN, EVANESCENT WAVES, EDGE AND CORNER WAVES, PISTON LOADING, ENERGY DENSITY DISTRIBUTION NEAR THE PISTON

ACOUSTIC SOURCES (CONT'D)

(IDEAL) POINT DIPOLE

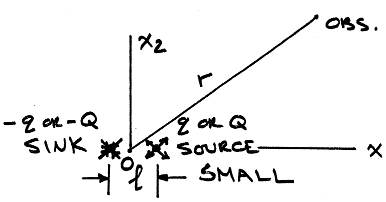
MATHEMATICAL DEFINITION: SINCE $\frac{\partial}{\partial x_i}$ COMMUTES WITH $\Box^2$ AND $H = \nabla^2 + k^2$, AND $q(t - r/c)/4\pi r$ AND $Qe^{-ikr}/4\pi r$ ARE SOLUTIONS OF $\Box^2 p = 0$ AND $H\underline{p} = 0$, RESPECTIVELY $\Rightarrow$

$$p(\vec{x}, t) = \frac{\partial}{\partial x_i}\left[\frac{q(t - r/c)}{4\pi r}\right] \quad \text{IS A SOLUTION OF } \Box^2 p = 0$$

$$\underline{p}(\vec{x}) = \frac{\partial}{\partial x_i}\left[\frac{Qe^{-ikr}}{4\pi r}\right] \quad \text{IS A SOLUTION OF } H\underline{p} = 0$$

PHYSICAL DEFINITION TAKE $i = 1$

WE HAVE A SOURCE AND A SINK NEAR EACH OTHER ALONG x_1-AXIS CLOSE TO THE ORIGIN

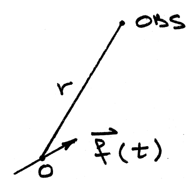


MODEL: AN OSCILLATING SPHERE OF RADIUS $a \ll \lambda$ OSCILLATING IN THE DIRECTION x_i WITH $q = -$ (FORCE ACTING ON THE FLUID)

ACOUSTIC SOURCES (CONT'D)

(IDEAL) POINT DIPOLE (CONT'D)

GENERALIZATION : A STATIONARY FORCE $\vec{f}(t)$, OR $\vec{F}$ STEADY STATE AMPLITUDE, ACTING ON THE FLUID



$$p(\vec{x},t) = -\frac{1}{4\pi}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[\frac{f_i(t-r/c)}{r}\right]$$

$$= \sum_{i=1}^{3}\frac{1}{4\pi}\left\{\frac{[\dot{f}_i\hat{r}_i]_{ret}}{cr} + \frac{[f_i\hat{r}_i]_{ret}}{r^2}\right\} \qquad \hat{r} = \vec{r}/r$$

$$\boxed{p(\vec{x},t) = \frac{1}{4\pi}\left\{\underbrace{\frac{[\dot{f}_r]_{ret}}{cr}}_{\text{FAR FIELD}} + \underbrace{\frac{[f_r]_{ret}}{r^2}}_{\text{NEAR FIELD}}\right\}}$$

$$P(\vec{x},t) = -\frac{1}{4\pi}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[\frac{i\, e^{-ikr}}{r}\right]$$

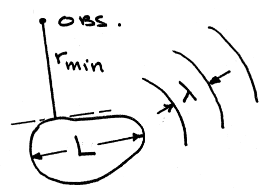
STEADY STATE $\boxed{P(\vec{x},t) = \frac{ik}{4\pi}\frac{F(kr)\,F_r\,e^{-ikr}}{r}}$

$$F(kr) = 1 - \frac{i}{kr}$$

$$F_r = \sum_{i=1}^{3} F_i\,\hat{r}_i$$

COMPACT (STATIONARY) SOURCE

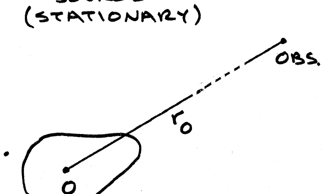
A SOURCE IS COMPACT IF IT CAN BE TREATED AS A POINT SOURCE FOR THE DETERMINATION OF THE RADIATION FIELD.



SOURCE (STATIONARY)

COMPACTNESS CONDITIONS

LET L BE THE MAXIMUM SOURCE DIMENSION, r_{min} BE THE MINIMUM DISTANCE OF THE OBSERVER FROM THE SOURCE. WE MUST HAVE $\boxed{L \ll r_{min}}$. THIS MEANS THAT WE CAN REPLACE $r = |\vec{x}-\vec{y}|$ BY r_o AS SHOWN ON THE RIGHT (NOTE POSITION OF ORIGIN O).

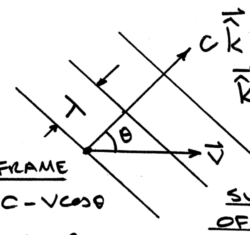


WE MUST ALSO HAVE MAXIMUM DIFFERENCE OF RETARDED (EMISSION) TIME OF THE POINTS ON THE SOURCE $\approx L/c \ll$ PERIOD $T = \frac{1}{f}$ THIS GIVES, USING $\lambda f = c$, $\boxed{L \ll \lambda}$. THIS CONDITION IS EQUIVALENT TO $\boxed{kL \ll 1}$ WHERE $k = \frac{\omega}{c}$

$\boxed{\begin{array}{l} L \ll r_{min} \\ L \ll \lambda \end{array}}$ OR $\boxed{\begin{array}{l} L \ll r_{min} \\ kL \ll 1 \end{array}}$ COMPACTNESS CONDITIONS

SOME WAVE KINEMATICS

FREQUENCY AND WAVE NUMBER RELATIONS FOR OBSERVER IN MOTION WITH VELOCITY $\vec{V}$ FOR PLANE WAVES



$\hat{\vec{k}} = \frac{\vec{k}}{k}$

$k = \frac{2\pi f}{c}$

SURFACES OF CONST. PHASE

REL. VELOCITY OF THE WAVE IN MOVING FRAME IN THE DIRECTION $\hat{\vec{k}}$: $c - \vec{V}\cdot\hat{\vec{k}} = c - V\cos\theta$

NO. OF PEAKS CROSSED /1 SEC. $= \frac{c - V\cos\theta}{\lambda} = f_{MO}$ FREQUENCY DETECTED BY MOVING OBSERVER

$$\boxed{f_{MO} = \frac{c - V\cos\theta}{c/f} = (1 - M\cos\theta)\, f} \qquad M = \frac{V}{c}$$

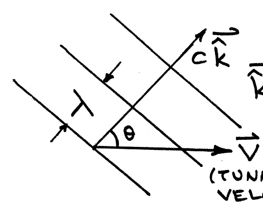
$1 - M\cos\theta$ IS KNOWN AS DOPPLER FACTOR. IF $1 - M\cos\theta < 1$ $\Rightarrow f_{MO} < f$. NOTE THAT f IS THE FREQUENCY OBSERVED BY AN OBSERVER STATIONARY IN THE MEDIUM.

THE WAVE NUMBER k DOES NOT CHANGE FOR MOVING OBSERVER

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{2\pi f_{MO}}{c - V\cos\theta}}$$

SOME WAVE KINEMATICS (CONT'D)

THE FREQUENCY AND WAVE NUMBER RELATIONS IN WIND TUNNEL FRAME FOR PLANE WAVES



$\hat{\vec{k}} = \frac{\vec{k}}{k}$

OBSERVER IS FIXED IN TUNNEL FRAME. THE ACOUSTIC WAVES RIDE THE FLOW WITH VELOCITY $\vec{V}$.

REL. VEL. OF A POINT ON SURFACE OF CONST. PHASE IN DIRECTION $\hat{\vec{k}} = c + V\cos\theta$

$$\boxed{f_T = \frac{c + V\cos\theta}{\lambda} = \frac{c + V\cos\theta}{c/f} = (1 + M\cos\theta)\, f}$$

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{2\pi f_T}{c + V\cos\theta}}$$

NOTE : THE WAVELENGTH λ DOES NOT CHANGE BECAUSE IT IS THE DISTANCE BETWEEN CONSECUTIVE PEAKS WHEN WE FREEZE TIME!

SOME WAVE KINEMATICS (CONT'D)

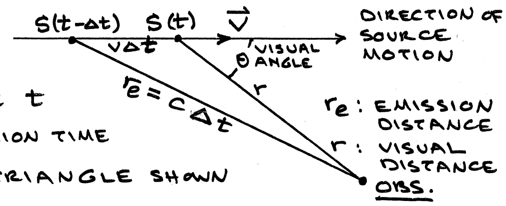
A POINT SOURCE IN MOTION, OBSERVER STATIONARY – SOURCE MOTION RECTILINEAR

$\vec{V}$: SOURCE VELOCITY, CONST.

S : SOURCE

$S(t)$ SOURCE POSITION AT TIME t

$S(t-\Delta t)$ " " " EMISSION TIME



USING COSINE LAW FOR THE TRIANGLE SHOWN

$$r_e^2 = r^2 + (V\Delta t)^2 + 2r\,V\Delta t\cos\theta$$

$$(1-M^2)\,r_e^2 - 2(rM\cos\theta)\,r_e - r^2 = 0 \quad , \quad M = V/C$$

$$r_e = \frac{r}{1-M^2}\left[M\cos\theta \pm \sqrt{M^2\cos^2\theta + 1 - M^2}\right]$$

$$= \frac{r}{1-M^2}\left[M\cos\theta \pm \sqrt{1-M^2\sin^2\theta}\right]$$

IF $M<1$, WE HAVE ONLY ONE SOLUTION $r_e>0$:

$$\boxed{r_e = \frac{r}{1-M^2}\left[M\cos\theta + \sqrt{1-M^2\sin^2\theta}\right]}$$

N

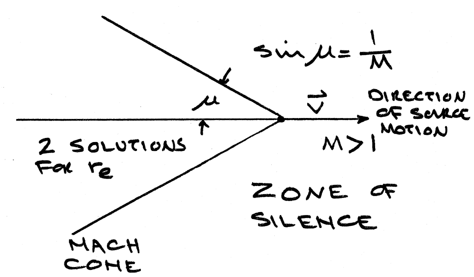
SOME WAVE KINEMATICS (CONT'D)

A POINT SOURCE IN MOTION, OBSERVER STATIONARY (CONT'D)

IF $M>1$, WE HAVE NO SOLUTION WHEN $1-M^2\sin^2\theta<0$

OR $\sin^2\theta > \frac{1}{M^2}$. WE ARE IN ZONE OF SILENCE

IF $1-M^2\sin^2\theta>0$, WE ARE INSIDE THE MACH CONE AND WE HAVE TWO SOLUTIONS FOR r_e .



$$\boxed{\text{EMISSION TIME } t_e = t - r_e/c}$$

THE TRIANGLE SHOWN ON PREVIOUS PAGE IS KNOWN AS THE GARRICK TRIANGLE.

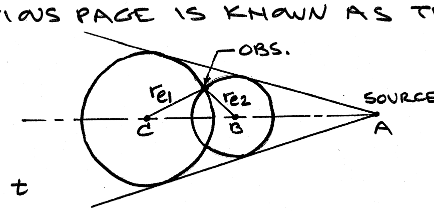
DESCRIPTION OF SYMBOLS, C: SPD OF SOUND

A : SOURCE AT TIME t

B : " " " $t - |AB|/V$

C : " " " $t - |AC|/V$

OBS. GETS TWO SIGNALS FROM B AND C AT t



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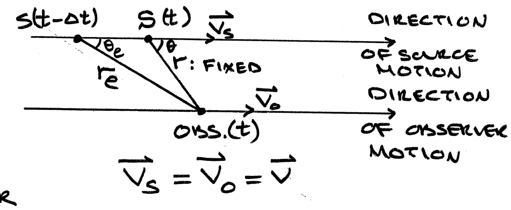
SOME WAVE KINEMATICS (CONT'D)

SOURCE AND OBSERVER IN MOTION RECTILINEARLY WITH THE SAME VELOCITY



AGAIN USING GARRICK TRIANGLE, WE GET THE SAME RELATION FOR r_e.

ALSO $t_e = t - r_e/c$

BUT r_e/c IS A CONSTANT FOR r (VISUAL DISTANCE) = FIXED $\Longrightarrow$

$$e^{i\omega t_e} = e^{-i r_e/c} \cdot e^{i\omega t}$$

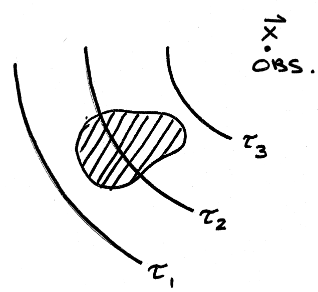
THIS MEANS THAT <u>THE OBSERVER HEARS THE SOURCE FREQUENCY WITHOUT ANY CHANGE</u>! HOWEVER, NOTE THAT THE SIGNAL RECEIVED IS <u>NOT</u> THE SAME AS WHEN $\vec{V} = 0$ BECAUSE r_e AND θ_e DETERMINE THE EMISSION DISTANCE AND ANGLE BOTH OF WHICH ARE <u>VELOCITY DEPENDENT</u>.

A VERY IMPORTANT SOLUTION OF WAVE EQUATION

THE SOLUTION OF $\Box^2 p = Q(\vec{x}, t)$

CAN BE WRITTEN AS

$$\boxed{4\pi\, p(\vec{x}, t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r=c(t-\tau)} Q(\vec{y}, \tau)\, d\Omega}$$



WHERE $d\Omega$ IS THE ELEMENT OF THE SURFACE OF THE SPHERE

Ω : $r = c(t-\tau)$ CENTER AT $\vec{x}$

$(\vec{x}, t)$ OBSERVER VARIABLES FIXED

SOURCE TIME τ: $-\infty < \tau \leq t$

τ : SOURCE TIME

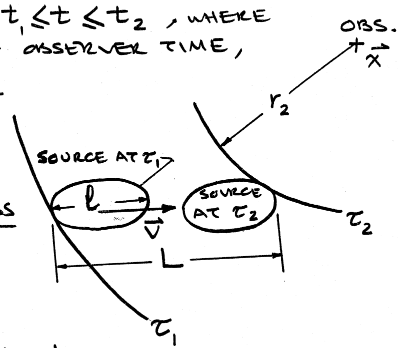
$\tau_1 < \tau_2 < \tau_3 < t$

THIS IS A VERY IMPORTANT SOLUTION OF THE WAVE EQUATION. WE NOTE THAT $Q(\vec{x}, t)$ CAN BE <u>A SOURCE IN MOTION</u>. WE WILL USE THIS RESULT FOR MANY PROBLEMS LATER, e.g. COMPACTNESS CONCEPT FOR MOVING SOURCES.

– Ω IS CALLED <u>THE COLLAPSING SPHERE</u>.

COMPACTNESS CONDITIONS FOR A MOVING SOURCE

ASSUME AN EXTENDED SOURCE OF TYPICAL LENGTH ℓ AND FREQUENCY f MOVING AT VELOCITY $\vec{V}$. WE ARE LOOKING FOR CONDITIONS THAT ALLOW US TO CONSIDER THE SOURCE AS A POINT SOURCE FOR $t_1 \leq t \leq t_2$, WHERE t IS THE OBSERVER TIME. FOR EACH OBSERVER TIME, FIND THE SOURCE TIMES τ_1 AND τ_2 WHEN THE COLLAPSING SPHERE ENTERS AND LEAVES THE SOURCE, RESPECTIVELY. LET L BE THE DISTANCE SHOWN IN THE FIGURE. THEN THE COMPACTNESS CONDITIONS ARE :



FOR ALL $t_1 \leq t \leq t_2$

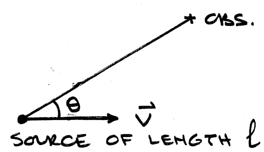
$$L \ll r_2$$

$$\tau_2 - \tau_1 \ll T \equiv \frac{1}{f}$$

NOTE: τ_1, τ_2, L ARE FUNCTIONS OF $(\vec{x}, t)$

THE SITUATION NOW LOOKS LIKE THE FIGURE ON THE RIGHT AND

$$\boxed{L \approx \frac{\ell}{|1 - M_r|}, \quad M_r = |\vec{V}| \cos\theta / c}$$

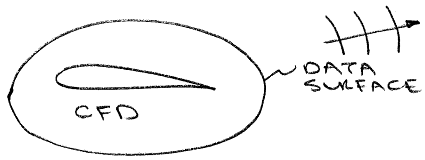


NOISE RADIATION FROM MOVING BODIES

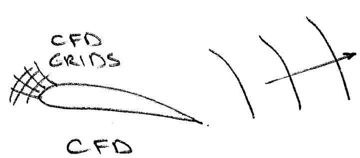
AT PRESENT WE HAVE TWO GENERAL METHOD FOR CALCULATING THE NOISE RADIATED FROM MOVING BODIES

i) THE ACOUSTIC ANALOGY : FFOWCS WILLIAMS - HAWKINGS (FW-H) EQUATION WITH PENETRABLE DATA SURFACE

$$\square^2 p' = Q$$



ii) CFD-BASED COMPUTATIONAL AEROACOUSTICS (CAA) : LINEARIZED EULER , NAVIER-STOKES , ETC.

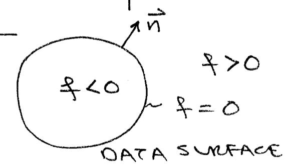


– THE ACOUSTIC ANALOGY BASED ON FW-H EQ. CAN BE CLASSIFIED AS KIRCHHOFF METHOD.

NOISE RADIATION FROM MOVING BODIES (CONT'D)

THE FW-H EQ. WITH PENETRABLE DATA SURFACE

$$\boxed{\begin{aligned}\Box^2 p' = &\frac{\partial}{\partial t}\left\{\left[\rho u_n - (\rho-\rho_0)v_n\right]\delta(f)\right\} \quad &\text{THICKNESS SOURCE}\\ &-\frac{\partial}{\partial x_i}\left\{\left[\rho(u_n - v_n)u_i + p n_i\right]\delta(f)\right\} \quad &\text{LOADING SOURCE}\\ &+\frac{\partial^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right] \quad &\text{QUADRUPOLE SOURCE}\end{aligned}}$$



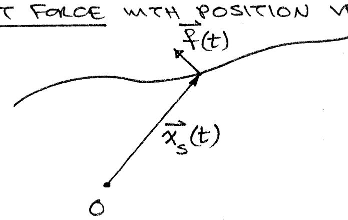
ρ : DENSITY
ρ_0 : DENSITY OF UNDISTURBED MEDIUM
u_n: FLUID VELOCITY NORMAL TO $f=0$
v_n: NORMAL VELOCITY OF $f=0$
p : PRESSURE (GAGE, i.e. $p - p_0$)
$p' = (\rho-\rho_0)c^2$, ACOUSTIC PRESSURE IN LINEAR REGION
$T_{ij} = \rho u_i u_j + [p' - (\rho-\rho_0)c^2]\delta_{ij}$ LIGHTHILL STRESS TENSOR
u_i : FLUID VELOCITY COMPONENT
$\delta(f)$: DIRAC DELTA FUNCTION
$H(f)$: HEAVISIDE FUNCTION

NOISE RADIATION FROM MOVING BODIES (CONT'D)

– THE LOWSON FORMULA IS THE SOLUTION OF

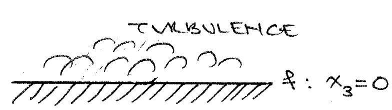
$$\boxed{\Box^2 p' = -\frac{\partial}{\partial x_i}\left\{f_i(t)\,\delta[\vec{x} - \vec{x}_s(t)]\right\}}$$

THIS MODELS A FLUCTUATING POINT FORCE WITH POSITION VECTOR $\vec{x}_s(t)$. $f_i(t)$ IS THE COMPONENT OF THE FORCE $\vec{f}(t)$ ACTING ON THE FLUID.



– THE CURLE FORMULA IS THE SOLUTION OF THE FW-H EQUATION FOR HALF SPACE $x_3>0$ AND $v_n = 0$

– THE RAYLEIGH FORMULA IS A SPECIAL SOLUTION OF THICKNESS TERM OF FW-H EQ.



– THE GREEN'S FUNCTION OF WAVE EQUATION IN THE UNBOUNDED SPACE IS

$$G(\vec{y},\tau;\vec{x},t) = \frac{\delta(g)}{4\pi r}, \quad g = \tau - t + r/c$$

SOME GENERALIZED FUNCTION THEORY

WE NEED TO LEARN HOW TO USE MULTIDIMENSIONAL DIRAC DELTA FUNCTION AND ITS DERIVATIVES. WE WILL LEARN THE MANIPULATION OF INTEGRALS INVOLVING $\delta(f)$ AND ITS DERIVATIVES WITHOUT FULL RIGOROUS JUSTIFICATION. THE JUSTIFICATION REQUIRES A LOT OF WORK. TO LEARN THIS SUBJECT IN DEPTH, SEE BOOKS BY THE FOLLOWING AUTHORS:

i) SEQUENTIAL APPROACH:
- M. J. LIGHTHILL (ONE DIMENSIONAL G.F.'S)
- D. S. JONES (2ND ED.)

ii) FUNCTIONAL APPROACH (OUR METHOD): (*)
- GELFAND & SHILOV, VOL. 1
- R. P. KANWAL (2ND ED.)
- A. H. ZEMANIAN (DOVER BOOKS)

G.F. THEORY HAS HELPED IN *i*) GREAT ADVANCES IN THE THEORY OF PDE'S, *ii*) EXTENDING THE POWER OF OPERATIONAL TECHNIQUES SUCH AS FOURIER TRANSFORM, *iii*) JUSTIFYING MANY AD HOC TECHNIQUES IN APPLIED MATHEMATICS SUCH AS FINITE PART OF DIVERGENT INTEGRALS AND THE USE OF DIVERGENT SERIES.

* SEE ALSO MY NASA TP 3428 AND NASA TM-110285

GENERALIZED FUNCTIONS (CONT'D)

CONVENTIONAL WAY OF THINKING ABOUT FUNCTIONS SUCH AS $f(x) = \sin x$: A TABLE OF ORDERED PAIRS $(x, f(x))$

A MORE GENERAL (NEW) WAY OF THINKING ABOUT FUNCTIONS IS AS THE TABLE: $\{F[\phi] = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx\}$ WHERE $\phi(x)$ IS IN A SPECIFIED SPACE OF FUNCTIONS D, CALLED TEST FUNCTION SPACE.

- GENERALLY, THE MOST USEFUL TEST FUNCTION SPACES ARE INFINITELY DIFFERENTIABLE.
- $F[\phi]$ IS CALLED A FUNCTIONAL BECAUSE IT MAPS D INTO SCALARS ($\mathbb{R}$ OR $\mathbb{C}$). FOR AN ORDINARY FUNCTION $f(x)$, THIS FUNCTIONAL IS LINEAR AND CONTINUOUS.
  - LINEARITY MEANS $F[\alpha_1\phi_1 + \alpha_2\phi_2] = \alpha_1 F[\phi_1] + \alpha_2 F[\phi_2]$, ϕ_1 AND ϕ_2 ARE IN D
  - CONTINUITY MEANS THAT IF $\phi_n \to \phi$ THEN $F[\phi_n] \to F[\phi]$ SIMILARLY IF $\partial\phi_n/\partial x_i \to \partial\phi/\partial x_n$, THEN $F[\partial\phi_n/\partial x_i] \to F[\partial\phi/\partial x_i]$. THIS IS A VERY NICE PROPERTY. (*)

(*) WE REQUIRE THIS PROPERTY TO HOLD FOR ANY DERIVATIVES OF ϕ_n AND ϕ. NOTE BOTH ϕ_n AND ϕ ARE IN D.

GENERALIZED FUNCTIONS (CONT'D)

NOTE THAT WE ARE NOW VIEWING ORDINARY FUNCTION $f(x)$ AS [A TABLE GENERATED BY] THE FUNCTIONAL $F[\phi]$ WHICH IS CONTINUOUS & LINEAR. WE WILL THINK AS $f(x)$ AND $F[\phi]$ AS THE SAME THING! NOW FUNCTIONAL IS A RULE MAPPING D INTO SCALARS AND THE RULE DOES NOT HAVE TO BE $\int f\phi\,dx$!

ARE THERE ANY OTHER CONTINUOUS & LINEAR FUNCTIONALS ON D? THE ANSWER IS YES!

EXAMPLE: TAKE THE FUNCTIONAL $\delta[\phi] = \phi(0)$. THIS IS A MAPPING OF D INTO REALS $\mathbb{R}$ IF ϕ IS REAL. THIS FUNCTIONAL IS CONTINUOUS AND LINEAR! WE SHOULD LEGITIMATELY CONSIDER THIS A "FUNCTION" FROM OUR NEW POINT OF VIEW. SUCH A "FUNCTION" IS A GENERALIZATION OF THE CONCEPT OF ORDINARY FUNCTION.

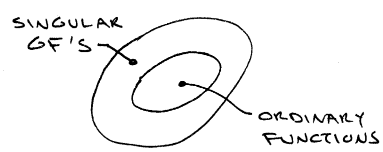
GENERALIZED FUNCTIONS: THE SPACE OF CONTINUOUS, LINEAR FUNCTIONALS [FUNCTIONS!] ON D

- ALL ORDINARY FUNCTIONS ARE G.F.'S (REGULAR G.F.'S)
- THERE ARE MANY FNS IN G.F. SPACE THAT ARE NOT ORDINARY FUNCTIONS, e.g. $\delta[\phi] = \phi(0)$! WE CALL SUCH FUNCTIONS SINGULAR G.F.'S.

GENERALIZED FUNCTIONS (CONT'D)

FOR SINGULAR GF'S SUCH AS $\delta[\phi]$, WE INTRODUCE "SYMBOLIC" FUNCTIONS, SUCH AS $\delta(x)$, AS FOLLOWS

$$\delta[\phi] = \int_{-\infty}^{\infty} \phi(x)\,\delta(x)\,dx = \phi(0)$$



THE SPACE OF GF'S

NOTE THAT "THE INTEGRAL" HERE IS NOT THE USUAL INTEGRAL (RIEMANN, LEBESGUE, ETC.) YOU LEARNED BEFORE! IT IS OBVIOUSLY WRITTEN SIMILAR TO THE FUNCTIONAL USED FOR ORDINARY FUNCTIONS BUT HERE IT IS USED FOR BOOK-KEEPING! IT STANDS FOR $\phi(0)$. BUT WITHOUT THE INTRODUCTION OF SYMBOLIC FUNCTIONS, WE WOULD HAVE TO USE AWKWARD EXPRESSION FOR DEFINING GREEN'S FUNCTION, ETC! WITHOUT A CLEAR UNDERSTANDING OF THIS POINT, YOU WILL HAVE MUCH PROBLEMS UNDERSTANDING THE MANIPULATIONS INVOLVING SINGULAR GF'S.

— SUMMARY: WE DID TWO THINGS i) WE LOOKED AT ORDINARY FUNCTIONS NOT AS ORDERED PAIRS $(x, f(x))$ BUT AS FUNCTIONALS $F[\phi] = \int f\phi\,dx$, ϕ IN TEST FN SPACE, ii) WE TAKE ANY CONTINUOUS LINEAR FUNCTIONAL AS A FUNCTION, i.e. G.F.!

GENERALIZED FUNCTIONS (CONT'D)

WE EXTEND THE OPERATIONS OF ORDINARY FNS TO ALL GF'S BY USING THE FUNCTIONAL DEFINITION

EXAMPLE: TAKE TEST FUNCTION SPACE D AS THE SPACE OF INFINITELY DIFFERENTIABLE FUNCTIONS THAT ARE ZERO OUTSIDE A FINITE INTERVAL. LET $f(x)$ BE AN ORDINARY DIFFERENTIABLE FUNCTION AND $F[\phi]=\int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx$, ϕ IN D. NOW, BY INTEGRATION BY PARTS AND THE FACT THAT ϕ IS ZERO BEYOND A FINITE INTERVAL, WE HAVE $\int_{-\infty}^{\infty} f'\phi\,dx = -\int_{-\infty}^{\infty} f\,\phi'\,dx \equiv -F[\phi']$

SINCE $F[\phi]$ DEFINES (IDENTIFIES, IS) $f(x)$, WE MUST IDENTIFY $f'(x)$ WITH $F'[\phi]$ WHICH WE HAVE SHOWN ABOVE TO BE $-F[\phi']$, i.e. $\boxed{F'[\phi] = -F[\phi']}$ WHICH MAKES SENSE FOR ALL GEN. FNS!

— WE NEXT WRITE THIS RESULT SYMBOLICALLY. BUT FIRST AN EXAMPLE:

THE DIRAC DELTA FN $\delta(x)$: $\delta[\phi] = \phi(0) = \int_{-\infty}^{\infty}\phi(x)\delta(x)\,dx$

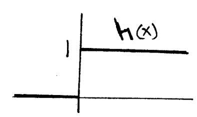
$$\delta'[\phi] = -\delta[\phi'] = -\phi'(0) = \underbrace{\int_{-\infty}^{\infty}\phi(x)\,\delta'(x)\,dx}$$

THIS INTEGRAL STANDS FOR $\delta'[\phi]$: A FUNCTIONAL

GENERALIZED FUNCTIONS (CONT'D)

GEN. DERIVATIVE OF HEAVISIDE FUNCTION

$$h(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$



NOTE THAT THE ORDINARY DERIVATIVE OF THIS FUNCTION $h'(x)=0$, (UNDEFINED AT $x=0$). THE GENERALIZED DERIVATIVE IS NOT ZERO. USING THE RULE OF PREVIOUS VUGRAPH:

$H[\phi] = \int_{-\infty}^{\infty} h(x)\,\phi(x)\,dx = \int_{0}^{\infty}\phi(x)\,dx$: THE FUNCTIONAL DEFINING $h(x)$

$$H'[\phi] = -H[\phi'] = -\int_0^{\infty}\phi'(x)\,dx = \phi(0) = \delta[\phi] = \int_{-\infty}^{\infty}\phi(x)\,\delta(x)\,dx$$

i.e. $\boxed{\text{GEN. DER. OF } h(x) \equiv \bar{h}'(x) = \delta(x) \neq h'(x) = 0}$ ORDINARY DERIVATIVE OF $h(x)$

WE USE A BAR OVER DERIVATIVE (OR $\bar{d}/dx$, $\bar{\partial}/\partial x_i$, ETC) TO SIGNIFY GENERALIZED DIFFERENTIATION WHENEVER THERE IS THE DANGER OF CONFUSION. FOR EXAMPLE, WE DON'T HAVE TO WRITE $\bar{\delta}'(x)$ BECAUSE, $\delta(x)$ ONLY HAS GENERALIZED DERIVATIVE.

— AN IMPORTANT RESULT. NOTE THAT $\int_{-\infty}^{x} h'(y)\,dy = 0$ BUT $\int_{-\infty}^{x}\bar{h}'(y)\,dx = \int_{-\infty}^{x}\delta(y)\,dy = h(x)$, i.e. GENERALIZED DIFFERENTIATION MAINTAINS THE MEMORY OF THE JUMP!

GENERALIZED FUNCTIONS (CONT'D)

SOME USEFUL RESULTS :

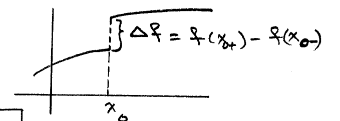
i) $\delta(x-x_0)$: $\boxed{\int_{-\infty}^{\infty} \phi(x)\,\delta(x-x_0)\,dx = \phi(x_0)}$ THIS IS SHIFTED DELTA FUNCTION

ii) nTH DERIVATIVE OF G.F. : $\boxed{F^{(n)}[\phi] = (-1)^n F[\phi^{(n)}]}$

EXAMPLE : $\int_{-\infty}^{\infty} \phi(x)\,\delta^{(n)}(x-x_0)\,dx = (-1)^n \phi^{(n)}(x_0)$

WHERE $\phi^{(n)}(x) = \frac{d^n\phi}{dx^n}$, ϕ IN D (C^∞ FNS WITH BOUNDED SUPPORT)

iii) f DIFFERENTIABLE WITH THE EXCEPTION OF $x = x_0$, f HAS A JUMP OF Δf AT x_0, THEN



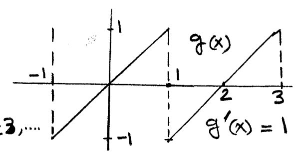
$\boxed{\text{GEN. DER. } f(x) \equiv \bar{f}'(x) = f'(x) + \Delta f\,\delta(x-x_0)}$

EXAMPLE $h(x)$ HEAVISIDE FN , $\bar{h}'(x) = \underbrace{\Delta h}_{=1}\delta(x-0) = \delta(x)$!

NOTE : $\int_{-\infty}^{x} \bar{f}'(x)\,dx = f(x) \neq \int_{-\infty}^{x} f'(x)\,dx$

EXAMPLE : $g(x)$ A RAMP FN, PERIODIC WITH PERIOD 2 $\Rightarrow$ $\bar{g}'(x) = 1 - 2\underbrace{\sum_{n=-\infty}^{\infty} \delta(x-2n-1)}_{\text{SHIFTED DELTA FNS AT } \pm1, \pm3, \dots}$

JUMP AT $x = \pm1, \pm3, \dots$



GENERALIZED FUNCTIONS (CONT'D)

WHEN WE WANT TO FIND THE GREEN'S FUNCTION OF AN ODE OR A PDE, THE PROBLEM MUST BE SET UP IN THE SPACE OF GENERALIZED FUNCTIONS AND ALL DERIVATIVES ARE THEN GENERALIZED DERIVATIVES.

EXAMPLE : TO FIND THE GREEN'S FUNCTION OF THE OPERATOR

$$Lu : \begin{cases} u'' & x \in [0,1] \\ u(0) + u(1) = 0 \\ 2u'(0) - u'(1) = 0 \end{cases} \text{BC}$$

, WE MUST SOLVE FOR $g(x,y)$ — $\boxed{\text{LINEAR \& HOMOGENEOUS}}$

SUCH THAT

$$\begin{cases} \frac{\bar{\partial}^2 g}{\partial x^2} = \delta(x-y) \quad (*) & \text{GENERALIZED DERIVATIVE HERE} \\ g(0,y) + g(1,y) = 0 \\ 2\frac{\partial g}{\partial x}(0,y) - \frac{\partial g}{\partial x}(1,y) = 0 & \text{ORDINARY DERIVATIVE HERE} \end{cases}$$

NOTE : GENERALLY, WE PREFER TO USE THE NOTATION $\frac{\bar{d}^2}{dx^2}$ & $\frac{d}{dx}$!

— TO SOLVE FOR u IN $u''(x) = f(x)$ + BC'S (ABOVE), WE USE $u(x) = \int_0^1 f(y)\,g(x,y)\,dy$

THIS SHOWS THE USEFULNESS OF GREEN'S FUNCTION

(*) THIS SAYS THAT $g(x,y)$ CAN NOT BE DISCONTINUOUS AT $x=y$ BUT $\partial g/\partial x$ HAS TO BE TO GET $\delta(x-y)$ ON THE RIGHT !

GENERALIZED FUNCTIONS (CONT'D)

DIRAC DELTA FUNCTION IN MULTIDIMENSIONAL SPACE (3D)

HERE WE CAN HAVE TWO SITUATIONS (CASES 2 & 3) THAT ARE NEW:

i) $\delta(\vec{x})$, A DIRAC DELTA FUNCTION CONCENTRATED AT A POINT

ii) $\delta(f)$, WHERE $f(\vec{x})=0$ IS A SURFACE IN 3D

iii) $\delta(f)\delta(g)$ WHERE $f(\vec{x})=0$ AND $g(\vec{x})=0$ ARE TWO SURFACES THAT INTERSECT ON A CURVE $C: f=g=0$

WE DISPOSE OF CASE i) FIRST

$$\int \phi(\vec{x})\,\delta(\vec{x})\,d\vec{x} = \phi(0) \qquad \text{(0 IS THE ORIGIN IN 3D)}$$

$$\int \phi(\vec{x})\,\delta(\vec{x}-\vec{x}_0)\,d\vec{x} = \phi(\vec{x}_0)$$

HERE $\int \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$. USE THREE INTEGRAL SIGNS ONLY WHEN YOU WANT TO IMPRESS PEOPLE!

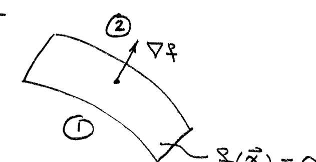
— CASE (ii) IS A DIRAC DELTA FUNCTION CONCENTRATED ON THE SURFACE $f=0$

— CASE (iii) IS A DIRAC DELTA FUNCTION CONCENTRATED ON THE CURVE $C: f=g=0$.

GENERALIZED FUNCTIONS (CONT'D)

HOW DOES $\delta(f)$ APPEAR IN MATHEMATICAL EXPRESSIONS?

GENERALIZED DERIVATIVE OF A SCALAR FUNCTION DISCONTINUOUS ACROSS THE SURFACE $f(\vec{x})=0$



LET $Q(\vec{x})$ BE DISCONTINUOUS ACROSS $f=0$. DEFINE $\Delta Q = Q(f_{2}) - Q(f_{1})$ WHERE REGION (2) IS WHERE ∇f POINTS INTO.

THEN WE CAN SHOW THAT

$$\boxed{\frac{\overline{\partial} Q}{\partial x_i} = \frac{\partial Q}{\partial x_i} + \Delta Q \frac{\partial f}{\partial x_i}\,\delta(f)} \quad \boxed{\overline{\nabla} Q = \nabla Q + \Delta Q\,\nabla f\,\delta(f)}$$

IF $\vec{Q}(\vec{x})$ IS A VECTOR FUNCTION DISCONTINUOUS ACROSS $f=0$, THEN

$$\boxed{\begin{aligned} \overline{\nabla}\cdot\vec{Q} &= \nabla\cdot\vec{Q} + \nabla f\cdot\Delta\vec{Q}\,\delta(f) \\ \overline{\nabla}\times\vec{Q} &= \nabla\times\vec{Q} + \nabla f\times\Delta\vec{Q}\,\delta(f) \end{aligned}}$$

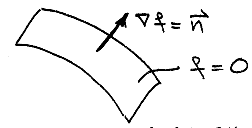
— IN APPLICATIONS, THE DISCONTINUITY SURFACE IS EITHER REAL (e.g, A SHOCK SURFACE) OR ARTIFICIALLY INTRODUCED (DERIVATIONS OF FW-H & KIRCHHOFF EQS.).

— WE CAN ALWAYS DEFINE $f(\vec{x})$ SUCH THAT $\nabla f = \vec{n}$ (UNIT NORMAL). IF THIS CONDITION DOES NOT HOLD, THEN DEFINE $\tilde{f}(\vec{x}) = f(\vec{x})/|\nabla f|$.

GENERALIZED FUNCTIONS (CONT'D)

IMPORTANT FACT: CONSERVATION LAWS ARE VALID IF WE TREAT THE DERIVATIVES AS GEN. DERIVATIVES. THIS MEANS THAT THE SHOCK JUMP CONDITIONS ARE INCLUDED IN CONSERVATION LAWS.

CONTINUITY EQ.:



MOVING SHOCK

$$0 = \frac{\bar{\partial} \rho}{\partial t} + \bar{\nabla}\cdot(\rho \vec{u}) = \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) + \left[\Delta \rho \frac{\partial f}{\partial t} + \Delta(\rho\vec{u})\cdot\vec{n}\right]\delta(f)$$

$\boxed{\frac{\partial f}{\partial t} = -v_n}$ NORMAL VELOCITY OF THE SHOCK

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \qquad \vec{x} \text{ NOT ON THE SHOCK}$$

$$\boxed{-\Delta\rho\, v_n + \Delta(\rho u_n) = \Delta\left[\rho(u_n - v_n)\right] = 0}$$

MOMENTUM EQUATION

$$\frac{\bar{\partial}}{\partial t}(\rho u_i) + \frac{\bar{\partial}}{\partial x_j}(\rho u_i u_j) + \frac{\bar{\partial} p}{\partial t} = \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial t} + \left[\Delta(\rho u_i)\frac{\partial f}{\partial t} + \Delta(\rho u_i u_j)\, n_j + \Delta p\, n_i\right]\delta(f)$$

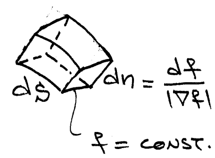
$$\Rightarrow \boxed{\Delta\left[\rho u_i (u_n - v_n)\right] + \Delta p\, n_i = 0}$$

GENERALIZED FUNCTIONS (CONT'D)

A USEFUL RESULT : $\int Q(\vec{x})\,\delta(f)\,d\vec{x} = ?$

WRITING $d\vec{x} = dn\, dS = \frac{df\, dS}{|\nabla f|}$

WE HAVE



$$\int Q(\vec{x})\,\delta(f)\,d\vec{x} = \int Q(\vec{x})\,\delta(f)\,\frac{df\, dS}{|\nabla f|} = \int_{f=0} \frac{Q(\vec{x})}{|\nabla f|}\, dS$$

IF $|\nabla f| = 1$, AS WE USUALLY ASSUME, THEN

$$\boxed{\int Q(\vec{x})\,\delta(f)\,d\vec{x} = \int_{f=0} Q(\vec{x})\, dS}$$

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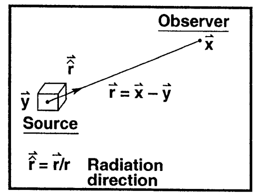
Things to Know About Green's Function of Wave Equation

- The Green's function of the wave equation in *the unbounded space* is

$$G(\vec{y}, \tau; \vec{x}, t) = \begin{cases} \dfrac{\delta(g)}{4\pi r} & \tau \le t \\ 0 & \tau > t \end{cases}$$



$g = \tau - t + \dfrac{r}{c}$ *outgoing wave*

$(\vec{y}, \tau)$ *source* space-time variables

$(\vec{x}, t)$ *observer* space-time variables

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Things to Know About Green's Function of Wave Equation

- There are many methods to derive $G(\vec{y}, \tau; \vec{x}, t)$ rigorously. It is easy to show that G depends on $\vec{x} - \vec{y}$ and $t - \tau$. Using $\vec{x} - \vec{y} = \vec{r}$, $\lambda = t - \tau$, take *spatial* Fourier transform of $\bar{\Box}^2_{(\vec{r}, \lambda)} G = \delta(\vec{r})\delta(\lambda)$ to get a simple problem involving finding the Green's function of an O.D.E. in λ. The inverse spatial Fourier transform of the Green's function of the O.D.E. gives Green's function of the wave equation for both the outgoing and incoming waves.

$$\boxed{\vec{r} = \vec{x} - \vec{y}, \quad r = |\vec{x} - \vec{y}|, \quad \hat{\vec{r}} = \frac{\vec{r}}{r}, \quad \frac{\partial r}{\partial x_i} = \hat{r}_i, \quad \frac{\partial r}{\partial y_i} = -\hat{r}_i}$$

Useful things to remember

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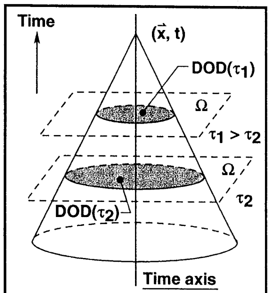
Things to Know About Green's Function of Wave Equation

The support of $\delta(g)$ is on the surface $g = 0$.
The surface $g = 0$ is $r = |\vec{x} - \vec{y}| = c(t - \tau)$.
This is the *characteristic cone* of the wave equation with vertex at $(\vec{x}, t)$. Since $\Box^2$ is a differential equation with constant coefficients, $g = 0$ is also the *characteristic conoid* with vertex at $(\vec{x}, t)$. This gives us the picture on the right. Note that we have drawn the 3D space Ω as a plane in the figure. Therefore, this figure is a 3D illustration of what happens in 4D (3D space + time).

- **Note**: $g = 0$ is a cone because if the 4-vector $\vec{A} = (\vec{x} - \vec{y}, t - \tau)$ lies on $g = 0 \Rightarrow \alpha\vec{A} = [\alpha(\vec{x} - \vec{y}), \alpha(t - \tau)]$ also lies on $g = 0$. This is the property of a cone.



Characteristic cone (Conoid) at $(\vec{x}, t)$. Ω is the unbounded 3-space. DOD is domain of dependence of $(\vec{x}, t)$ which depends on τ

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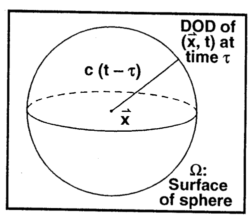
Things to Know About Green's Function of Wave Equation

- *Visualization of domain of dependence of* $(\vec{x}, t)$ *in four dimensions.*

Fix $(\vec{x}, t)$ and $\tau \Rightarrow r = c(t - \tau)$ is a sphere with center at $\vec{x}$ and radius $c(t - \tau)$. Any source on this sphere at time τ, contributes to $\vec{x}$ at time t. As τ increases, the radius shrinks, hence we have a *collapsing sphere*. Radius becomes zero at $\tau = t$.



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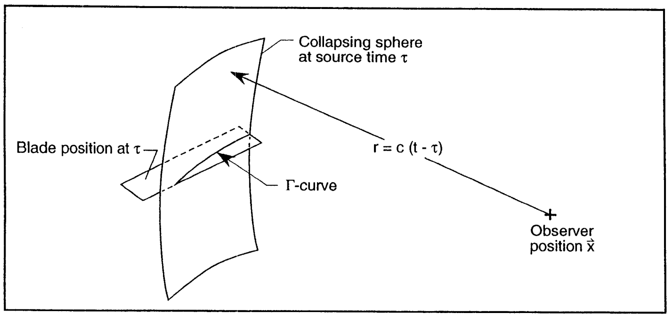
The Collapsing Sphere Concept

Equation of collapsing sphere: $r = c(t - \tau)$, $(\vec{x}, t)$ fixed



The Σ-surface is the locus of Γ-curves in space. If the blade surface is described by $f(\vec{y}, \tau) = 0$, the equation of the Σ-surface is:

$$F(\vec{y}; \vec{x}, t) = [f(\vec{y}, \tau)]_{\text{ret}} = f(\vec{y}, t - r/c) = 0, (\vec{x}, t) \text{ fixed}$$

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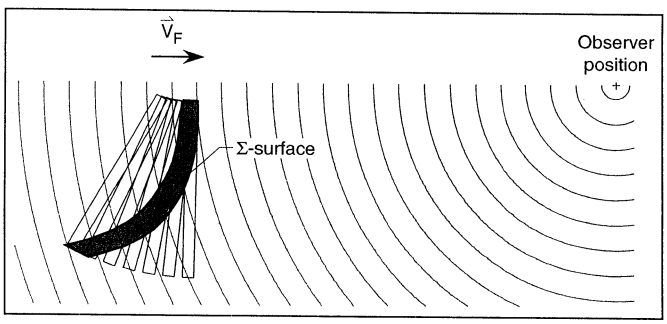
Construction of Σ-Surface for a Helicopter Rotor Blade



In this construction, we have taken a rotor blade of zero thickness rotating with rotational Mach number 0.67 and forward Mach number 0.15. The observer is in the rotor plane. The circles are the intersection of the collapsing sphere with the plane containing the rotor. The circles are drawn at equal source time intervals. The observer time is $t = \tau + r/c$ where r is the radius of the collapsing sphere at τ. Note that t is fixed for the above Σ-surface.

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The Two Forms of the Solution of Wave Equation (Volume Sources)

We want to find the solution of $\Box^2\phi = Q(\vec{x}, t)$

$$4\pi\phi(\vec{x}, t) = \int \frac{1}{r}Q(\vec{y}, \tau)\delta(g)d\vec{y}d\tau$$

All volume integrals are over unbounded 3 space and all time integrals are over $(-\infty, t)$.

i) Let $\tau \rightarrow g \Rightarrow \frac{\partial g}{\partial \tau} = 1$ and $4\pi\phi(\vec{x}, t) = \int \frac{1}{r}Q\left(\vec{y}, g + t - \frac{r}{c}\right)\delta(g)dgd\vec{y}$

Integrate with respect to g to get

$$\boxed{4\pi\phi(\vec{x}, t) = \int \frac{1}{r}Q\left(\vec{y}, t - \frac{r}{c}\right)d\vec{y} = \int \frac{[Q]_{\text{ret}}}{r}d\vec{y}}$$

Retarded Time Solution

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The Two Forms of the Solution of Wave Equation (Volume Sources) (Cont'd)

ii) Let $y_3 \rightarrow g \Rightarrow \frac{\partial g}{\partial y_3} = -\frac{1}{c}\hat{r}_3$

$$4\pi\phi(\vec{x}, t) = \int \frac{cQ(\vec{y}, \tau)}{r}\delta(g)\ dg\frac{dy_1dy_2}{|\hat{r}_3|}\ d\tau$$

Since in the inner integrals $(\vec{x}, t)$ and τ are fixed, then $\frac{dy_1dy_2}{|\hat{r}_3|} = d\Omega$ element of surface area of sphere $r = c(t-\tau)$. Integrate with respect to g to get:

$$\boxed{4\pi\phi(\vec{x}, t) = \int_{-\infty}^{t}\frac{d\tau}{t-\tau}\int_{r = c(t-\tau)} Q(\vec{y}, \tau)\ d\Omega}$$

Collapsing Sphere Solution

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NOISE GENERATION FROM MOVING SOURCES

LOWSON'S FORMULA (THE SOUND FROM A MOVING DIPOLE) - 1965

ASSUME THAT A COMPACT (POINT) FORCE OF STRENGTH $\vec{F}(t)$ MOVES SUBSONICALLY. LET THE POSITION OF THE FORCE BE GIVEN BY $\vec{x}_s(t)$. NOTE THAT $\vec{F}(t)$ IS THE FORCE ACTING ON THE MEDIUM. LOWSON PROPOSED THAT THE NOISE FROM THIS SOURCE IS DESCRIBED BY:

$$\boxed{\Box^2 p' = -\frac{\partial}{\partial x_i}\{F_i(t)\,\delta[\vec{x}-\vec{x}_s(t)]\}}$$

WHERE p' IS THE ACOUSTIC PRESSURE. WE CAN USE VECTOR NOTATION FOR THIS EQUATION AS FOLLOWS:

$$\boxed{\Box^2 p' = -\nabla\cdot\{\vec{F}(t)\,\delta[\vec{x}-\vec{x}_s(t)]\}}$$

THIS IS THE GENERALIZATION OF LAMB'S DIFFERENTIAL EQUATION WHICH WE DEDUCE FROM HIS SOLUTION OF THE SOUND FROM A STATIONARY COMPACT FORCE:

$$\Box^2 p' = -\nabla\cdot[\vec{F}(t)\,\delta(\vec{x})] \qquad (*)$$

LAMB DERIVED THE FOLLOWING SOLUTION BY STUDYING THE ACOUSTIC FIELD OF AN OSCILLATING SPHERE AND THEN LETTING ITS RADIUS GO TO ZERO:

$$4\pi p'(\vec{x},t) = -\nabla\cdot\left[\frac{\vec{F}(t-r/c)}{r}\right]$$

THIS IS THE SOLUTION OF THE PDE (*). LAMB DID NOT GIVE THE PDE!

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NOISE GENERATION FROM MOVING SOURCES (CONT'D)

LOWSON'S FORMULA (CONT'D)

SOLUTION OF THE WAVE EQ. $\Box^2 p' = -\nabla\cdot\{\vec{F}(t)\,\delta[\vec{x}-\vec{x}_s(t)]\}$ (*)

THERE ARE MANY WAYS TO FIND p' SOME (MANY!) OF WHICH REQUIRE A LOT OF ALGEBRAIC MANIPULATIONS. WE USE SEVERAL TRICKS TO GET THE SOLUTION MORE ELEGANTLY.

— IF $\vec{\phi}_1(\vec{x},t)$ IS THE SOLUTION OF $\Box^2\vec{\phi}_1 = \vec{Q}(\vec{x},t) \Rightarrow$

$\phi_2(\vec{x},t) = \nabla\cdot\vec{\phi}_1$ IS THE SOLUTION OF $\Box^2\phi_2 = \nabla\cdot\vec{Q}$

PROOF: $\nabla\cdot\Box^2\vec{\phi}_1 = \Box^2[\nabla\cdot\vec{\phi}_1] = \nabla\cdot\vec{Q} \quad \therefore \quad \phi_2 = \nabla\cdot\vec{\phi}_1 + \phi_3$ WHERE $\Box^2\phi_3 = 0$. THE SOLUTION OF $\Box^2\phi_3=0$ IS $\phi_3=0$!

THE SOLUTION OF EQ. (*) IS, THEREFORE

$$4\pi p'(\vec{x},t) = -\nabla_x\cdot\int\frac{1}{r}\vec{F}(\tau)\,\delta[\vec{y}-\vec{x}_s(\tau)]\,\delta(g)\,d\vec{y}\,d\tau$$

LET $\vec{z} = \vec{y}-\vec{x}_s(\tau)$, $\vec{y} = \vec{z}+\vec{x}_s(\tau) \Rightarrow d\vec{y} = d\vec{z}$

$$g = \tau - t + |\vec{x}-\vec{z}-\vec{x}_s(\tau)|/c$$

$$4\pi p'(\vec{x},t) = -\nabla_x\cdot\int\frac{1}{r}\vec{F}(\tau)\,\delta(\vec{z})\,\delta[g(\vec{z},\tau;\vec{x},t)]\,d\vec{z}\,d\tau$$

$$= -\nabla_x\cdot\int\left[\frac{1}{r}\vec{F}(\tau)\,\delta(g)\right]_{\vec{z}=0}d\tau$$

NOISE GENERATION FROM MOVING SOURCES (CONT'D)
LOWSON'S FORMULA (CONT'D)

$$g\big|_{\vec{\eta}=0} = \tau - t + |\vec{x} - \vec{x}_s(\tau)|/c \quad , \quad r\big|_{\vec{\eta}=0} = |\vec{x} - \vec{x}_s(\tau)|$$

NOW LET $\tau \to g$, WE HAVE THE JACOBIAN OF TRANSFORMATION

$$\frac{1}{|\partial g/\partial \tau|} = \frac{1}{|1 - M_r|} \quad , \quad M_r = \frac{1}{c}\,\dot{\vec{x}}_s(\tau)\cdot\hat{r} \,, \quad \hat{r} = \frac{\vec{r}}{r}$$

NOTE $\dfrac{\partial g}{\partial \tau} = 1 - \dfrac{1}{c}\dfrac{\partial}{\partial \tau}|\vec{x} - \vec{x}_s(\tau)| = 1 - \dfrac{1}{c}\dot{\vec{x}}_s(\tau)\cdot\hat{r}$!

$$4\pi p'(\vec{x},t) = -\nabla_x\cdot\int\left[\frac{1}{r|1-M_r|}\vec{F}(\tau)\,\delta(g)\right]_{\vec{\eta}=0} dg$$

$$= -\nabla_x\cdot\left[\frac{\vec{F}(\tau)}{r|1-M_r|}\right]_{\substack{\vec{\eta}=0\\ g=0}}$$

NOW $g\big|_{\vec{\eta}=0} = 0$ MEANS THAT $\tau - t + |\vec{x} - \vec{x}_s(\tau)|/c = 0$.

THIS GIVES A SOLUTION FOR RETARDED TIME τ^* IF THE FORCE MOVES SUBSONICALLY. NOTE $\tau^* = \tau^*(\vec{x},t)$.

$$\boxed{4\pi p'(\vec{x},t) = -\nabla_x\cdot\left[\frac{\vec{F}(\tau)}{r|1-M_r|}\right]_{\tau^*}}$$

IF WE TAKE THE DIVERGENCE, WE GET LOWSON'S FORMULA

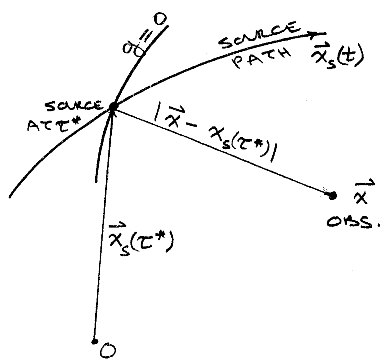
NOISE GENERATION FROM MOVING SOURCES (CONT'D)
LOWSON'S FORMULA (CONT'D)

INTERPRETATION OF $\tau^* = \tau^*(\vec{x},t)$

$g = \tau - t + |\vec{x} - \vec{y}|/c = 0$
IS THE COLLAPSING SPHERE FOR $(\vec{x},t)$

τ^* IS THE SOLUTION OF
$\tau - t + |\vec{x} - \vec{x}_s(\tau)|/c = 0$
i.e. WHEN THE COLLAPSING SPHERE INTERSECTS THE SOURCE!

IN GENERAL FINDING τ^* INVOLVES THE SOLUTION OF A NONALGEBRAIC (TRANSCENDENTAL) EQUATION



TO GET LOWSON'S FORMULA, WE MUST TAKE THE DIVERGENCE OF $[F(\tau)/(r|1-M_r|)]_{\tau^*}$. IT IS A LONG MESSY RESULT. WE NEED THE FAR FIELD RESULT WHICH CAN BE OBTAINED EASILY FROM A RESULT OF FARASSAT. IT CAN BE SHOWN THAT THE ACOUSTIC PRESSURE CAN BE WRITTEN ALSO AS (EXACT RESULT!):

$$\boxed{4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{F_r}{r|1-M_r|}\right]_{\tau^*} + \left[\frac{F_r}{r^2|1-M_r|}\right]_{\tau^*}}$$

WHERE $F_r = \vec{F}\cdot\hat{r}$, i.e. COMPONENT OF FORCE IN RADIATION DIRECTION

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

LOWSON'S FORMULA IN THE FAR FIELD

IN THE FAR FIELD LOWSON'S FORMULA SIMPLIFIES CONSIDERABLY.

$$\boxed{4\pi p'(\vec{x},t) = \underbrace{\left[\frac{\dot{F}_r}{cr(1-M_r)^2}\right]_{\tau^*}}_{\text{FORCE FLUCTUATION}} + \underbrace{\left[\frac{F_r \dot{M}_r}{cr(1-M_r)^3}\right]_{\tau^*}}_{\text{FORCE ACCELERATION}}}$$

NOTE DEPENDENCE ON $1-M_r$!

WHERE $\dot{F}_r = \dot{\vec{F}}\cdot\hat{r}$, $\dot{M}_r = \dot{\vec{M}}\cdot\hat{r}$ WHERE $(\dot{\ }) = \partial/\partial\tau$.

WE HAVE $\frac{\partial}{\partial t}[(\cdot)]_{\tau^*} = \left[\frac{\partial\tau^*}{\partial t}\frac{\partial}{\partial\tau}(\cdot)\right]_{\tau^*}$. FROM

$\tau^* - t + |\vec{x} - x_s(\tau^*)|/c = 0$, WE GET $\frac{\partial\tau^*}{\partial t} = \frac{1}{1-M_r}$

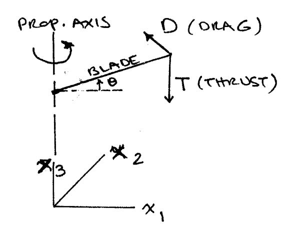
LOWSON'S FORMULA TELLS US THAT, IN THE FAR FIELD, $p'(\vec{x},t)$ DEPENDS ON FORCE FLUCTUATION AND ACCELERATION IN RADIATION DIRECTION AT THE MOMENT OF EMISSION OF ACOUSTIC SIGNAL.

— WE CAN SHOW THAT $\tau - t + |\vec{x} - \vec{x}_s(\tau)|/c = 0$ HAS ONLY ONE SOLUTION FOR SUBSONIC MOTION OF THE FORCE. IT HAS AN ODD NUMBER OF SOLUTIONS FOR τ^* IF THE SOURCE MOVES SUPERSONICALLY.

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

GUTIN'S RESULT (1930'S)

GUTIN CALCULATED THE NOISE FROM STEADY ROTATING FORCES FROM A PROPELLER. HE DERIVED THE RESULT FOR NONCOMPACT SOURCE DISTIBUTION BUT BECAUSE OF THE LACK OF COMPUTERS IN 1930'S, HE MADE APPROXIMATIONS EQUIVALENT TO ROTATING POINT FORCES AS SHOWN. WE ARE SHOWING THE DRAG AND LIFT FORCES ACTING ON THE MEDIUM. GUTIN USED LAMB'S RESULT BY REPRESENTING THE PROPELLER FORCES BY STATIONARY PERIODIC FORCES IN PROPELLER DISK. LET Ω BE THE SHAFT FREQUENCY. THEN



$\vec{F} = (-D\sin\theta, D\cos\theta, -T)$

$\dot{\vec{F}} = (-D\Omega\cos\theta, -D\Omega\sin\theta, 0)$

$$\hat{p}_m(\vec{x}) = \frac{1}{T}\int_0^T p(\vec{x},t)\, e^{im\Omega t}\,dt \qquad T = \frac{1}{f} = \frac{2\pi}{\Omega}$$

WE CAN SHOW THAT $dt = |1-M_r|\,d\tau$

$$\hat{p}_m(\vec{x}) = \frac{1}{T}\int_0^T \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{F_r}{r|1-M_r|}\right]_{\tau^*} e^{im\Omega t}\,dt = -\frac{im\Omega}{cT}\int_0^T \left[\frac{F_r}{r|1-M_r|}\right]_{\tau^*} e^{im\Omega t}\,dt$$

$$= -\frac{im\Omega}{cT}\int_0^T \frac{F_r(\tau)}{r}\, e^{im\Omega(\tau + r/c)}\,d\tau \qquad \text{(FAR FIELD)}$$

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

GUTIN'S RESULT (CONT'D)

$$r^2 = (x_1 - R\cos\theta)^2 + (x_2 - R\sin\theta)^2 + x_3^2 \quad , \quad R << \|x\| \equiv r_0$$

$$= r_0^2 + R^2 - 2R(x_1\cos\theta + x_2\sin\theta)$$

PUT THE OBSERVER ON $x_1 x_3$-PLANE. THIS DOES NOT AFFECT THE PRESSURE AMPLITUDE BUT DOES INFLUENCE PHASE.

$$r \approx r_0 - \frac{R}{r_0} x_1 \cos\theta = r_0 - R\sin\psi\cos\theta \quad , \quad \sin\psi = \frac{x_1}{r_0}$$

$$\hat{\vec{r}} = (\cos\psi, 0, \sin\psi)$$

$$F_r = -D\cos\psi\sin\theta - T\sin\psi \quad , \quad \theta = \Omega\tau \ , \ d\theta = \Omega d\tau$$

$$\hat{P}_m(\vec{x}) = -\frac{im}{cTr_0}\int_0^{2\pi} F_r(\theta)\, e^{im\Omega r_0}\, e^{im\Omega(\theta - \frac{R}{c}\sin\psi\cos\theta)}\, d\theta$$

$$= \frac{im\Omega}{2\pi c r_0} e^{im\Omega r_0} \underbrace{\int_0^{2\pi} (D\cos\psi\sin\theta + T\sin\psi)\, e^{im(\theta - \frac{R\sin\psi}{c}\cos\theta)\Omega}\, d\theta}_{\text{ANALYTIC RESULT IS AVAILABLE!}}$$

$$(\alpha D\cos\psi + \beta T\sin\psi)\, J_m\!\left(\frac{mR\Omega}{c}\sin\psi\right)$$

BESSEL FN OF 1ST KIND AND ORDER m

$$\boxed{\text{FOR } B \text{ BLADES } m = nB, \ n = 1, 2, \ldots}$$

The Governing Wave Equation for Deriving Kirchhoff Formulas

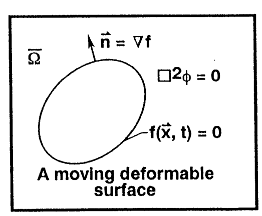
We consider *the exterior problem* here.

$\bar{\Omega}$: The exterior unbounded space

$$\text{Let } \tilde{\phi}(\vec{x}, t) = \begin{cases} \phi(\vec{x}, t) & \vec{x} \in \bar{\Omega} \\ 0 & \vec{x} \notin \bar{\Omega} \end{cases} \Rightarrow \Box^2\tilde{\phi} = 0 \text{ everywhere}$$

$$\frac{\bar{\partial}\tilde{\phi}}{\partial t} = \frac{\partial\tilde{\phi}}{\partial t} + \phi\frac{\partial f}{\partial t}\delta(f) = \frac{\partial\tilde{\phi}}{\partial t} - v_n\phi\delta(f)$$

where $v_n = -\frac{\partial f}{\partial t}$ is the local normal velocity on $f = 0$



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The Governing Wave Equation for Deriving Kirchhoff Formulas (Cont'd)

Next take the second time derivative of $\tilde{\phi}$:

$$\frac{\bar{\partial}^2\tilde{\phi}}{\partial t^2} = \frac{\partial^2\tilde{\phi}}{\partial t^2} + \frac{\partial \phi}{\partial t}\frac{\partial f}{\partial t}\delta(f) - \frac{\partial}{\partial t}[v_n \phi \delta(f)] = \frac{\partial^2\tilde{\phi}}{\partial t^2} - v_n \phi_t \delta(f) - \frac{\partial}{\partial t}[v_n \phi \delta(f)]$$

Similarly for the space derivatives we have:

$$\bar{\nabla}\tilde{\phi} = \nabla\tilde{\phi} + \phi \hat{n}\delta(f), \qquad \bar{\nabla}^2\tilde{\phi} = \nabla^2\tilde{\phi} + \phi_n \delta(f) + \nabla\cdot[\phi\hat{n}\delta(f)]$$

83 The above results give:

$$\bar{\Box}^2\tilde{\phi} = \frac{1}{c^2}\frac{\bar{\partial}^2\tilde{\phi}}{\partial t^2} - \bar{\nabla}^2\tilde{\phi} = \Box^2\tilde{\phi} - \left(\frac{v_n \phi_t}{c^2} + \phi_n\right)\delta(f)$$

$$-\frac{1}{c^2}\frac{\partial}{\partial t}[v_n \phi \delta(f)] - \nabla\cdot[\phi \hat{n}\delta(f)]$$

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The Governing Wave Equation for Deriving Kirchhoff Formulas (Cont'd)

Since $\Box^2\tilde{\phi} = 0$, and using $M_n = v_n/c$, we get

$$\boxed{\bar{\Box}^2\tilde{\phi} = -\left(\phi_n + \frac{1}{c}M_n\phi_t\right)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}[M_n\phi\,\delta(f)] - \nabla\cdot[\phi\,\hat{n}\,\delta(f)]}$$

We now solve this wave equation for stationary, subsonic and supersonic surfaces.

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Derivation of the Classical Kirchhoff Formula

The Kirchhoff surface $f(\vec{x})$ is now stationary so that $M_n = 0$. The governing wave equation is $\boxed{\bar{\Box}^2\tilde{\phi} = -\phi_n\delta(f) - \nabla\cdot[\phi\hat{n}\delta(f)]}$

$$4\pi\tilde{\phi}(\vec{x},t) = -\int\frac{\phi_n}{r}\delta(f)\delta(g)d\vec{y}d\tau - \nabla_{\vec{x}}\cdot\int\frac{\phi\hat{n}}{r}\delta(f)\delta(g)d\vec{y}d\tau$$

where ϕ_n and ϕ in the integrands are functions of $(\vec{y},\tau)$. Now let $\tau \rightarrow g$, $\frac{\partial g}{\partial \tau} = 1$, and integrate with respect to g, to get

$$4\pi\tilde{\phi}(\vec{x},t) = -\int\frac{\phi_n(\vec{y}, t-r/c)}{r}\delta(f)d\vec{y} - \nabla_{\vec{x}}\cdot\int\frac{\phi(\vec{y}, t-r/c)\hat{n}}{r}\delta(f)d\vec{y}$$

We have dealt with these integrals before. The integration of $\delta(f)$ gives

$$4\pi\tilde{\phi}(\vec{x},t) = -\int_{f=0}\frac{1}{r}\phi_n(\vec{y}, t-r/c)\,dS - \nabla_{\vec{x}}\cdot\int_{f=0}\frac{\hat{n}}{r}\phi(\vec{y}, t-r/c)\,dS$$

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Derivation of the Classical Kirchhoff Formula (Cont'd)

Taking the divergence operator in and using subscript ret for retarded time, we get *the classical Kirchhoff formula*

$$\boxed{4\pi\tilde{\phi}(\vec{x},t) = \int_{f=0}\frac{[c^{-1}\dot{\phi}\cos\theta - \phi_n]_{\text{ret}}}{r}dS + \int_{f=0}\frac{\cos\theta}{r^2}[\phi]_{\text{ret}}dS}$$

In this equation $\cos\theta = \hat{n}\cdot\hat{r}$. Again, our method tells that $\tilde{\phi}(\vec{x},t) = 0$ in the interior of $f = 0$ which is not obvious from classical derivation.

Note: Only r is a function of $\vec{x}$ in the integrands of the integrals in previous vugraph. We assume $\vec{x}$ is not on S and S is piecewise smooth. The justification for bringing the divergence operator inside the integral follows from classical analysis.

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NOISE GENERATION FROM MOVING SOURCES (CONT'D)

METHOD OF DESCENT : FROM $\Box_3^2 \rightarrow \Box_2^2$ (D'ALEMBERTIAN IN 2D)

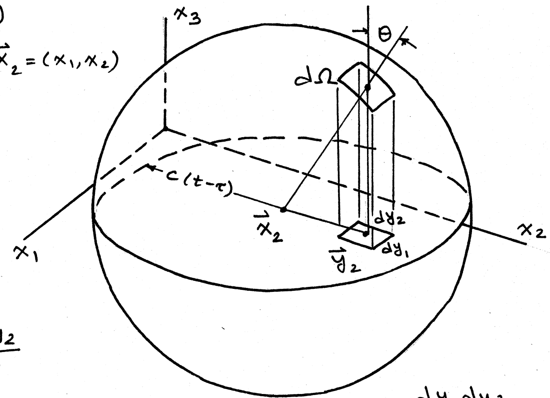
TO SOLVE $\Box_2^2 \phi = Q(\vec{x}_2, t)$

WHERE $\Box_2^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla_2^2$, $\vec{x}_2 = (x_1, x_2)$

WE SOLVE

$$\Box_3^2 \phi = Q(\vec{x}_2, t)$$

i.e. WE ASSUME Q DOES NOT DEPEND ON x_3.



$$4\pi\phi(\vec{x}_2, t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r=c(t-\tau)} Q(\vec{y}_2, \tau)\, d\Omega$$

$$= 2\int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r_2=c(t-\tau)} Q(\vec{y}_2, \tau)\, \frac{dy_1 dy_2}{\cos\theta}$$

$$= 2c \int_{-\infty}^{t} d\tau \int_{r_2=c(t-\tau)} \frac{Q(\vec{y}_2,\tau)\, dy_1 dy_2}{\sqrt{c^2(t-\tau)^2 - r_2^2}}$$

$$\boxed{2\pi\phi(\vec{x}_2, t) = c\int_{-\infty}^{t} d\tau \int_{r_2=c(t-\tau)} \frac{Q(\vec{y}_2,\tau)\, dy_1 dy_2}{\sqrt{c^2(t-\tau)^2 - r_2^2}}}$$

$$d\Omega = \frac{dy_1 dy_2}{\cos\theta}$$

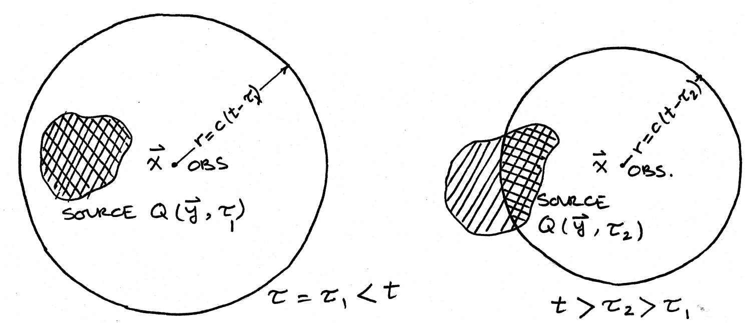
SPHERE WITH RADIUS $r = c(t-\tau)$

$$\cos\theta = \frac{\sqrt{c^2(t-\tau)^2 - |\vec{x}_2 - \vec{y}_2|^2}}{c(t-\tau)}$$

$$r_2 = |\vec{x}_2 - \vec{y}_2|$$

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

METHOD OF DESCENT (CONT'D)

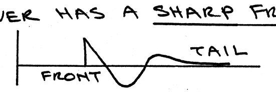


$\tau = \tau_1 < t$ $\qquad$ $t > \tau_2 > \tau_1$

WE ARE NOW IN 2D SO WE DROP SUBSCRIPT 2 FROM $\vec{x}_2$, $\vec{y}_2$ AND r_2.

REGION OF INTEGRATION OVER THE SOURCE REGION IS DOUBLE CROSS-HATCHED.

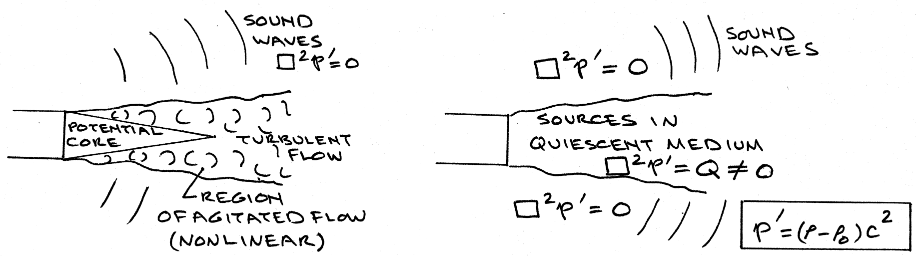
— FOR ANY FINITE SOURCE DISTRIBUTION, AND EVEN FINITE DURATION, THE SIGNAL RECEIVED BY AN OBSERVER HAS A SHARP FRONT BUT A DECAYING TAIL.



— AS $\tau \rightarrow t$, WE HAVE AN IMPROPER CONVERGENT INTEGRAL FOR $Q(\vec{y}, \tau)$ A CONTINUOUS FUNCTION.

THE ACOUSTIC ANALOGY (AA)

THE ACOUSTIC ANALOGY WAS INTRODUCED BY M.J. LIGHTHILL IN 1952 IN AEROACOUSTICS. LIGHTHILL'S 1ST PAPER "ON SOUND GENERATED AERODYNAMICALLY. I. GENERAL THEORY", PROC. ROY. SOC. OF LOND., VOL. 211A, 564-587, CONCERNS THE STUDY OF JET NOISE. SUPERFICIALLY, THE IDEA OF AA IS VERY SIMPLE



A REAL JET | A JET MODELED BY AA

LIGHTHIL SHOWED THAT $Q = \dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$ WHERE

$$\boxed{T_{ij} = \rho u_i u_j + [(p - p_0) - c^2(\rho - \rho_0)]\delta_{ij}}$$ LIGHTHILL STRESS TENSOR

u_i: COMPONENT OF VELOCITY, $[\ ]_0$: UNDISTURBED MEDIUM CONDITIONS

THE ACOUSTIC ANALOGY (AA) (CONT'D)

LIGHTHILL OBTAINED HIS JET NOISE EQUATION BY MANIPULATING MASS CONTINUITY AND MOMENTUM EQS. WE WILL DO THIS FOR FW-H EQ. WE GIVE SOME MATHEMATICAL RESULTS NOW.

i) THE SOLUTION OF $\square^2 p' = \dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$ $\quad V_J$: JET VOLUME

$$4\pi p'(\vec{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{V_J} \frac{[T_{ij}(\vec{y}, \tau)]_{ret}}{r} d\vec{y}$$

$$= \frac{1}{c^2 r_0} \int_{V_J} \left[\frac{\partial^2}{\partial \tau^2} T_{rr}(\vec{y}, \tau)\right]_{ret} d\vec{y} \quad \text{(FAR FIELD)}$$

r_0 : MEAN DISTANCE OF OBSERVER FROM THE JET

$T_{rr} = T_{ij} \hat{r}_i \hat{r}_j$, $\hat{r} = \dfrac{\vec{r_0}}{r_0}$ RADIATION DIRECTION

WE CAN WRITE THE LAST RESULT AS

$$4\pi p'(\vec{x}, t) = \frac{1}{c^2 r_0} \int_{V_J} \frac{\partial^2}{\partial t^2} T_{rr}(\vec{y}, t - \tfrac{r}{c}) \, d\vec{y}$$

WE NOW INTRODUCE AUTOCORRELATION OF PRESSURE

STATIONARY ASSUMPTION USED HERE!

$$I(\vec{x}, \tau) = \frac{1}{\rho_0 c} \langle p'(\vec{x}, t) p'(\vec{x}, t + \tau) \rangle \quad \text{TIME AVERAGE}$$

WHERE τ IS NOW JUST A TIME VARIABLE!

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THE ACOUSTIC ANALOGY (CONT'D)

$$I(\vec{x},\tau) = \frac{1}{16\pi^2 \rho_0 c^5 r_0^2} \int_{V_j}\int_{V_j} \left\langle \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_1, t-\frac{r_1}{c}) \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_2, t+\tau-\frac{r_2}{c}) \right\rangle d\vec{y}_1 d\vec{y}_2$$

$$r_1 = |\vec{x}-\vec{y}_1| , \quad r_2 = |\vec{x}-\vec{y}_2| .$$

WE CAN SHOW THAT

$$\left\langle \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_1, t-\frac{r_1}{c}) \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_2, t+\tau-\frac{r_2}{c}) \right\rangle = \frac{\partial^4 G}{\partial \tau^4}$$

WHERE $G = \langle T_{rr}(\vec{y}_1, t-r_1/c) T_{rr}(\vec{y}_2, t+\tau-r_2/c) \rangle$

$= \langle T_{rr}(\vec{y}_1, t) T_{rr}(\vec{y}, t+\tau+(r_1-r_2)/c) \rangle$

WE HAVE AGAIN USED THE ASSUMPTION OF STATIONARY PROCESS HERE.

$$\frac{r_1-r_2}{c} \approx \frac{1}{c}\hat{r}\cdot(\vec{y}_2-\vec{y}_1) \equiv \frac{1}{c}\hat{r}\cdot\vec{\eta}$$

DEFINE TWO POINT CROSS CORRELATION

$$R(\vec{y},\vec{\eta},\tau) = \langle T_{rr}(\vec{y},t) T_{rr}(\vec{y}+\vec{\eta}, t+\tau) \rangle$$

$$I(\vec{x},\tau) = \frac{1}{16\pi^2 \rho_0 c^5 r_0^2} \frac{\partial^4}{\partial \tau^4} \int_{V_j}\int_{V_j} R(\vec{y},\vec{\eta},\tau+\hat{r}\cdot\vec{\eta}/c)\, d\vec{y}\, d\vec{\eta}$$

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THE ACOUSTIC ANALOGY (CONT'D)

FROM THIS RESULT, WE CAN GET THE SPECTRAL DENSITY OF THE SOUND BY FOURIER TRANSFORM:

$$S(\vec{x},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} I(\vec{x},\tau) e^{i\omega\tau} d\tau$$

$$= \frac{\pi\omega^4}{2\rho_0 c^5 r_0^2} \int_{V_j} H[\vec{y}, \frac{\omega}{c}\hat{r}, (1-M_r)\omega]\, d\vec{y}$$

WHERE

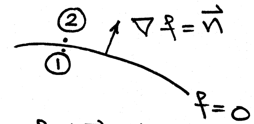
$$H(\vec{y},\vec{k},\omega) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{V_j} e^{i(\omega\tau-\vec{k}\cdot\vec{\eta})} R(\vec{y},\vec{\eta},\tau)\, d\vec{\eta}\, d\tau$$

HERE M_r IS BASED ON CONVECTION SPEED OF TURBULENT EDDIES. THE NEXT STEP IN THE GAME IS COMING UP WITH A MODEL FOR $R(\vec{y},\vec{\eta},\tau)$. VARIOUS MODELS ARE USED. THE PROCESS IS CONTINUING!

THE ACOUSTIC ANALOGY (CONT'D)

PREPARATION FOR DERIVATION OF FW-H EQ.

$q(\vec{x},t)$ DISCONTINUOUS ACROSS $f=0$



$$\frac{\bar{\partial} q}{\partial t} = \frac{\partial q}{\partial t} + \Delta q \frac{\partial f}{\partial t}\delta(f)$$
$$= \frac{\partial q}{\partial t} - v_n \Delta q\, \delta(f)$$
$$\frac{\bar{\partial} q}{\partial x_i} = \frac{\partial q}{\partial x_i} + n_i \Delta q\, \delta(f)$$

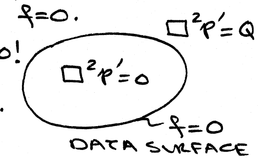
$f(\vec{x},t)=0$

$\Delta q = (q_2 - q_1)_{f=0}$

$\left\{ v_n = -\frac{\partial f}{\partial t} \right.$ LOCAL NORMAL VELOCITY OF $f=0$

THE BASIC IDEA BEHIND DERIVATION OF FW-H EQ.

WE WANT TO DERIVE A LINEAR WAVE EQUATION VALID IN THE ENTIRE 3D SPACE SO THAT WE USE THE GREEN'S FUNCTION OF WAVE EQUATION IN UNBOUNDED SPACE. WE MUST, THEREFORE, EXTEND $p' = c^2\rho'$ TO INSIDE OF DATA SURFACE $f=0$. WE SELECT THE SIMPLEST p' SATISFYING $\square^2 p' = 0$! THIS WOULD INTRODUCE DISCONTINUITIES IN FLUID PARAMETERS ACROSS THE SURFACE $f=0$. THE PROPER SETTING FOR SUCH A PROBLEM IS IN GENERALIZED FUNCTION SPACE.



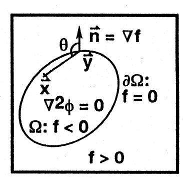
Use of Green's Functions for Discontinuous Solutions

Green's function can be used to find discontinuous solutions if the derivatives in the differential equation are treated as generalized derivatives. This adds to usefulness of Green's function.

Example: *Green's Identity for Laplace Equation*

Let $\tilde{\phi}(\vec{x}) = \begin{cases} \phi(\vec{x}) & \vec{x}\in\Omega \\ 0 & \vec{x}\notin\Omega \end{cases} \Rightarrow \nabla^2\tilde{\phi} = 0$ everywhere.

$$\bar{\nabla}\tilde{\phi} = \nabla\tilde{\phi} + \Delta\tilde{\phi}\vec{n}\delta(f) = \nabla\tilde{\phi} - \phi\vec{n}\delta(f)$$
$$\bar{\nabla}^2\tilde{\phi} = \nabla^2\tilde{\phi} - \nabla\phi\cdot\vec{n}\delta(f) - \nabla\cdot[\phi\vec{n}\delta(f)]$$
$$= -\frac{\partial\phi}{\partial n}\delta(f) - \nabla\cdot[\phi\vec{n}\delta(f)]$$



Interior Problem

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Use of Green's Functions for Discontinuous Solutions (Cont'd)

Since this equation is valid in the unbounded space, we can use the Green's function $-\frac{1}{4\pi r}$ to get the Green's identity

$$4\pi\tilde{\phi}(\vec{x}) = \int \frac{1}{r}\frac{\partial\phi}{\partial n}\delta(f)d\vec{y} + \nabla_{\vec{x}}\cdot\int\frac{\phi\vec{n}}{r}\delta(f)d\vec{y}$$

$$= \int_{f=0}\frac{1}{r}\frac{\partial\phi}{\partial n}dS + \nabla_{\vec{x}}\cdot\int_{f=0}\frac{\phi\vec{n}}{r}dS = \int_{f=0}\frac{\phi_n}{r}dS - \int_{f=0}\frac{\phi\cos\theta}{r^2}dS$$

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This method tells us that when $\vec{x}\notin\Omega$, $\tilde{\phi} = 0$ which is not obvious from the classical derivation. The exterior problem is similar.

Note: $r = |\vec{x}-\vec{y}|$ is the only term in the integrands of the above integrals which is a function of $\vec{x}$. We assume that $\vec{x}$ is not located on S and S is piecewise smooth. The justification for the exchange of the divergence and integral operators follows from classical analysis.

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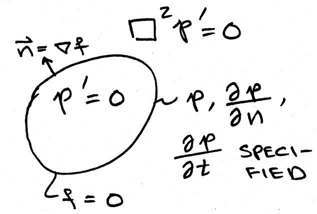
THE ACOUSTIC ANALOGY (CONT'D)

THE K EQUATION (K: KIRCHHOFF)



WE NEED TO FIND WHAT $\bar{\Box}^2 p'$ IS BECAUSE p' IS DISCONTINUOUS ACROSS $f=0$

$v_n = -\frac{\partial f}{\partial t}, M_n = \frac{v_n}{c}$

$f = f(\vec{x},t) = 0$ DEFORMABLE !

$$\frac{\bar{\partial} p'}{\partial t} = \frac{\partial p'}{\partial t} + (p'-0)\frac{\partial f}{\partial t}\delta(f) = \frac{\partial p'}{\partial t} - v_n p'\delta(f)$$

$$\frac{\bar{\partial}^2 p'}{\partial t^2} = \frac{\partial^2 p'}{\partial t^2} - v_n\frac{\partial p'}{\partial t}\delta(f) - \frac{\partial}{\partial t}[v_n p'\delta(f)]$$

$$\bar{\nabla} p' = \nabla p' + p'\vec{n}\,\delta(f)$$

$$\bar{\nabla}^2 p' = \nabla^2 p' + \nabla p'\cdot\vec{n}\,\delta(f) + \nabla\cdot[p'\vec{n}\,\delta(f)]$$

$$= \nabla^2 p' + \frac{\partial p}{\partial n}\delta(f) + \nabla\cdot[p'\vec{n}\,\delta(f)]$$

$$\bar{\Box}^2 p' = \underbrace{\Box^2 p'}_{\text{0 EVERYWHERE!}} - \left(\frac{M_n}{c}\frac{\partial p'}{\partial t} + \frac{\partial p}{\partial n}\right)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}[M_n p'\delta(f)]$$

$$- \nabla\cdot[p'\vec{n}\,\delta(f)]$$

$$\boxed{\bar{\Box}^2 p' = -\left(\frac{M_n}{c}\frac{\partial p'}{\partial t} + \frac{\partial p}{\partial n}\right)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}[M_n p'\delta(f)] - \nabla\cdot[p'\vec{n}\,\delta(f)]}$$

THE SOLUTION OF THIS EQ., THE K EQ. GIVES KIRCHHOFF FORMULA!

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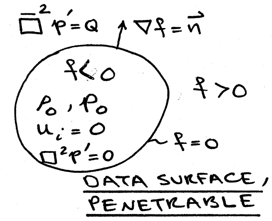
THE ACOUSTIC ANALOGY (CONT'D)

DERIVATION OF THE FFOWCS WILLIAMS-HAWKINGS (FW-H) EQ.

WE TAKE THE DATA SURFACE AS A SURFACE OF DISCONTINUITY BY EXTENDING $p' = c^2(\rho - \rho_0)$ TO INSIDE OF THE DATA SURFACE $f = 0$. IT IS CLEAR THAT $p' = 0$ INSIDE $f = 0$ AND, THEREFORE, $\Box^2 p' = 0$ THERE. WE ASSUME THAT INSIDE $f = 0$, FLUID AT CONDITIONS OF UNDISTURBED MEDIUM EXISTS WITH $\vec{u} = 0$, WHERE $\vec{u}$ IS FLUID VELOCITY. WE WANT TO DERIVE AN EQ. OF THE FORM $\bar{\Box}^2 p' = Q$, WHERE $\bar{\Box}^2$ IS THE WAVE EQ. WITH GENERALIZED DERIVATIVES. v_n : LOCAL NORMAL VELOCITY OF DATA SURFACE



MASS CONTINUITY EQ.

$$\frac{\bar{\partial}\rho}{\partial t} + \frac{\bar{\partial}}{\partial x_i}(\rho u_i) = \underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)}_{0} + \Big[(\rho - \rho_0)\underbrace{\frac{\partial f}{\partial t}}_{-v_n} + \underbrace{\rho u_i n_i}_{u_n}\Big]\delta(f)$$

$$= \rho_0 v_n \delta(f) + \rho(u_n - v_n)\delta(f) \equiv Q$$

MOMENTUM EQ. (INVISCID)

$$\frac{\bar{\partial}}{\partial t}(\rho u_i) + \frac{\bar{\partial}}{\partial x_j}(\rho u_i u_j) + \frac{\bar{\partial} P}{\partial x_i} = \underbrace{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial P}{\partial x_i}}_{0}$$

$$+ \Big[\rho u_i \underbrace{\frac{\partial f}{\partial t}}_{-v_n} + \rho u_i \underbrace{u_j n_j}_{u_n} + (P - P_0) n_i\Big]\delta(f) = \big[\rho u_i(u_n - v_n) + (P - P_0) n_i\big]\delta(f)$$

$$\equiv Q_i$$

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THE ACOUSTIC ANALOGY (CONT'D)

DERIVATION OF THE FW-H EQ. (CONT'D)

NOW FORM $\frac{\bar{\partial}}{\partial t}(\text{L.S. OF MASS CON.}) - \frac{\bar{\partial}}{\partial x_i}(\text{L.S. OF MOM. EQ.}) = \frac{\bar{\partial} Q}{\partial t} - \frac{\bar{\partial} Q_i}{\partial x_i}$

WE GET

$$\frac{\bar{\partial}^2 \rho}{\partial t^2} - \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\underbrace{(\rho u_i u_j + P\delta_{ij})} = \frac{\bar{\partial} Q}{\partial t} - \frac{\bar{\partial} Q_i}{\partial x_i}$$

NOW ADD AND SUBTRACT $\bar{\nabla}^2 \rho$ TO BOTH SIDES, REPLACE ρ BY $\rho' = \rho - \rho_0$, KEEP ONLY $\bar{\Box}^2(\rho' c^2)$ ON THE LEFT SIDE TO GET

$$\boxed{\begin{aligned} &\bar{\Box}^2(\rho' c^2) = \frac{\bar{\partial} Q}{\partial t} - \frac{\bar{\partial} Q_i}{\partial x_i} + \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\underbrace{\big[\rho u_i u_j + [(P - P_0) - c^2(\rho - \rho_0)]\delta_{ij}\big]}_{T_{ij} \cdot H(f)} \\ &Q = \rho_0 v_n \delta(f) + \rho(u_n - v_n)\delta(f) \\ &Q_i = \big[\rho u_i(u_n - v_n) + (P - P_0) n_i\big]\delta(f) \end{aligned}}$$

THIS IS THE FW-H EQ. FOR PENETRABLE DATA SURFACE

$H(f)$: HEAVISIDE FN, $H(f) = \begin{cases} 0 & f < 0 \quad \text{INSIDE DATA SURFACE} \\ 1 & f > 0 \quad \text{OUTSIDE " "} \end{cases}$

WE USE $p' \equiv \rho' c^2$ IN OUR WORK.

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THE ACOUSTIC ANALOGY (CONT'D)

COMMENTS OF FW-H EQ.

i) WE HAVE EMBEDDED AN EXTERNAL PROBLEM (FOR $f>0$) INTO AN UNBOUNDED SPACE BECAUSE OF SIMPLICITY OF THE GREEN'S FUNCTION IN UNBOUNDED SPACE. TO CONVINCE YOURSELF THAT THIS IS LEGITIMATE, SEE THE EXAMPLE FOR AN ODE IN MY NASA TR-R450, 1975 (APPENDIX)

ii) IF $f=0$ IS AN IMPENETRABLE SURFACE, *i.e.* $u_n = v_n$, THEN THE FW-H EQ. BECOMES

$$\boxed{\bar{\Box}^2 p' = \underbrace{\frac{\partial}{\partial t}\left[\rho_0 v_n \delta(f)\right]}_{\text{THICKNESS}} - \underbrace{\nabla\cdot\left[p\vec{n}\,\delta(f)\right]}_{\text{LOADING}} + \underbrace{\frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right]}_{\text{QUADRUPOLES}}}$$

iii) THICKNESS AND LOADING TERMS OF FW-H EQ. CAN BE USED TO SOLVE THE THICKNESS AND LOADING PROBLEMS OF AERODYNAMICS.

iv) THE NOISE GENERATED BY ALL DISCONTINUITIES IN THE FLOW, SHOCKS, WAKE AND VORTICES ARE INCLUDED IN QUADRUPOLE SOURCE TERM. THE EASIEST WAY TO SEE THIS AND FIND THE SOURCE STRENGTHS IS BY USING GENERALIZED FUNCTIONS!

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THE ACOUSTIC ANALOGY (CONT'D)

COMMENTS ON FW-H EQ.

V) TYPICAL SOURCE TERMS IN FW-H & K EQ.

SURFACE TERMS $\quad \frac{\partial}{\partial t}\left[Q\delta(f)\right], \quad \nabla\cdot\left[\vec{Q}\,\delta(f)\right], \quad Q\delta(f)$

VOLUME TERM (FW-H) $\quad \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right]$

THE SUBSONIC SOLUTIONS ARE EASY TO GIVE!

V*i*) WE ALWAYS TRY TO GET THE EXACT SOLUTION FOR EXACT GEOMETRY OF DATA SURFACE $f=0$ AND EXACT KINEMATICS

V*ii*) THERE ARE MANY EQUIVALENT SOLUTIONS OF FW-H EQ. WITH DIFFERENT DEGREES OF COMPLEXITY AND USEFULNESS FOR CODE DEVELOPMENT. IN GENERAL, ONE SHOULD SPEND A LOT OF TIME THINKING ABOUT ALGORITHMS FOR CODE DEVELOPMENT.

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How Does $\delta(f)$ Appear in Applications? (Cont'd)

- In our work the discontinuities in functions are either real (e.g., shock waves) or artificial (e.g., across blade surface in derivation of FW-H eq.).
- **Example**: *Shock surface sources* in Lighthill jet noise theory. Let the shock surfaces be defined by $f(\vec{x}, t) = 0$. We can show that Lighthill's equation is valid in presence of shocks if we interpret the derivatives of the source term as generalized derivatives:

$$\Box^2 p' = \frac{\bar{\partial}^2 T_{ij}}{\partial x_i \partial x_j}$$

$$= \frac{\bar{\partial}}{\partial x_i}\left[\frac{\partial T_{ij}}{\partial x_j} + \Delta T_{ij}\frac{\partial f}{\partial x_j}\delta(f)\right]$$

$$= \underbrace{\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}}_{\text{Turbulence Source}} + \underbrace{\Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right)\frac{\partial f}{\partial x_i}\delta(f) + \frac{\bar{\partial}}{\partial x_i}\left[\Delta T_{ij}\frac{\partial f}{\partial x_j}\delta(f)\right]}_{\text{Shock Surface Sources}}$$

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SOLUTION OF FW-H EQUATION

THICKNESS AND LOADING TERMS

WE HAVE TERMS OF THE FOLLOWING TYPES

$\frac{\partial}{\partial t}[Q(\vec{x},t)\delta(f)]$ THICKNESS , $\nabla \cdot [\vec{Q}\,\delta(f)]$ LOADING

IF $f=0$ IS THE DATA SURFACE THAT IS IMPENETRABLE, THEN $Q = \rho_0 v_n$ AND $\vec{Q} = p\vec{n}$

1. SOLUTION OF $\Box^2 p' = \frac{\partial}{\partial t}[Q\,\delta(f)]$

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int \frac{Q(\vec{y},\tau)}{r}\,\delta(g)\,\delta(f)\,d\vec{y}\,d\tau$$

ASSUMING $f=0$ IS RIGID (VERY COMMON), FIX AN $\vec{\eta}$-FRAME TO THIS SURFACE SUCH THAT $f(\vec{y}(\vec{\eta},\tau),\tau) \equiv \bar{f}(\vec{\eta})$, i.e. TIME INDEPENDENT. WE HAVE $d\vec{y} = d\vec{\eta}$ BUT, IN GENERAL, $\vec{y} = \vec{y}(\vec{\eta},\tau)$, i.e., WE HAVE $\vec{y}$ NOW A FUNCTION OF SOURCE TIME. WE HAVE

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int \frac{Q[\vec{y}(\vec{\eta},\tau),\tau)]}{r(\vec{\eta},\tau;\vec{x})}\,\delta(g)\,\delta(\bar{f})\,d\vec{\eta}\,d\tau$$

NOTE: $\int \equiv \int_{-\infty}^{t}\iiint_{-\infty}^{\infty} \cdots\cdots d\vec{y}\,d\tau$

SOLUTION OF FW-H EQ. (CONT'D)

NEXT LET $\tau \to g$, WE HAVE $\quad d\tau = \dfrac{dg}{|\partial g/\partial \tau|_{\vec{\eta}}}$

$$g = \tau - t + |\vec{x} - \vec{y}(\vec{\eta},\tau)|/c \leftarrow \underline{\text{NOTE}}\ r = |\vec{x} - \vec{y}(\vec{\eta},\tau)| = r(\vec{\eta},\tau;\vec{x})$$

$$\left.\frac{\partial g}{\partial \tau}\right|_{\vec{\eta}} = 1 - \frac{1}{c}\underbrace{\frac{\partial \vec{y}}{\partial \tau}}_{\vec{v}\ \text{LOCAL SURFACE VELOCITY}} \cdot \hat{r} = 1 - M_r \ , \quad M_r = \frac{\vec{v}}{c}\cdot\hat{r}$$

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int \left[\frac{Q[\vec{y}(\vec{\eta},\tau),\tau]}{r|1-M_r|}\right]_{g=0} \delta(\bar{f})\, d\vec{\eta}$$

WHAT IS THE MEANING OF $[\cdots\cdot]_{g=0}$? IT MEANS FIND THE EMISSION TIME τ^* BY SOLVING

$$g = \tau^* - t + |\vec{x} - \vec{y}(\vec{\eta},\tau^*)|/c = 0$$

$$i.e. \quad \tau^* = \tau^*(\vec{\eta};\vec{x},t) \qquad \text{[CALLED ALSO RETARDED TIME]}$$

NOW INTEGRATE THE DELTA FUNCTION $\delta(\bar{f})$

$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int_{\bar{f}=0}\left[\frac{Q[\vec{y}(\vec{\eta},\tau),\tau]}{r|1-M_r|}\right]_{\tau^*} dS}$$

THIS IS THE THICKNESS NOISE PART OF FORMULATION 1.

SOLUTION OF FW-H EQ.

WE WILL WORK MORE ON THIS EQUATION LATER. WE NOW MENTION A FEW FACTS.

i) IN GENERAL, $\partial/\partial t$ IS TAKEN NUMERICALLY

ii) IF ALL PARTS OF THE SURFACE MOVE SUBSONICALLY, EACH POINT ON THE SURFACE HAS ONLY ONE EMISSION TIME τ^*. FOR SUPERSONIC PARTS OF THE SURFACE, EACH POINT ON THE SURFACE AN ODD NUMBER OF EMISSION TIMES, MULTIPLICITY OF THE ROOTS OF $g=0$ INCLUDED.

iii) THE ABOVE EQUATION IS A POOR CANDIDATE FOR PREDICTING THE THICKNESS NOISE OF SUPERSONIC SURFACES.

iv) IT IS COMMON TO WRITE $f=0$ INSTEAD OF $\bar{f}=0$ IN THE ABOVE RESULT, AS WE HAVE DONE. THIS IS DONE TO REDUCE THE NUMBER OF SYMBOLS USED. REMEMBER THE DISTINCTION BETWEEN $f(\vec{y},\tau)$ AND $\bar{f}(\vec{\eta}) = f[\vec{y}(\vec{\eta},\tau),\tau]$.

SOLUTION OF FW-H EQ. (CONT'D

BRINGING $\partial/\partial t$ INTO THE INTEGRAL

REMEMBERING THAT f (ACTUALLY $\bar{f}$!) IS INDEPENDENT OF TIME t, WE HAVE

$$\frac{\partial}{\partial t}\int_{f=0}\frac{Q[\vec{y}(\vec{\eta},\tau^*),\tau^*]}{r(\vec{\eta},\tau;\vec{x})|1-M_r(\vec{\eta},\tau;\vec{x})|}\,dS = \int_{f=0}\frac{\partial}{\partial\tau^*}\left[\frac{Q}{r|1-M_r|}\right]\frac{\partial\tau^*}{\partial t}\,dS$$

FROM $\tau^* - t + |\vec{x}-\vec{y}(\vec{\eta},\tau^*)|/c = 0$, WE GET $\frac{\partial\tau^*}{\partial t} = \left[\frac{1}{1-M_r}\right]_{\tau^*}$

$$\boxed{4\pi\, p'(\vec{x},t) = \int_{f=0}\left\{\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left[\frac{Q[\vec{y}(\vec{\eta},\tau),\tau]}{r|1-M_r|}\right]\right\}_{\tau^*} dS}$$

$$\left.\frac{\partial}{\partial\tau}\right|_{\vec{\eta}} \underbrace{Q[\vec{y}(\vec{\eta},\tau),\tau]}_{\bar{Q}(\vec{\eta},\tau)} = \frac{\partial Q}{\partial y_i}\frac{\partial y_i}{\partial \tau} + \frac{\partial Q}{\partial \tau} \equiv \dot{\bar{Q}}$$ RATE OF CHANGE OF Q AS MEASURED BY AN OBSERVER ON $f=0$

$$\frac{\partial}{\partial\tau}r = \frac{\partial}{\partial\tau}|\vec{x}-\vec{y}(\vec{\eta},\tau)| = -\hat{r}\cdot\vec{v} = -v_r$$ SURFACE VELOCITY IN RADIATION DIRECTION

WHEN THESE ARE SUBSTITUTED IN THE ABOVE INTEGRAL, WE GET THE THICKNESS NOISE PART OF FORMULATION 1A USED IN WOPWOP

SOLUTION OF FW-H EQ. (CONT'D

2. SOLUTION OF $\square^2 p' = -\nabla\cdot[\vec{Q}\,\delta(f)]$

FORMAL SOLUTION

$$4\pi\, p'(\vec{x},t) = -\nabla_x\cdot\int\frac{\vec{Q}(\vec{y},\tau)}{r}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

THE FIRST THOUGHT COMING TO OUR MIND IS TO USE NUMERICAL DIFFERENTIATION. BUT THIS REQUIRES EVALUATION OF A SURFACE INTEGRAL SIX TIMES! NOT VERY EFFICIENT METHOD. CAN WE CONVERT THE SPACE DERIVATIVES TO OBSERVER TIME DERIVATIVE EXACTLY? THE ANSWER IS YES!

$$\nabla_x\cdot\int\vec{Q}(\vec{y},\tau)\,\delta(f)\,\frac{\delta(g)}{r}\,d\vec{y}\,d\tau = \int\vec{Q}\,\delta(f)\cdot\nabla_x\frac{\delta(g)}{r}\,d\vec{y}\,d\tau$$

$$\boxed{\begin{aligned}\nabla_x\frac{\delta(g)}{r} &= \frac{\hat{r}}{c}\frac{\delta'(g)}{r} - \hat{r}\frac{\delta(g)}{r^2}\\ &= -\frac{\hat{r}}{cr}\frac{\partial}{\partial t}\delta(g) - \hat{r}\frac{\delta(g)}{r^2}\end{aligned}}$$

$$\boxed{Q_r = \vec{Q}\cdot\hat{r}}$$

$$= -\int\left[\frac{Q_r}{cr}\,\delta(f)\frac{\partial}{\partial t}\delta(g) + \frac{Q_r}{r^2}\,\delta(f)\,\delta(g)\right]d\vec{y}\,d\tau$$

$$= -\frac{\partial}{\partial t}\int\frac{Q_r}{cr}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau - \int\frac{Q_r}{r^2}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

SOLUTION OF FW-H EQ. (CONT'D)

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int \frac{Q_r}{cr}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau + \int \frac{Q_r}{r^2}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int_{f=0}\left[\frac{Q_r}{cr|1-M_r|}\right]_{\tau^*} dS + \int_{f=0}\left[\frac{Q_r}{r^2|1-M_r|}\right]_{\tau^*} dS}$$

THIS IS THE LOADING PART OF FORMULATION 1. AGAIN, WE CAN TAKE THE DERIVATIVE WRT t INSIDE THE FIRST INTEGRAL TO GET THE LOADING PART OF FORMULATION 1A

$$\boxed{4\pi p'(\vec{x},t) = \int_{f=0}\left\{\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\left[\frac{Q_r}{cr|1-M_r|}\right]\right\}_{\tau^*} dS + \int_{f=0}\left[\frac{Q_r}{r^2|1-M_r|}\right]_{\tau^*} dS}$$

REMEMBER $Q_r = \vec{Q}(\vec{\eta},\tau)\cdot\hat{r}(\vec{\eta},\tau;\vec{x})$ SO THAT

$$\frac{\partial}{\partial \tau}Q_r = \dot{\vec{Q}}\cdot\hat{r} + \vec{Q}\cdot\frac{\partial \hat{r}}{\partial \tau}$$

THIS EXPLAINS WHY FORMULATION 1A HAS SO MANY TERMS!

SOLUTION OF FW-H EQ. (CONT'D)

SUMMARY

THE SOLUTION OF $\square^2 p' = \frac{\partial}{\partial t}[Q\delta(f)] - \nabla\cdot[\vec{Q}\,\delta(f)]$

WHERE $Q = \rho_0 v_n + \rho(u_n - v_n)$, $\vec{Q} = \rho(u_n - v_n)\vec{u} + (p - p_0)\vec{n}$

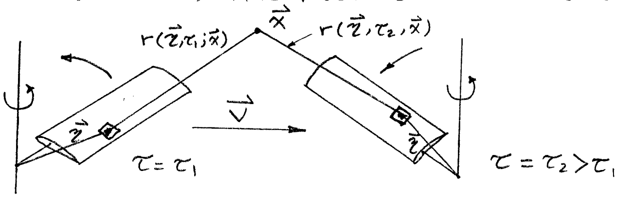
IS:

$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{c\,\partial t}\int_{f=0}\left[\frac{cQ + Q_r}{r(1-M_r)}\right]_{ret} dS + \int_{f=0}\left[\frac{Q_r}{r^2(1-M_r)}\right]_{ret} dS}$$

$Q_r = \vec{Q}\cdot\hat{r}$, $\hat{r} = (\vec{x}-\vec{y})/|\vec{x}-\vec{y}|$, $M_r = \vec{M}\cdot\hat{r}$, $\frac{\partial}{\partial t}$ TAKEN NUMERICALLY

THIS IS FORMULATION 1.

NOTE: $\partial/\partial t = \partial/\partial t|_{\vec{x}}$, THE $\vec{x}$-FRAME IS FIXED TO THE UNDISTURBED MEDIUM. $r = |\vec{x} - \vec{y}(\vec{\eta},\tau)|$, $\vec{\eta}$ IS THE LAGRANGIAN VARIABLE IN A FRAME FIXED TO $S: f=0$. THEREFORE $r = r(\vec{\eta},\tau;\vec{x})$ AND MUST BE INSIDE THE SQ. BRACKETS.



SOLUTION OF FW-H EQ. (CONT'D)

FORMULATION 1A

SOLUTION OF $\Box^2 \rho'_T = \frac{\partial}{\partial t}[Q\,\delta(f)]$ (THICKNESS SOURCE)

$$\boxed{4\pi\rho'_T(\vec{x},t) = \int_{f=0}\left[\frac{\dot{Q}}{r(1-M_r)^2}\right]_{ret} dS + \int_{f=0}\left[\frac{Q(r\dot{M}_r + cM_r - cM^2)}{r^2(1-M_r)^3}\right]_{ret} dS}$$

SOLUTION OF $\Box^2 \rho'_L = -\nabla\cdot[\vec{Q}\,\delta(f)]$ (LOADING SOURCE)

$$\boxed{\begin{aligned} 4\pi\rho'_L(\vec{x},t) = {} & \frac{1}{c}\int_{f=0}\left[\frac{\dot{Q}_r}{r(1-M_r)^2}\right]_{ret} dS + \int_{f=0}\left[\frac{Q_r - \vec{M}\cdot\vec{Q}}{r^2(1-M_r)^2}\right]_{ret} dS \\ & + \frac{1}{c}\int_{f=0}\left[\frac{Q_r(r\dot{M}_r + cM_r - cM^2)}{r^2(1-M_r)^3}\right]_{ret} dS \end{aligned}}$$

DOT $(\dot{\ }) \equiv \left.\frac{\partial}{\partial\tau}\right|_{\vec{\eta}}$, $\dot{M}_r = \dot{M}_i \cdot \hat{r}_i$, $\dot{Q}_r = \dot{Q}_i \hat{r}_i$, $Q_r = \vec{Q}\cdot\hat{r}$

$\left.\frac{\partial}{\partial\tau}\right|_{\vec{\eta}}$ MEANS RATE OF CHANGE OF A QUANTITY AS MEASURED BY AN INSTRUMENT FIXED TO THE SURFACE $S: f=0$. WE ARE USING THE SUMMATION CONVENTION : $a_i b_i = \sum_{i=1}^{3} a_i b_i$.

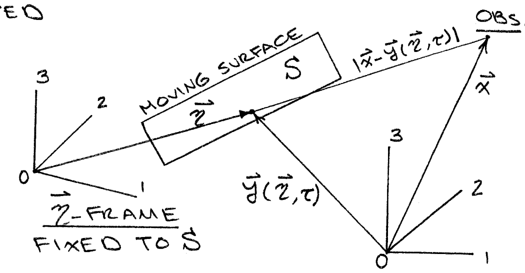
SOLUTION OF FW-H EQ. (CONT'D)

TRICKS OF THE TRADE

i) RETARDED TIME IS CALCULATED FROM

$$\tau - t + |\vec{x} - \vec{y}(\vec{\eta},\tau)|/c = 0$$

IN GENERAL THIS IS A TRANSCENTAL (NOT A POLYNOMIAL) EQUATION. USE NUMERICAL METHOD TO FIND τ. NOTE THAT t, $\vec{x}$ AND $\vec{\eta}$ ALWAYS HAVE A NUMERICAL VALUE.



$\vec{x}$ - FRAME
FIXED TO UN-DISTURBED MEDIUM

THE SHOOTING TECHNIQUE IS A GOOD METHOD OF SOLVING THIS EQUATION FOR SUBSONIC SURFACES. FOR SUPERSONIC SURFACES, LOTS OF COMPLICATIONS APPEAR. MOST PROBLEMS ARE ASSOCIATED WITH MULTIPLE ROOTS OR NEARLY EQUAL ROOTS. ONE DISCOVERS QUICKLY THAT SMALL ERRORS IN RETARDED TIME CAUSES LARGE ERRORS IN $\rho'(\vec{x},t)$.

– FOR UNIFORM RECTILINEAR MOTION OF S, WE HAVE ANALYTIC SOLUTION FOR EMISSION TIME – ONE FOR SUBSONIC, TWO OR NONE FOR SUPERSONIC. USE THE GARRICK TRIANGLE TO FIND THE QUADRATIC EQUATION YOU NEED TO SOLVE.

SOLUTION OF FW-H EQ. (CONT'D)
TRICKS OF THE TRADE (CONT'D)

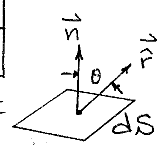
REMEMBER THAT A VECTOR IS A MATHEMATICAL OBJECT THAT HAS A MEANING EVEN WHEN NO FRAME OF REFERENCE IS SPECIFIED. THEREFORE, $\tau - t + |\vec{x} - \vec{y}(\vec{\eta},\tau)|/c = 0$ CAN BE WRITTEN IN EITHER THE FIXED OR MOVING FRAMES. THIS COMMENT ALSO HOLDS FOR ALL DOT PRODUCTS SUCH AS $Q_r = \vec{Q}\cdot\hat{r}$, $M_r = \vec{M}\cdot\hat{r}$, ETC.

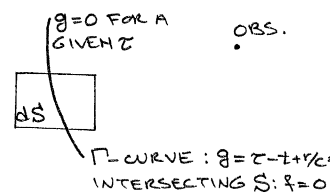
ii) INTEGRATION : USE SMART INTEGRATION METHODS SUCH AS GAUSS-LEGENDRE TECHNIQUE. AVOID RECTANGULAR OR SIMPSON RULES. SMART INTEGRATION METHODS CAN LEAD TO SUBSTANTIAL SAVING IN EXECUTION TIME AND IN INCREASED ACCURACY.

iii) VERY IMPORTANT RELATIONS

$$\boxed{\frac{dS}{|1-M_r|} = \frac{c\, d\Gamma\, d\tau}{\sin\theta} = \frac{d\Sigma}{\Lambda}}$$

$$\boxed{\Lambda^2 = 1 + M_n^2 - 2M_n\cos\theta}$$





$d\Sigma$ IS THE ELEMENT OF THE SURFACE AREA OF $F(\vec{y};\vec{x},t) = f(\vec{y}, t - r/c) = 0$. THIS IS THE LOCUS OF Γ-CURVES FOR $(\vec{x},t)$ FIXED.

SOLUTION OF FW-H EQ. (CONT'D)
TRICKS OF THE TRADE (CONT'D)

TWO OTHER WAYS OF WRITING FORMULATION 1

$$\boxed{\begin{aligned} 4\pi p'(\vec{x},t) &= \frac{\partial}{c\,\partial t}\int_{-\infty}^{t}\frac{d\tau}{t-\tau}\int_{r=c(t-\tau)}\left[cQ(\vec{y},\tau)+Q_r(\vec{y},\tau)\right]\frac{d\Gamma}{\sin\theta} \\ &+ \frac{1}{c}\int_{-\infty}^{t}\frac{d\tau}{(t-\tau)^2}\int_{r=c(t-\tau)} Q_r(\vec{y},\tau)\frac{d\Gamma}{\sin\theta}\end{aligned}}$$

THE COLLAPSING SPHERE METHOD USED HERE.

– NOTE THAT WE ARE NOT USING LAGRANGIAN VARIABLE HERE. THE VARIABLE $\vec{y}$ IS THE SOURCE POSITION IN A FRAME FIXED TO UNDISTURBED MEDIUM.

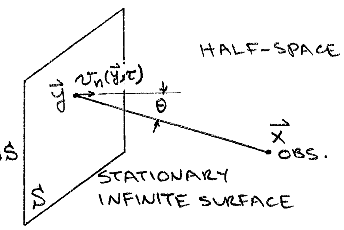
$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{c\,\partial t}\int_{F=0}\frac{1}{r}\left[\frac{cQ+Q_r}{\Lambda}\right]_{ret} d\Sigma + \int_{F=0}\frac{1}{r^2}\left[\frac{Q_r}{\Lambda}\right]_{ret} d\Sigma}$$

– NOTE THAT $r = |\vec{x}-\vec{y}|$ AND SHOULD NOT BE INSIDE THE SQUARE BRACKETS BECAUSE r IS NOT A FUNCTION OF SOURCE TIME. BOTH OF THE ABOVE EQUATIONS ARE VALID FOR SUBSONIC AND SUPERSONIC MOTION OF $S: f=0$. NUMERICAL TIME DIFFERENTIATION CAN CAUSE ERRORS IN $p'(\vec{x},t)$.

SOLUTION OF FW-H EQ. (CONT'D)

RAYLEIGH'S PISTON IN THE WALL REVISITED

WE ARE INTERESTED IN RADIATION INTO THE HALF-SPACE BY VOLUME SOURCES ON THE INFINITE PLANE. WE HAVE SHOWN THAT (RAYLEIGH)



$$2\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int_S \frac{[\rho_0 v_n]_{ret}}{r}\, dS = \int_S \frac{[\rho_0 \dot{v}_n]_{ret}}{r}\, dS$$

USING FW-H EQ., WE HAVE

$$4\pi p'(\vec{x},t) = \underbrace{\frac{\partial}{\partial t}\int_S \frac{[\rho_0 v_n]_{ret}}{r}\, dS}_{\text{RAYLEIGH}:\ 2\pi p'(\vec{x},t)} - \nabla_x \cdot \int_S \frac{[p\vec{n}]_{ret}}{r}\, dS$$

$\Longrightarrow$

$$2\pi p'(\vec{x},t) = -\nabla_x \cdot \int_S \frac{[p\vec{n}]_{ret}}{r}\, dS \qquad \text{TAKE DIVERGENCE INSIDE}$$

$$\boxed{2\pi p'(\vec{x},t) = \int_S \left\{ \frac{[\dot{p}]_{ret}\cos\theta}{cr} + \frac{[p]_{ret}\cos\theta}{r^2} \right\} dS} \qquad \text{RAYLEIGH'S FIRST INTEGRAL}$$

$\cos\theta = \vec{n}\cdot\hat{r}$

SOLUTION OF FW-H EQ. (CONT'D)

THE CURLE FORMULA : TURBULENT FLOW OVER A STATIONARY SURFACE $S : f(\vec{x},t) = 0$

ESSENTIALLY, CURLE GAVE THE SOLUTION OF FW-H EQ. FOR A FLAT SURFACE WITH QUADRUPOLES IN THE VICINITY OF THE SURFACE. THE SOLUTION OF FW-H EQ. (NOT RESTRICTED TO FLAT SURFACES) IN THIS CASE IS :

$$\boxed{4\pi p'(\vec{x},t) = -\nabla_x \cdot \int_S \frac{[p\vec{n}]_{ret}}{r}\, dS + \frac{\partial^2}{\partial x_i \partial x_j}\int_{f>0} \frac{[T_{ij}]_{ret}}{r}\, d\vec{y}}$$

IN THE FAR FIELD, WE HAVE THE FOLLOWING SIMPLE RESULT

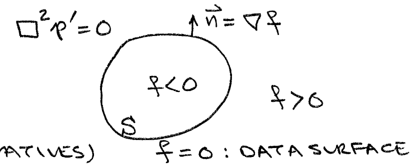
$$\boxed{4\pi p'(\vec{x},t) = \int_S \frac{[\dot{p}]_{ret}\cos\theta}{cr}\, dS + \int_{f>0} \frac{[\ddot{T}_{rr}]}{c^2 r}\, d\vec{y}}$$

$T_{rr} = T_{ij}\,\hat{r}_i \hat{r}_j$

SOLUTION OF THE FW-H EQ. (CONT'D)

THE KIRCHHOFF FORMULA FOR MOVING SURFACES

THIS RESULT IS MOST SIMPLY OBTAINED BY FIRST EXTENDING p' TO INSIDE THE SURFACE AS FOLLOWS: $\tilde{p}' = \begin{cases} p' & f > 0 \\ 0 & f < 0 \end{cases}$



$\Rightarrow \Box^2 \tilde{p}' = 0$ (ORDINARY DERIVATIVES)

WE NEXT FIND $\bar{\Box}^2 p'$ (GENERALIZED DERIVATIVES)

SEE NASA TM-110285 (1996) :

$$\boxed{\bar{\Box}^2 \tilde{p}' = -\left(\frac{\partial p'}{\partial n} + \frac{1}{c} M_n \frac{\partial p'}{\partial t}\right)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}\left[M_n p' \delta(f)\right] - \nabla \cdot \left[p' \vec{n}\, \delta(f)\right]}$$

NOTE THAT p', $\partial p'/\partial t$ AND $\partial p'/\partial n$ ON THE RIGHT SIDE ARE EVALUATED ON $f = 0+$, i.e. ON THE EXTERIOR SIDE OF $S: f=0$.

- WE HAVE GIVEN THE SOLUTION OF THE WAVE EQUATION WITH SOURCES OF THE TYPES ABOVE. THE FORMAL SOLUTION OF THIS EQUATION IS THE KIRCHHOFF FORMULA FOR MOVING SURFACES.

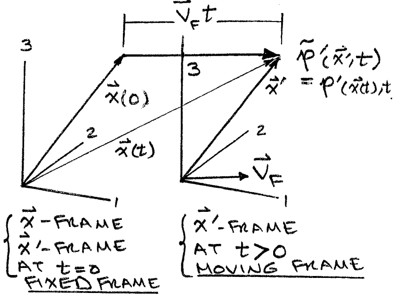
SOLUTION OF THE FW-H EQ. (CONT'D)

THE KIRCHHOFF FORMULA FOR MOVING SURFACES (CONT'D)

- USE THE KIRCHHOFF FORMULA IF YOU ARE SURE $\Box^2 p' = 0$ IN THE EXTERIOR REGION OF THE DATA SURFACE $f=0$. OTHERWISE USE THE FW-H EQ. WITH PENETRABLE DATA SURFACE.
- NOTE THAT $p'=0$ INSIDE THE SURFACE $f=0$ FOR BOTH THE KIRCHHOFF FORMULA AND THE SOLUTION OF THE FW-H EQUATION. THIS GIVES A FOOL-PROOF METHOD OF TESTING YOUR COMPUTER CODE AND THE ACCURACY OF THE INPUT DATA. IF YOU HAVE DONE EVERYTHING CORRECTLY, THEN YOU SHOULD GET $p'=0$ TO MACHINE ACCURACY INSIDE THE DATA SURFACE $f=0$. TRY THIS FOR MANY POINTS ARBITRARILY CHOSEN
- FROM THE COMPUTATIONAL EXPERIMENTS OF BRENTNER AND FARASSAT (AIAA J., VOL. 36(8), 1998, 1379-1386), IT WAS FOUND THAT IN THE NEAR FIELD OF MOVING SURFACES, THE FW-H EQ. CHANGES AT A SMALLER RATE THAN THE KIRCHHOFF FORMULA AS THE DATA SURFACE IS ENLARGED AND MOVED FARTHER. ALSO THE FW-H EQ. GIVES MUCH CLOSER RESULT THAN THE KIRCHHOFF FORMULA TO MEASURED DATA IN THE NEAR FIELD. MORE WORK ON THIS IS NEEDED.

SOLUTION OF THE FW-H EQ.

MOVING OBSERVER CALCULATIONS



FOR AN OBSERVER MOVING WITH THE AIRCRAFT, THE ACOUSTIC PRESSURE $\tilde{p}'(\vec{x}',t)$ IS CALCULATED AS FOLLOWS. HERE $\vec{x}'$ IS A FRAME FIXED TO THE AIRCRAFT SUCH THAT THE $\vec{x}$ AND $\vec{x}'$ FRAME COINCIDE AT $t=0$. LET THE FLIGHT VELOCITY BE $\vec{V}_F$. NOTE THAT $\vec{x}'$ IS A FIXED OBSERVER POSITION IN THE MOVING FRAME. NOW $\vec{x}$ IS A FUNCTION OF OBSERVER TIME:

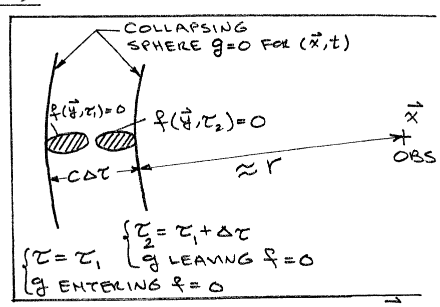
$$\vec{x}(t) = \vec{x}(0) + \vec{V}_F t = \vec{x}' + \vec{V}_F(t) \quad , \quad \vec{x}(0) = \vec{x}'$$

$$\therefore \quad \boxed{\tilde{p}'(\vec{x}',t) = p'(\vec{x}' + \vec{V}_F(t), t)}$$

WHERE $p'(\vec{x},t)$ IS THE ACOUSTIC PRESSURE IN THE FRAME FIXED TO THE UNDISTURBED MEDIUM. IN PRACTICE, WE CALCULATE p' AT DISCRETE TIME POINTS WITH TIME GRID Δt. WE USE $\boxed{\tilde{p}'(\vec{x}',t) = p'(\vec{x}' + (n\Delta t)\vec{V}_F, n\Delta t)}$.

SOLUTION OF THE FW-H EQ. (CONT'D)

LOWSON'S FORMULA REVISITED



WE HAVE SEEN THAT LOWSON PROPOSED, BASED ON RESULTS OF LAMB AND LIGHTHILL, THAT THE NOISE FROM A MOVING UNSTEADY FORCE $\vec{F}(\quad)$ IS THE SOLUTION OF THE WAVE EQUATION

$$\Box^2 p' = -\nabla \cdot \{\vec{F}\,\delta[\vec{x} - \vec{x}_s(t)]\}$$

WHERE $\vec{x}_s(t)$ IS THE POSITION VECTOR OF THE POINT FORCE $\vec{F}(t)$. THIS REQUIRES A PROOF BECAUSE IT IS NOT OBVIOUS. WE START BY ASSUMING THAT THE FORCE IS GENERATED BY A FINITE BODY $f(\vec{y},\tau)=0$. FROM FORMULATION 1

$$4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int_{f=0}\left[\frac{p\cos\theta}{r|1-M_r|}\right]_{ret} dS + \int_{f=0}\left[\frac{p\cos\theta}{r^2|1-M_r|^2}\right]_{ret} dS$$

NOW IF WE ASSUME $\boxed{c\Delta\tau \ll r \;,\; \Delta\tau \ll T}$ WHERE T IS THE TYPICAL PERIOD OF FLUCTUATION OF p ON THE SURFACE, THEN

$$4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{r}}{r|1-M_r|}\cdot\underbrace{\int_{f=0} p\vec{n}\,dS}_{\vec{F}(\tau)}\right]_{ret} + \left[\frac{\hat{r}}{r^2|1-M_r|^2}\cdot\underbrace{\int_{f=0} p\vec{n}\,dS}_{\vec{F}(\tau)}\right]_{ret} \quad \text{(CONT'D)}$$

SOLUTION OF THE FW-H EQ. (CONT'D)

LOWSON'S FORMULA REVISITED (CONT'D)

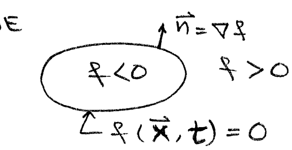
$$\Longrightarrow \quad 4\pi\, p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{F_r}{r|1-M_r|}\right]_{ret} + \left[\frac{F_r}{r^2|1-M_r|}\right]_{ret}$$

WHERE $F_r = \vec{F}\cdot\hat{r}$. BUT THIS IS THE SOLUTION OF

$$\Box^2 p' = -\nabla\cdot\{\vec{F}(t)\,\delta[\vec{x}-\vec{x}_s(t)]\}$$

ANOTHER VIEW OF THICKNESS NOISE TERM

LET $H(f) = \begin{cases} 1 & f>0 \\ 0 & f<0 \end{cases}$ BE THE HEAVISIDE FUNCTION. THEN $1-H(f) = \begin{cases} 1 & f<0 \\ 0 & f>0 \end{cases}$.



$$\frac{\partial}{\partial t}[1-H(f)] = -\frac{\partial f}{\partial t}\delta(f) = v_n\,\delta(f)$$

$$\boxed{Q_T = \frac{\partial^2}{\partial t^2}\{\rho_0[1-H(f)]\} = \frac{\partial}{\partial t}[\rho_0 v_n \delta(f)]}$$ THICKNESS SOURCE !

$\therefore$ THE SOLUTION OF $\Box^2 p'_T(\vec{x},t) = Q_T$ IS

$$\boxed{\begin{aligned} 4\pi\, p'_T(\vec{x},t) &= \frac{\partial^2}{\partial t^2}\int \rho_0[1-H(f)]\frac{\delta(g)}{r}\,d\vec{y}\,d\tau = \frac{\partial^2}{\partial t^2}\int_{f<0}\left[\frac{\rho_0}{r|1-M_r|}\right]_{ret} d\vec{y} \\ &= \frac{\partial^2}{\partial t^2}\int_{f<0}\frac{\rho_0}{r}\,d\vec{y} \end{aligned}}$$

SOLUTION OF THE FW-H EQ. (CONT'D

SUCCI'S THICKNESS NOISE FORMULA

SUCCI'S FORMULA GIVES THE THICKNESS NOISE FOR A COMPACT SOURCE. IT IS GENERALLY APPLIED TO A RIGID SURFACE. THE ASSUMPTION OF COMPACTNESS IS $\boxed{c\Delta\tau \ll r}$ HERE.

FROM PREVIOUS PAGE

$$\boxed{\begin{aligned} 4\pi\, p'_T(\vec{x},t) &= \frac{\partial^2}{\partial t^2}\left[\frac{\rho_0}{r|1-M_r|}\underbrace{\int_{f<0} d\vec{y}}_{\text{VOLUME } V \text{ INSIDE } f=0}\right]_{ret} \\ &= \frac{\partial^2}{\partial t^2}\left[\frac{\rho_0 V}{r|1-M_r|}\right]_{ret} \\ &= \underbrace{\left\{\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left[\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left(\frac{\rho_0 V}{r|1-M_r|}\right)\right]\right\}_{ret}}_{\text{TAKE THE TIME DERIVATIVES!}} \end{aligned}}$$

IN APPLICATION TO PROPELLERS, SUCCI DIVIDED THE BLADE INTO SMALL VOLUME ELEMENTS V_i AND SUMMED THE CONTRIBUTION OF ALL VOLUME ELEMENTS TO p'_T. IT WORKS! REMEMBER $r = |\vec{x}-\vec{y}(\vec{\eta},\tau)|$, $\vec{M}$ AND $\hat{r}$ IN $M_r = \vec{M}\cdot\hat{r}$ ARE ALL FUNCTIONS OF TIME. SO THE FINAL RESULT HAS MANY TERMS.

SOLUTION OF THE FW-H EQ. (CONT'D)

ISOM'S THICKNESS NOISE RESULT

WE KNOW THAT $1 - H(f) = \begin{cases} 0 & f>0 \quad \text{OUTSIDE THE SURFACE} \\ 1 & f<0 \quad \text{INSIDE THE SURFACE} \end{cases}$

LET US DEFINE $\phi(\vec{x},t) = \rho_0 c^2 [1 - H(f)] = \begin{cases} 0 & f>0 \\ \rho_0 c^2 & f<0 \end{cases}$

$$\bar{\Box}^2 \phi = \underbrace{-\frac{\bar{\partial}^2}{\partial t^2}[\rho_0 H(f)]}_{\text{THICKNESS NOISE TERM}} + \underbrace{\bar{\nabla}^2 [\rho_0 c^2 H(f)]}_{\nabla \cdot [\rho_0 c^2 \vec{n}\, \delta(f)]}$$

MINUS LOADING TERM WITH $p = \rho_0 c^2$

$$4\pi \phi(\vec{x},t) = \text{THICKNESS NOISE} + \nabla \cdot \int \frac{\rho_0 c^2 \vec{n}}{r} \delta(f)\delta(g)\, d\vec{y}\, d\tau$$

$$= 0 \quad \text{OUTSIDE THE BODY}$$

$$\Rightarrow \text{THICKNESS NOISE} = -\nabla \cdot \int \frac{\rho_0 c^2 \vec{n}}{r} \delta(f)\delta(g)\, d\vec{y}\, d\tau$$

i.e. THICKNESS NOISE IS EQUIVALENT TO THE LOADING NOISE FROM A UNIFORM PRESSURE LOADING OF MAGNITUDE $\rho_0 c^2 \approx 140{,}000$ Pa! THIS IS ISOM'S RESULT. THE ABOVE PROOF IS BY FFOWCS WILLIAMS BUT PUBLISHED BY FARASSAT. ISOM SHOWED THIS RESULT IN THE FAR FIELD NUMERICALLY. FARASSAT PROVED THE RESULT FOR THE FAR FIELD AND FW GAVE THE FULL RESULT.

SOLUTION OF THE FW-H EQ. (CONT'D)

WHAT IS HIDDEN IN THE VOLUME TERM OF FW-H EQUATION

REF.: F. FARASSAT & M.K. MYERS "AN ANALYSIS OF THE QUADRUPOLE NOISE SOURCE OF HIGH SPEED ROTATING BLADES", COMPUTATIONAL ACOUSTICS, VOL. 2, LEE, CAKMAK & VICHNEVETSKY (EDS.) ELSEVIER SCIENCE PUBL., 1990.

WE NOW LOOK AT THE EQUATION $\Box^2 p' = \frac{\bar{\partial}^2}{\partial x_i \partial x_j}[T_{ij} H(f)]$.

FORMALLY, THE SOLUTION IS

$$4\pi p'(\vec{x},t) = \frac{\bar{\partial}^2}{\partial x_i \partial x_j} \int_{F>0} \frac{[T_{ij}]_{ret}}{r} d\vec{y}$$

WHERE $F(\vec{y};\vec{x},t) = f(\vec{y}, t - r/c) = [f(\vec{y},\tau)]_{ret}$. IN THIS FORM, THIS SOLUTION IS NOT OF MUCH USE! THE QUADRUPOLE CAN HAVE MANY REAL OR IDEALIZED DISCONTINUITIES EACH OF WHICH IS A SOURCE OF SOUND! IT IS NOT A GOOD IDEA TO TAKE THE SPACE DERIVATIVES $\bar{\partial}^2/\partial x_i \partial x_j$ INSIDE THE INTEGRAL. THIS CAUSES UNNECESSARY COMPLICATIONS! IT IS BETTER TO WORK WITH THE SOURCE TERM AND USE GENERALIZED FUNCTION THEORY.

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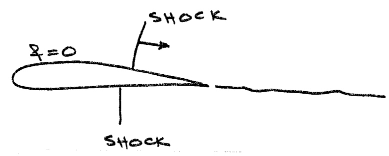
SOLUTION OF FW-H EQ. (CONT'D)

WHAT IS HIDDEN IN THE VOLUME TERM OF FW-H EQUATION



$$E = \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\, [T_{ij}H(f)] = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}\, H(f)$$ pure quadrupole term

$$+ [2\nabla_2 \cdot \vec{Q}_T + \frac{\partial Q_n}{\partial n} - 4H_f Q_n - Q_G]\,\delta(f)$$
$$+ \hat{Q}_n \delta'(f)$$ blade surface terms

$$+ \left[2\nabla_2 \cdot \vec{q}_T + \Delta\left(\frac{\partial Q'_{n'}}{\partial n}\right) - 4H_k q_{n'} - q_G\right]\delta(k)$$
$$+ \hat{q}_{n'}\delta'(k)$$ shock surface terms

$$+ [Q_\nu]\,\delta(\tilde{f})\delta(f).$$ trailing edge term

SMALL PERTURBATION EQUATIONS FOR ACOUSTIC WAVES IN FLUIDS WITH MEAN FLOW

NOTATION : MEAN (BACKGROUND) QUANTITIES DENOTED BY SUBSCRIPT O, PERTURBATION VALUES BY A PRIME. ALL QUANTITIES DIMENSIONAL

$$\rho = \rho_0 + \rho', \quad p = p_0 + p', \quad \vec{V} = \vec{V}_0 + \vec{V}'$$

NOTE : $\rho_0 = \rho_0(\vec{x},t)$, $p_0 = p_0(\vec{x},t)$, $\vec{V}_0 = \vec{V}_0(\vec{x},t)$

MASS CONTINUITY EQUATIONS

$$\frac{\partial \rho_0}{\partial t} + \nabla\cdot(\rho_0 \vec{V}_0) = 0 \qquad \text{MEAN} \qquad (1)$$

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot(\rho'\vec{V}_0 + \rho_0 \vec{V}') = 0 \qquad \text{ACOUSTIC} \qquad (2)$$

MOMENTUM EQUATIONS

$$\frac{\partial \vec{V}_0}{\partial t} + \vec{V}_0\cdot\nabla\vec{V}_0 = -\frac{\nabla p_0}{\rho_0} \qquad \text{MEAN} \qquad (3)$$

$$\frac{\partial \vec{V}'}{\partial t} + \vec{V}_0\cdot\nabla\vec{V}' + \vec{V}'\cdot\nabla\vec{V}_0 = -\frac{\nabla p'}{\rho_0} + \frac{\rho'\nabla p_0}{\rho_0^2} \qquad \text{ACOUSTIC} \qquad (4)$$

EQUATION OF STATE $\quad p = A\rho^{\gamma}\ , \quad A = p_r\,\rho_r^{-\gamma}$

WHERE p_r AND ρ_r ARE REFERENCE CONDITIONS

$$\begin{cases} p_0 = A\,\rho_0^{\gamma} & \text{MEAN} \quad (5) \\ p' = c_0^2\,\rho' & \text{ACOUSTIC} \quad \text{WHERE } c_0^2 = \gamma R T_0 = \dfrac{\gamma p_0}{\rho_0} \end{cases} \quad (6)$$

ENERGY EQUATIONS (ISENTROPIC) $\quad dp - c^2 d\rho = 0$

$$\frac{Dp}{Dt} - \frac{\gamma p}{\rho}\frac{D\rho}{Dt} = 0 \implies \frac{Dp}{Dt} - \gamma p\,\nabla\cdot\vec{V} = 0 \quad (7)$$

$$\begin{cases} \dfrac{\partial p_0}{\partial t} + \vec{V}_0\cdot\nabla p_0 - \gamma p_0\,\nabla\cdot\vec{V}_0 = 0 \quad \text{MEAN} & (8) \\ \dfrac{\partial p'}{\partial t} + \vec{V}_0\cdot\nabla p' + \vec{V}'\cdot\nabla p_0 - \gamma p_0\,\nabla\cdot\vec{V}' - \gamma p'\,\nabla\cdot\vec{V}_0 = 0 & (9) \\ \qquad\qquad \text{ACOUSTIC} \end{cases}$$

THESE EQUATIONS SIMPLIFY IF WE HAVE IRROTATIONAL FLOW. THERE ARE SITUATIONS OF INTEREST IN DUCTED FANS WHERE WE DO NOT HAVE IRROTIONAL FLOW.

IRROTATIONAL FLOW $\quad \nabla\times\vec{V} = 0$

$$\begin{cases} \nabla\times\vec{V}_0 = 0 & \text{MEAN} \quad (10) \\ \nabla\times\vec{V}' = 0 & \text{ACOUSTIC} \quad (11) \end{cases}$$

$\phi = \phi_0 + \phi' \quad$ VELOCITY POTENTIAL

IN TERMS OF ϕ, THE MOMENTUM EQUATION BECOMES

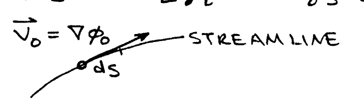
$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{\gamma-1}c^2 = \frac{1}{2}V_\infty^2 + \frac{1}{\gamma-1}c_\infty^2 \quad (12)$$

ASSUMING THAT V_∞ AND c_∞ ARE TIME INDEPENDENT

$$\begin{cases} \dfrac{\partial\phi_0}{\partial t} + \dfrac{1}{2}|\nabla\phi_0|^2 + \dfrac{1}{\gamma-1}c_0^2 = \dfrac{1}{2}V_\infty^2 + \dfrac{1}{\gamma-1}c_\infty^2 & \text{MEAN} \quad (13) \\ \dfrac{\partial\phi'}{\partial t} + \nabla\phi_0\cdot\nabla\phi' + c_0^2\dfrac{\rho'}{\rho_0} = 0 & \text{ACOUSTIC} \quad (14) \end{cases}$$

— FROM $p' = c_0^2\rho'$ AND THE ABOVE EQUATION

$$p' = -\rho_0\left[\frac{\partial\phi'}{\partial t} + \nabla\phi_0\cdot\nabla\phi'\right] = -\rho_0\left[\frac{\partial\phi'}{\partial t} + V_0\frac{\partial\phi'}{\partial s}\right] \quad (15)$$



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IF THE MEAN FLOW IS TIME INDEPENDENT, $\frac{\partial \phi_0}{\partial t} = 0$

$$c_0^2 = \frac{\gamma - 1}{2}\left[V_\infty^2 - |\nabla \phi_0|^2\right] + c_\infty^2 \qquad (16)$$

NOTE THAT THE MOMENTUM EQUATION IS WRITTEN IN AN INERTIAL FRAME. A ROTATING FRAME IS NOT AN INERTIAL FRAME.

THE GOVERNING EQUATION FOR ϕ'

FROM THE ACOUSTIC MOMENTUM EQUATION, WE GET

$$\rho' = -\frac{\rho_0}{c_0^2}\left[\frac{\partial \phi'}{\partial t} + \nabla\phi_0 \cdot \nabla\phi'\right] \equiv -\frac{\rho_0}{c_0^2} L\phi' \qquad (17)$$

NOW SUBSTITUTE FOR ρ' IN THE ACOUSTIC MASS CONTINUITY EQUATION (2), TO GET THE DESIRED EQUATION FOR ϕ':

$$-\frac{\partial}{\partial t}\left[\frac{\rho_0}{c_0^2} L\phi'\right] - \nabla \cdot \left[\left(\frac{\rho_0}{c_0^2} L\phi'\right)\nabla\phi_0\right] + \rho_0 \nabla^2 \phi' + \nabla\rho_0 \cdot \nabla\phi' = 0 \qquad (18)$$

A SECOND ORDER LINEAR EQUATION, BUT A COMPLICATED EQUATION!

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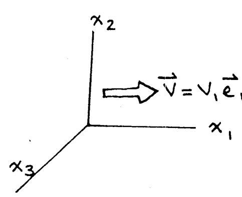
<u>TIME INDEPENDENT UNIFORM MEAN FLOW</u> : $\nabla \phi_0 = V_1 \vec{e}_1$



$$\rho' = -\frac{\rho_0}{c_0^2}\left[\frac{\partial \phi'}{\partial t} + V_1 \frac{\partial \phi'}{\partial x_1}\right] \qquad (19)$$

ρ_0 AND c_0 CONSTANT

THE EQUATION FOR ϕ' IS

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x_1}\right)^2 \phi' - \nabla^2 \phi' = 0 \qquad (20)$$

NOTE THAT THE LINEAR OPERATOR L IN EQ.(17) IS

$$L = \frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x_1} \qquad (21)$$

THE ACOUSTIC PRESSURE p' ALSO SATISFIES AN EQUATION SIMILAR TO EQ. (20) :

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x_1}\right)^2 p' - \nabla^2 p' = 0 \qquad (22)$$

OR

$$\left(\frac{1}{c_0}\frac{\partial}{\partial t} + M \frac{\partial}{\partial x_1}\right)^2 p' - \nabla^2 p' = 0 \qquad (23)$$

WHERE $M = V_1/c_0$ IS THE MACH NUMBER OF THE MEAN FLOW.

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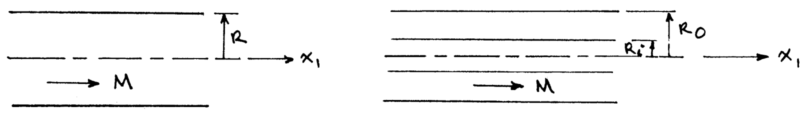
ACOUSTICS OF A CYLINDRICAL DUCT WITH UNIFORM FLOW



CIRCULAR DUCT

BC : $\left.\frac{\partial p'}{\partial r}\right|_{r=R} = 0$

ANNULAR DUCT

BC : $\left.\frac{\partial p'}{\partial r}\right|_{r=R_i} = \left.\frac{\partial p'}{\partial r}\right|_{r=R_o} = 0$

GOVERNING EQ. : $\left(\frac{1}{c_o}\frac{\partial}{\partial t} + M\frac{\partial}{\partial x_1}\right)^2 p' - \nabla^2 p' = 0$ (23)

ASSUME A SOLUTION OF THE FORM

$$p' = P(x_1, r, \theta)\, e^{i\omega t} \qquad (24)$$

THE EQUATION FOR P IS

$$(1-M^2)\frac{\partial^2 P}{\partial x_1^2} + \nabla_c^2 P - 2iMk\frac{\partial P}{\partial x_1} + k^2 P = 0 \qquad (25)$$

$$\nabla_c^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad (26)$$

$$k = \frac{\omega}{c_o} \quad \text{WAVE NUMBER}$$

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TO SOLVE EQ. (25), WE USE SEPARATION OF VARIABLE TECHNIQUE. WE CONFINE OURSELVES TO CIRCULAR DUCT. THE EIGENFUNCTIONS, INCLUDING THE TIME DEPENDENCE ARE OF THE FOLLOWING FORM:

$$\begin{cases} p'_{mn} = A_{mn} J_m(k_{r,mn} r) \exp i(\omega t - m\theta - k_{a,mn} x_1) \\ J_m(\cdot) \text{ BESSEL FN OF FIRST KIND \& ORDER } m \end{cases} \qquad (27)$$

WHERE $k_{r,mn}$ AND $k_{a,mn}$ ARE THE RADIAL AND AXIAL WAVE NUMBERS. WHEN m IS A GIVEN POSITIVE INTEGER, $k_{r,mn}$ FOR $n = 1, 2, \ldots$ ARE OBTAINED FROM

$$J'_m(k_{r,mn} R) = 0, \qquad (28)$$

i.e., THEY ARE $\frac{1}{R} \times$ ZEROS OF $J'_m(x)$. THIS RESULT FOLLOWS FROM THE APPLICATION OF THE BC: $\left.\frac{\partial p'}{\partial r}\right|_{r=R} = 0$ TO p'_{mn}, EQ. (27).

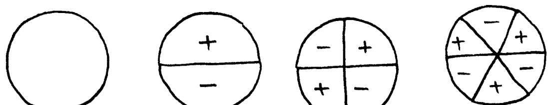
THE RELATION BETWEEN $k_{a,mn}$ AND $k_{r,mn}$ IS GIVEN BY

$$k_{a,mn} = \frac{k}{\beta^2}\left[-M \pm \sqrt{1 - (\beta k_{r,mn}/k)^2}\right] \qquad (29)$$

WHERE $\beta^2 = 1 - M^2$. WE WILL GIVE AN INTERESTING GRAPHICAL INTERPRETATION OF THIS RELATION LATER. FIRST LET US NOTICE A FEW THINGS FROM EQ. (27):

– i) AT A FIXED x_1, THE AXIAL DISTANCE, $p'_{mn} \propto e^{i(\omega t - m\theta)}$, i.e. THE PATTERN IS ROTATING WITH ANGULAR VELOCITY $\dot{\theta} = \frac{\omega}{m}$. $\therefore$ IF $\omega > 0$ (AS WE ALWAYS ASSUME), SIGN $(\dot{\theta})$ = SIGN(m).

– ii) FOR A FIXED (x_1, t), $p'_m \propto A(r)\, e^{-im\theta}$, i.e. m IS THE <u>CIRCUMFERENTIAL MODE</u>. WE DRAW THE ANTINODES (WRT θ ONLY!) BELOW FOR SOME m

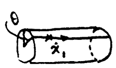


, ETC !

$m=0$ $m=1$ $m=2$ $m=3$

NOTE CAREFULLY THAT THE ORDER OF THE BESSEL FUNCTION IN p'_{mn} IS PRECISELY m.

– iii) ASSUMING $k_{a,mn}$ REAL AND NONZERO, FOR A FIXED r AND θ, $p'_{mn} \propto e^{i(\omega t - k_{a,mn} x_1)}$. THIS MEANS THAT THE POINTS OF CONSTANT PHASE TRAVEL AXIALLY WITH SPEED $\dot{x}_1 = \frac{\omega}{k_{a,mn}}$ AND SIGN$(\dot{x}_1)$ = SIGN$(k_{a,mn})$.

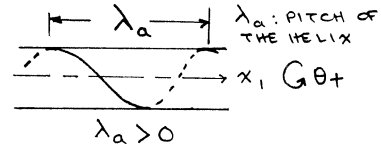


THUS IF $k_{a,mn} > 0$, THE POINTS OF CONSTANT PHASE MOVE WITH THE SPEED $\dot{x}_1$ ALONG POSITIVE x_1-AXIS AND NEGATIVE x_1-AXIS IF $k_{a,mn} < 0$. WE WILL DISCUSS THE CONDITION FOR WHICH k_{mn} IS REAL. IN THIS CASE WE SAY THAT THE MODE (m, n) IS <u>PROPAGATING</u>.

– iv) FOR (r, t) FIXED AND $k_{a,mn}$ REAL, THE CURVES OF CONSTANT PHASE ON THE CYLINDER $r = a$ ARE HELICES $m\theta + k_{a,mn} x_1 = \text{CONST.}$

$$\underbrace{m\,\Delta\theta}_{2\pi} + k_{a,mn} \underbrace{\Delta x}_{\lambda_a} = 0$$

$$\lambda_a = -\frac{2\pi}{k_{a,mn}}$$



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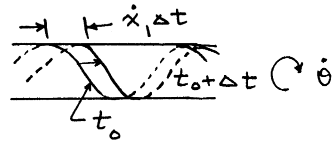
WHAT IS THE MEANING OF THE NEGATIVE SIGN? ASSUMING $m>0$ AND $k_{a,mn}>0$, IF WE TRACE THE HELIX OF CONSTANT PHASE, FOR INCREASING θ, WE MOVE TOWARD THE NEGATIVE AXIS. IN THE FIGURE ON P9, WE HAVE SHOWN THE CASE OF $\lambda_a>0$ SO THAT $\mathrm{SIGN}(m)\cdot\mathrm{SIGN}(k_{a,mn})<0$.

V) COMBINING THE ABOVE PICTURE OF THE CURVE ON CONSTANT PHASE WITH THE RESULT OF (ii), P8, WE SEE THAT FOR $k_{a,mn}$ REAL, i.e. <u>PROPAGATING WAVE</u>, THE HELICES OF CONSTANT PHASE ROTATE AT ANGULAR VELOCITY $\dot{\theta} = \frac{\omega}{m}$ THUS APPEAR AS BARBER POLE AS SHOWN BELOW:



— NOTE THAT UPTO HERE ω, m AND n ARE ARBITRARY WE WILL LATER USE PERIODICITY OF BLADES AND VANES TO RESTRICT EIGENFUNCTIONS.

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Vi) NOW LET $k_{a,mn}$ BE COMPLEX, i.e. LET

$$k_{a,mn} = \kappa_{a,mn} + i\,\mu_{a,mn}$$

THEN $\quad p_{mn} \propto e^{+\mu_{a,mn} x_1}$, i.e. DECAYING IF $\mu_{a,mn} < 0$ AND x_1 IS INCREASING. WE HAVE A <u>DECAYING</u> MODE.

<u>THE CONCEPT OF MODE CUT-OFF</u>

TO HAVE A PROPAGATING MODE, WE NEED REAL $k_{a,mn}$. LET US REITERATE WHAT THE PROCEDURE FOR FINDING $k_{a,mn}$ IS:

$k = \frac{\omega}{c_0} \Rightarrow$ DECIDE ABOUT CIRCUMFERENTIAL MODE $m (=0,1,2,\ldots)$

$\Rightarrow$ SOLVE $J_m'(x) = 0$, GET $\alpha_1, \alpha_2, \alpha_3 \ldots$ IN INCREASING ORDER,

THEN $k_{r,mn} = \frac{\alpha_n}{R} \Rightarrow$ PLUG IN EQ.(29) TO SEE IF

$k_{a,mn}$ IS REAL OR COMPLEX, i.e. MODE (m,n) PROPAGATING OR DECAYING

THIS PROCEDURE WILL EVENTUALLY COME TO AN END SINCE $k_{r,mn}$ IS AN INCREASING SEQUENCE OF POSITIVE NUMBERS

THIS ALREADY SHOWS THAT FOR ANY GIVEN CIRCUMFERENTIAL MODE m, THERE IS A RADIAL MODE n SUCH THAT FOR ALL RADIAL MODES GREATER THAN n, $k_{a,mn}$ IS COMPLEX AND THEREFORE DECAYING, i.e. NONPROPAGATING.

LET US DEFINE THE CUT-OFF RATIO β_{mn}

$$\beta_{mn} = \frac{k}{\beta k_{r,mn}} \qquad (30) \qquad (*)$$

FROM EQ. (29)

IF $\beta_{mn} < 1$, MODE (m,n) DECAYING

IF $\beta_{mn} > 1$, MODE (m,n) PROPAGATING

LET US NOTICE A FEW IMPORTANT FACTS ABOUT THE CUT-OFF RATIO :

– i) SINCE k AND $k_{r,mn}$ ARE INDEPENDENT OF MACH NUMBER OF THE FLOW, THE DEPENDENCE OF β_{mn} ON

(*) EQ.(29) FOR $k_{a,mn}$ CAN BE WRITTEN IN TERMS OF β_{mn}, SEE EQ.(33), P 19

M IS THROUGH $\beta = \sqrt{1-M^2}$.

– ii) SINCE $|M| < 1$ ALWAYS IN OUR ANALYSIS, $\beta \downarrow$ AS $M \uparrow$. THEREFORE $\beta_{mn} \uparrow$ AS $M \uparrow$, i.e. IF THE MODE (m,n) IS PROPAGATING FOR FLOW MACH NUMBER M_1, IT WILL SURELY PROPAGATE FOR $M > M_1$. SIMILARLY, A NONPROPAGATING MODE MAY PROPAGATE IF M IS INCREASED. ALL MODES PROPAGATE AT $M = 1$!

– iii) $k_{r,mn}$ IS DEPENDENT ON DUCT GEOMETRY ONLY.

BEFORE WE GIVE THE GRAPHICAL CONSTRUCTION OF EQ.(29), LET US SAY SOMETHING ABOUT THE ZEROES OF BESSEL FUNCTIONS. THERE IS NO SIMPLE FORMULA FOR SOLUTIONS OF $J'_m(x) = 0$. THE FOLLOWING ARE SOME USEFUL FACTS TO REMEMBER :

i) FOR $m = 0, 1, 2, \ldots$, $J'_m(x) = 0$ HAS AN INFINITE NUMBER OF SOLUTIONS ON THE POSITIVE REAL AXIS.

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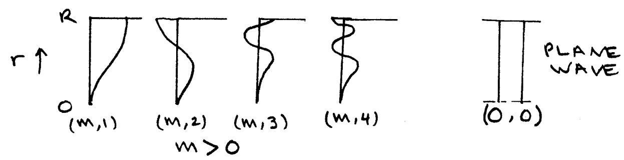
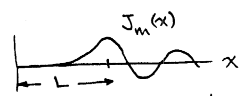
ii) FOR LARGE x, THE ZEROES ARE SPACED π APART.

iii) ALL p'_{mn}, $m>0$ ARE ZERO ON THE AXIS. THE RADIAL DISTRIBUTION OF p'_{mn} IS EASY TO CONSTRUCT.



iv) I FIND THAT MATHEMATICA IS THE EASIEST SOFTWARE FOR GETTING THE ZEROES OF $J'_m(x)$. THERE IS ALSO A PACKAGE FOR GETTING THE ZEROES OF BESSEL FUNCTIONS IN MATHEMATICA.



v) HERE IS A GOOD RULE OF THUMB.
FOR $m>1$, THE DISTANCE L WHERE THE FIRST PEAK OF $J_m(x)$ APPEARS, OR THE FIRST ZERO $J'_m(x)$ APPEARS, IS APPROXIMATELY m, i.e. THE THE ORDER OF THE BESSEL FUNCTION.

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<u>MODE CUT-ON AND CUT-OFF</u>

WE KNOW THAT $k_{r,mn}$'s ARE GEOMETRIC QUANTITIES. WE CAN THINK ABOUT THE CUT-OFF PHENOMENON IN TWO WAYS:

i) GIVEN k AND A CIRCUMFERENTIAL MODE m, WE KNOW THAT $k_{r,m1} < k_{r,m2} < k_{r,m3} \dots$. THEREFORE, THERE IS AN n SUCH THAT $\beta_{mn} = \dfrac{k}{\beta k_{r,mn}} > 1$ BUT $\beta_{m,n+1} = \dfrac{k}{\beta k_{r,m,n+1}} < 1$

WE SAY MODES $(m,1)$, $(m,2)$ ... , (m,n) ARE <u>CUT ON</u> WHILE MODE $(m,n+1)$, $(m,n+2)$ .... ARE ALL <u>CUT OFF</u>.

ii) GIVEN A MODE (m,n), WE ASK FOR THE FREQUENCY f_{mn} SUCH THAT THIS MODE IS CUT ON <u>ABOVE</u> f_{mn}, i.e.

$$\frac{2\pi(f_{mn}/c_0)}{\beta\, k_{r,mn}} = \beta_{mn} = 1 \quad \therefore \quad f_{mn} = \frac{\beta c_0 k_{r,mn}}{2\pi} \qquad (31)$$

THIS CALLED THE <u>CUT-OFF FREQUENCY</u> OF MODE (m,n).

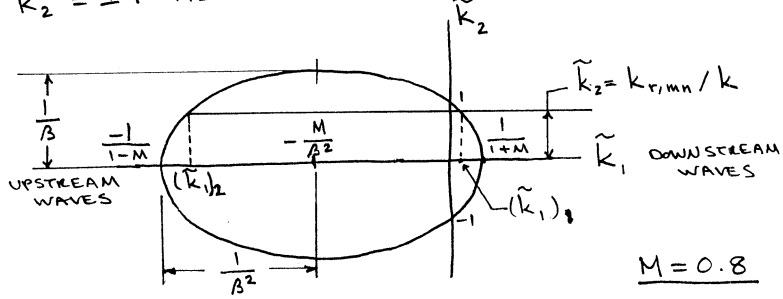
— NOTE FOR A PLANE WAVE, i.e. MODE $(0,0)$, $\beta_{00} = \infty > 1$, i.e. THIS MODE ALWAYS PROPAGATES.

A GRAPHICAL CONSTRUCTION

LET US WRITE $\tilde{k}_1 = k_{a,mn}/k$ AND $\tilde{k}_2 = k_{r,mn}/k$. THEN EQ. (29) CAN BE WRITTEN AS

$$\frac{(\tilde{k}_1 + M/\beta^2)^2}{1/\beta^4} + \frac{\tilde{k}_2^{\,2}}{1/\beta^2} = 1 \qquad (32)$$

THIS IS THE EQUATION OF AN ELLIPSE WITH CENTER AT $\tilde{k}_1 = -\frac{M}{\beta^2}$, SEMIMAJOR AXIS $= \frac{1}{\beta^2}$, SEMIMINOR AXIS $= \frac{1}{\beta}$, CROSSING $\tilde{k}_1$-AXIS AT $\tilde{k}_1 = \frac{-1}{1-M}$ AND $\tilde{k}_1 = \frac{1}{1+M}$ AND $\tilde{k}_2$-AXIS AT $\tilde{k}_2 = \pm 1$ AS SHOWN BELOW.



HERE ARE A FEW FACTS ABOUT THIS ELLIPSE:

i) FOR EACH MACH NUMBER, THERE IS A UNIQUE ELLIPSE IN $\tilde{k}_1\tilde{k}_2$-PLANE. EXACTLY THE SAME ELLIPSE CAN BE USED FOR STUDYING WAVE PROPAGATION IN 2D, 3D RECTANGULAR AND ANNULAR DUCT.

ii) THE ASPECT RATIO OF THE ELLIPSE IS $\frac{1}{\beta}$, i.e. THE ELLIPSE LOOKS MORE ELONGATED AS M INCREASES. ALSO BOTH SEMIMAJOR AND SEMIMINOR AXES INCREASE AS M INCREASES AND THE CENTER MOVES FARTHER FROM THE ORIGIN ALONG NEGATIVE $\tilde{k}_1$-AXIS.

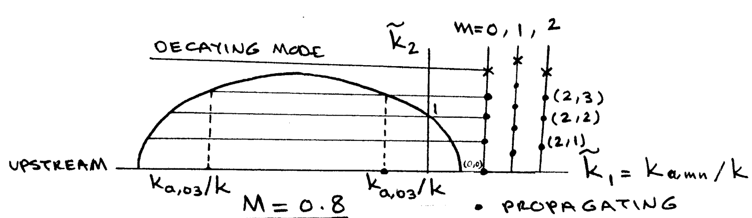
iii) THE <u>REAL</u> SOLUTIONS FOR $k_{a,mn}$ FROM EQ. (29) ARE OBTAINED BY DRAWING A HORIZONTAL LINE AT $\tilde{k}_2 = k_{r,mn}/k$ AS SHOWN IN THE FIGURE ON P16. IF THIS LINE INTERSECTS THE ELLIPSE AT $(\tilde{k}_1)_1$ AND $(\tilde{k}_1)_2$ THEN THE CORRESPONDING $k_{a,mn}$ ARE $k(\tilde{k}_1)_1$ AND $k(\tilde{k}_1)_2$. IN THIS CASE MODE (m,n) IS PROPAGATING. NOTE THAT THE ELLIPSE ALWAYS INTERSECTS $\tilde{k}_2$-AXIS AT -1 AND 1. IF $\tilde{k}_2 > 1$, WE HAVE $(\tilde{k}_1)_1 < 0$ AND $(\tilde{k}_1)_2 < 0$.

iv) FROM THIS FIGURE, IF $\tilde{k}_2 = \frac{k_{r,mn}}{k} > \frac{1}{\beta}$ (SEMIMINOR AXIS),

THEN THE HORIZONTAL LINE DOES NOT INTERSECT THE ELLIPSE, i.e. WE HAVE NO REAL SOLUTIONS OF EQ. (29) AND THEREFORE THE MODE IS DECAYING. BUT THE CONDITION $\frac{k_{r,mn}}{k} > \frac{1}{\beta}$ IS EXACTLY $\beta_{mn} = \frac{k}{\beta k_{r,mn}} < 1$, i.e. IN THIS CASE THE CUT-OFF RATIO IS LESS THAN ONE.

v) FROM THE FACT THAT THE SEMIMINOR AXIS SIZE INCREASES AS M INCREASES, AND THE ABOVE GEOMETRIC CONSTRUCTION, WE FIND THAT IF THE MODE (m,n) PROPAGATES FOR M_1, IT WILL ALSO PROPAGATE FOR $M > M_1$. REMEMBER $k_{r,mn}$ DOES NOT DEPEND ON MACH NUMBER.

vi) HERE IS THE WAY WE GRAPHICALLY CAN GET ALL PROPAGATING MODES IN A DUCT. FOR CIRCUMFERENTIAL MODE $m = 0, 1, 2, \ldots$, MARK OFF $\tilde{k}_2 = k_{r,mn}/k$ ON A VERTICAL AXIS, DRAW HORIZONTAL LINES THAT CROSS THE ELLIPSE. IT IS EASIER IF WE DRAW A SEPARATE VERTICAL LINE FOR EACH CIRCUMFERENTIAL MODE. THIS IS SHOWN ON THE NEXT PAGE.

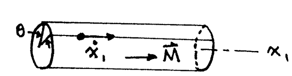


NOTE: AT M=0, THE ELLIPSE BECOMES A CIRCLE OF RADIUS 1 WITH CENTER AT THE ORIGIN.

PHASE AND GROUP VELOCITY

FROM THE RELATION PHASE $= \omega t - m\theta - k_{a,mn} x_1$

WE HAVE SHOWN THAT FOR θ FIXED, THE AXIAL PHASE SPEED IS $\dot{x}_1 = \frac{\omega}{k_{a,mn}}$ $\therefore$ $\mathrm{SIGN}(\dot{x}_1) = \mathrm{SIGN}(k_{a,mn})$.



LET US WRITE EQ. (29) AS FOLLOWS

$$k_{a,mn\pm} = \frac{k}{\beta^2}\left[-M \pm \sqrt{1 - 1/\beta_{mn}^2}\right] \qquad (33)$$

WE ALWAYS HAVE $|k_{a,mn+}| < |k_{a,mn-}|$ FOR $M \geq 0$. BUT

AS SEEN FROM THE FIGURE ON THE TOP OF P19, ALTHOUGH $k_{a,mn-} < 0$, $k_{a,mn+}$ CAN BE BOTH POSITIVE OR NEGATIVE.

IF i) $k_{a,mn+} > 0 \Rightarrow \dot{x}_{1+} > 0$ DOWNSTREAM MOVING PHASE,

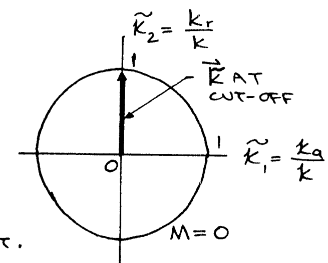
ii) $k_{a,mn+} < 0 \Rightarrow \dot{x}_{1+} < 0$ UPSTREAM MOVING PHASE,

iii) $k_{a,mn-} < 0$ (ALWAYS) $\Rightarrow \dot{x}_{1-} < 0$ UPSTREAM MOVING PHASE.

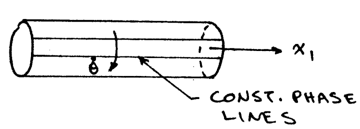
HOWEVER, IN CASE ii) SINCE $\dot{x}_{1-} < \dot{x}_{1+} < 0$, THE WAVE ASSOCIATED WITH $k_{a,mn-}$ MOVES UPSTREAM <u>FASTER</u> THAN THE WAVE ASSOCIATED WITH $k_{a,mn+}$.

ENERGY TRANSFER IS ASSOCIATED WITH THE GROUP VELOCITY. EVERSMAN HAS SHOWN THAT REGARDLESS OF THE SIGN OF $k_{a,mn+}$, THE WAVES ASSOCIATED WITH $k_{a,mn+}$ AND $k_{a,mn-}$ TRANSFER ENERGY TO DOWNSTREAM AND UPSTREAM REGIONS OF THE DUCT, RESPECTIVELY. WE WILL EXPLAIN THIS FURTHER BELOW.

<u>WHAT IS THE CUT-OFF PHENOMENON?</u>



FOR A DUCT WITH NO FLOW, $\beta_{mn} = \dfrac{k}{k_r}$ AND AT CUT-OFF $\beta_{mn} = 1$, i.e. $k_r = k$ AND $k_a = 0$, i.e. WE HAVE NO AXIAL PROPAGATION AND HAVE A STANDING WAVE IN RADIAL DIRECTION. THE CONSTANT PHASE LINES ON $r =$ CONST. SURFACES ARE PARALLEL TO THE DUCT AXIS AND THE WHOLE PRESSURE PATTERN ROTATES AT $\dot{\theta} = \dfrac{\omega}{m}$.



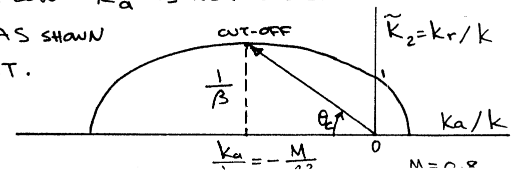
$$p'_{mn} = A_{mn} J_m(k_{r,mn} r)\, e^{i(\omega t - m\theta)} \qquad (34)$$

(NO x_1 DEPENDENCE!)

THE ENERGY FLUX IN x_1 DIRECTION IS ZERO.

IN THE CASE OF DUCT WITH FLOW k_a IS NOT ZERO AT CUT-OFF, IN FACT, $k_a < 0$ AS SHOWN IN THE FIGURE ON THE RIGHT.

WHAT IS SO PARTICULAR ABOUT $k_a/k = -M/\beta^2$?

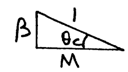


THE ACOUSTIC ENERGY TRAVELS AT GROUP VELOCITY $\vec{V}_G$

$$\vec{V}_G = \vec{V} + c\,\vec{e}_{k'} = c\,(\vec{M} + \vec{e}_k) \qquad (35)$$

WHERE $\vec{e}_{k'}$ IS THE UNIT VECTOR ALONG WAVE NUMBER VECTOR $\vec{k}'$. HERE $\vec{V}$ AND $\vec{M}$ ARE THE FLUID VELOCITY AND MACH NUMBER VECTORS, RESPECTIVELY. ON PREVIOUS PAGE, FROM THE BOTTOM FIGURE, WE HAVE

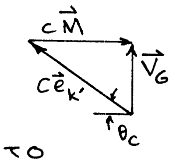
$$\tan\theta_c = \frac{1/\beta}{M/\beta^2} = \frac{\beta}{M} \qquad (36)$$



$\boxed{\cos\theta_c = M}$

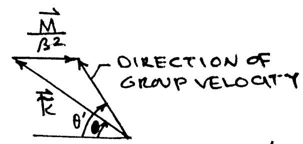
THE COMPONENT OF $\vec{V}_G$ ALONG x_1-AXIS, i.e. THE DUCT AXIS, IS

$$\vec{V}_G \cdot \vec{e}_1 = c\,(M - \cos\theta_c) = c\,(M - M) = 0\,!$$



THIS MEANS FOR THE ENERGY FLUX VECTOR IS NORMAL TO THE WALL AND THUS CUT-OFF TO THE OUTSIDE.

TO FIND THE DIRECTION OF GROUP VELOCITY IN OUR GRAPHICAL CONSTRUCTION, FOLLOW THE DIAGRAM ON THE RIGHT. NOTE THAT $\frac{-M}{\beta^2}$ IS THE $(k_a/k)_{\text{cut-off}}$.



$$\tilde{k} = \sqrt{\tilde{k}_1^2 + \tilde{k}_2^2} = \frac{k'}{k}$$

WHY DO WE TREAT $(k_a, k_r) = \vec{k}'$ AS PROPAGATION VECTOR?

CONSIDER $\quad p' = A\, J_m(k_r r)\, \exp i\,[\omega t - m\theta - k_a x_1]$

LET US WRITE $k_r = \frac{\alpha_n}{r_0}$ AND ASSUME α_n IS LARGE. REMEMBER α_n IS THE nTH ZERO OF $J_m'(x) = 0$. THEN

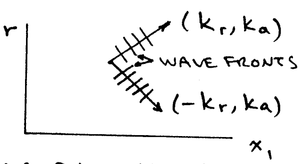
$$J_m(k_r r) = J_m\!\left(\frac{r}{r_0}\alpha_n\right) \approx \sqrt{\frac{2 r_0}{\pi r \alpha_n}}\, \cos\!\left(\frac{\alpha_n r}{r_0} - \frac{\pi}{4} - \frac{m\pi}{2}\right) \qquad (37)$$

FOR r NEAR r_0 (DUCT WALL)

LET $\frac{\pi}{4} + \frac{m\pi}{2} = \psi$ (PHASE), THEN USING $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, WE HAVE (AFTER SUBSTITUTING $k_r = \frac{\alpha_n}{r_0}$):

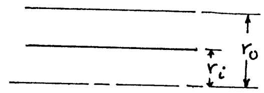
$$p' \approx \frac{A}{2}\sqrt{\frac{2}{\pi k_r r}}\Big\{\exp i\,[\omega t + k_r r - m\theta - k_a x - \psi] + \exp i\,[\omega t - k_r r - m\theta - k_a x + \psi]\Big\} \qquad (38)$$

NOW IF WE TAKE θ = FIXED, i.e. IN (r, x_1)-PLANE, WE HAVE TWO WAVES AS SHOWN WHICH LOOK ESSENTIALLY THE SAME AS THE PICTURE OF WAVE NUMBER VECTORS IN A 2D DUCT. BUT REMEMBER THAT OUR PICTURE IN A CIRCULAR DUCT IS APPROXIMATE.



ANNULAR DUCT

THE DUCT EIGENFUNCTIONS NOW INCLUDE BESSEL FUNCTIONS OF 2ND KIND :



$$p'_{mn} = A_{mn}\left[J_m(k_{r,mn}r) + Q\,Y_m(k_{r,mn})\right] \exp i\left[\omega t - m\theta - k_{a,mn}x_1\right]$$

k_r'S AND Q ARE FOUND FROM

$$\begin{cases} J'_m(r_o z) + Q\,Y'_m(r_o z) = 0 \\ J'_m(r_i z) + Q\,Y'_m(r_i z) = 0 \end{cases} \Rightarrow \boxed{J'_m(r_o z)Y'_m(r_i z) - Y'_m(r_o z)J'_m(r_i z) = 0} \quad (39)$$

$$Q = -\frac{J'_m(r_o z)}{Y'_m(r_o z)} = -\frac{J'_m(r_i z)}{Y'_m(r_i z)} \quad (40)$$

AS THE CASE OF CIRCULAR DUCT, THE BOXED EQ. ABOVE HAS AN INFINITE NUMBER OF ROOTS $z_1 < z_2 < z_3 < \cdots < z_n < \cdots$

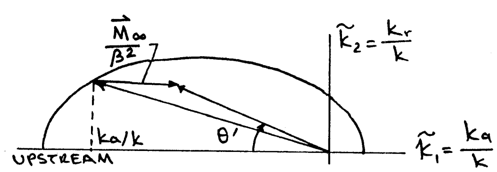
AND $k_{r,mn} = z_n$, $n = 1, 2, \ldots$. <u>NOTE</u>, USE z_n IN Q , i.e. $Q = Q(z_n)$.

BECAUSE Y'_m HAS A NASTY SINGULARITY NEAR THE AXIS, THE SOLUTION OF THE ABOVE EQUATION IS SUBJECT TO ERRORS FOR SMALL r_i/r_o , i.e. SMALL HUB-TO-TIP RATIOS EVEN ON A MAIN FRAME COMPUTER . TO USE MATHEMATICA , PLOT $f(z) = J'_m(r_o z)Y'_m(r_i z) - Y'_m(r_o z)J'_m(r_i z)$ AND USE FindRoot[f(z), {z, z1, z2}] TO GET THE ROOT IN THE INTERVAL (z_1, z_2).

DIRECTION OF RADIATION PEAK FOR A GIVEN MODE

IF THE TUNNEL OR FLIGHT MACH NUMBER IS M_∞ , RICE, HEIDMANN AND SOFRIN (AIAA-79-0183) SUGGEST THAT THE DIRECTION OF PEAK RADIATION FOR A MODE IS THE DIRECTION OF GROUP VELOCITY BASED ON M_∞ . HERE IS THE CONSTRUCTION BASED ON OUR GRAPHICAL METHOD. NOTE THAT THE ELLIPSE IN $\tilde{k}_1\tilde{k}_2$-PLANE IS BASED ON THE <u>DUCT</u> MACH NUMBER.

θ' IS THE ANGLE OF PEAK RADIATION. WE CAN CALCULATE THIS ANGLE FROM



$$\tan\theta' = \frac{\tilde{k}_2}{|\tilde{k}_1 + M_\infty/\beta^2|} = \frac{k_r/k}{|M_\infty - M - \sqrt{1 - 1/\beta^2_{mn}}\,|/\beta^2} = \frac{\beta}{|(M_\infty - M)\beta_{mn} - \sqrt{\beta^2_{mn} - 1}\,|} \quad (41)$$

THIS IS EQUIVALENT TO WHAT RICE ET AL HAVE GIVEN . THEIR EXPRESSION LOOKS MORE COMPLICATED SINCE THEY CALCULATED $\cos\theta'$. OUR GRAPHICAL METHOD HAS SIMPLIFIED THE ANALYSIS . NOTE THAT β_{mn} IS THE CUT-OFF RATIO OF THE MODE .

MODES IN A DUCT WITH ROTOR AND EGV

ASSUME THAT WE HAVE B BLADES ON THE ROTOR AND V VANES ON EGV AND THE ROTOR IS TURNING WITH ANGULAR VELOCITY Ω. WE ASSUME THE BLADES AND VANES ARE CIRCUMFERENTIALLY SPACED $2\pi/B$ AND $2\pi/V$, RESPECTIVELY.

ROTOR ALONE

IN THE PHASE RELATION $\omega t - m\theta - k_{a,mn}x_1$, WE WANT TO FIND HOW ω AND m ARE RELATED TO Ω AND B. FOR A MICROPHONE IN THE DUCT, WE DETECT A PERIODIC SIGNAL IN TIME WITH FUNDAMENTAL FREQUENCY $B\Omega$. USING FOURIER TRANSFORMATION, THIS PERIODIC SIGNAL CAN BE DECOMPOSED INTO COMPONENTS WITH FREQUENCIES WHICH ARE MULTIPLES OF $B\Omega$.

$\therefore$ ω IS A MULTIPLE OF $B\Omega$: $B\Omega, 2B\Omega, 3B\Omega \cdots\cdots$

THE ACOUSTIC DISTURBANCE MUST ROTATE WITH ANGULAR VELOCITY Ω, i.e. WITH THE ROTOR. THIS MEANS THAT $\dot{\theta} = \Omega$ AND $\theta = \Omega t + \text{CONST}$. THEREFORE, $\Omega t - \theta$ MUST APPEAR IN THE PHASE RELATION. WE HAVE DISCOVERED THAT FOR THE ROTOR ALONE CASE $\boxed{\text{PHASE} = mB(\Omega t - \theta) - k_{a,mB,n}x_1}$ (42)

THIS MEANS THAT CIRCUMFERENTIAL MODES OF ORDER $B, 2B,$

$3B, \cdots, mB, \cdots$ CAN BE GENERATED. NOTE THAT CORRESPONDING TO EACH FREQUENCY $mB\Omega$, WE HAVE ONLY ONE CIRCUMFERENTIAL MODE mB. ALSO m CANNOT BE ZERO SINCE TIME DEPENDENCE DISAPPEARS. THE EIGENFUNCTIONS HAVE THE FOLLOWING FORM:

$$p' = A\, J_{mB}(k_{r,mB,n}r)\exp i\left[mB(\Omega t - \theta) - k_{a,mB,n}x_1\right] \qquad (43)$$

WE CAN WRITE $k_{r,mn} \equiv k_{r,mB,n}$, $k_{a,mn} = k_{a,mB,n}$.

NOTE THAT FOR mBTH CIRCUMFERENTIAL MODE, k IN EQ.(29) IS $k_m = \dfrac{mB\Omega}{c_0}$. WE CAN HAVE SEVERAL RADIAL MODES CORRESPONDING TO A CIRCUMFERENTIAL MODE mB. NOW THIS IS THE PROCEDURE WE USE TO FIND PROPAGATING AND DECAYING MODES:

FOR CIRCUMFERENTIAL MODE mB, $m = 1, 2, 3, \cdots$, FIND THE ZEROES OF $J'_{mB}(x) = 0$: $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \cdots\cdots$

$$\Longrightarrow \quad k_{r,mn} = \frac{\alpha_n}{R} \Longrightarrow \beta_{mn} = \frac{k_m}{\beta\, k_{r,mn}}$$

WHERE $k_m = \dfrac{mB\Omega}{c_0} \Longrightarrow$ IF $\beta_{mn} > 1$, PROPAGATING; IF $\beta_{mn} < 1$,

DECAYING $\Rightarrow$ FIND $k_{a,mn} = \dfrac{k_m}{\beta^2}\left[-M \pm \sqrt{1 - 1/\beta_{mn}^2}\right]$

OUR GRAPHICAL ANALYSIS STAYS THE SAME. REMEMBER TO CHECK IF YOU HAVE USED THE RIGHT $k = k_m$ FOR EACH CIRCUMFERENTIAL MODE mB.

A USEFUL RULE OF THUMB. $\beta_{mn} = \dfrac{k_m}{\beta k_{r,mn}} = \dfrac{mB\Omega / c_0}{\beta k_{r,mn}}$

LET R BE THE DUCT RADIUS, THEN $R\Omega/c_0$ IS THE BLADE TIP MACH NUMBER M_T, ALSO $k_{r,mn}R = \alpha_n$, THE NTH ZERO OF $J'_{mB}(x)$. USING THESE TWO PARAMETERS, WE CAN WRITE β_{mn} AS

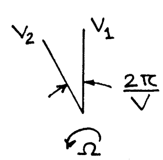
$$\beta_{mn} = \frac{mB\,M_T}{\beta\,\alpha_n} \qquad (44)$$

$$\beta_{m1} = \frac{mB\,M_T}{\beta\,\alpha_1} < \frac{M_T}{\beta} \quad \text{SINCE } \alpha_1 > mB \text{ (USUALLY } \alpha_1 >> mB)$$

$\therefore$ FOR SMALL M, i.e. SMALL DUCT MACH NUMBER WHEN $\beta \approx 1$, $\beta_{m1} < M_T$. THIS MEANS THAT FOR ROTORS WITH SUBSONIC TIP SPEED AND SMALL DUCT MACH NUMBERS, NO MODE CAN PROPAGATE OUT. THIS IS A VERY USEFUL RESULT.

INTERACTION MODES

THE STUDY OF INTERACTION MODES CAN BE BASED MAINLY ON KINEMATICS. WE HAVE V VANES SPACED $\frac{2\pi}{V}$ RADIANS. LET US HAVE B BLADES ROTATING WITH ANGULAR VELOCITY Ω. CONCENTRATING ON VANE NUMBER 1, V_1, FIRST, THE WAKES FROM B BLADES HIT THIS VANE AT TIME INTERVALS $\frac{2\pi}{B\Omega}$ SECONDS, i.e. WITH FREQUENCY $B\Omega$ WHICH IS THE FUNDAMENTAL FREQUENCY OF OSCILLATIONS IN TIME. ASSUME V_1 IS AT $\theta = 0$. A LINE PERTURBATION (RADIAL) AT VANE 1 HAS THE REPRESENTATION



$$\delta(\theta) \sum_1^{\infty} A_m e^{imB\Omega t} \qquad (45)$$

NOW THE NTH VANE AT $\theta = \frac{2n\pi}{V}$ SEES A PERTURBATION AT V_1, EXACTLY $\frac{2\pi n}{V\Omega}$ SECONDS LATER. THIS MEANS THAT THE PERTURBATION AT THE NTH VANE HAS THE REPRESENTATION

$$\delta\left(\theta - \frac{2\pi n}{V}\right) \sum_1^{\infty} A_m \exp i\left[mB\Omega\left(t - \frac{2\pi n}{V\Omega}\right)\right] \qquad (46)$$

30

THEREFORE, V VANES PRODUCE A PERTURBATION OF THE FORM

$$\sum_{m=1}^{\infty} \sum_{n=0}^{V-1} A_m \, \delta\left(\theta - \frac{2\pi n}{V}\right) \exp i \left[mB\Omega\left(t - \frac{2\pi n}{V\Omega}\right)\right] \qquad (47)$$

A CIRCUMFERENTIAL FOURIER TRANSFORM, SUMMATION ON n AND NOTING THAT UNLESS SOME SPECIAL COMBINATIONS OF B AND V ARE TAKEN, THE RESULTING FOURIER COMPONENTS ADD UP TO ZERO, WE FIND ONLY THE FOLLOWING CIRCUMFERENTIAL MODES CAN EXIST

$$\boxed{mB + nV = q} \qquad \begin{array}{l} m = 1, 2, 3, \ldots \\ n = 0, \pm 1, \pm 2 \\ q \text{ CIRCUMFERENTIAL MODE} \end{array} \qquad (48)$$

WE NOTE THAT q CAN BE A POSITIVE OR A NEGATIVE INTEGER. IF $q > 0$, THE MODE IS ROTATING IN THE DIRECTION OF THE ROTOR. OTHERWISE, IT IS ROTATING IN OPPOSITE DIRECTION TO THE ROTOR.

WE MENTION SOME IMPORTANT FACTS HERE.

i) m IN EQ.(48) INDICATES THE MULTIPLE OF BLADE PASSAGE FREQUENCY. THEREFORE, THE CIRCUMFERENTIAL MODES

31

CORRESPONDING TO BPF, i.e. $m = 1$, ARE

$$\ldots\ldots B-2V \quad B-V \quad B \quad B+V \quad B+2V \quad \ldots\ldots$$

AND FOR 2BPF, i.e. $m=2$, ARE

$$\ldots\ldots 2B-2V \quad 2B-V \quad 2B \quad 2B+V \quad 2B+2V \quad \ldots\ldots$$

ii) LET $\tilde{C}$ BE GREATEST COMMON DIVISOR OF B AND V, THEN $q = mB + nV$ CAN BE ONLY A MULTIPLE OF $\tilde{C}$. THEREFORE TO PRODUCE <u>ALL</u> CIRCUMFERENTIAL MODES, THEREFORE B AND V MUST BE RELATIVELY PRIME.

iii) THE EIGENFUNCTIONS FOR INTERACTION MODES ARE OF THE FORM:

$$p'_{mq} = A_{mq} \, J_q(k_{r,qn} r) \exp i \left[mB\Omega t - q\theta - k_{a,qn} x_1 \right] \qquad (49)$$

<u>NOTE</u> $k = k_m = \dfrac{mB\Omega}{c_o}$, THE ORDER OF BESSEL FUNCTION IS q, i.e. THE CIRCUMFERENTIAL MODE NUMBER.

31

CORRESPONDING TO BPF, i.e. $m=1$, ARE

$$\ldots\ldots B-2V \quad B-V \quad B \quad B+V \quad B+2V \quad \ldots\ldots$$

AND FOR 2BPF, i.e. $m=2$, ARE

$$\ldots\ldots 2B-2V \quad 2B-V \quad 2B \quad 2B+V \quad 2B+2V \quad \ldots$$

ii) LET $\tilde{c}$ BE GREATEST COMMON DIVISOR OF B AND V, THEN $q = mB + nV$ CAN BE ONLY A MULTIPLE OF $\tilde{c}$. THEREFORE TO PRODUCE <u>ALL</u> CIRCUMFERENTIAL MODES, B AND V MUST BE RELATIVELY PRIME.

iii) THE EIGENFUNCTIONS FOR INTERACTION MODES ARE OF THE FORM:

$$p'_{mq} = A_{mq} J_q(k_{r,qn} r) \exp i \left[mB\Omega t - q\theta - k_{a,qn} x_1 \right] \quad (49)$$

<u>NOTE</u> $k = k_m = \frac{mB\Omega}{c_0}$, THE ORDER OF BESSEL FUNCTION IS q, i.e. THE CIRCUMFERENTIAL MODE NUMBER.

32

WE HAVE COME UP WITH THE FOLLOWING <u>PROCEDURE</u>

DECIDE MULTIPLE OF BPF, i.e. mPF $\Longrightarrow k_m = \frac{mB\Omega}{c_0}$

$\Longrightarrow$ FIND ALL CIRCUMFERENTIAL MODES $mB+nV$, $n=0,\pm1,\ldots$

$\Longrightarrow$ FOR EACH CIRCUMFERENTIAL MODE q, FIND ROOTS OF $J'_{|q|}(x) = 0$ GETTING $\alpha_1, \alpha_2, \ldots \alpha_n, \ldots$.

$\Longrightarrow k_{r,qn} = \frac{\alpha_n}{R}$ $\Longrightarrow$ FIND CUT-OFF RATIO $\beta_{qn} = \frac{k_m}{\beta k_{r,qn}}$

$\Longrightarrow$ FIND $k_{a,qn} = \frac{k_m}{\beta^2} \left[-M \pm \sqrt{1 - \frac{1}{\beta_{qn}^2}} \right]$

<u>NOTE</u> $k_{r,qn}$ DEPENDS ON q AND DOES <u>NOT</u> DEPEND ON M AND m.

iv) THE SAME CIRCUMFERENTIAL MODE CAN BE PRODUCED BY MANY MULTIPLES OF BPF'S. THIS CAN BE SEEN EASILY FROM $q = mB + nV = (m + p\frac{V}{\tilde{c}})B + (n - p\frac{B}{\tilde{c}})V$ WHERE $\tilde{c}$ IS GCD OF B AND m AND p IS AN INTEGER WHICH MAKES $m' = m + p\frac{V}{\tilde{c}} > 0$.

33

V) WHY WORRY ABOUT INTERACTION MODES? WE KNOW THAT k_r'S COME FROM SOLUTION OF $J'_q(x) = 0$. THE SMALLER q IS, THE LOWER IS THE FIRST ROOT OF THIS EQUATION AND THE CUTOFF RATIO IS

$$\beta_{qn} = \frac{k_m}{\beta k_{r,qn}}$$

NOW β_{qn} CAN BE ABOVE ONE AND THUS MODE (q,n) WILL BE PROPAGATING. NOTE THAT WE SHOULD SAY MODE (q,n) FOR MBPF, i.e. THE MULTIPLE OF BLADE PASSAGE FREQUENCY FOR THE MODE (q,n) MUST BE MENTIONED.

<u>EXAMPLE</u> $B = 16$, $V = 20$

$$q = -4 = 16 - 20 = (1 + \tfrac{20}{4})16 + (-1 - \tfrac{16}{4})20$$
$$= 6 \times 16 - 5 \times 20$$

$\therefore$ 4TH CIRCUMFERENTIAL MODE IS PRODUCED, AMONGST OTHERS, BY BPF AND 6BPF.

34

<u>HOW DOES A ROTATING MICROPHONE SEPARATE MODES?</u>

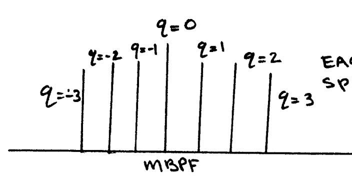
WE CONSIDER THE PHASE OF A PROPAGATING MODE (q,n) FOR MBPF, DISREGARDING $k_{a,qn} x_1$

$$\text{PHASE} = mB\Omega t - q\theta$$

FOR A ROTATING MICROPHONE WITH ANGULAR VELOCITY $\alpha\Omega$, WE HAVE $\theta = \alpha\Omega t$ SO THAT

$$\text{PHASE} = (mB + \alpha q)\Omega t \qquad (50)$$

THIS MEANS THAT A NARROWBAND FILTER WILL GIVE A SPECTRUM AS SHOWN BELOW



EACH FREQUENCY SPACED $\frac{\alpha}{B} \times$ BPF APART

IN LEWIS ROTATING MICROPHONE DESIGN $\alpha = \frac{1}{250}$.

5 Collected Lectures on Aeroacoustics

LEC. 1/1

LEC. 1/1

SPEED OF SOUND IN A GAS

$C = \sqrt{\gamma R T}$

γ = RATIO OF SPECIFIC HEATS
= 1.4 FOR AIR (OR ANY DIATOMIC GAS)

$R = \frac{8314}{M}$ GAS CONSTANT

M = MOLECULAR MASS (= 29 FOR AIR)

T = TEMPERATURE °K
= 273 + T°C

C = 344 m/s AT 20°C

DENSITY OF AIR AT SEA LEVEL AND 20°C = 1.17 kg/m³

IN ACOUSTICS WE USE SI UNITS ALWAYS.

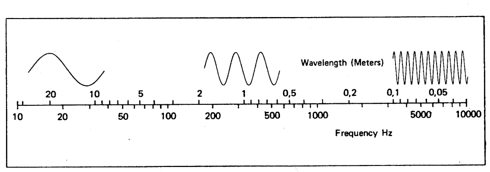
WAVELENGTH VS FREQUENCY

LEC. 1/2

AUDIO FREQUENCY RANGE: 20 Hz TO 20 kHz

FROM $\lambda = \frac{c}{f}$, WE GET

WAVELENGTH RANGE (FOR AUDIO FREQ.) : 17 m TO 1.7 cm



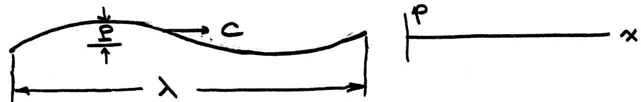
Wavelength in air versus frequency under normal conditions

- HUMAN EAR IS MOST SENSITIVE IN 1 kHz TO 5 kHz RANGE
(34 cm TO 7 cm WAVELENGTH RANGE

THE GOVERNING EQUATIONS

WE NEED SOME SIMPLE RELATIONS TO GET THE GOV. EQS.

PLANE WAVE



PRESSURE WAVE AT A FIXED TIME

$\boxed{\lambda f = c}$ SPEED OF SOUND m/s

WAVELENGTH m — λ; FREQUENCY HZ (CYCLES/S) — f

THIS MEANS THAT THE WAVES IN A CYLINDER OF LENGTH C m WILL PASS THROUGH A FIXED POINT PRODUCING $f = \frac{c}{\lambda}$ CYCLES IN ONE SECOND !

PERIOD $\boxed{T = \frac{1}{f}}$ SECONDS $\boxed{\lambda = cT}$

$\boxed{\frac{\partial p}{\partial t} \approx \frac{P}{T} = \frac{cP}{\lambda}}$, $\boxed{\frac{\partial p}{\partial x} \approx \frac{P}{\lambda}}$ $\Big\{$ WE USE SIMILAR APPROXIMATIONS FOR ρ, $\vec{u}$, ETC.

P : AMPLITUDE

THE GOVERNING EQUATIONS (CONT'D)

MASS CONTINUITY $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$, $\boxed{\vec{u}_0 = 0, \text{i.e. MEDIUM IS AT REST}}$

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\sim \frac{\tilde{\rho}'}{T} = \frac{\tilde{\rho}' c}{\lambda}} + \underbrace{\rho\,\nabla\cdot\vec{u}}_{\sim \frac{\rho_0 U}{\lambda}} + \underbrace{\vec{u}\cdot\nabla\rho}_{\sim \frac{\tilde{\rho}' U}{\lambda}} = 0$$

$\tilde{\rho}'$ AMPLITUDE OF DENSITY PERTURBATION, U AMP. OF $|\vec{u}|$

ASSUME $\boxed{\tilde{\rho}' \ll \rho_0}$ (≈ 1.2 kg/m^2)

$\Rightarrow \frac{\rho_0 U}{\lambda} \gg \frac{\tilde{\rho}' U}{\lambda}$ $\therefore$ IGNORE $\vec{u}\cdot\nabla\rho$ COMPARED TO $\rho\,\nabla\cdot\vec{u}$. APPROXIMATE ρ AS ρ_0 .

$$\boxed{\frac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot\vec{u} = 0} \qquad \boxed{\rho' = \rho - \rho_0}$$

MASS CONTINUITY

– NOTE THAT BALANCING $\partial\rho/\partial t$ WITH $\rho_0 \nabla\cdot\vec{u}$ WILL GIVE

$$\boxed{\tilde{\rho}' \sim \rho_0 \frac{U}{c}}$$

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THE GOVERNING EQUATIONS (CONT'D)

MOMENTUM EQUATION $\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{\nabla p}{\rho}$

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{\sim \frac{cU}{\lambda}} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{\sim \frac{U^2}{\lambda}} = -\underbrace{\frac{\nabla p}{\rho}}_{\frac{P}{\rho_0 \lambda}} \qquad (P \text{ AMP. OF } p)$$

IF $U \ll c$ (SP. OF SOUND), WE CAN NEGLECT $\vec{u}\cdot\nabla\vec{u}$ COMPARED TO $\partial\vec{u}/\partial t$.

$$\boxed{\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p' = 0} \quad \text{MOMENTUM} \quad \left\{ \begin{array}{l} \text{CONDITION OF VALIDITY} \\ \boxed{\rho' \ll \rho_0 ,\ |\vec{u}| \ll c} \end{array} \right.$$

$\boxed{p' = p - p_0}$

– NOTE THAT BALANCING $\rho_0 \partial\vec{u}/\partial t$ WITH ∇p GIVES

$\boxed{P \sim \rho_0 c\, U}$ $\boxed{\rho_0 c \approx 407 \text{ RAYLS}}$

– THE RELATION BETWEEN $p' = p - p_0$ & $\rho' = \rho - \rho_0$ $\boxed{p' = c^2 \rho'}$

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THE GOVERNING EQUATIONS (CONT'D)

OTHER RELATIONS

$$\begin{array}{l} \frac{\partial}{\partial t} \\ \nabla\cdot \end{array} \left\{ \begin{array}{l} \frac{\partial \rho'}{\partial t} + \rho_0 \nabla\cdot\vec{u} = 0 \\ \rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla(c^2 \rho') = 0 \end{array} \right. \Rightarrow \underbrace{\left\{ \begin{array}{l} \frac{\partial^2 \rho'}{\partial t^2} + \rho_0 \frac{\partial}{\partial t}\nabla\cdot\vec{u} = 0 \\ \rho_0 \frac{\partial}{\partial t}\nabla\cdot\vec{u} + c^2 \nabla^2 \rho' = 0 \end{array} \right.}_{\text{SUBTRACT, DIVIDE BY } c^2}$$

$$\Rightarrow \boxed{\frac{1}{c^2}\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 \rho' = \Box^2 \rho' = 0}$$

$\Box^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$: D'ALEMBERTIAN, WAVE OPERATOR

MULTIPLY ABOVE RESULT BY c^2, USE $p' = c^2 \rho'$ TO GET

$$\boxed{\Box^2 p' = 0}$$

– LET $\vec{\xi} = \nabla\times\vec{u}$ VORTICITY, TAKE CURLE OF THE MOMENTUM EQ : $\rho_0 \frac{\partial}{\partial t}(\nabla\times\vec{u}) + \underbrace{\nabla\times\nabla p'}_{=0} = \rho_0 \frac{\partial \vec{\xi}}{\partial t} = 0$

i.e. $\partial\vec{\xi}/\partial t = 0$. SINCE $\vec{\xi}(\vec{x},0) = 0 \Rightarrow \boxed{\vec{\xi} = 0}$ FOR ALL TIME!

THE GOVERNING EQUATIONS (CONT'D)

THE CONDITION $\vec{\xi} = 0$ IMPLIES THAT THERE IS A VELOCITY POTENTIAL $\phi(\vec{x}, t)$ SUCH THAT $\vec{u} = \nabla\phi$. THE MOMENTUM EQ. GIVES $\nabla\left[\rho_o \frac{\partial\phi}{\partial t} + p'\right] = 0 \Rightarrow \rho_o\frac{\partial\phi}{\partial t} + p' = f(t)$

WE CAN TAKE $f(t) = 0$ BECAUSE $\phi(\vec{x}, t)$ IS ALWAYS DETERMINED UPTO A FUNCTION OF TIME : IF ϕ IS A VELOCITY POTENTIAL $\Rightarrow$ $\phi_1(\vec{x}, t) = \phi(\vec{x}, t) + F(t)$ IS ALSO A VELOCITY POTENTIAL.

$\Rightarrow \boxed{p' = -\rho_o \frac{\partial\phi}{\partial t}}$, $\Box^2 p' = 0$ GIVES $\boxed{\Box^2\phi = 0}$ (*)

TAKE THE GRADIENT OF THIS TO GET $\boxed{\Box^2\vec{u} = 0}$

$\therefore$ $\boxed{p', \rho', \vec{u} \text{ SATISFY THE WAVE EQUATION}}$

— (*) $\Box^2\phi = 0$ PUTS A RESTRICTION ON $F(t)$, i.e. $F(t)$ CAN ONLY BE A LINEAR FUNCTION $at + b \Rightarrow p' = -\rho_o\partial\phi/\partial t$ CAN BE FOUND UPTO A CONSTANT $-\rho_o a$ WHICH IS SET TO ZERO SINCE $\langle p' \rangle = 0$, IN GENERAL.

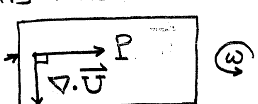
THE GOVERNING EQUATIONS (CONT'D)

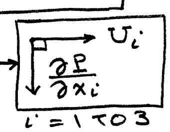
STEADY STATE CASE

OFTEN THE TIME DEPENDENCE IS PERIODIC. WE TAKE THE TIME DEPENDENCE AS $e^{i\omega t}$, $\omega = 2\pi f$ (ANGULAR FREQ.) rad/s. THEN $\partial/\partial t = i\omega$ AND MASS CONTINUITY AND MOMENTUM EQS. BECOME :

$$\boxed{\begin{aligned} i\omega P + \rho_o c^2 \nabla\cdot\vec{U} &= 0 \\ i\omega\rho_o\vec{U} + \nabla P &= 0 \end{aligned}}$$

PHASOR DIAGRAMS





$$p' = P(\vec{x})e^{i\omega t}, \quad \vec{u} = U(\vec{x})e^{i\omega t}$$

$\underbrace{\Box^2 p' = 0}_{\text{HYPERBOLIC}}$ BECOMES $\underbrace{\boxed{\nabla^2 P + \frac{\omega^2}{c^2}P = 0}}_{\text{ELLIPTIC}}$ HELMHOLTZ EQUATION

— NOTE THAT P AND U ARE COMPLEX QUANTITIES CALLED PHASORS. THE QUANTITIES OF INTEREST ARE $\mathrm{Re}[Pe^{i\omega t}]$ AND $\mathrm{Re}[\vec{U}e^{i\omega t}]$.

SOME SIMPLE RESULTS

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

LET $a(t) = A e^{i\omega t}$, $b(t) = B e^{i\omega t}$, A AND B COMPLEX, $T = \frac{1}{f}$, $\omega = 2\pi f$, THEN

$$\boxed{\frac{1}{T}\int_0^T \mathrm{Re}\, a(t)\cdot \mathrm{Re}\, b(t)\, dt = \frac{1}{2}\mathrm{Re}\,(AB^*)}$$

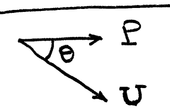
B^* = COMPLEX CONJUGATE OF B

EXAMPLE : $I = \frac{1}{T}\int_0^T \mathrm{Re}\, p\ \mathrm{Re}\, u\, dt$

$$= \frac{1}{2}\mathrm{Re}\,(P U^*) = \frac{1}{2}|P||U|\cos\theta$$

$p = P e^{i\omega t}$, $u = U e^{i\omega t}$

$|P|$ ABS. VALUE OF P



PHASOR DIAGRAM

THE DECIBEL SCALE FOR NOISE

THE RANGE OF THE PRESSURE VARIATION IN ACOUSTICS IS VERY WIDE. WE, THEREFORE, DEFINE A LOGARITHMIC SCALE CALLED THE DECIBEL SCALE :

$$\boxed{L_p = \text{SOUND PRESSURE LEVEL} = 20 \log_{10}\frac{P_{rms}}{P_{ref}}} \quad \text{DECIBELS}$$

$$P_{rms}^2 = \frac{1}{T}\int_0^T p^2(t)\,dt \quad , \quad P_{rms} = \frac{P}{\sqrt{2}} \quad \begin{cases}\text{SINUSOIDAL WAVE}\\ P = \text{AMP. OF WAVE}\end{cases}$$

$$\boxed{P_{ref} = 20\ \mu Pa} \quad , \quad Pa\ (\text{PASCAL}) = 1\ N/m^2$$

ATMOSPHERIC PRESSURE (SEA LEVEL) $\approx 10^5$ Pa

— WE HAVE $P_{rms} = 20 \times 10^{-6} \times 10^{L_p/20}$ Pa

$$\boxed{L_W = \text{SOUND POWER LEVEL} = 10 \log_{10}\frac{W}{W_{ref}}} \quad \text{DECIBELS}$$

$$W = \int_S p\, \vec{u}\cdot\vec{n}\, dS \quad (\text{FAR FIELD}) \ , \quad \boxed{W_{ref} = 10^{-12}\ \text{WATT}}$$

W IS THE SOUND POWER IN WATTS

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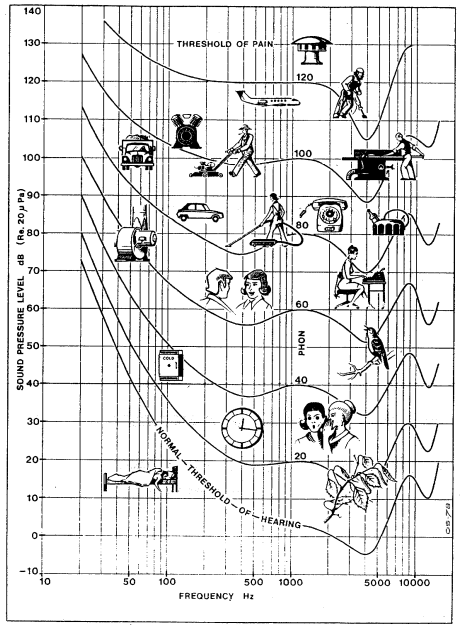
CONSTANT LOUDNESS CONTOURS



Typical sound pressure levels of common noise sources

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SOUND POWER RANGE

Power (Watts)	Power Level (dB re 10^{-12} W)
100 000 000	200
1 000 000	180
10 000	160
100	140
1	120
0,01	100
0,000,1	80
0,000,001	60
0,000,000,01	40
0,000,000,000,1	20
0,000,000,000,001	0

Saturn rocket	(50,000,000 W)
4 Jet Airliner	(50,000 W)
Large orchestra	(10 W)
Chipping hammer	(1 W)
Shouted speech	(0,001 W)
Conversational speech	(20×10^{-6} W)
Whisper	(10^{-9} W)

Sound Power output of some typical noise sources

CHANGE IN LEVEL dB	SUBJECTIVE EFFECT
3	just perceptible
5	clearly perceptible
10	twice as loud

THE USE OF COMPLEX NUMBERS IN ACOUSTICS

THE GOVERNING EQS. OF ACOUSTICS ARE LINEAR. LET US USE THE SUBSCRIPTS c, r AND i FOR COMPLEX, REAL AND IMAGINARY: $p_c = p_r + i\,p_i$, ETC. NOW IF p_i AND $\vec{u}_i$ SATISFY THE GOVERNING EQS. OF ACOUSTICS, LIKE p_r AND $\vec{u}_r$ $\Rightarrow$

$$\frac{\partial p_c}{\partial t} + \rho_0 \nabla\cdot\vec{u}_c = 0 \quad , \quad \rho_0 \frac{\partial \vec{u}_c}{\partial t} + \nabla p_c = 0$$

NOTE: WE DROP PRIME FROM p_c' AND $\vec{u}_c'$!

IT IS AT TIMES EASIER TO SOLVE THIS PROBLEM FOR p_c AND $\vec{u}_c$ AND TAKE THE REAL PARTS p_r AND $\vec{u}_r$ AS THE SOLUTION TO THE ORIGINAL PROBLEM. THIS METHOD IS CALLED COMPLEXIFICATION OF THE PROBLEM. THE MOST USEFUL FORM OF COMPLEXIFICATION IS BY TAKING $p_c = P\,e^{i\omega t}$, $\vec{u}_c = \vec{U}\,e^{i\omega t}$ WHERE P AND $\vec{U}$ ARE COMPLEX AMPLITUDES. THIS WAS INTRODUCED BY RAYLEIGH (THEORY OF SOUND). P AND $\vec{U}$ SATISFY

$$\boxed{\begin{aligned} i\omega P + \rho_0 c^2 \nabla\cdot\vec{U} &= 0 \\ i\rho_0\omega\vec{U} + \nabla P &= 0 \end{aligned}} \qquad \begin{aligned} P &= P(\vec{x}) \\ \vec{U} &= \vec{U}(\vec{x}) \end{aligned}$$

STEADY STATE CONDITION

SOME USEFUL RESULTS

$$P = |P|\,e^{i\phi} \quad , \quad p_c(\vec{x},t) = P\,e^{i\omega t} = |P|\,e^{i(\omega t+\phi)}$$

$$p \equiv p_r = \mathrm{Re}\; p_c = |P|\cos(\omega t+\phi) \qquad (\text{FROM: } e^{i\theta} = \cos\theta + i\sin\theta)$$

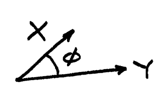
$|P|$: AMPLITUDE OF p , ϕ : PHASE OF p

$$p_{rms}^2 = \frac{1}{T}\int_0^T p^2\,dt = \frac{1}{2}|P|^2 \quad , \quad \boxed{p_{rms} = \frac{|P|}{\sqrt{2}}}$$

LET $x = X\,e^{i\omega t}$, $y = Y\,e^{i\omega t}$, THE TIME AVERAGE OF $x_r y_r$ IS:

$$\underbrace{\langle x_r, y_r\rangle}_{\text{ALSO WRITTEN } \overline{x_r y_r}} = \frac{1}{T}\int_0^T x_r\,y_r\,dt = \frac{1}{2}\mathrm{Re}(XY^*)$$

$$= \frac{1}{2}|X||Y|\cos\phi$$



PHASOR DIAGRAM

$\langle x_r, y_r\rangle = 0$ IF $\phi = 90^\circ$

$\langle x_r, y_r\rangle$ IS MAXIMUM FOR $\phi = 0$.

$$\boxed{\langle x_r, y_r\rangle = \frac{1}{2}\mathrm{Re}(XY^*)}$$

STEADY STATE CONDITION

WE HAVE S.S. CONDITION IF

i) THE BOUNDARY CONDITIONS ARE PERIODIC IN TIME

ii) WE ALLOW ALL TRANSIENTS TO DIE, i.e. $t \to \infty$

— A PERIODIC FUNCTION $f(t)$ CAN BE WRITTEN AS

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-in\omega t}\, dt$$

— $f(t)$, IN GENERAL IS REAL $\Rightarrow c_{-n} = c_n^*$ AND

$\mathrm{Re}(c_{-n} e^{-in\omega t}) = \mathrm{Re}(c_n e^{in\omega t}) \Rightarrow$ WE CAN COMPLEXIFY THE PROBLEM FOR $n > 0$ BUT USE BC $2c_n e^{in\omega t}$.

— BY SUPERPOSITION PRINCIPLE, WE CAN SUM THE EFFECTS OF ALL HARMONICS OVER $n = 1, 2, 3, \ldots$. IN GENERAL, $\langle f(t) \rangle = 0$ SO THAT $c_0 = 0$.

TIME DOMAIN VS FREQUENCY DOMAIN

FOR THE STEADY STATE CASE, BY USING $p = P e^{i\omega t}$, WE GET FROM $\Box^2 p = 0$, $\mathcal{H}P = \nabla^2 P + \frac{\omega^2}{c^2} P = 0$, $P = P(\vec{x})$.

IN GENERAL, USING FOURIER TRANSFORM IN TIME, $t \to \omega$

$$\hat{f}(\vec{x}, \omega) = \int_{-\infty}^{\infty} f(\vec{x}, t)\, e^{-i\omega t}\, dt$$

WE TRANSFORM $\Box^2 p = Q(\vec{x}, t)$ TO

HELMHOLTZ OPERATOR $\quad \mathcal{H}\hat{p}(\vec{x}, \omega) = \nabla^2 \hat{p} + \frac{\omega^2}{c^2} \hat{p} = -\hat{Q}(\vec{x}, \omega)$

THE BC'S ARE ALSO TRANSFORMED TO FUNCTIONS OF $(\vec{x}, \omega)$. WE THEN SOLVE FOR $\hat{p}(\vec{x}, \omega)$ FOR ALL $|\omega| < \infty$. FROM $\hat{p}$ WE FIND $p(\vec{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{p}(\vec{x}, \omega)\, e^{i\omega t}\, d\omega$

THIS IS FREQUENCY DOMAIN METHOD. IF WE SOLVE DIRECTLY FOR $p(\vec{x}, t)$ FROM $\Box^2 p = Q(\vec{x}, t) +$ BC'S, WE ARE WORKING IN TIME DOMAIN. FREQUENCY DOMAIN METHOD IS MOSTLY APPLIED TO STEADY STATE PROBLEMS.

TIME VS FREQUENCY DOMAIN (CONT'D)

TIME DOMAIN AND FREQUENCY DOMAIN APPROACHES ARE COMPLEMENTARY. NEITHER METHOD IS SUPERIOR TO THE OTHER. WE LEARN SOMETHING NEW FROM EACH METHOD. OFTEN THE METHOD ONE USES DEPENDS ON THE EXPERIENCE AND THE TRADITION IN THE FIELD.

THESE ARE SOME CHARACTERISTICS OF F.D. AND T.D. METHODS:

- MANY PROBLEMS OF ENGINEERING ARE STEADY STATE SO THAT F.D. METHOD IS AN OPTION,
- MANY SIMPLE SOURCE AND PROPAGATION MODELS ARE AVAILABLE USING F.D. METHOD,
- IN F.D. METHOD, BY USING $t \rightarrow \omega$, WE HAVE ESSENTIALLY A 3 DIMENSIONAL PROBLEM. SOME PEOPLE FEEL MORE COMFORTABLE TO WORK IN 3D THAN FOUR DIMENSIONAL T.D. METHOD. HOWEVER, BECAUSE OF THE PARTICULARLY SIMPLE GREEN'S FUNCTION OF THE WAVE OPERATOR, ONE CAN ALSO LEARN TO WORK JUST AS EASILY IN T.D.

TIME DOMAIN VS FREQUENCY DOMAIN (CONT'D)

- IT APPEARS THAT MORE ANALYTIC (CLOSED FORM) SOLUTIONS ARE AVAILABLE USING F.D. APPROACH. MOST OFTEN, THIS HAS BEEN ACHIEVED BY SOME APPROXIMATIONS. APPROXIMATIONS IN GEOMETRY, OBSERVER DISTANCE AND SOURCE MOTION MAY NOT BE ACCEPTABLE IN SOME AEROACOUSTIC PROBLEMS.
- FOR SOME PROBLEMS OF AEROACOUSTICS, e.g. HIGH SPEED HELICOPTER ROTOR NOISE AND PROPELLER NOISE PREDICTION, BOTH T.D. AND F.D. METHODS CAN BE TIME CONSUMING ON A COMPUTER. HOWEVER, EXECUTION TIME ON A COMPUTER IS VERY MUCH DEPENDENT ON THE SKILL AND EXPERIENCE OF CODE DEVELOPER. SO AFTER YOU SELECT A METHOD, SPEND A LOT OF TIME THINKING ABOUT ALGORITHMS YOU USE IN YOUR CODE. THE DIFFERENCE BETWEEN A GOOD AND A BAD ALGORITHM COULD BE A FACTOR OF 100 IN EXECUTION TIME ON A COMPUTER!
- T.D. ANALYSIS IS MORE RECENT IN AEROACOUSTICS. EXPERIENCE HERE IS LIMITED!

ENERGY RELATIONS (STATIONARY MEDIUM)

STARTING WITH MOMENTUM EQ. $\rho_0 \frac{\partial \vec{u}}{\partial t} + \nabla p = 0$, TAKE DOT PRODUCT OF THIS WITH $\vec{u}$:

$$\rho_0 \vec{u} \cdot \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla p = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 u^2\right) + \nabla \cdot (p\vec{u}) - p\nabla \cdot \vec{u} = 0 \quad (1)$$

WHERE $u = |\vec{u}(\vec{x},t)|$ VEC. NORM!

MASS CONT. EQ : $\frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0 \Rightarrow p\nabla \cdot \vec{u} = -\frac{1}{\rho_0 c^2} p \frac{\partial p}{\partial t}$

$$= -\frac{1}{2\rho_0 c^2}\frac{\partial p^2}{\partial t}$$

SUBSTITUTE IN (1) TO GET

$$\frac{\partial}{\partial t}\underbrace{\left[\frac{1}{2}\rho_0 u^2 + \frac{1}{2\rho_0 c^2} p^2\right]}_{e\,:\,\text{ACOUSTIC ENERGY DENSITY}} + \nabla \cdot \underbrace{(p\vec{u})}_{\vec{I}\,:\,\text{ACOUSTIC INTENSITY VECTOR}} = 0$$

$$\boxed{\frac{\partial e}{\partial t} + \nabla \cdot \vec{I} = 0}$$

UNITS OF e JOULES
UNITS OF $\vec{I}$ W/m^2

NOTE : $\vec{I}$ IS ALWAYS A REAL QUANTITY IN THIS COURSE!

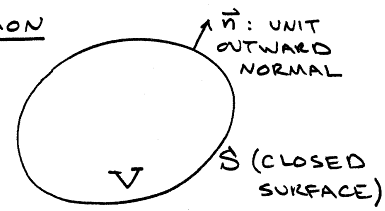
ENERGY RELATIONS (CONT'D)

$\frac{1}{2}\rho_0 u^2$ IN e IS THE INSTANTANEOUS KINETIC ENERGY/UNIT VOL.

$\frac{p^2}{2\rho_0 c^2}$ IN e IS THE INSTANTANEOUS POTENTIAL ENERGY STORED BY COMPRESSION/UNIT VOL.

ANOTHER VIEW OF ENERGY RELATION

$$\underbrace{\frac{\partial}{\partial t}\int_V e\, dv}_{\text{RATE OF INCREASE OF ENERGY IN } V} = -\underbrace{\int_S \vec{I}\cdot\vec{n}\, dS}_{\text{RATE OF ENERGY LEAVING } S}$$



$$\langle e \rangle = \frac{1}{4}\rho_0 |\vec{U}|^2 + \frac{1}{4\rho_0 c^2}|P|^2 \quad \text{FOR HARMONIC WAVES}$$

WHERE P AND $\vec{U}$ ARE COMPLEX AMPLITUDES OF p & $\vec{u}$

$$\langle \vec{I} \rangle = \frac{1}{2}\mathrm{Re}(P\vec{U}^*)$$

NOTE : $|\vec{U}|^2 = \underbrace{\vec{U}\cdot\vec{U}^*}_{\text{VEC. DOT PRODUCT}} = |U_1|^2 + |U_2|^2 + |U_3|^2$, U_1, U_2, U_3 COMPLEX.

ENERGY RELATIONS (CONT'D)

CONSIDER AGAIN STEADY STATE CASE : $\frac{\partial e}{\partial t} + \nabla \cdot \vec{I} = 0$

$$\boxed{\left\langle \frac{\partial e}{\partial t} \right\rangle = \frac{1}{T}\int_0^T \frac{\partial e}{\partial t}\, dt = \frac{e(T)-e(0)}{T} = 0 \;!}$$

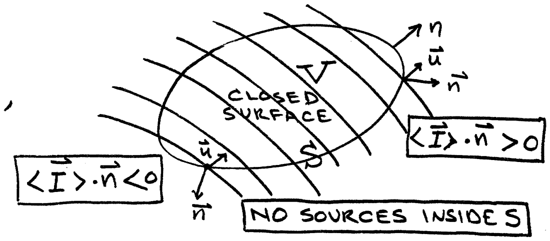
i.e. IN STEADY STATE CASE NO ACCUMULATION OF ACOUSTIC ENERGY ON AVERAGE IS POSSIBLE ! $\Rightarrow \langle \nabla \cdot \vec{I} \rangle = 0$

$$\Rightarrow \boxed{\left\langle \int_S \vec{I}\cdot\vec{n}\, dS \right\rangle = \int_S \langle \vec{I} \rangle \cdot \vec{n}\, dS = 0} \quad (*)$$

i.e. ON AVERAGE, THE ENERGY ENTERING A CLOSED SURFACE IS EQUAL TO ENERGY LEAVING THE SURFACE
∴ NO ACCUMULATION OF ENERGY IN VOLUME V WITHIN THE SURFACE !

(*) NOTE THAT, IN GENERAL,

$$\boxed{\langle \vec{I} \rangle \neq 0}$$



SIMPLE MODELS OF WAVES

i) PLANE WAVES

$$p(\vec{x},t) = A\, e^{i(\omega t - \vec{k}\cdot\vec{x})} = \underline{P}(\vec{x})\, e^{i\omega t}$$

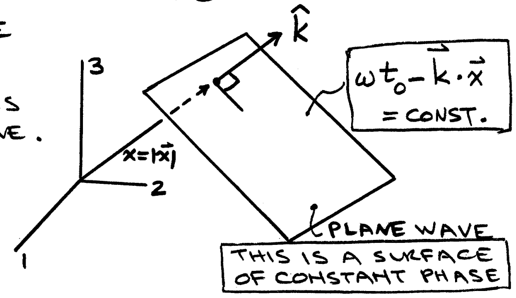
$\underline{P}(\vec{x}) = A\, e^{-i\vec{k}\cdot\vec{x}}$. ASSUME $\vec{k} = (k_1, k_2, k_3)$ REAL.

TO SATISFY $\nabla^2 \underline{P} + \frac{\omega^2}{c^2}\underline{P} = 0$, WE MUST HAVE

$\frac{\omega^2}{c^2} = k_1^2 + k_2^2 + k_3^2 = |\vec{k}|^2$ OR $\boxed{k = \frac{\omega}{c}}$. THE DIRECTION OF $\vec{k}$ IS ARBITRARY. LET $\hat{\vec{k}} = \frac{\vec{k}}{k}$ AND $\vec{x} = \hat{\vec{k}}\, x$

$$\Rightarrow p(\vec{x},t) = A\, e^{i(\omega t - kx)} = A\, e^{-ik(x-ct)}$$

i.e. WE HAVE A PLANE WAVE TRAVELLING WITH SPEED c NORMAL TO THE PLANE. $\hat{\vec{k}}$ IS THE UNIT NORMAL TO THE PLANE.
THE PLANE WAVE WITH $\vec{k}$ REAL IS A PROPAGATING WAVE WITHOUT DECAY.



SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

FROM $i\rho_0\omega\vec{U} + \nabla P = 0$, WE GET

$$\boxed{\vec{U} = \frac{P}{\rho_0 c}\hat{\vec{k}}} \Rightarrow \boxed{\langle\vec{I}\rangle = \frac{1}{2\rho_0 c}|P|^2\hat{\vec{k}} = \frac{p_{rms}^2}{\rho_0 c}\hat{\vec{k}}}$$

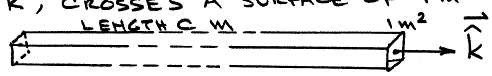
THIS MEANS THAT $\vec{U} \parallel \hat{\vec{k}}$ AND COMPONENTS OF $\vec{U}$ VIEWED AS PHASORS ARE IN PHASE WITH P $\boxed{\vec{U_i} \longrightarrow P}$

ACOUSTIC ENERGY DENSITY

$$\langle e\rangle = \underbrace{\frac{1}{4}\rho_0|\vec{U}|^2}_{=\frac{1}{4\rho_0 c}|P|^2 \ (\text{USING } \vec{U} = \frac{P}{\rho_0 c}\hat{\vec{k}})} + \frac{1}{4\rho_0 c^2}|P|^2 = \frac{|P|^2}{2\rho_0 c^2} = \frac{p_{rms}^2}{\rho_0 c^2} = \frac{|\langle\vec{I}\rangle|}{c}$$

WE HAVE EQUIPARTITION OF KINETIC AND POTENTIAL ENERGY DENSITIES.

$\boxed{|\langle\vec{I}\rangle| = c\langle e\rangle}$ MEANS THAT THE ACOUSTIC ENERGY IN A CYLINDER OF UNIT AREA ($1\,m^2$) AND LENGTH c, WITH AXIS PARALLEL TO $\hat{\vec{k}}$, CROSSES A SURFACE OF $1\,m^2$ AND $\perp$ TO $\hat{\vec{k}}$!



SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

ii) EVANESCENT WAVES $\quad p = A\,e^{i(\omega t - \vec{k}\cdot\vec{x})}$

THESE ARE ALSO PLANE WAVES BUT $\vec{k} = \vec{k}_R + i\,\vec{k}_i$ WHERE $\vec{k}_R$ AND $\vec{k}_i$ ARE VECTORS WITH REAL COMPONENTS. TO SATISFY HELMHOLTZ EQ., WE MUST HAVE

$$k_1^2 + k_2^2 + k_3^2 = \underbrace{|\vec{k}_r|^2 - |\vec{k}_i|^2}_{\text{REAL}} + \underbrace{2i\vec{k}_r\cdot\vec{k}_i}_{\text{IMAGINARY}} = \underbrace{\frac{\omega^2}{c^2}}_{\text{REAL}}$$

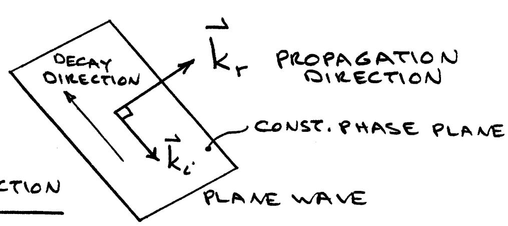
$$\Rightarrow \vec{k}_r\cdot\vec{k}_i = 0 \quad i.e.\ \vec{k}_r \perp \vec{k}_i$$

$$|\vec{k}_r|^2 - |\vec{k}_i|^2 = \frac{\omega^2}{c^2} > 0, \quad i.e.\ \boxed{|\vec{k}_r|^2 = \frac{\omega^2}{c^2} + |\vec{k}_i|^2}$$

$$P = A\,e^{-i\vec{k}\cdot\vec{x}} = A\,e^{\vec{k}_i\cdot\vec{x}}\,e^{-i\vec{k}_r\cdot\vec{x}}$$

$$|P| = |A|\,e^{\vec{k}_i\cdot\vec{x}}$$

DECAYS IN $-\vec{k}_i$ DIRECTION



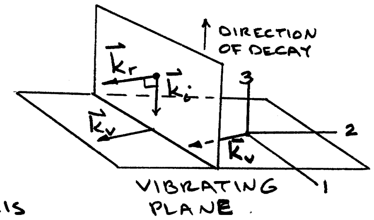
SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

EXAMPLE OF EVANESCENT WAVES

WAVES IN THE VICINITY OF A VIBRATING PLANE FOR WHICH WE HAVE

$k_1^2 + k_2^2 > \frac{\omega^2}{c^2}$. HERE $(k_1, k_2, 0) = \vec{k}_v = \vec{k}_r$

WHERE $\vec{k}_v$ THE WAVE NUMBER VECTOR OF THE VIBRATION. $\vec{k}_i$ IS PARALLEL TO x_3-AXIS



– PROPAGATION SPEED OF EVANESCENT WAVE $= \frac{\omega}{|\vec{k}_r|} < c$

i.e. EVAN. WAVES TRAVEL AT SUBSONIC SPEED IN THE DIRECTION OF $\vec{k}_r$.

– $\vec{U} = \frac{P}{\rho_0 \omega} \vec{k} = \frac{P}{\rho_0 \omega}(\vec{k}_r + i\vec{k}_i)$, P AND $\vec{U}$ ARE NO LONGER IN PHASE , $\vec{U}$ ALSO DECAYS IN THE DIRECTION $-\vec{k}_i$ (SEE NEXT SLIDE)

– $\langle \vec{I} \rangle = \frac{1}{2} Re(P \vec{U}^*) = \frac{|P|^2}{2\rho_0 \omega} \vec{k}_r$

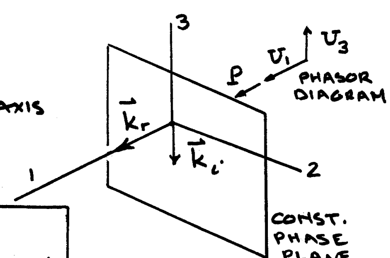
ENERGY FLOWS IN THE DIRECTION OF $\vec{k}_r$ ONLY !

BECAUSE OF THE DECAY, ENERGY DOES NOT PROPAGATE TO INFINITY BUT STAYS IN THE VICINITY OF THE VIBRATING PLANE.

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

EVANESCENT WAVES (CONT'D)

NOW LET US DEFINE x_1-AXIS ALONG $\vec{k}_r$, x_3-AXIS ALONG $-\vec{k}_i$ AND x_2-AXIS IN SUCH A WAY THAT THE $\vec{x}$-FRAME IS RIGHT HANDED. THEN



$\vec{k}_r$ & $\vec{k}_i$ REAL VECTORS

$$\boxed{U_1 = \frac{P}{\rho_0 \omega} |\vec{k}_r| \qquad \xrightarrow{U_1} P \quad \text{PHASOR DIAGRAM}}$$

$$\boxed{\langle I_1 \rangle = \frac{|P|^2}{2\rho_0 \omega} |\vec{k}_r| \quad , \langle I_2 \rangle = \langle I_3 \rangle = 0}$$

$$\boxed{U_3 = \frac{iP}{\rho_0 \omega} |\vec{k}_i| \qquad \text{PHASOR DIAGRAM}} \Rightarrow \langle I_3 \rangle = \frac{1}{2} Re(P U_3^*) = 0$$

$$\boxed{U_2 = 0 \ \text{NO VELOCITY COMPONENT IN } x_2 \text{ DIRECTION}} \Rightarrow \langle I_2 \rangle = 0$$

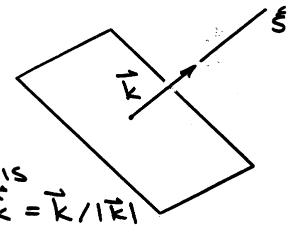
ENERGY FLOWS IN x_1 DIRECTION ONLY. $\boxed{|P| = |A| e^{-|\vec{k}_i| x_3}}$

i.e. THE AMPLITUDE $|P|$ DECAYS IN x_3 DIRECTION.

– THE ENERGY DENSITY DECAYS IN x_3 DIRECTION ALSO.

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

PHASE VELOCITY



$\theta = \vec{k}\cdot\vec{x} - \omega t$ IS CALLED THE PHASE OF THE PLANE WAVE

$p(\vec{x},t) = A\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

FOR THIS PLANE WAVE, WE DEFINE AXIS ξ ALONG $\vec{k}$, TAKING $\vec{\xi} = \xi\hat{k}$, $\hat{k} = \vec{k}/|\vec{k}|$

$\theta = \vec{k}\cdot\hat{k}\xi - \omega t = |\vec{k}|\xi - \omega t$. NOW WE CAN STUDY THE VELOCITY OF THE SURFACE OF CONSTANT PHASE $\theta =$ CONST. ALONG ξ-AXIS : $\dot{\theta} = 0 = |\vec{k}|\dot{\xi} - \omega$, $\dot{\xi} = \omega/|\vec{k}|$. FOR A PROPAGATING PLANE WAVE, WE HAVE FROM HELMHOLTZ EQ. $|\vec{k}|^2 = \omega^2/c^2$ OR $\boxed{\dot{\xi} = \omega/|\vec{k}| = c}$ SPEED OF SOUND

FOR AN EVANESCENT WAVE : $p(\vec{x},t) = A\,e^{\vec{k}_i\cdot\vec{x}}\cdot e^{i(\vec{k}_r\cdot\vec{x}-\omega t)}$

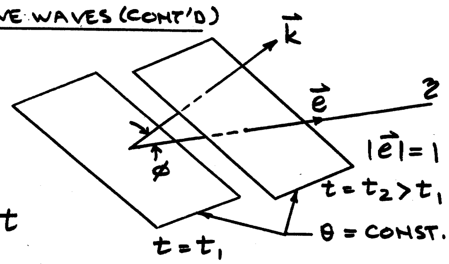
$\theta = \vec{k}_r\cdot\vec{x} - \omega t = \vec{k}_r\cdot\hat{k}_r\xi - \omega t = |\vec{k}_r|\xi - \omega t$

$\dot{\theta} = 0 = |\vec{k}_r|\dot{\xi} - \omega$, $\boxed{\dot{\xi} = \dfrac{\omega}{|\vec{k}_r|}}$, $|\vec{k}_r|^2 = \dfrac{\omega^2}{c^2} + |\vec{k}_i|^2 > \dfrac{\omega^2}{c^2}$

$\Rightarrow \boxed{\dot{\xi} < c}$ i.e. THE PHASE VELOCITY OF AN EVANESCENT WAVE IS SUBSONIC

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

TRACE VELOCITY



NOW DEFINE A FIXED AXIS η BY SPECIFYING THE UNIT VECTOR $\vec{e}$ ALONG IT. WE HAVE

$\theta = \vec{k}\cdot\vec{x} - \omega t = \vec{k}\cdot\vec{e}\,\eta - \omega t$
$= (|\vec{k}|\cos\phi)\eta - \omega t$

$\dot{\theta} = 0 = (|\vec{k}|\cos\phi)\dot{\eta} - \omega$, $\boxed{\dot{\eta} = \dfrac{\omega}{|\vec{k}|\cos\phi} > \dfrac{\omega}{|\vec{k}|}}$

THIS IS THE TRACE VELOCITY OF THE PLANE WAVE $p = A\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ ALONG η-AXIS. $\boxed{\dot{\eta} = \dfrac{c}{\cos\phi}}$. NOTE THAT WE ARE FOLLOWING THE SAME PLANE WAVE INTERSECTING η-AXIS AND $\dot{\eta}$ GIVES USE THE SPEED OF THE POINT OF INTERSECTION FOR AN OBSERVER FIXED TO THE UNDISTURBED MEDIUM. NOTE ALSO THAT $\cos\phi$ IS THE DIRECTION COSINE OF $\vec{k}$ WRT η-AXIS

TRACE SPEED OF $p = A\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ ALONG x_1, x_2 AND x_3-AXES

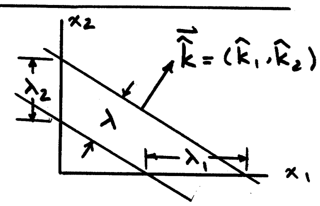
$\boxed{\dot{x}_i = \dfrac{c}{\hat{k}_i}}$, $\hat{k} = \vec{k}/|\vec{k}|$ UNIT VEC. ALONG $\vec{k}$

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

RELATIONS BETWEEN WAVELENGTH AND WAVE NUMBER

$f\lambda = c$ SPEED OF SOUND

$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{\lambda f} = \frac{\omega}{c}$ WAVE NO.



f = THE NO. OF PEAKS/SEC. PASSING OVER A POINT $\Rightarrow$ f DOES NOT CHANGE ANYWHERE FOR AN OBSERVER FIXED TO THE UNDISTURBED MEDIUM.

$t = t_1$ FIXED

FROM THE FIGURE

$$\lambda_1 = \frac{\lambda}{\hat{k}_1}, \quad \frac{2\pi}{\lambda_1} = k_1 = \frac{2\pi f}{\lambda_1 f} = \frac{\omega}{(\lambda f)/\hat{k}_1} = \frac{\omega}{c/\hat{k}_1} = \frac{\omega}{\dot{x}_1}$$

WHERE $\dot{x}_1$ IS TRACE VELOCITY ALONG x_1-AXIS. WE HAVE

$$\boxed{\lambda_i = \frac{\lambda}{\hat{k}_i}} \qquad \boxed{k_i = \frac{2\pi}{\lambda_i} = \frac{\omega}{\dot{x}_i}} \qquad \vec{k} = (k_1, k_2, k_3)$$

THESE RELATIONS ARE FOR AN OBSERVER FIXED TO THE MEDIUM.

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

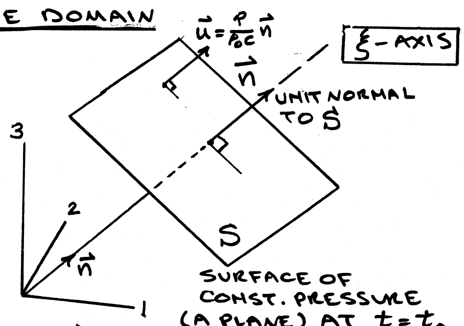
PROPAGATING PLANE WAVES IN TIME DOMAIN

$p = f(\vec{n}\cdot\vec{x} - ct)$

$\vec{n}$ PROPAGATION DIRECTION

$|\vec{n}| = 1$, f ARBITRARY FUNCTION

LET $\xi = \vec{n}\cdot\vec{x} \Rightarrow \xi - ct = \vec{n}\cdot\vec{x} - ct$ = CONST. IS A PLANE SURFACE



– FROM MOM. EQ.

$$\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p = -f'(\vec{n}\cdot\vec{x} - ct)\,\vec{n}$$

$\vec{u} = \frac{\vec{n}}{\rho_0 c} f(\vec{n}\cdot\vec{x} - ct) + \vec{g}(\vec{x})$, g ARBITRARY. SINCE $\vec{u} = 0$ WHEN $p = 0 \Rightarrow \vec{g}(\vec{x}) = 0$.

$$\boxed{\begin{aligned}\vec{u} &= \frac{\vec{n}}{\rho_0 c} f(\vec{n}\cdot\vec{x} - ct) \\ &= \frac{p}{\rho_0 c}\,\vec{n}\end{aligned}}$$

– NOTE THAT BOTH p AND u ARE FUNCTIONS OF $\xi - ct$, $\xi = \vec{x}\cdot\vec{n}$ (SEE ABOVE FIG.)

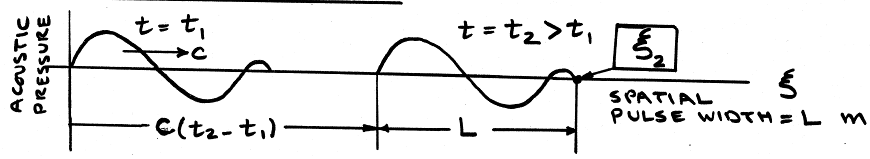
SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

PROPAGATING PLANE WAVES IN TIME DOMAIN (CONT'D)

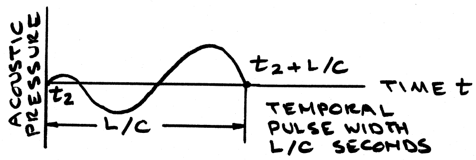
THE INSTANTANEOUS ACOUSTIC INTENSITY $\vec{I}(\vec{x},t)$ IS

$$\boxed{\vec{I} = \frac{p^2(\vec{x},t)}{\rho_0 c}\,\vec{n}} \Rightarrow \boxed{\langle \vec{I} \rangle = \frac{p^2_{rms}}{\rho_0 c}\,\vec{n}}$$

– SOMETHING TO REMEMBER



THE SPATIAL DISTRIBUTION OF A PRESSURE PULSE AT TWO TIMES t_1 AND $t_2 > t_1$



ACOUSTIC PRESSURE SIGNAL MEASURED BY A MICROPHONE AT THE POINT ξ_2

SIMPLE MODELS OF WAVES – PLANE WAVES (CONT'D)

PARTICLE DISPLACEMENT $\vec{d}$: $\vec{u} = \frac{\partial \vec{d}}{\partial t}$, $\vec{d} = \vec{D}(\vec{x})\,e^{i\omega t}$

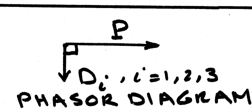
FROM MOMENTUM EQ., WE GET FOR STEADY STATE

$$\boxed{\vec{D}(\vec{x}) = \frac{1}{\rho_0 \omega^2}\nabla \underline{P}}$$

FOR PLANE WAVES $\underline{P} = A\,e^{-i\vec{k}\cdot\vec{x}}$, $\nabla \underline{P} = -i\,\underline{P}\,\vec{k}$, $|\vec{k}| = \frac{\omega}{c}$

$$\vec{D}(\vec{x}) = -\frac{i\underline{P}}{\rho_0 c \omega}\hat{k}$$

$$|\vec{D}| = \frac{|\underline{P}|}{\rho_0 c \omega} = \frac{|\vec{U}|}{\omega}$$



$$\hat{k} = \frac{\vec{k}}{|\vec{k}|}$$

ORDER OF MAGNITUDE OF ACOUSTIC QUANTITIES (PLANE WAVES)

FREQUENCY	LEVEL dB re: 20 μPa	$\lvert P \rvert$ pa	$\lvert \vec{U} \rvert$ m/s	$\lvert \vec{D} \rvert$ m
1 kHz	60	0.028	6.88×10^{-5}	1.1×10^{-8}
1 kHz	120 TRESHOLD OF PAIN	28.2	6.88×10^{-2}	1.1×10^{-5}
10 kHz	60	0.028	6.88×10^{-5}	1.1×10^{-9}
10 kHz	120	28.2	6.88×10^{-2}	1.1×10^{-6}

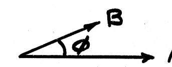
NOTE VERY SMALL VALUES OF $|\vec{D}|$!

COMBINATION OF ACOUSTIC PRESSURES IN STEADY STATE

WE NEED TO EVALUATE p^2_{rms} TO FIND THE DECIBEL LEVEL OF SOUND.

i) TWO WAVES OF THE SAME FREQUENCY

$$p_c = A e^{i\omega t} + B e^{i\omega t} = (A+B)e^{i\omega t}$$



PHASOR DIAGRAM

$$\boxed{\langle p_r^2 \rangle = \frac{1}{2}|P|^2 = \frac{1}{2}|A+B|^2 = \frac{1}{2}\left[|A|^2 + |B|^2 + 2|A||B|\cos\phi\right]}$$

$|P|$ IS MAXIMUM IF $\phi = 0$, $i.e.$ $B = \alpha A$, $\alpha > 0$, α REAL

$|P|$ IS MINIMUM IF $\phi = \pi$, $i.e.$ $B = \alpha A$, $\alpha < 0$, α REAL

$|P| = 0$ IF $B = -A$, $i.e.$ $|A| = |B|$ AND $\phi = \pi$

THIS IS THE IDEA BEHIND ANTI-NOISE.

ii) $p_c = A_1 e^{i\omega t} + A_2 e^{2i\omega t} + \cdots + A_n e^{ni\omega t} = p_r(\vec{x},t) + i p_i(\vec{x},t)$

$\boxed{\langle p_r^2 \rangle = \frac{1}{2}\sum_{i=1}^{n} |A_i|^2}$. THE REASON IS AS FOLLOWS:

$$\langle p_r^2 \rangle = \frac{1}{4}\langle (P + P_c^*)^2 \rangle = \frac{1}{4}\left[\langle P_c^2 \rangle + \langle P_c^{*2} \rangle + 2\langle P_c P_c^* \rangle\right]$$

$$= \frac{1}{2}\langle |P_c|^2 \rangle = \frac{1}{2}\sum_{i=1}^{n} |A_i|^2 + \frac{1}{2}\sum_{\substack{j,k=1 \\ j\neq k}}^{n} \underbrace{\langle A_j A_k^* e^{i(j-k)\omega t} \rangle}_{=0}$$

{ WE CAN SHOW $\langle P_c^2 \rangle = \langle P_c^{*2} \rangle = 0$.
THIS FOLLOWS FROM $\langle e^{im\omega t} \rangle = 0$, $m \neq 0$.

COMBINATION OF ACOUSTIC PRESSURES IN STEADY STATE (CONT'D)

WE SUMMARIZE THE ABOVE RESULTS, GIVING A NEW RESULT(ii):

i) FOR TWO WAVES OF IDENTICAL FREQUENCY, $\langle p_r^2 \rangle$ DEPENDS ON THE PHASOR ADDITION OF COMPLEX AMPLITUDES WHEN THE PHASE BETWEEN THE TWO WAVES IS KNOWN.

ii) FROM TIME SERIES ANALYSIS, IT CAN BE SHOWN THAT WHEN THE PHASE BETWEEN TWO WAVES OF IDENTICAL FREQUENCY IS RANDOM, THEN $\langle p_r^2 \rangle = \frac{1}{2}(|A_1|^2 + |A_2|^2)$, $i.e.$ OF THE TWO WAVES ADD UP.

iii) FOR WAVES OF DIFFERENT FREQUENCIES, THE MEAN SQ. OF THE RESULTING PRESSURE IS THE SUM OF THE MEAN SQ. OF THE COMPONENTS:

$$p^2_{rms} = \langle p_r^2 \rangle = \frac{1}{2}\sum_{i=1}^{n} |A_i|^2 = \sum_{i=1}^{n} (p^2_{rms})_i.$$

$$\Rightarrow \langle \vec{I} \rangle = \frac{\langle p_r^2 \rangle}{\rho_0 c}\vec{n} = \sum_{i=1}^{n} \langle \vec{I}_i \rangle$$ PLANE WAVE PROPAGATING IN THE DIRECTION $\vec{n}$.

LEC. 3/6
3/6

SIMPLE MODELS OF WAVES - SPHERICAL WAVES

TIME DOMAIN : p SPHERICALLY SYMMETRIC IF $p(\vec{x},t) = p(r,t)$

r DISTANCE FROM ORIGIN. WE CAN SHOW THAT IN THIS CASE

$$\square^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp) = 0 \Rightarrow$$

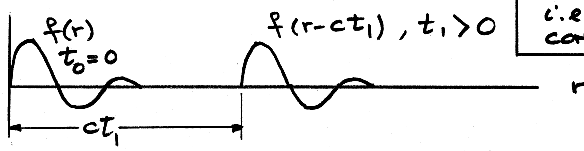
$$\boxed{\frac{1}{c^2}\frac{\partial^2 (rp)}{\partial t^2} - \frac{\partial^2 (rp)}{\partial r^2} = 0} \Rightarrow \boxed{p(r,t) = \underbrace{\frac{f(r-ct)}{r}}_{\text{OUTGOING WAVE}} + \underbrace{\frac{g(r+ct)}{r}}_{\text{INCOMING WAVE}}}$$

f AND g ARBITRARY FUNCTIONS

— WE WILL CONCENTRATE ON OUTGOING WAVES.

— INTERPRETATION OF $f(r-ct)$

NOTE : FOR r = CONST. WE HAVE p = CONST. i.e. SURFACES OF CONST. p ARE SPHERES



— THE r IN THE DENOMINATOR OF p IS THE SPHERICAL ATTENUATION. A FINITE PRESSURE ATTENUATES TO ZERO AS $r \to \infty$.

LEC. 3/7
3/7

SIMPLE MODELS OF WAVES - SPHERICAL WAVES (CONT'D)

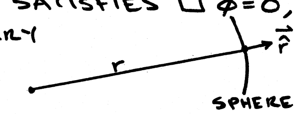
TIME DOMAIN (CONT'D)

WE NOTE THAT SINCE VEL. POTENTIAL ALSO SATISFIES $\square^2 \phi = 0$, WE HAVE $\phi(r,t) = \frac{f(r-ct)}{r}$, f ARBITRARY

$$p(r,t) = -\rho_0 \frac{\partial \phi}{\partial t} = \frac{\rho_0 c}{r} f'(r-ct)$$

$$\vec{u}(r,t) = \nabla \phi = \underbrace{\frac{\hat{\vec{r}}}{r} f'(r-ct)}_{\text{FAR FIELD}} - \underbrace{\frac{\hat{\vec{r}}}{r^2} f(r-ct)}_{\text{NEAR FIELD}}$$

$$\hat{\vec{r}} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{r} \quad (*)$$



WE NOTE THAT IN THE FAR FIELD $\boxed{\vec{u} = \frac{p}{\rho_0 c}\hat{\vec{r}}}$ WE HAVE ONLY RADIAL COMPONENT

THIS RELATION IS LOCALLY LIKE PLANE WAVE PROPAGATING IN THE DIRECTION $\hat{\vec{r}}$. BUT NOTE THAT IN THE FAR FIELD BOTH p AND $\vec{u}$ FALL OFF AS $1/r$, i.e. BOTH HAVE SPHERICAL ATTENUATION.

— WE CAN LEARN MORE FROM FREQUENCY ANALYSIS. IN PARTICULAR THE ENERGY DENSITY AND ACOUSTIC INTENSITY RELATIONS CAN BE STUDIED MORE EASILY IN FREQ. DOMAIN.

(*) — NOTE $|\hat{\vec{r}}| = 1$, i.e. $\hat{\vec{r}}$ IS UNIT OUTWARD NORMAL TO SPHERE r = CONST.

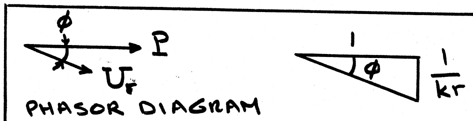
SIMPLE MODELS OF WAVES – SPHERICAL WAVES (CONT'D)

FREQUENCY DOMAIN : $p(r,t) = \frac{A}{r} e^{i(\omega t - kr)}$

$P(r) = \frac{A}{r} e^{-ikr}$, $k = \frac{\omega}{c}$ WAVE NUMBER , $k = \frac{2\pi}{\lambda} =$

λ = WAVE LENGTH m , FROM MOMENTUM EQ. WE GET

$$\boxed{\begin{aligned} \vec{U} &= \frac{P}{\rho_o c}\left(1 - \frac{i}{kr}\right)\hat{r} \\ &= U_r \hat{r} \\ U_r &= \frac{P}{\rho_o c}\left(1 - \frac{i}{kr}\right) \end{aligned}}$$



PHASOR DIAGRAM

TO GET THIS RESULT, REMEMBER THAT

$\nabla f(r) = f'(r) \nabla r$, $\nabla r = \frac{\vec{x}}{|\vec{x}|} = \hat{r}$

FOR OUTGOING WAVES , U_r ALWAYS LAGS P BY $\phi = \tan^{-1}(1/kr)$.

$\boxed{\phi \to \frac{\pi}{2} \text{ IF } kr \to 0 \ , \ \phi \to 0 \text{ IF } kr \to \infty}$

$kr \to 0$ IF $k \to 0$ (i.e. $\omega \to 0$) FOR A FIXED r

$kr \to 0$ IF $r \to 0$ FOR A FIXED k (OR ω)

$kr \to \infty$ IF $k \to \infty$ (i.e. $\omega \to \infty$ HIGH FREQ.) FOR A FIXED r

$kr \to \infty$ IF $r \to \infty$ FOR ALL FINITE k

SIMPLE MODELS OF WAVES – SPHERICAL WAVES (CONT'D)

AVERAGE ACOUSTIC INTENSITY

$\boxed{\langle \vec{I} \rangle = \frac{1}{2} Re\,(P \vec{U}^*) = \frac{|A|^2 \hat{r}}{2\rho_o c\, r^2}}$ VARIES AS r^{-2}

ON A SPHERE OF RADIUS r : $\int_S \langle \vec{I} \rangle \cdot \hat{r}\, d\vec{S} = \frac{2\pi |A|^2}{\rho_o c}$ = CONST.

AS EXPECTED .

ACOUSTIC ENERGY DENSITY (AVE.) $\langle e \rangle = \langle \frac{1}{2} \rho_o u_r^2 \rangle + \langle \frac{p_r^2}{2\rho_o c^2} \rangle$
K.E. DENS. P.E. DENS.

$\boxed{\langle \frac{1}{2} \rho_o u_r^2 \rangle = \frac{|A|^2}{4 \rho_o c^2 r^2}\left(1 + \frac{1}{k^2 r^2}\right)}$

$\boxed{\langle \frac{p_r^2}{2\rho_o c^2} \rangle = \frac{|A|^2}{4\rho_o c^2 r^2}}$ $\Rightarrow$ $\boxed{\langle \frac{1}{2} \rho_o u_r^2 \rangle = \left(1 + \frac{1}{k^2 r^2}\right) \langle \frac{p_r^2}{2\rho_o c^2} \rangle}$

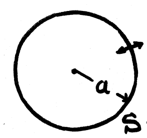
- AVE. POT. ENERGY DENSITY VARIES AS r^{-2}
- AVE. K.E. DENSITY = AVE P.E. DENSITY AS $kr \to \infty$
- IF $kr \ll 1 \Rightarrow$ AVE. K.E. DENSITY $\gg$ AVE. P.E. DENSITY
  THIS MAKES SENSE BECAUSE $\phi \to \pi/2$ AND LARGE VELOCITY $|U_r|$ IS NEEDED TO SEND OUT ANY ACOUSTIC ENERGY.

WE WILL LEARN MORE FROM THE FOLLOWING PROBLEM.

SIMPLE MODELS OF WAVES – SPHERICAL WAVES (CONT'D)

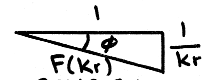
A PULSATING SPHERE: RADIUS a

LET THE RADIAL VELOCITY AMPLITUDE BE U_0 (REAL) SUCH THAT THE DISPLACEMENT AMPLITUDE $|D_0| = \frac{U_0}{\omega} \ll a$.



LET $\boxed{F(kr) = 1 - \frac{i}{kr}}$, THEN THE RESULTS OF THE PREVIOUS SLIDE CAN BE USED TO SHOW THAT

$$\boxed{\begin{aligned} p(r) &= \frac{\rho_0 c\, a\, U_0}{r\, F(ka)}\, e^{-ik(r-a)} \\ \vec{U}(r) &= \frac{a\, F(kr)}{r\, F(ka)}\, U_0\, e^{-ik(r-a)}\, \hat{r} \end{aligned}}$$



$$\boxed{\langle \vec{I} \rangle = \frac{\rho_0 c\, a^2 U_0^2}{2 r^2 |F(ka)|^2}\, \hat{r}} \Rightarrow \boxed{\langle W \rangle = \frac{\rho_0 c\, S\, U_0^2}{2 |F(ka)|^2}}$$

MEAN RADIATED ACOUSTIC POWER

$$\boxed{\begin{aligned} \langle W \rangle &\approx \tfrac{1}{2} \rho_0 c (ka)^2 U_0^2 S \quad \text{IF } ka \ll 1 \\ \langle W \rangle &\approx \tfrac{1}{2} \rho_0 c\, U_0^2 S \quad \text{IF } ka \gg 1 \end{aligned}}$$

$\leftarrow \phi \approx \pi/2$

$\leftarrow \phi \approx 0$

THUS $\langle W \rangle$ THE TOTAL AVE. ACOUSTIC POWER RADIATED IS A FUNCTION OF ka. FOR SMALL $ka (\ll 1)$, THE PULSATING SPHERE IS A VERY INEFFICIENT RADIATOR. $\langle W \rangle_{MAX} = \frac{1}{2} \rho_0 c\, U_0^2 S$

SIMPLE MODELS OF WAVES – SPHERICAL WAVES (CONT'D)

STATIONARY MONOPOLE: ASSUME $ka \ll 1$

$$p(r) \approx \frac{i \rho_0 c k a^2}{r} U_0\, e^{-ik(r-a)}$$

$$= \frac{i \rho_0 \omega}{4\pi r} (4\pi a^2 U_0)\, e^{-ik(r-a)} = \frac{i \rho_0 \omega S U_0}{4\pi r} e^{-ik(r-a)}$$

IF $q(t) = \rho_0 S U_0 e^{i\omega t}$ IS THE RATE OF MASS INJECTION, THEN $\dot{q}(t) = i \rho_0 \omega S U_0 e^{i\omega t}$, i.e. $i \rho_0 \omega S U_0 = \dot{Q} = i\omega Q$ IS THE COMPLEX AMPLITUDE OF $\dot{q}(t)$. NOW LET $a \to 0$ KEEPING Q FINITE, THEN

$$\boxed{p(r) = \frac{\dot{Q}}{4\pi r} e^{-ikr} = \frac{i\omega Q}{4\pi r} e^{-ikr}} \quad \boxed{\vec{U}(r) = \frac{i\omega Q F(kr)}{4\pi \rho_0 c\, r} e^{-ikr}\, \hat{r}}$$

IS THE SOLUTION OF THE HELMHOLTZ EQUATION

IDEAL OR POINT MONOPOLE $\quad \boxed{\nabla^2 p + \frac{\omega^2}{c^2} p = -i\omega Q\, \delta(\vec{x}) = -\dot{Q}\, \delta(\vec{x})}$

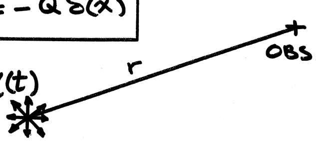
IN TIME DOMAIN THE SOLUTION OF

$$\boxed{\Box^2 p = \dot{q}(t)\, \delta(\vec{x})}$$

IS

$$\boxed{p(r,t) = \frac{\dot{q}(t - r/c)}{4\pi r}}$$

RAYLEIGH'S SOLUTION



Q: RATE OF MASS INJECTION AMPLITUDE $\}$ F.D.

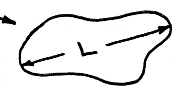
SIMPLE MODELS OF WAVES — SPHERICAL WAVES (CONT'D)

STATIONARY MONOPOLE (CONT'D)

i) $|P(r)| = \text{CONST.}$, $|\vec{U}(r)| = \text{CONST.}$ ON SPHERE $r = \text{CONST.}$

i.e. A POINT MONOPOLE HAS SPHERICALLY SYMMETRIC DIRECTIVITY

ii) FOR $ka \gg 1$, i.e. $a \gg \lambda$, $\langle W \rangle \approx \frac{1}{2}\rho_o c\, U_o^2 S$ IMPLIES THAT EACH PART OF THE SPHERE OF RADIUS a ACTS LIKE A PLANAR SURFACE. WE CAN SHOW THAT THIS RESULT IS TRUE FOR ANY PULSATING SHAPE IF $L \gg \lambda$ WHERE L IS THE LENGTH SCALE OF THE BODY.



iii) FOR $ka \ll 1$, $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{p}{r}\right) \sim \frac{\omega^2 p}{r c^2} = \frac{k^2 p}{r} \sim \frac{p}{r\lambda^2}$

$\nabla^2\left(\frac{p}{r}\right) \sim \frac{\partial^2}{\partial r^2}\left(\frac{p}{r}\right) \sim \frac{p}{r^3}$. IF $a \lesssim r \ll \lambda \Rightarrow \frac{p}{r\lambda^2} \ll \frac{p}{r^3}$

THIS MEANS THAT FOR $a \lesssim r \ll \lambda$, $ka \ll 1$, p AND $u(\vec{x},t)$ SATISFY LAPLACIAN EQS. $\nabla^2 p = 0$, $\nabla^2 \vec{u} = 0$, i.e. INCOMPRESSIBLE FLOW EQS. THIS MEANS THAT WE CAN TREAT THE PROBLEM AS QUASI-STEADY WITH t AS A PARAMETER OF THE PROBLEM.

12/05/00

ACOUSTIC SOURCES

MONOPOLE SOURCE (IDEAL). BY CONVENTION, THIS IS DEFINED AS:

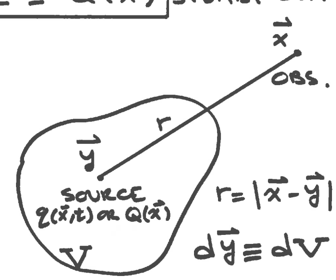
$$\boxed{\begin{aligned}\Box^2 p &= q(t)\,\delta(\vec{x})\\ 4\pi p(\vec{x},t) &= \frac{q(t-r/c)}{r}\end{aligned}} \qquad \boxed{\begin{aligned}HP &= Q\,\delta(\vec{x})\\ 4\pi P(\vec{x}) &= -\frac{Q\,e^{-ikr}}{r}\end{aligned}}$$

STEADY STATE $H = \nabla^2 + k^2$, $k = \omega/c$

THESE ARE (IDEAL) POINT MONOPOLE SOURCES. FROM THESE WE FIND THE SOLUTION TO THE FOLLOWING TWO IMPORTANT PDE'S : $\boxed{\Box^2 p = q(\vec{x},t)}$, $\boxed{HP = Q(\vec{x})}$ STEADY STATE

$$\boxed{4\pi\, p(\vec{x},t) = \int_V \frac{q(t-r/c)}{r}\, d\vec{y}}$$

$$\boxed{4\pi\, P(\vec{x}) = -\int_V \frac{Q(\vec{y})\, e^{-ikr}}{r}\, d\vec{y}} \quad \text{STEADY STATE}$$



THESE ARE VERY IMPORTANT RESULTS.

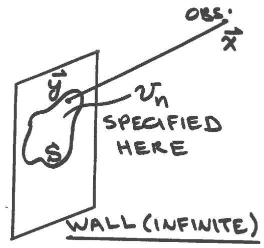
MONOPOLE DISTRIBUTION

WE GAVE THE MODEL OF AN IDEAL POINT MONOPOLE IN THE LAST LECTURE AS A PULSATING SPHERE (NOTE CHANGE OF NOTATION HERE!).

ACOUSTIC SOURCES (CONT'D)

RAYLEIGH PISTON IN THE WALL FORMULA

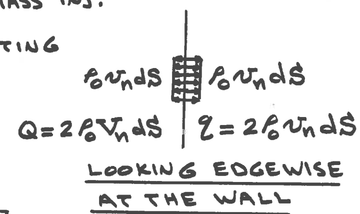
WE HAVE A NORMAL VELOCITY DISTRIBUTION ON AN INFINITE WALL. RAYLEIGH REASONED THAT q AND Q FOR AN ELEMENT OF THE SURFACE AREA dS ARE



$q = 2\rho_0 v_n(\vec{x},t)\,dS$ RATE OF MASS INJECTION

$Q = 2\rho_0 V_n(\vec{x})\,dS$ AMP. OF RATE OF MASS INJ.

* (WE ARE USING THE NOTATION FOR PULSATING SPHERE AGAIN!)



$$\boxed{p(\vec{x},t) = \frac{\rho_0}{2\pi}\int_S \frac{\dot{v}_n(\vec{y},t-r/c)}{r}\,dS}$$

$$\boxed{P(\vec{x}) = \frac{i\omega\rho_0}{2\pi}\int_S \frac{V_n(\vec{y})e^{-ikr}}{r}\,dS}$$ STEADY STATE

THESE ARE ALSO VERY IMPORTANT RESULTS.

ACOUSTIC SOURCES (CONT'D)

VELOCITY POTENTIAL FOR RAYLEIGH'S PISTON IN THE WALL FORMULA

$\phi(\vec{x},t)$ VEL. POT. $\Rightarrow p = -\rho_0 \frac{\partial\phi}{\partial t}$

$\Phi(\vec{x})$ VEL. POT. (STEADY STATE) $\Rightarrow P = -i\omega\rho_0\Phi$

FROM FORMULAS ON PREVIOUS PAGE

$$\boxed{\phi(\vec{x},t) = -\frac{1}{2\pi}\int_S \frac{v_n(\vec{y},t-r/c)}{r}\,dS}$$

$$\boxed{\Phi(\vec{x}) = -\frac{1}{2\pi}\int_S \frac{V_n(\vec{y})e^{-ikr}}{r}\,dS}$$ STEADY STATE

FROM $\nabla_x r = \hat{r} = \vec{r}/r$, $\vec{r} = \vec{x}-\vec{y}$, WE GET

$$\boxed{\vec{u}(\vec{x},t) = \nabla_x\phi = \frac{1}{2\pi}\int_S\left\{\frac{[\dot{v}_n]_{ret}}{cr} + \frac{[v_n]_{ret}}{r^2}\right\}\hat{r}\,dS}$$

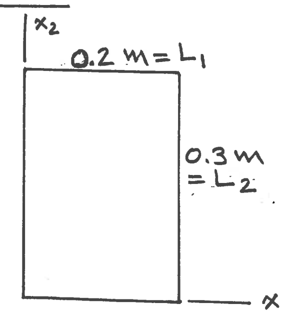
$$\vec{U}(\vec{x}) = \nabla_x\Phi = \frac{1}{2\pi}\int_S\left\{\frac{ik}{r} + \frac{1}{r^2}\right\}V_n(\vec{y})\hat{r}\,dS$$

STEADY STATE $$\boxed{\vec{U}(\vec{x}) = \frac{ik}{2\pi}\int_S \frac{F(kr)}{r}V_n(\vec{y})\hat{r}\,dS}$$ $F(kr) = 1 - \frac{i}{kr}$

ACOUSTIC SOURCES (CONT'D)

RAYLEIGH PISTON IN THE WALL FORMULA (CONT'D)

LET US CONSIDER A PISTON IN AN INFINITE WALL, 0.2 X 0.3 m AS SHOWN ON THE RIGHT.
LET $f = 3000$ HZ AND



$\omega = 2\pi f$

$$V_n(x_1, x_2) = \sin\left(\frac{j\pi x_1}{L_1}\right)\sin\left(\frac{l\pi x_2}{L_2}\right)$$

$$j = 1, 2, \dots \quad , \quad l = 1, 2, \dots$$

FIND $\underline{P}(\vec{x})$ NUMERICALLY. TAKE $c = 340$ m/s. PLOT $|\underline{P}|$ IN PLANES $x_3 =$ CONST. FROM SMALL TO LARGE x_3.

PLOT $|\underline{P}|$ IN THE PLANES $x_1 = 0, L_1/2, L_1$ AND PLANES $x_2 = 0, L_2/2$ AND L_2. PLOT VECTORS $\vec{\underline{U}}(\vec{x})$ IN THESE PLANES ALSO. STUDY ACOUSTIC ENERGY DENSITY NEAR THE PISTON.

WHAT WE ARE LOOKING FOR: FRESNEL & FRAUNHOFER ZONES, DIRECTIVITY PATTERN, EVANESCENT WAVES, EDGE AND CORNER WAVES, PISTON LOADING, ENERGY DENSITY DISTRIBUTION NEAR THE PISTON

ACOUSTIC SOURCES (CONT'D)

(IDEAL) POINT DIPOLE

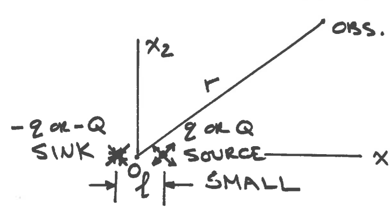
MATHEMATICAL DEFINITION: SINCE $\frac{\partial}{\partial x_i}$ COMMUTES WITH $\square^2$ AND $H = \nabla^2 + k^2$, AND $q(t - r/c)/4\pi r$ AND $Q e^{-ikr}/4\pi r$ ARE SOLUTIONS OF $\square^2 p = 0$ AND $H\underline{P} = 0$, RESPECTIVELY $\Rightarrow$

$$p(\vec{x}, t) = \frac{\partial}{\partial x_i}\left[\frac{q(t - r/c)}{4\pi r}\right] \quad \text{IS A SOLUTION OF } \square^2 p = 0$$

$$\underline{P}(\vec{x}) = \frac{\partial}{\partial x_i}\left[\frac{Q e^{-ikr}}{4\pi r}\right] \quad \text{IS A SOLUTION OF } H\underline{P} = 0$$

PHYSICAL DEFINITION TAKE $i = 1$

WE HAVE A SOURCE AND A SINK NEAR EACH OTHER ALONG x_1-AXIS CLOSE TO THE ORIGIN

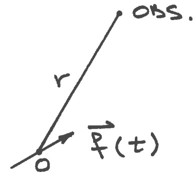


MODEL: AN OSCILLATING SPHERE OF RADIUS $a \ll \lambda$ OSCILLATING IN THE DIRECTION x_i WITH $q = -$ (FORCE ACTING ON THE FLUID)

ACOUSTIC SOURCES (CONT'D)

(IDEAL) POINT DIPOLE (CONT'D)

GENERALIZATION : A STATIONARY FORCE $\vec{f}(t)$, OR $\vec{F}$ STEADY STATE AMPLITUDE, ACTING ON THE FLUID



$$p(\vec{x},t) = -\frac{1}{4\pi}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[\frac{f_i(t-r/c)}{r}\right]$$

$$= \sum_{i=1}^{3}\frac{1}{4\pi}\left\{\frac{[\dot{f}_i \hat{r}_i]_{ret}}{cr} + \frac{[f_i \hat{r}_i]_{ret}}{r^2}\right\} \qquad \hat{r} = \vec{r}/r$$

$$\boxed{p(\vec{x},t) = \frac{1}{4\pi}\left\{\underbrace{\frac{[\dot{f}_r]_{ret}}{cr}}_{\text{FAR FIELD}} + \underbrace{\frac{[f_r]_{ret}}{r^2}}_{\text{NEAR FIELD}}\right\}}$$

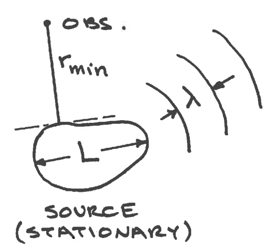
$$P(\vec{x},t) = -\frac{1}{4\pi}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[\frac{i\, e^{-ikr}}{r}\right]$$

STEADY STATE $\boxed{P(\vec{x},t) = \frac{ik}{4\pi}\,\frac{F(kr)\,F_r\, e^{-ikr}}{r}}$ $\qquad F(kr) = 1 - \frac{i}{kr}$

$$F_r = \sum_{i=1}^{3} F_i \hat{r}_i$$

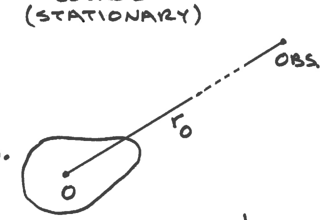
COMPACT (STATIONARY) SOURCE

A SOURCE IS COMPACT IF IT CAN BE TREATED AS A POINT SOURCE FOR THE DETERMINATION OF THE RADIATION FIELD.



COMPACTNESS CONDITIONS

LET L BE THE MAXIMUM SOURCE DIMENSION, r_{min} BE THE MINIMUM DISTANCE OF THE OBSERVER FROM THE SOURCE. WE MUST HAVE $\boxed{L \ll r_{min}}$. THIS MEANS THAT WE CAN REPLACE $r = |\vec{x} - \vec{y}|$ BY r_o AS SHOWN ON THE RIGHT (NOTE POSITION OF ORIGIN O).



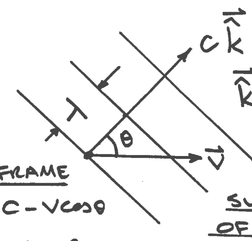
WE MUST ALSO HAVE MAXIMUM DIFFERENCE OF RETARDED (EMISSION) TIME OF THE POINTS ON THE SOURCE $\approx L/c \ll$ PERIOD $T = \frac{1}{f}$

THIS GIVES, USING $\lambda f = c$, $\boxed{L \ll \lambda}$. THIS CONDITION IS EQUIVALENT TO $\boxed{kL \ll 1}$ WHERE $k = \frac{\omega}{c}$

$\boxed{\begin{matrix} L \ll r_{min} \\ L \ll \lambda \end{matrix}}$ OR $\boxed{\begin{matrix} L \ll r_{min} \\ kL \ll 1 \end{matrix}}$ COMPACTNESS CONDITIONS

SOME WAVE KINEMATICS

FREQUENCY AND WAVE NUMBER RELATIONS FOR OBSERVER IN MOTION WITH VELOCITY $\vec{V}$ FOR PLANE WAVES



$\hat{k} = \dfrac{\vec{k}}{k}$

$k = \dfrac{2\pi f}{c}$

SURFACES OF CONST. PHASE

REL. VELOCITY OF THE WAVE IN MOVING FRAME IN THE DIRECTION $\hat{k}$: $c - \vec{V}\cdot\hat{k} = c - V\cos\theta$

NO. OF PEAKS CROSSED / 1 SEC. $= \dfrac{c - V\cos\theta}{\lambda} = f_{MO}$ FREQUENCY DETECTED BY MOVING OBSERVER

$$\boxed{f_{MO} = \frac{c - V\cos\theta}{c/f} = (1 - M\cos\theta)\, f} \qquad M = \frac{V}{c}$$

$1 - M\cos\theta$ IS KNOWN AS DOPPLER FACTOR. IF $1 - M\cos\theta < 1$ $\Rightarrow f_{MO} < f$. NOTE THAT f IS THE FREQUENCY OBSERVED BY AN OBSERVER STATIONARY IN THE MEDIUM.

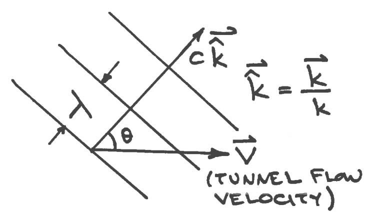
THE WAVE NUMBER k DOES NOT CHANGE FOR MOVING OBSERVER

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \left[\cancel{\frac{2\pi f_{MO}}{1 - M\cos\theta}}\right]}$$

$$\left[\frac{2\pi f_{MO}}{c - V\cos\theta}\right]$$

SOME WAVE KINEMATICS (CONT'D)

THE FREQUENCY AND WAVE NUMBER RELATIONS IN WIND TUNNEL FRAME FOR PLANE WAVES



$\hat{k} = \dfrac{\vec{k}}{k}$

OBSERVER IS FIXED IN TUNNEL FRAME. THE ACOUSTIC WAVES RIDE THE FLOW WITH VELOCITY $\vec{V}$.

REL. VEL. OF A POINT ON SURFACE OF CONST. PHASE IN DIRECTION $\hat{k} = c + V\cos\theta$

$$\boxed{f_T = \frac{c + V\cos\theta}{\lambda} = \frac{c + V\cos\theta}{c/f} = (1 + M\cos\theta)\, f}$$

$$\boxed{k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{2\pi f_T}{c + V\cos\theta}}$$

NOTE : THE WAVELENGTH λ DOES NOT CHANGE BECAUSE IT IS THE DISTANCE BETWEEN CONSECUTIVE PEAKS WHEN WE FREEZE TIME!

SOME WAVE KINEMATICS (CONT'D)

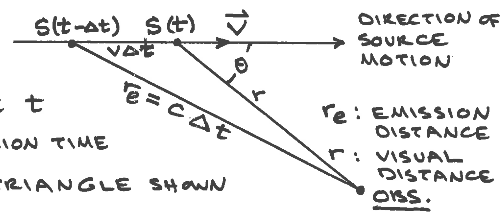
A POINT SOURCE IN MOTION, OBSERVER STATIONARY – SOURCE MOTION RECTILINEAR

$\vec{V}$: SOURCE VELOCITY, CONST.

S : SOURCE

$S(t)$ SOURCE POSITION AT TIME t

$S(t-\Delta t)$ " " " EMISSION TIME



USING COSINE LAW FOR THE TRIANGLE SHOWN

$$r_e^2 = r^2 + (V\Delta t)^2 + 2r\,V\Delta t\cos\theta$$

$$(1-M^2)r_e^2 - 2(rM\cos\theta)\,r_e - r^2 = 0 \quad , \quad M = V/C$$

$$r_e = \frac{r}{1-M^2}\left[M\cos\theta \pm \sqrt{M^2\cos^2\theta + 1 - M^2}\right]$$

$$= \frac{r}{1-M^2}\left[M\cos\theta \pm \sqrt{1 - M^2\sin^2\theta}\right]$$

IF $M < 1$, WE HAVE ONLY ONE SOLUTION

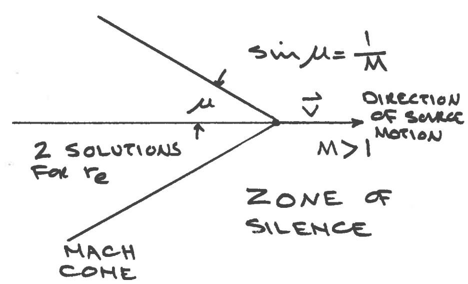
$$r_e = \frac{r}{1-M^2}\left[M\cos\theta + \sqrt{1 - M^2\sin^2\theta}\right]$$

SOME WAVE KINEMATICS (CONT'D)

A POINT SOURCE IN MOTION, OBSERVER STATIONARY (CONT'D)

IF $M > 1$, WE HAVE NO SOLUTION WHEN $1 - M^2\sin^2\theta < 0$ OR $\sin^2\theta > \frac{1}{M^2}$. WE ARE IN ZONE OF SILENCE

IF $1 - M^2\sin^2\theta > 0$, WE ARE INSIDE THE MACH CONE AND WE HAVE TWO SOLUTIONS FOR r_e .

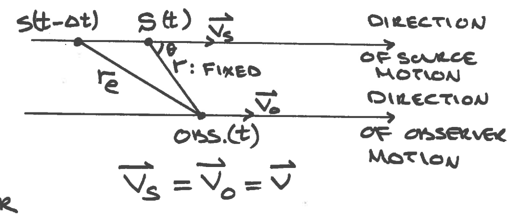


$$\boxed{\text{EMISSION TIME } t_e = t - r_e/c}$$

THE TRIANGLE SHOWN ON PREVIOUS PAGE IS KNOWN AS THE GARRICK TRIANGLE.

SOME WAVE KINEMATICS (CONT'D)

SOURCE AND OBSERVER IN MOTION RECTILINEARLY WITH THE SAME VELOCITY



AGAIN USING GARRICK TRIANGLE, WE GET THE SAME RELATION FOR r_e.

ALSO $t_e = t - r_e/c$

BUT r_e/c IS A CONSTANT FOR r (VISUAL DISTANCE) = FIXED $\Rightarrow$

$$e^{i\omega t_e} = e^{-i r_e/c} \cdot e^{i\omega t}$$

THIS MEANS THAT THE OBSERVER HEARS THE SOURCE FREQUENCY WITHOUT ANY CHANGE !

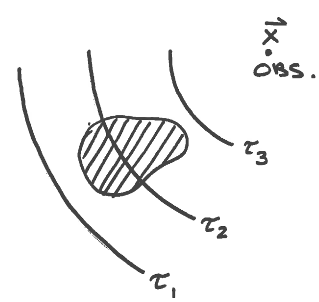
12/19/00

A VERY IMPORTANT SOLUTION OF WAVE EQUATION

THE SOLUTION OF $\Box^2 p = Q(\vec{x}, t)$

CAN BE WRITTEN AS

$$\boxed{4\pi\, p(\vec{x},t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r=c(t-\tau)} Q(\vec{y},\tau)\, d\Omega}$$



τ : SOURCE TIME

$\tau_1 < \tau_2 < \tau_3 < t$

WHERE $d\Omega$ IS THE ELEMENT OF THE SURFACE OF THE SPHERE

$\Omega : r = c(t-\tau)$ CENTER AT $\vec{x}$

$(\vec{x}, t)$ OBSERVER VARIABLES FIXED

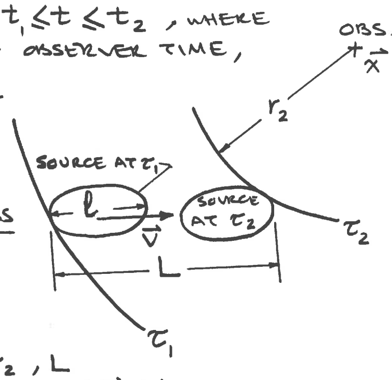
SOURCE TIME τ: $-\infty < \tau \leq t$

THIS IS A VERY IMPORTANT SOLUTION OF THE WAVE EQUATION. WE NOTE THAT $Q(\vec{x},t)$ CAN BE A SOURCE IN MOTION. WE WILL USE THIS RESULT FOR MANY PROBLEMS LATER, e.g. COMPACTNESS CONCEPT FOR MOVING SOURCES.

- Ω IS CALLED THE COLLAPSING SPHERE.

COMPACTNESS CONDITIONS FOR A MOVING SOURCE

ASSUME AN EXTENDED SOURCE OF TYPICAL LENGTH ℓ AND FREQUENCY f MOVING AT VELOCITY $\vec{V}$. WE ARE LOOKING FOR CONDITIONS THAT ALLOW US TO CONSIDER THE SOURCE AS A POINT SOURCE FOR $t_1 \leq t \leq t_2$, WHERE t IS THE OBSERVER TIME. FOR EACH OBSERVER TIME, FIND THE SOURCE TIMES τ_1 AND τ_2 WHEN THE COLLAPSING SPHERE ENTERS AND LEAVES THE SOURCE, RESPECTIVELY. LET L BE THE DISTANCE SHOWN IN THE FIGURE. THEN THE COMPACTNESS CONDITIONS ARE :



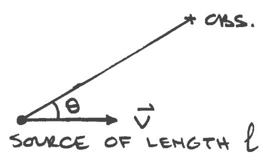
FOR ALL $t_1 \leq t \leq t_2$

$$L \ll r_2$$
$$\tau_2 - \tau_1 \ll T \equiv \frac{1}{f}$$

NOTE: τ_1, τ_2, L ARE FUNCTIONS OF $(\vec{x}, t)$

THE SITUATION NOW LOOKS LIKE THE FIGURE ON THE RIGHT AND

$$\boxed{L \approx \frac{\ell}{|1 - M_r|}, \quad M_r = |\vec{V}|\cos\theta / c}$$

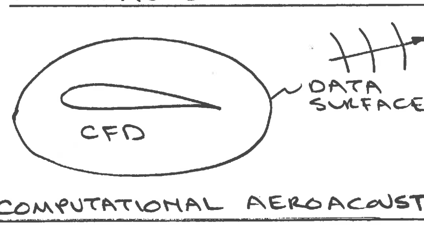


NOISE RADIATION FROM MOVING BODIES

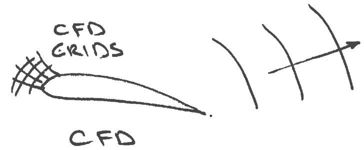
AT PRESENT WE HAVE TWO GENERAL METHOD FOR CALCULATING THE NOISE RADIATED FROM MOVING BODIES

i) THE ACOUSTIC ANALOGY : FFOWCS WILLIAMS - HAWKINGS (FW-H) EQUATION WITH PENETRABLE DATA SURFACE

$$\Box^2 p' = Q$$



ii) CFD-BASED COMPUTATIONAL AEROACOUSTICS (CAA) : LINEARIZED EULER, NAVIER-STOKES, ETC.



— THE ACOUSTIC ANALOGY BASED ON FW-H EQ. CAN BE CLASSIFIED AS KIRCHHOFF METHOD.

NOISE RADIATION FROM MOVING BODIES (CONT'D)

THE FW-H EQ. WITH PENETRABLE DATA SURFACE

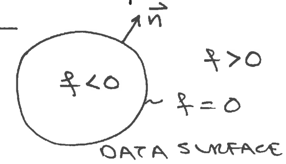
$$\boxed{\begin{aligned}\Box^2 p' &= \frac{\partial}{\partial t}\{[\rho u_n - (\rho-\rho_0) v_n]\,\delta(f)\} \\ &\quad - \frac{\partial}{\partial x_i}\{[\rho(u_n - v_n)u_i + p n_i]\,\delta(f)\} \\ &\quad + \frac{\partial^2}{\partial x_i \partial x_j}[T_{ij} H(f)]\end{aligned}}$$

THICKNESS SOURCE

LOADING SOURCE

QUADRUPOLE SOURCE

for penetrable surface, = 0 on solid surface



ρ : DENSITY

ρ_0 : DENSITY OF UNDISTURBED MEDIUM

u_n : FLUID VELOCITY NORMAL TO $f=0$

v_n : NORMAL VELOCITY OF $f=0$

p : PRESSURE (GAGE, i.e. $p - p_0$)

$p' = (\rho - \rho_0)c^2$, ACOUSTIC PRESSURE IN LINEAR REGION

$T_{ij} = \rho u_i u_j + [p - (\rho - \rho_0)c^2]\,\delta_{ij}$ LIGHTHILL STRESS TENSOR

u_i : FLUID VELOCITY COMPONENT

$\delta(f)$: DIRAC DELTA FUNCTION (3D delta fn)

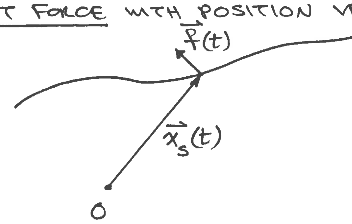
$H(f)$: HEAVISIDE FUNCTION

NOISE RADIATION FROM MOVING BODIES (CONT'D)

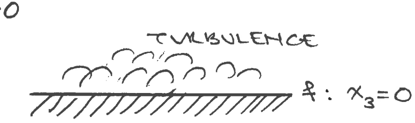
– THE LOWSON FORMULA IS THE SOLUTION OF

$$\boxed{\Box^2 p' = -\frac{\partial}{\partial x_i}\{f_i(t)\,\delta[\vec{x} - \vec{x}_s(t)]\}}$$

THIS MODELS A FLUCTUATING POINT FORCE WITH POSITION VECTOR $\vec{x}_s(t)$. $f_i(t)$ IS THE COMPONENT OF THE FORCE $\vec{f}(t)$ ACTING ON THE FLUID .



– THE CURLE FORMULA IS THE SOLUTION OF THE FW-H EQUATION FOR HALF SPACE $x_3 > 0$ AND $v_n = 0$



– THE RAYLEIGH FORMULA IS A SPECIAL SOLUTION OF THICKNESS TERM OF FW-H EQ.

– THE GREEN'S FUNCTION OF WAVE EQUATION IN THE UNBOUNDED SPACE IS

$$G(\vec{y}, \tau; \vec{x}, t) = \frac{\delta(g)}{4\pi r}, \quad g = \tau - t + r/c$$

1/16/2001

LEC. 6/4

SOME GENERALIZED FUNCTION THEORY

WE NEED TO LEARN HOW TO USE MULTIDIMENSIONAL DIRAC DELTA FUNCTION AND ITS DERIVATIVES. WE WILL LEARN THE MANIPULATION OF INTEGRALS INVOLVING $\delta(f)$ AND ITS DERIVATIVES WITHOUT FULL RIGOROUS JUSTIFICATION. THE JUSTIFICATION REQUIRES A LOT OF WORK. TO LEARN THIS SUBJECT IN DEPTH, SEE BOOKS BY THE FOLLOWING AUTHORS:

i) SEQUENTIAL APPROACH:
- M. J. LIGHTHILL (ONE DIMENSIONAL G.F.'S)
- D.S. JONES (2ND ED.)

ii) FUNCTIONAL APPROACH (OUR METHOD): (*)
- GELFAND & SHILOV, VOL. 1
- R. P. KANWAL (2ND ED.)
- A. H. ZEMANIAN (DOVER BOOKS)

G.F. THEORY HAS HELPED IN i) GREAT ADVANCES IN THE THEORY OF PDE'S, ii) EXTENDING THE POWER OF OPERATIONAL TECHNIQUES SUCH AS FOURIER TRANSFORM, iii) JUSTIFYING MANY AD HOC TECHNIQUES IN APPLIED MATHEMATICS SUCH AS FINITE PART OF DIVERGENT INTEGRALS AND THE USE OF DIVERGENT SERIES.

* SEE ALSO MY NASA TP 3428 AND NASA TM-110285

LEC. 6/5

GENERALIZED FUNCTIONS (CONT'D)

CONVENTIONAL WAY OF THINKING ABOUT FUNCTIONS SUCH AS $f(x) = \sin x$: A TABLE OF ORDERED PAIRS $(x, f(x))$

A MORE GENERAL (NEW) WAY OF THINKING ABOUT FUNCTIONS IS AS THE TABLE: $\{F[\phi] = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx\}$ WHERE $\phi(x)$ IS IN A SPECIFIED SPACE OF FUNCTIONS D, CALLED TEST FUNCTION SPACE.

- GENERALLY, THE MOST USEFUL TEST FUNCTION SPACES ARE INFINITELY DIFFERENTIABLE.
- $F[\phi]$ IS CALLED A FUNCTIONAL BECAUSE IT MAPS D INTO SCALARS ($\mathbb{R}$ OR $\mathbb{C}$). FOR AN ORDINARY FUNCTION $f(x)$, THIS FUNCTIONAL IS LINEAR AND CONTINUOUS.
  - LINEARITY MEANS $F[\alpha_1\phi_1 + \alpha_2\phi_2] = \alpha_1 F[\phi_1] + \alpha_2 F[\phi_2]$, ϕ_1 AND ϕ_2 ARE IN D
  - CONTINUITY MEANS THAT IF $\phi_n \to \phi$ THEN $F[\phi_n] \to F[\phi]$ SIMILARLY IF $\partial\phi_n/\partial x_i \to \partial\phi/\partial x_i$, THEN $F[\partial\phi_n/\partial x_i] \to F[\partial\phi/\partial x_i]$. THIS IS A VERY NICE PROPERTY. (*)

(*) WE REQUIRE THIS PROPERTY TO HOLD FOR ANY DERIVATIVES OF ϕ_n AND ϕ. NOTE BOTH ϕ_n AND ϕ ARE IN D.

LEC. 6/6

GENERALIZED FUNCTIONS (CONT'D)

NOTE THAT WE ARE NOW VIEWING ORDINARY FUNCTION $f(x)$ AS [A TABLE GENERATED BY] THE FUNCTIONAL $F[\phi]$ WHICH IS CONTINUOUS & LINEAR. WE WILL THINK AS $f(x)$ AND $F[\phi]$ AS THE SAME THING! NOW FUNCTIONAL IS A RULE MAPPING $\mathcal{D}$ INTO SCALARS AND THE RULE DOES NOT HAVE TO BE $\int f\phi\,dx$! ARE THERE ANY OTHER CONTINUOUS & LINEAR FUNCTIONALS ON $\mathcal{D}$? THE ANSWER IS YES!

EXAMPLE: TAKE THE FUNCTIONAL $\delta[\phi] = \phi(0)$. THIS IS A MAPPING OF $\mathcal{D}$ INTO REALS $\mathbb{R}$ IF ϕ IS REAL. THIS FUNCTIONAL IS CONTINUOUS AND LINEAR! WE SHOULD LEGITIMATELY CONSIDER THIS A "FUNCTION" FROM OUR NEW POINT OF VIEW. SUCH A "FUNCTION" IS A GENERALIZATION OF THE CONCEPT OF ORDINARY FUNCTION.

GENERALIZED FUNCTIONS: THE SPACE OF CONTINUOUS, LINEAR FUNCTIONALS [FUNCTIONS!] ON $\mathcal{D}$

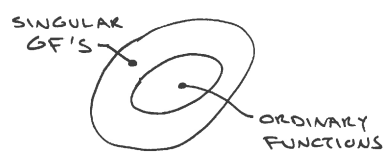
- ALL ORDINARY FUNCTIONS ARE G.F.'S (REGULAR G.F.'S)
- THERE ARE MANY FNS IN G.F. SPACE THAT ARE NOT ORDINARY FUNCTIONS, e.g. $\delta[\phi] = \phi(0)$! WE CALL SUCH FUNCTIONS SINGULAR G.F.'S.

LEC. 6/7.

GENERALIZED FUNCTIONS (CONT'D)

FOR SINGULAR GF'S SUCH AS $\delta[\phi]$, WE INTRODUCE "SYMBOLIC" FUNCTIONS, SUCH AS $\delta(x)$, AS FOLLOWS

$$\delta[\phi] = \int_{-\infty}^{\infty} \phi(x)\,\delta(x)\,dx = \phi(0)$$



THE SPACE OF GF'S

NOTE THAT "THE INTEGRAL" HERE IS NOT THE USUAL INTEGRAL (RIEMANN, LEBESGUE, ETC.) YOU LEARNED BEFORE! IT IS OBVIOUSLY WRITTEN SIMILAR TO THE FUNCTIONAL USED FOR ORDINARY FUNCTIONS BUT HERE IT IS USED FOR BOOK-KEEPING! IT STANDS FOR $\phi(0)$. BUT WITHOUT THE INTRODUCTION OF SYMBOLIC FUNCTIONS, WE WOULD HAVE TO USE AWKWARD EXPRESSION FOR DEFINING GREEN'S FUNCTION, ETC! WITHOUT A CLEAR UNDERSTANDING OF THIS POINT, YOU WILL HAVE MUCH PROBLEMS UNDERSTANDING THE MANIPULATIONS INVOLVING SINGULAR GF'S.

— SUMMARY: WE DID TWO THINGS i) WE LOOKED AT ORDINARY FUNCTIONS NOT AS ORDERED PAIRS $(x, f(x))$ BUT AS FUNCTIONALS $F[\phi] = \int f\phi\,dx$, ϕ IN TEST FN SPACE, ii) WE TAKE ANY CONTINUOUS LINEAR FUNCTIONAL AS A FUNCTION, i.e. G.F.!

GENERALIZED FUNCTIONS (CONT'D)

WE EXTEND THE OPERATIONS OF ORDINARY FNS TO ALL GF'S BY USING THE FUNCTIONAL DEFINITION

EXAMPLE : TAKE TEST FUNCTION SPACE D AS THE SPACE OF INFINITELY DIFFERENTIABLE FUNCTIONS THAT ARE ZERO OUTSIDE A FINITE INTERVAL. LET $f(x)$ BE AN ORDINARY DIFFERENTIABLE FUNCTION AND $F[\phi] = \int_{-\infty}^{\infty} f(x)\,\phi(x)\,dx$, ϕ IN D. NOW, BY INTEGRATION BY PARTS AND THE FACT THAT ϕ IS ZERO BEYOND A FINITE INTERVAL, WE HAVE $\int_{-\infty}^{\infty} f'\phi\,dx = -\int_{-\infty}^{\infty} f\,\phi'\,dx \equiv -F[\phi']$

SINCE $F[\phi]$ DEFINES (IDENTIFIES, IS) $f(x)$, WE MUST IDENTIFY $f'(x)$ WITH $F'[\phi]$ WHICH WE HAVE SHOWN ABOVE TO BE $-F[\phi']$, i.e. $\boxed{F'[\phi] = -F[\phi']}$ WHICH MAKES SENSE FOR ALL GEN. FNS !

— WE NEXT WRITE THIS RESULT SYMBOLICALLY. BUT FIRST AN EXAMPLE :

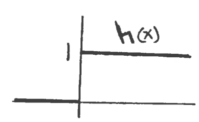
THE DIRAC DELTA FN $\delta(x)$: $\delta[\phi] = \phi(0) = \int_{-\infty}^{\infty}\phi(x)\delta(x)\,dx$

$$\delta'[\phi] = -\delta[\phi'] = -\phi'(0) = \left.\int_{-\infty}^{\infty}\phi(x)\,\delta'(x)\,dx\right\}$$ THIS INTEGRAL STANDS FOR $\delta'[\phi]$: A FUNCTIONAL

GENERALIZED FUNCTIONS (CONT'D)

GEN. DERIVATIVE OF HEAVISIDE FUNCTION

$$h(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$



NOTE THAT THE ORDINARY DERIVATIVE OF THIS FUNCTION $h'(x) = 0$, (UNDEFINED AT $x=0$). THE GENERALIZED DERIVATIVE IS NOT ZERO. USING THE RULE OF PREVIOUS VUGRAPH :

$H[\phi] = \int_{-\infty}^{\infty} h(x)\,\phi(x)\,dx = \int_{0}^{\infty}\phi(x)\,dx$: THE FUNCTIONAL DEFINING $h(x)$

$$H'[\phi] = -H[\phi'] = -\int_{0}^{\infty}\phi'(x)\,dx = \phi(0) = \delta[\phi] = \int_{-\infty}^{\infty}\phi(x)\,\delta(x)\,dx$$

i.e. $\boxed{\text{GEN. DER. OF } h(x) \equiv \bar{h}'(x) = \delta(x) \neq \underbrace{h'(x)}_{\text{ORDINARY DERIVATIVE OF } h(x)} = 0}$

WE USE A BAR OVER DERIVATIVE (OR $\bar{d}/dx$, $\bar{\partial}/\partial x_i$, ETC) TO SIGNIFY GENERALIZED DIFFERENTIATION WHENEVER THERE IS THE DANGER OF CONFUSION. FOR EXAMPLE, WE DON'T HAVE TO WRITE $\bar{\delta}'(x)$ BECAUSE, $\delta(x)$ ONLY HAS GENERALIZED DERIVATIVE.

— AN IMPORTANT RESULT. NOTE THAT $\int_{-\infty}^{x} h'(y)\,dy = 0$ BUT

$\int_{-\infty}^{x}\bar{h}'(y)\,dx = \int_{-\infty}^{x}\delta(y)\,dy = h(x)$, i.e. GENERALIZED DIFFERENTIATION MAINTAINS THE MEMORY OF THE JUMP !

GENERALIZED FUNCTIONS (CONT'D)

SOME USEFUL RESULTS :

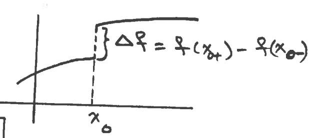
i) $\delta(x-x_0)$: $\boxed{\int_{-\infty}^{\infty} \phi(x)\,\delta(x-x_0)\,dx = \phi(x_0)}$ THIS IS SHIFTED DELTA FUNCTION

ii) NTH DERIVATIVE OF G.F. : $\boxed{F^{(n)}[\phi] = (-1)^n F[\phi^{(n)}]}$

EXAMPLE : $\int_{-\infty}^{\infty} \phi(x)\,\delta^{(n)}(x-x_0)\,dx = (-1)^n \phi^{(n)}(x_0)$

WHERE $\phi^{(n)}(x) = \dfrac{d^n \phi}{dx^n}$, ϕ IN D (C^∞ FNS WITH BOUNDED SUPPORT)

iii) f DIFFERENTIABLE WITH THE EXCEPTION OF $x=x_0$, f HAS A JUMP OF Δf AT x_0, THEN



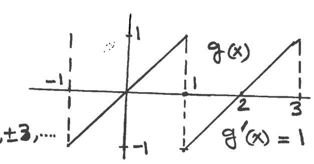
$\boxed{\text{GEN. DER. } f(x) \equiv \bar{f}'(x) = f'(x) + \Delta f\,\delta(x-x_0)}$

EXAMPLE $h(x)$ HEAVISIDE FN , $\bar{h}'(x) = \underbrace{\Delta h}_{=1}\,\delta(x-0) = \delta(x)$!

NOTE : $\int_{-\infty}^{x} \bar{f}'(x)\,dx = f(x) \neq \int_{-\infty}^{x} f'(x)\,dx$

EXAMPLE : $g(x)$ A RAMP FN, PERIODIC WITH PERIOD 2 $\Rightarrow$ $\bar{g}'(x) = 1 + 2\sum_{n=-\infty}^{\infty} \underbrace{\delta(x-2n-1)}_{\text{SHIFTED DELTA FNS AT } \pm 1, \pm 3, \dots}$

JUMP AT $x = \pm 1, \pm 3, \dots$



GENERALIZED FUNCTIONS (CONT'D)

WHEN WE WANT TO FIND THE GREEN'S FUNCTION OF AN ODE OR A PDE, THE PROBLEM MUST BE SET UP IN THE SPACE OF GENERALIZED FUNCTIONS AND ALL DERIVATIVES ARE THEN GENERALIZED DERIVATIVES.

EXAMPLE : TO FIND THE GREEN'S FUNCTION OF THE OPERATOR

$$Lu : \begin{cases} u'' & x \in [0,1] \\ u(0) + u(1) = 0 \\ 2u'(0) - u'(1) = 0 \end{cases} \text{BC}$$

, WE MUST SOLVE FOR $g(x,y)$

$\boxed{\text{LINEAR \& HOMOGENEOUS}}$

SUCH THAT

$$\begin{cases} \dfrac{\bar{\partial}^2 g}{\partial x^2} = \delta(x-y) \quad (*) & \text{GENERALIZED DERIVATIVE HERE} \\ g(0,y) + g(1,y) = 0 & \\ 2\dfrac{\partial g}{\partial x}(0,y) - \dfrac{\partial g}{\partial x}(1,y) = 0 & \text{ORDINARY DERIVATIVE HERE} \end{cases}$$

NOTE : GENERALLY, WE PREFER TO USE THE NOTATION $\dfrac{\bar{d}^2}{dx^2}$ & $\dfrac{d}{dx}$!

— TO SOLVE FOR u IN $u''(x) = f(x)$ + BC'S (ABOVE), WE USE $u(x) = \int_0^1 f(y)\,g(x,y)\,dy$

THIS SHOWS THE USEFULNESS OF GREEN'S FUNCTION

(*) THIS SAYS THAT $g(x,y)$ CAN NOT BE DISCONTINUOUS AT $x=y$ BUT $\partial g/\partial x$ HAS TO BE TO GET $\delta(x-y)$ ON THE RIGHT !

GENERALIZED FUNCTIONS (CONT'D)

DIRAC DELTA FUNCTION IN MULTIDIMENSIONAL SPACE (3D)

HERE WE CAN HAVE TWO SITUATIONS (CASES 2 & 3) THAT ARE NEW:

i) $\delta(\vec{x})$, A DIRAC DELTA FUNCTION CONCENTRATED AT A POINT

ii) $\delta(f)$, WHERE $f(\vec{x})=0$ IS A SURFACE IN 3D

iii) $\delta(f)\delta(g)$ WHERE $f(\vec{x})=0$ AND $g(\vec{x})=0$ ARE TWO SURFACES THAT INTERSECT ON A CURVE $C: f=g=0$

WE DISPOSE OF CASE i) FIRST

$$\int \phi(\vec{x})\,\delta(\vec{x})\,d\vec{x} = \phi(0) \qquad \text{(0 IS THE ORIGIN IN 3D)}$$

$$\int \phi(\vec{x})\,\delta(\vec{x}-\vec{x}_0)\,d\vec{x} = \phi(\vec{x}_0)$$

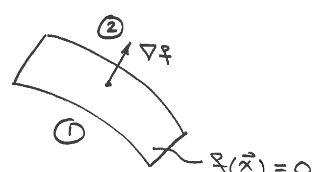
HERE $\int \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$. USE THREE INTEGRAL SIGNS ONLY WHEN YOU WANT TO IMPRESS PEOPLE!

- CASE (ii) IS A DIRAC DELTA FUNCTION CONCENTRATED ON THE SURFACE $f=0$
- CASE (iii) IS A DIRAC DELTA FUNCTION CONCENTRATED ON THE CURVE $C: f=g=0$.

GENERALIZED FUNCTIONS (CONT'D)

HOW DOES $\delta(f)$ APPEAR IN MATHEMATICAL EXPRESSIONS?

GENERALIZED DERIVATIVE OF A SCALAR FUNCTION DISCONTINUOUS ACROSS THE SURFACE $f(\vec{x})=0$



LET $Q(\vec{x})$ BE DISCONTINUOUS ACROSS $f=0$. DEFINE $\Delta Q = Q(f_2) - Q(f_1)$ WHERE REGION ② IS WHERE ∇f POINTS INTO.

THEN WE CAN SHOW THAT

$$\boxed{\frac{\overline{\partial} Q}{\partial x_i} = \frac{\partial Q}{\partial x_i} + \Delta Q \frac{\partial f}{\partial x_i}\delta(f)} \quad \boxed{\overline{\nabla} Q = \nabla Q + \Delta Q \nabla f\, \delta(f)}$$

IF $\vec{Q}(\vec{x})$ IS A VECTOR FUNCTION DISCONTINUOUS ACROSS $f=0$, THEN

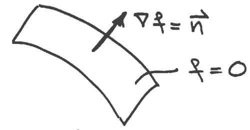
$$\boxed{\begin{aligned} \overline{\nabla}\cdot\vec{Q} &= \nabla\cdot\vec{Q} + \nabla f\cdot\Delta\vec{Q}\,\delta(f) \\ \overline{\nabla}\times\vec{Q} &= \nabla\times\vec{Q} + \nabla f\times\Delta\vec{Q}\,\delta(f) \end{aligned}}$$

- IN APPLICATIONS, THE DISCONTINUITY SURFACE IS EITHER REAL (e.g., A SHOCK SURFACE) OR ARTIFICIALLY INTRODUCED (DERIVATIONS OF FW-H & KIRCHHOFF EQS.).
- WE CAN ALWAYS DEFINE $f(\vec{x})$ SUCH THAT $\nabla f = \vec{n}$ (UNIT NORMAL). IF THIS CONDITION DOES NOT HOLD, THEN DEFINE $\tilde{f}(\vec{x}) = f(\vec{x})/|\nabla f|$.

GENERALIZED FUNCTIONS (CONT'D)

IMPORTANT FACT: CONSERVATION LAWS ARE VALID IF WE TREAT THE DERIVATIVES AS GEN. DERIVATIVES. THIS MEANS THAT THE SHOCK JUMP CONDITIONS ARE INCLUDED IN CONSERVATION LAWS

CONTINUITY EQ.:



MOVING SHOCK

$$0 = \frac{\bar{\partial} \rho}{\partial t} + \bar{\nabla} \cdot (\rho \vec{u}) = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) + \left[\Delta \rho \frac{\partial f}{\partial t} + \Delta(\rho \vec{u}) \cdot \vec{n} \right] \delta(f)$$

$\boxed{\frac{\partial f}{\partial t} = -v_n}$ NORMAL VELOCITY OF THE SHOCK

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \qquad \vec{x} \text{ NOT ON THE SHOCK}$$

$$\boxed{-\Delta \rho \, v_n + \Delta(\rho u_n) = \Delta[\rho(u_n - v_n)] = 0}$$

MOMENTUM EQUATION

$$\frac{\bar{\partial}}{\partial t}(\rho u_i) + \frac{\bar{\partial}}{\partial x_j}(\rho u_i u_j) + \frac{\bar{\partial} p}{\partial t} = \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial t} + \left[\Delta(\rho u_i) \frac{\partial f}{\partial t} + \Delta(\rho u_i u_j) n_j + \Delta p \, n_i \right] \delta(f)$$

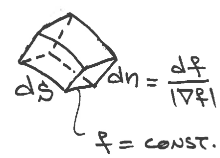
$$\Rightarrow \quad \boxed{\Delta[\rho u_i (u_n - v_n)] + \Delta p \, n_i = 0}$$

GENERALIZED FUNCTIONS (CONT'D)

A USEFUL RESULT: $\int Q(\vec{x}) \delta(f) \, d\vec{x} = ?$ (volume)

WRITING $d\vec{x} = dn \, dS = \frac{df \, dS}{|\nabla f|}$

WE HAVE



$$\int Q(\vec{x}) \delta(f) \, d\vec{x} = \int Q(\vec{x}) \delta(f) \frac{df \, dS}{|\nabla f|} = \int_{f=0} \frac{Q(\vec{x})}{|\nabla f|} dS$$

IF $|\nabla f| = 1$, AS WE USUALLY ASSUME, THEN

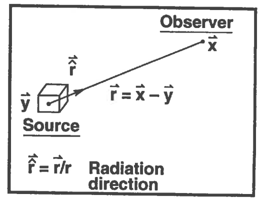
$$\boxed{\int Q(\vec{x}) \delta(f) \, d\vec{x} = \int_{f=0} Q(\vec{x}) \, dS}$$

LEC. 6/16

Things to Know About Green's Function of Wave Equation

- The Green's function of the wave equation in *the unbounded space* is

$$G(\vec{y}, \tau; \vec{x}, t) = \begin{cases} \dfrac{\delta(g)}{4\pi r} & \tau \le t \\ 0 & \tau > t \end{cases}$$



$g = \tau - t + \dfrac{r}{c}$ *outgoing wave*

$(\vec{y}, \tau)$ *source* space-time variables

$(\vec{x}, t)$ *observer* space-time variables

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LEC 6/17

Things to Know About Green's Function of Wave Equation

- There are many methods to derive $G(\vec{y}, \tau; \vec{x}, t)$ rigorously. It is easy to show that G depends on $\vec{x} - \vec{y}$ and $t - \tau$. Using $\vec{x} - \vec{y} = \vec{r}$, $\lambda = t - \tau$, take *spatial* Fourier transform of $\bar{\Box}^2_{(\vec{r},\lambda)} G = \delta(\vec{r})\delta(\lambda)$ to get a simple problem involving finding the Green's function of an O.D.E. in λ. The inverse spatial Fourier transform of the Green's function of the O.D.E. gives Green's function of the wave equation for both the outgoing and incoming waves.

$$\boxed{\vec{r} = \vec{x} - \vec{y}, \quad r = |\vec{x} - \vec{y}|, \quad \hat{r} = \frac{\vec{r}}{r}, \quad \frac{\partial r}{\partial x_i} = \hat{r}_i, \quad \frac{\partial r}{\partial y_i} = -\hat{r}_i}$$

Useful things to remember

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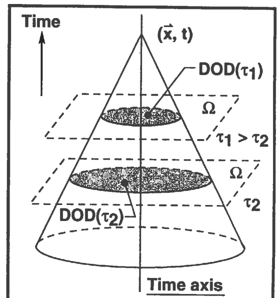
Things to Know About Green's Function of Wave Equation

"where it's non-zero"

The support of $\delta(g)$ is on the surface $g = 0$. The surface $g = 0$ is $r = |\vec{x} - \vec{y}| = c(t - \tau)$. This is the *characteristic cone* of the wave equation with vertex at $(\vec{x}, t)$. Since $\Box^2$ is a differential equation with constant coefficients, $g = 0$ is also the *characteristic conoid* with vertex at $(\vec{x}, t)$. This gives us the picture on the right. Note that we have drawn the 3D space Ω as a plane in the figure. Therefore, this figure is a 3D illustration of what happens in 4D (3D space + time).

- **Note**: $g = 0$ is a cone because if the 4-vector $\vec{A} = (\vec{x} - \vec{y}, t - \tau)$ lies on $g = 0 \Rightarrow \alpha\vec{A} = [\alpha(\vec{x} - \vec{y}), \alpha(t - \tau)]$ also lies on $g = 0$. This is the property of a cone.



Characteristic cone (Conoid) at $(\vec{x}, t)$. Ω is the unbounded 3-space. DOD is domain of dependence of $(\vec{x}, t)$ which depends on τ

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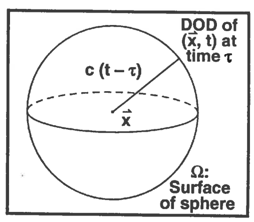
Things to Know About Green's Function of Wave Equation

- *Visualization of domain of dependence of $(\vec{x}, t)$ in four dimensions.*

Fix $(\vec{x}, t)$ and $\tau \Rightarrow r = c(t - \tau)$ is a sphere with center at $\vec{x}$ and radius $c(t - \tau)$. Any source on this sphere at time τ, contributes to $\vec{x}$ at time t. As τ increases, the radius shrinks, hence we have a *collapsing sphere*. Radius becomes zero at $\tau = t$.



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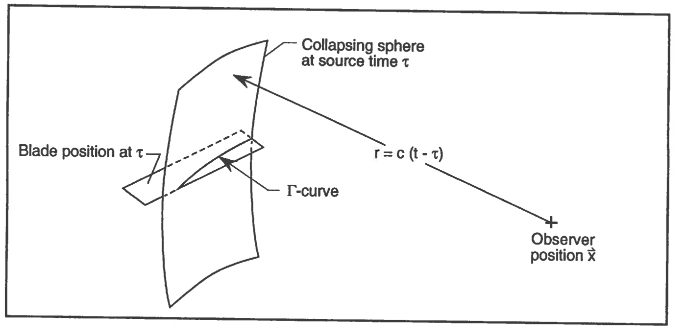
The Collapsing Sphere Concept

Equation of collapsing sphere: $r = c(t - \tau)$, $(\vec{x}, t)$ fixed



The Σ-surface is the locus of Γ-curves in space. If the blade surface is described by $f(\vec{y}, \tau) = 0$, the equation of the Σ-surface is:

$$F(\vec{y}; \vec{x}, t) = [f(\vec{y}, \tau)]_{\text{ret}} = f(\vec{y}, t - r/c) = 0, \ (\vec{x}, t) \text{ fixed}$$

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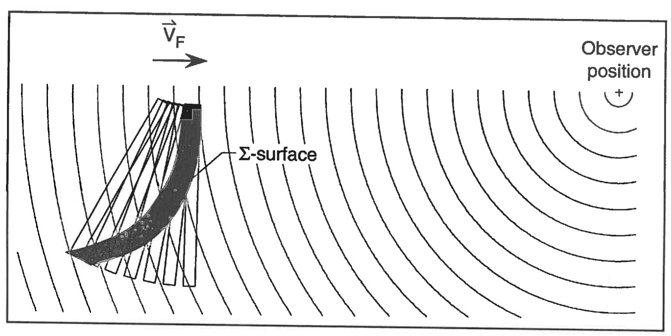
Construction of Σ-Surface for a Helicopter Rotor Blade



In this construction, we have taken a rotor blade of zero thickness rotating with rotational Mach number 0.67 and forward Mach number 0.15. The observer is in the rotor plane. The circles are the intersection of the collapsing sphere with the plane containing the rotor. The circles are drawn at equal source time intervals. The observer time is $t = \tau + r/c$ where r is the radius of the collapsing sphere at τ. Note that t is fixed for the above Σ-surface.

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The Two Forms of the Solution of Wave Equation (Volume Sources)

We want to find the solution of $\Box^2\phi = Q(\vec{x}, t)$

$$4\pi\phi(\vec{x}, t) = \int \frac{1}{r}Q(\vec{y}, \tau)\delta(g)d\vec{y}d\tau$$

All volume integrals are over unbounded 3 space and all time integrals are over $(-\infty, t)$.

i) Let $\tau \to g \Rightarrow \frac{\partial g}{\partial \tau} = 1$ and $4\pi\phi(\vec{x}, t) = \int \frac{1}{r}Q\left(\vec{y}, g + t - \frac{r}{c}\right)\delta(g)dg d\vec{y}$

Integrate with respect to g to get

$$\boxed{4\pi\phi(\vec{x}, t) = \int \frac{1}{r}Q\left(\vec{y}, t - \frac{r}{c}\right)d\vec{y} = \int \frac{[Q]_{\text{ret}}}{r}d\vec{y}}$$

Retarded Time Solution

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The Two Forms of the Solution of Wave Equation (Volume Sources) (Cont'd)

ii) Let $y_3 \to g \Rightarrow \frac{\partial g}{\partial y_3} = -\frac{1}{c}\hat{r}_3$

$$4\pi\phi(\vec{x}, t) = \int \frac{cQ(\vec{y}, \tau)}{r}\delta(g)\, dg \frac{dy_1 dy_2}{|\hat{r}_3|}\, d\tau$$

Since in the inner integrals $(\vec{x}, t)$ and τ are fixed, then $\frac{dy_1 dy_2}{|\hat{r}_3|} = d\Omega$ element of surface area of sphere $r = c(t-\tau)$. Integrate with respect to g to get:

$$\boxed{4\pi\phi(\vec{x}, t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r = c(t-\tau)} Q(\vec{y}, \tau)\, d\Omega}$$

Collapsing Sphere Solution

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NOISE GENERATION FROM MOVING SOURCES

LOWSON'S FORMULA (THE SOUND FROM A MOVING DIPOLE) - 1965

ASSUME THAT A COMPACT (POINT) FORCE OF STRENGTH $\vec{F}(t)$ MOVES SUBSONICALLY. LET THE POSITION OF THE FORCE BE GIVEN BY $\vec{x}_s(t)$. NOTE THAT $\vec{F}(t)$ IS THE FORCE ACTING ON THE MEDIUM. LOWSON PROPOSED THAT THE NOISE FROM THIS SOURCE IS DESCRIBED BY:

$$\boxed{\Box^2 p' = -\frac{\partial}{\partial x_i}\{F_i(t)\,\delta[\vec{x}-\vec{x}_s(t)]\}}$$

WHERE p' IS THE ACOUSTIC PRESSURE. WE CAN USE VECTOR NOTATION FOR THIS EQUATION AS FOLLOWS:

$$\boxed{\Box^2 p' = -\nabla\cdot\{\vec{F}(t)\,\delta[\vec{x}-\vec{x}_s(t)]\}}$$

THIS IS THE GENERALIZATION OF LAMB'S DIFFERENTIAL EQUATION WHICH WE DEDUCE FROM HIS SOLUTION OF THE SOUND FROM A STATIONARY COMPACT FORCE:

$$\Box^2 p' = -\nabla\cdot[\vec{F}(t)\,\delta(\vec{x})] \qquad (*)$$

LAMB DERIVED THE FOLLOWING SOLUTION BY STUDYING THE ACOUSTIC FIELD OF AN OSCILLATING SPHERE AND THEN LETTING ITS RADIUS GO TO ZERO:

$$4\pi p'(\vec{x},t) = -\nabla\cdot\left[\frac{\vec{F}(t-r/c)}{r}\right]$$

THIS IS THE SOLUTION OF THE PDE (*). LAMB DID NOT GIVE THE PDE!

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

LOWSON'S FORMULA (CONT'D)

SOLUTION OF THE WAVE EQ. $\Box^2 p' = -\nabla\cdot\{\vec{F}(t)\,\delta[\vec{x}-\vec{x}_s(t)]\}$ (*)

THERE ARE MANY WAYS TO FIND p' SOME (MANY!) OF WHICH REQUIRE A LOT OF ALGEBRAIC MANIPULATIONS. WE USE SEVERAL TRICKS TO GET THE SOLUTION MORE ELEGANTLY.

— IF $\vec{\phi}_1(\vec{x},t)$ IS THE SOLUTION OF $\Box^2\vec{\phi}_1 = \vec{Q}(\vec{x},t) \Rightarrow$

$\phi_2(\vec{x},t) = \nabla\cdot\vec{\phi}_1$ IS THE SOLUTION OF $\Box^2\phi_2 = \nabla\cdot\vec{Q}$

PROOF: $\nabla\cdot\Box^2\vec{\phi}_1 = \Box^2[\nabla\cdot\vec{\phi}_1] = \nabla\cdot\vec{Q} \quad \therefore \phi_2 = \nabla\cdot\vec{\phi}_1$ $+\ \phi_3$ WHERE $\Box^2\phi_3 = 0$. THE SOLUTION OF $\Box^2\phi_3 = 0$ IS $\phi_3 = 0$!

THE SOLUTION OF EQ. (*) IS, THEREFORE

$$4\pi p'(\vec{x},t) = -\nabla_x\cdot\int\frac{1}{r}\vec{F}(\tau)\,\delta[\vec{y}-\vec{x}_s(\tau)]\,\delta(g)\,d\vec{y}\,d\tau$$

LET $\vec{z} = \vec{y}-\vec{x}_s(\tau)$, $\vec{y} = \vec{z}+\vec{x}_s(\tau) \Rightarrow d\vec{y} = d\vec{z}$

$g = \tau - t + |\vec{x}-\vec{z}-\vec{x}_s(\tau)|/c$

$$4\pi p'(\vec{x},t) = -\nabla_x\cdot\int\frac{1}{r}\vec{F}(\tau)\,\delta(\vec{z})\,\delta[g(\vec{z},\tau;\vec{x},t)]\,d\vec{z}\,d\tau$$

$$= -\nabla_x\cdot\int\left[\frac{1}{r}\vec{F}(\tau)\,\delta(g)\right]_{\vec{z}=0}d\tau$$

NOISE GENERATION FROM MOVING SOURCES (CONT'D)
LOWSON'S FORMULA (CONT'D)

$$g\big|_{\vec{\eta}=0} = \tau - t + |\vec{x} - \vec{x}_s(\tau)|/c \quad , \quad r\big|_{\vec{\eta}=0} = |\vec{x} - \vec{x}_s(\tau)|$$

NOW LET $\tau \to g$, WE HAVE THE JACOBIAN OF TRANSFORMATION

$$\frac{1}{|\partial g/\partial \tau|} = \frac{1}{|1 - M_r|} \quad , \quad M_r = \frac{1}{c}\,\dot{\vec{x}}_s(\tau)\cdot\hat{r}, \quad \hat{r} = \frac{\vec{r}}{r}$$

NOTE $\frac{\partial g}{\partial \tau} = 1 - \frac{1}{c}\frac{\partial}{\partial \tau}|\vec{x} - \vec{x}_s(\tau)| = 1 - \frac{1}{c}\dot{\vec{x}}_s(\tau)\cdot\hat{r}$!

$$4\pi\, p'(\vec{x}, t) = -\nabla_x\cdot \int \left[\frac{1}{r|1 - M_r|}\vec{F}(\tau)\,\delta(g)\right]_{\vec{\eta}=0} dg$$

$$= -\nabla_x\cdot \left[\frac{\vec{F}(\tau)}{r|1 - M_r|}\right]_{\substack{\vec{\eta}=0\\ g=0}}$$

NOW $g\big|_{\vec{\eta}=0} = 0$ MEANS THAT $\tau - t + |\vec{x} - \vec{x}_s(\tau)|/c = 0$.

THIS GIVES A SOLUTION FOR RETARDED TIME τ^* IF THE FORCE MOVES SUBSONICALLY. NOTE $\tau^* = \tau^*(\vec{x}, t)$.

$$\boxed{4\pi\, p'(\vec{x}, t) = -\nabla_x\cdot\left[\frac{\vec{F}(\tau)}{r|1 - M_r|}\right]_{\tau^*}}$$

IF WE TAKE THE DIVERGENCE, WE GET LOWSON'S FORMULA

NOISE GENERATION FROM MOVING SOURCES (CONT'D)
LOWSON'S FORMULA (CONT'D)

INTERPRETATION OF $\tau^* = \tau^*(\vec{x}, t)$

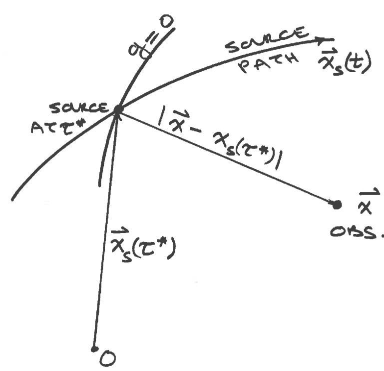
$g = \tau - t + |\vec{x} - \vec{y}|/c = 0$
IS THE COLLAPSING SPHERE FOR $(\vec{x}, t)$

τ^* IS THE SOLUTION OF

$\tau - t + |\vec{x} - \vec{x}_s(\tau)|/c = 0$

i.e. WHEN THE COLLAPSING SPHERE INTERSECTS THE SOURCE!

IN GENERAL FINDING τ^* INVOLVES THE SOLUTION OF A NONALGEBRAIC (TRANSCENDENTAL) EQUATION



TO GET LOWSON'S FORMULA, WE MUST TAKE THE DIVERGENCE OF $[F(\tau)/(r|1-M_r|)]_{\tau^*}$. IT IS A LONG MESSY RESULT. WE NEED THE FAR FIELD RESULT WHICH CAN BE OBTAINED EASILY FROM A RESULT OF FARASSAT. IT CAN BE SHOWN THAT THE ACOUSTIC PRESSURE CAN BE WRITTEN ALSO AS (EXACT RESULT!):

$$\boxed{4\pi\, p'(\vec{x}, t) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{F_r}{r|1 - M_r|}\right]_{\tau^*} + \left[\frac{F_r}{r^2|1 - M_r|}\right]_{\tau^*}}$$

WHERE $F_r = \vec{F}\cdot\hat{r}$, i.e. COMPONENT OF FORCE IN RADIATION DIRECTION

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

LOWSON'S FORMULA IN THE FAR FIELD

IN THE FAR FIELD LOWSON'S FORMULA SIMPLIFIES CONSIDERABLY.

$$\boxed{4\pi p'(\vec{x},t) = \underbrace{\left[\frac{\dot{F}_r}{cr(1-M_r)^2}\right]_{\tau^*}}_{\text{FORCE FLUCTUATION}} + \underbrace{\left[\frac{F_r \dot{M}_r}{cr(1-M_r)^3}\right]_{\tau^*}}_{\text{FORCE ACCELERATION}}}$$

NOTE DEPENDENCE ON $1-M_r$!

WHERE $\dot{F}_r = \dot{\vec{F}} \cdot \hat{r}$, $\dot{M}_r = \dot{\vec{M}} \cdot \hat{r}$ WHERE $(\dot{\ }) = \partial/\partial\tau$.

WE HAVE $\frac{\partial}{\partial t}\left[(\cdot)\right]_{\tau^*} = \left[\frac{\partial \tau^*}{\partial t}\frac{\partial}{\partial \tau}(\cdot)\right]_{\tau^*}$. FROM

$\tau^* - t + |\vec{x} - x_s(\tau^*)|/c = 0$, WE GET $\frac{\partial \tau^*}{\partial t} = \frac{1}{1-M_r}$

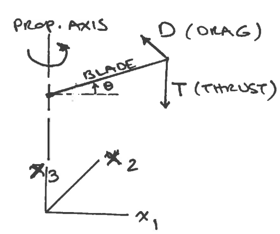
LOWSON'S FORMULA TELLS US THAT, IN THE FAR FIELD, $p'(\vec{x},t)$ DEPENDS ON FORCE FLUCTUATION AND ACCELERATION IN RADIATION DIRECTION AT THE MOMENT OF EMISSION OF ACOUSTIC SIGNAL.

— WE CAN SHOW THAT $\tau - t + |\vec{x} - \vec{x}_s(\tau)|/c = 0$ HAS ONLY ONE SOLUTION FOR SUBSONIC MOTION OF THE FORCE. IT HAS AN ODD NUMBER OF SOLUTIONS FOR τ^* IF THE SOURCE MOVES SUPERSONICALLY.

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

GUTIN'S RESULT (1930'S)

GUTIN CALCULATED THE NOISE FROM STEADY ROTATING FORCES FROM A PROPELLER. HE DERIVED THE RESULT FOR NONCOMPACT SOURCE DISTIBUTION BUT BECAUSE OF THE LACK OF COMPUTERS IN 1930'S, HE MADE APPROXIMATIONS EQUIVALENT TO ROTATING POINT FORCES AS SHOWN. WE ARE SHOWING THE DRAG AND LIFT FORCES ACTING ON THE MEDIUM. GUTIN USED LAMB'S RESULT BY REPRESENTING THE PROPELLER FORCES BY STATIONARY PERIODIC FORCES IN PROPELLER DISK. LET Ω BE THE SHAFT FREQUENCY. THEN



$\vec{F} = (-D\sin\theta, D\cos\theta, -T)$

$\dot{\vec{F}} = (-D\Omega\cos\theta, -D\Omega\sin\theta, 0)$

$$\hat{p}_m(\vec{x}) = \frac{1}{T}\int_0^T p(\vec{x},t)\, e^{im\Omega t}\, dt \qquad T = \frac{1}{f} = \frac{2\pi}{\Omega}$$

WE CAN SHOW THAT $dt = |1-M_r|\, d\tau$

$$\hat{p}_m(\vec{x}) = \frac{1}{T}\int_0^T \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{F_r}{r|1-M_r|}\right]_{\tau^*} e^{im\Omega t}\, dt = -\frac{im\Omega}{cT}\int_0^T \left[\frac{F_r}{r|1-M_r|}\right]_{\tau^*} e^{im\Omega t}\, dt$$

$$= -\frac{im\Omega}{cT}\int_0^T \frac{F_r(\tau)}{r}\, e^{im\Omega(\tau + r/c)}\, d\tau \qquad \text{(FAR FIELD)}$$

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

GUTIN'S RESULT (CONT'D)

$$r^2 = (x_1 - R\cos\theta)^2 + (x_2 - R\sin\theta)^2 + x_3^2 \quad , \quad R << \|x\| \equiv r_0$$

$$= r_0^2 + R^2 - 2R(x_1\cos\theta + x_2\sin\theta)$$

PUT THE OBSERVER ON x_1x_3-PLANE. THIS DOES NOT AFFECT THE PRESSURE AMPLITUDE BUT DOES INFLUENCE PHASE.

$$r \approx r_0 - \frac{R}{r_0}x_1\cos\theta = r_0 - R\sin\psi\cos\theta \quad , \quad \sin\psi = \frac{x_1}{r_0}$$

$$\vec{\hat{r}} = (\cos\psi, 0, \sin\psi)$$

$$F_r = -D\cos\psi\sin\theta - T\sin\psi \quad , \quad \theta = \Omega\tau \quad , \quad d\theta = \Omega d\tau$$

$$\hat{p}_m(\vec{x}) = -\frac{im}{cTr_0}\int_0^{2\pi} F_r(\theta)\, e^{im\Omega r_0}\, e^{im(\theta - \frac{R}{c}\sin\psi\cos\theta)}\, d\theta$$

$$= \frac{im\Omega}{2\pi c r_0} e^{im\Omega r_0} \underbrace{\int_0^{2\pi} (D\cos\psi\sin\theta + T\sin\psi)\, e^{im(\theta - \frac{R\sin\psi}{c}\cos\theta)}\, d\theta}_{\text{ANALYTIC RESULT IS AVAILABLE!}}$$

$$(\alpha D\cos\psi + \beta T\sin\psi)\, J_m\left(\frac{mR\sin\psi}{c}\right)$$

BESSEL FN OF 1ST KIND AND ORDER m

$\boxed{\text{FOR } B \text{ BLADES } m = nB, \; n = 1, 2, \ldots}$

The Governing Wave Equation for Deriving Kirchhoff Formulas

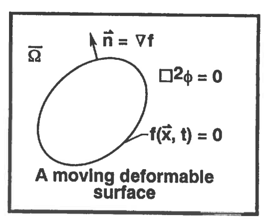
We consider *the exterior problem* here.

$\bar{\Omega}$: The exterior unbounded space

$$\text{Let } \tilde{\phi}(\vec{x}, t) = \begin{cases} \phi(\vec{x}, t) & \vec{x} \in \bar{\Omega} \\ 0 & \vec{x} \notin \bar{\Omega} \end{cases} \Rightarrow \Box^2\tilde{\phi} = 0 \text{ everywhere}$$

$$\frac{\bar{\partial}\tilde{\phi}}{\partial t} = \frac{\partial\tilde{\phi}}{\partial t} + \phi\frac{\partial f}{\partial t}\delta(f) = \frac{\partial\tilde{\phi}}{\partial t} - v_n\phi\delta(f)$$

where $v_n = -\frac{\partial f}{\partial t}$ is the local normal velocity on $f = 0$



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The Governing Wave Equation for Deriving Kirchhoff Formulas (Cont'd)

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Next take the second time derivative of $\tilde{\phi}$:

$$\frac{\bar{\partial}^2\tilde{\phi}}{\partial t^2} = \frac{\partial^2\tilde{\phi}}{\partial t^2} + \frac{\partial\phi}{\partial t}\frac{\partial f}{\partial t}\delta(f) - \frac{\partial}{\partial t}[v_n\phi\delta(f)] = \frac{\partial^2\tilde{\phi}}{\partial t^2} - v_n\phi_t\delta(f) - \frac{\partial}{\partial t}[v_n\phi\delta(f)]$$

Similarly for the space derivatives we have:

$$\bar{\nabla}\tilde{\phi} = \nabla\tilde{\phi} + \phi\hat{n}\delta(f), \qquad \bar{\nabla}^2\tilde{\phi} = \nabla^2\tilde{\phi} + \phi_n\delta(f) + \nabla\cdot[\phi\hat{n}\delta(f)]$$

The above results give:

$$\bar{\Box}^2\tilde{\phi} = \frac{1}{c^2}\frac{\bar{\partial}^2\tilde{\phi}}{\partial t^2} - \bar{\nabla}^2\tilde{\phi} = \Box^2\tilde{\phi} - \left(\frac{v_n\phi_t}{c^2} + \phi_n\right)\delta(f)$$

$$-\frac{1}{c^2}\frac{\partial}{\partial t}[v_n\phi\delta(f)] - \nabla\cdot[\phi\hat{n}\delta(f)]$$

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The Governing Wave Equation for Deriving Kirchhoff Formulas (Cont'd)

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Since $\Box^2\tilde{\phi} = 0$, and using $M_n = v_n/c$, we get

$$\boxed{\bar{\Box}^2\tilde{\phi} = -\left(\phi_n + \frac{1}{c}M_n\phi_t\right)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}[M_n\phi\,\delta(f)] - \nabla\cdot[\phi\,\hat{n}\,\delta(f)]}$$

We now solve this wave equation for stationary, subsonic and supersonic surfaces.

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Derivation of the Classical Kirchhoff Formula (Cont'd)

Taking the divergence operator in and using subscript ret for retarded time, we get *the classical Kirchhoff formula*

$$\boxed{4\pi\tilde{\phi}(\vec{x}, t) = \int_{f=0} \frac{[c^{-1}\dot{\phi}\cos\theta - \phi_n]_{\text{ret}}}{r} dS + \int_{f=0} \frac{\cos\theta}{r^2}[\phi]_{\text{ret}} dS}$$

In this equation $\cos\theta = \hat{n}\cdot\hat{r}$. Again, our method tells that $\tilde{\phi}(\vec{x}, t) = 0$ in the interior of $f = 0$ which is not obvious from classical derivation.

Note: Only r is a function of $\vec{x}$ in the integrands of the integrals in previous vugraph. We assume $\vec{x}$ is not on S and S is piecewise smooth. The justification for bringing the divergence operator inside the integral follows from classical analysis.

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Derivation of the Classical Kirchhoff Formula

The Kirchhoff surface $f(\vec{x})$ is now stationary so that $M_n = 0$. The governing wave equation is $\boxed{\bar{\Box}^2\tilde{\phi} = -\phi_n\delta(f) - \nabla\cdot[\phi\hat{n}\delta(f)]}$

$$4\pi\tilde{\phi}(\vec{x}, t) = -\int \frac{\phi_n}{r}\delta(f)\delta(g)d\vec{y}d\tau - \nabla_{\vec{x}}\cdot\int \frac{\phi\hat{n}}{r}\delta(f)\delta(g)d\vec{y}d\tau$$

where ϕ_n and ϕ in the integrands are functions of $(\vec{y}, \tau)$. Now let $\tau \rightarrow g$, $\frac{\partial g}{\partial\tau} = 1$, and integrate with respect to g, to get

$$4\pi\tilde{\phi}(\vec{x}, t) = -\int \frac{\phi_n(\vec{y}, t - r/c)}{r}\delta(f)d\vec{y} - \nabla_{\vec{x}}\cdot\int \frac{\phi(\vec{y}, t - r/c)\hat{n}}{r}\delta(f)d\vec{y}$$

We have dealt with these integrals before. The integration of $\delta(f)$ gives

$$4\pi\tilde{\phi}(\vec{x}, t) = -\int_{f=0} \frac{1}{r}\,\phi_n(\vec{y}, t - r/c)\,dS - \nabla_{\vec{x}}\cdot\int_{f=0} \frac{\hat{n}}{r}\,\phi(\vec{y}, t - r/c)\,dS$$

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NOISE GENERATION FROM MOVING SOURCES (CONT'D)

METHOD OF DESCENT : FROM $\Box_3^2 \rightarrow \Box_2^2$ (D'ALEMBERTIAN IN 2D)

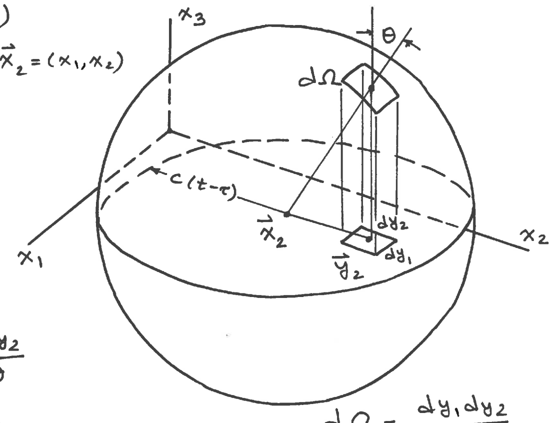
TO SOLVE $\Box_2^2 \phi = Q(\vec{x}_2, t)$

WHERE $\Box_2^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla_2^2$, $\vec{x}_2 = (x_1, x_2)$

WE SOLVE

$$\Box_3^2 \phi = Q(\vec{x}_2, t)$$

i.e. WE ASSUME Q DOES NOT DEPEND ON x_3.



$$4\pi\phi(\vec{x}_2,t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r=c(t-\tau)} Q(\vec{y}_2,\tau)\, d\Omega$$

$$= 2\int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r_2=c(t-\tau)} Q(\vec{y}_2,\tau)\, \frac{dy_1 dy_2}{\cos\theta}$$

$$= 2c \int_{-\infty}^{t} d\tau \int_{r_2=c(t-\tau)} \frac{Q(\vec{y}_2,\tau)\, dy_1 dy_2}{\sqrt{c^2(t-\tau)^2 - r_2^2}}$$

$$\boxed{2\pi\phi(\vec{x}_2,t) = c\int_{-\infty}^{t} d\tau \int_{r_2=c(t-\tau)} \frac{Q(\vec{y}_2,\tau)\, dy_1 dy_2}{\sqrt{c^2(t-\tau)^2 - r_2^2}}}$$

$$d\Omega = \frac{dy_1 dy_2}{\cos\theta}$$

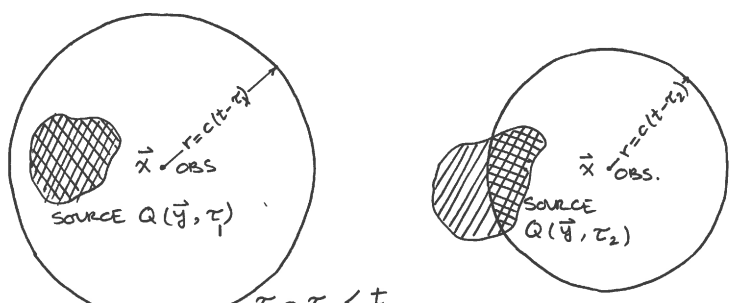
SPHERE WITH RADIUS $r = c(t-\tau)$

$$\cos\theta = \frac{\sqrt{c^2(t-\tau)^2 - |\vec{x}_2 - \vec{y}_2|^2}}{c(t-\tau)}$$

$$r_2 = |\vec{x}_2 - \vec{y}_2|$$

NOISE GENERATION FROM MOVING SOURCES (CONT'D)

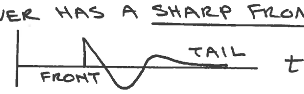
METHOD OF DESCENT (CONT'D)



$\tau = \tau_1 < t$ $\qquad$ $t > \tau_2 > \tau_1$

WE ARE NOW IN 2D SO WE DROP SUBSCRIPT 2 FROM $\vec{x}_2$, $\vec{y}_2$ AND r_2. REGION OF INTEGRATION OVER THE SOURCE REGION IS DOUBLE CROSS-HATCHED.

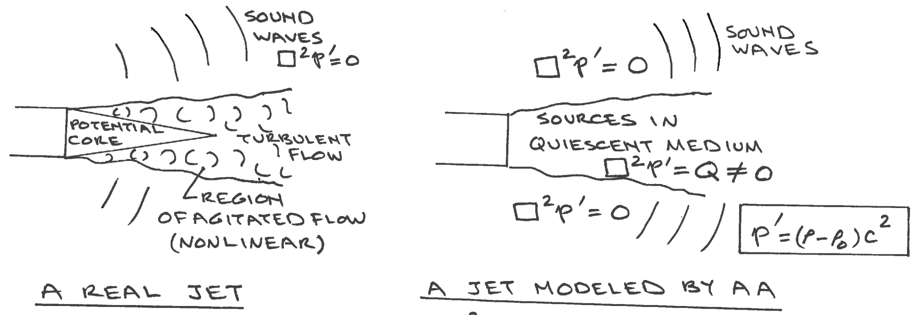
— FOR ANY FINITE SOURCE DISTRIBUTION, AND EVEN FINITE DURATION, THE SIGNAL RECEIVED BY AN OBSERVER HAS A SHARP FRONT BUT A DECAYING TAIL.



— AS $\tau \rightarrow t$, WE HAVE AN IMPROPER CONVERGENT INTEGRAL FOR $Q(\vec{y},\tau)$ A CONTINUOUS FUNCTION.

THE ACOUSTIC ANALOGY (AA)

THE ACOUSTIC ANALOGY WAS INTRODUCED BY M.J. LIGHTHILL IN 1952 IN AEROACOUSTICS. LIGHTHILL'S 1ST PAPER "ON SOUND GENERATED AERODYNAMICALLY. I. GENERAL THEORY", PROC. ROY. SOC. OF LOND., VOL. 211A, 564-587, CONCERNS THE STUDY OF JET NOISE. SUPERFICIALLY, THE IDEA OF AA IS VERY SIMPLE



LIGHTHIL SHOWED THAT $Q = \dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$ WHERE

$$\boxed{T_{ij} = \rho u_i u_j + [(p - p_0) - c^2(\rho - \rho_0)]\delta_{ij}}$$ LIGHTHILL STRESS TENSOR

u_i: COMPONENT OF VELOCITY, $[\ ]_0$: UNDISTURBED MEDIUM CONDITIONS

THE ACOUSTIC ANALOGY (AA) (CONT'D)

LIGHTHILL OBTAINED HIS JET NOISE EQUATION BY MANIPULATING MASS CONTINUITY AND MOMENTUM EQS. WE WILL DO THIS FOR FW-H EQ. WE GIVE SOME MATHEMATICAL RESULTS NOW.

i) THE SOLUTION OF $\Box^2 p' = \dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$ $\qquad V_J$: JET VOLUME

$$4\pi p'(\vec{x}, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{V_J} \frac{[T_{ij}(\vec{y}, \tau)]_{ret}}{r} d\vec{y}$$

$$= \frac{1}{c^2 r_0} \int_{V_J} \left[\frac{\partial^2}{\partial \tau^2} T_{rr}(\vec{y}, \tau)\right]_{ret} d\vec{y} \quad \text{(FAR FIELD)}$$

r_0: MEAN DISTANCE OF OBSERVER FROM THE JET

$T_{rr} = T_{ij} \hat{r}_i \hat{r}_j$, $\hat{r} = \dfrac{\vec{r}_0}{r_0}$ RADIATION DIRECTION

WE CAN WRITE THE LAST RESULT AS

$$4\pi p'(\vec{x}, t) = \frac{1}{c^2 r_0} \int_{V_J} \frac{\partial^2}{\partial t^2} T_{rr}\left(\vec{y}, t - \frac{r}{c}\right) d\vec{y}$$

WE NOW INTRODUCE AUTOCORRELATION OF PRESSURE

STATIONARITY ASSUMPTION USED HERE! $\qquad I(\vec{x}, \tau) = \dfrac{1}{\rho_0 c} \langle p'(\vec{x}, t) p'(\vec{x}, t+\tau) \rangle$ TIME AVERAGE

WHERE τ IS NOW JUST A TIME VARIABLE!

THE ACOUSTIC ANALOGY (CONT'D)

$$I(\vec{x},\tau) = \frac{1}{16\pi^2 \rho_0 c^5 r_0^2} \int_{V_j}\int_{V_j} \left\langle \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_1, t - \frac{r_1}{c}) \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_2, t+\tau-\frac{r_2}{c}) \right\rangle d\vec{y}_1 d\vec{y}_2$$

$$r_1 = |\vec{x} - \vec{y}_1| \;, \quad r_2 = |\vec{x} - \vec{y}_2| \;.$$

WE CAN SHOW THAT

$$\left\langle \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_1, t-\frac{r_1}{c}) \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_2, t+\tau-\frac{r_2}{c}) \right\rangle = \frac{\partial^4 G}{\partial \tau^4}$$

WHERE $G = \langle T_{rr}(\vec{y}_1, t - r_1/c)\, T_{rr}(\vec{y}_2, t+\tau - r_2/c) \rangle$

$$= \langle T_{rr}(\vec{y}_1, t)\, T_{rr}(\vec{y}, t+\tau+(r_1-r_2)/c) \rangle$$

WE HAVE AGAIN USED THE ASSUMPTION OF STATIONARY PROCESS HERE.

$$\frac{r_1 - r_2}{c} \approx \frac{1}{c}\, \hat{r}\cdot(\vec{y}_2 - \vec{y}_1) \equiv \frac{1}{c}\, \hat{r}\cdot\vec{\eta}$$

DEFINE TWO POINT CROSS CORRELATION

$$R(\vec{y},\vec{\eta},\tau) = \langle T_{rr}(\vec{y},t)\, T_{rr}(\vec{y}+\vec{\eta}, t+\tau) \rangle$$

$$I(\vec{x},\tau) = \frac{1}{16\pi^2 \rho_0 c^5 r_0^2} \frac{\partial^4}{\partial \tau^4} \int_{V_j}\int_{V_j} R(\vec{y},\vec{\eta},\tau+\hat{r}\cdot\vec{\eta}/c)\, d\vec{y}\, d\vec{\eta}$$

THE ACOUSTIC ANALOGY (CONT'D)

FROM THIS RESULT, WE CAN GET THE SPECTRAL DENSITY OF THE SOUND BY FOURIER TRANSFORM:

$$S(\vec{x},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} I(\vec{x},\tau)\, e^{i\omega\tau} d\tau$$

$$= \frac{\pi\omega^4}{2\rho_0 c^5 r_0^2} \int_{V_j} H[\vec{y}, \frac{\omega}{c}\hat{r}, (1-M_r)\omega]\, d\vec{y}$$

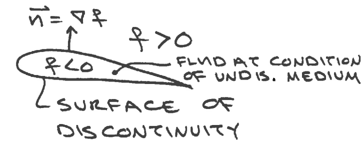
WHERE

$$H(\vec{y},\vec{k},\omega) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty}\int_{V_j} e^{i(\omega\tau - \vec{k}\cdot\vec{\eta})} R(\vec{y},\vec{\eta},\tau)\, d\vec{\eta}\, d\tau$$

HERE M_r IS BASED ON CONVECTION SPEED OF TURBULENT EDDIES. THE NEXT STEP IN THE GAME IS COMING UP WITH A MODEL FOR $R(\vec{y},\vec{\eta},\tau)$. VARIOUS MODELS ARE USED. THE PROCESS IS CONTINUING!

THE ACOUSTIC ANALOGY (CONT'D)

DERIVATION OF THE FFOWCS WILLIAMS-HAWKINGS (FW-H) EQ.



MASS CONT. :

$$\frac{\bar\partial \rho}{\partial t} + \frac{\bar\partial}{\partial x_i}(\rho u_i) =$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)}_{=0} + \Big(\overset{(\rho-\rho_0)}{\cancel{\rho}}\frac{\partial f}{\partial t} + \rho \underbrace{u_i n_i}_{u_n}\Big)\delta(f)$$

$$= \rho(u_n - v_n)\delta(f) + \rho_0 v_n \delta(f) \qquad (1)$$

$$\frac{\partial f}{\partial t} = -v_n$$

v_n LOCAL NORMAL VELOCITY OF THE SURFACE

MOM. EQ.

$$\frac{\bar\partial}{\partial t}(\rho u_i) + \frac{\bar\partial}{\partial x_j}(\rho u_i u_j) + \frac{\bar\partial p}{\partial x_i} = \qquad \text{(INVIS.)}$$

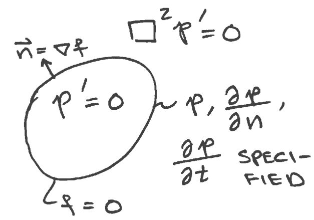
$$\underbrace{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i}}_{=0} + \Big[\rho u_i \frac{\partial f}{\partial t} + \rho u_i u_j n_j + (p-p_0)n_i\Big]\delta(f)$$

$$= \big[\rho u_i (u_n - v_n) + (p - p_0) n_i\big]\delta(f) \qquad (2)$$

TAKE $\bar\partial/\partial t$ OF BOTH SIDES OF EQ.(1), $\bar\partial/\partial x_i$ (SUMMED OVER i) OF BOTH SIDES OF EQ.(2), SUBTRACT THE RESULT OF EQ.(2) FROM RESULT OF EQ.(1). THIS GIVES $\bar\partial^2\rho/\partial t^2 + \dots = \dots$. SUBTRACT $\nabla^2\rho$ FROM BOTH SIDES, DEFINE $p' = c^2(\rho-\rho_0)$, REARRANGE TO GET THE FW-H EQ.

THE ACOUSTIC ANALOGY (CONT'D)

THE K EQUATION (K: KIRCHHOFF)



WE NEED TO FIND WHAT $\bar\Box^2 p'$ IS BECAUSE p' IS DISCONTINUOUS ACROSS $f = 0$

$$v_n = -\frac{\partial f}{\partial t}$$

$f = f(\vec{x}, t) = 0$ DEFORMABLE!

$$\frac{\bar\partial p'}{\partial t} = \frac{\partial p'}{\partial t} + (p' - 0)\frac{\partial f}{\partial t}\delta(f) = \frac{\partial p'}{\partial t} - v_n p' \delta(f)$$

$$\frac{\bar\partial^2 p'}{\partial t^2} = \frac{\partial^2 p'}{\partial t^2} - v_n \frac{\partial p'}{\partial t}\delta(f) - \frac{\partial}{\partial t}\big[v_n p' \delta(f)\big]$$

$$\bar\nabla p' = \nabla p' + p' \vec{n}\,\delta(f)$$

$$\bar\nabla^2 p' = \nabla^2 p' + \nabla p' \cdot \vec{n}\,\delta(f) + \nabla\cdot\big[p'\vec{n}\,\delta(f)\big]$$

$$= \nabla^2 p' + \frac{\partial p}{\partial n}\delta(f) + \nabla\cdot\big[p'\vec{n}\,\delta(f)\big]$$

$$\bar\Box^2 p' = \underbrace{\Box^2 p'}_{\text{0 EVERYWHERE!}} - \Big(\frac{M_n}{c}\frac{\partial p'}{\partial t} + \frac{\partial p}{\partial n}\Big)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}\big[M_n p' \delta(f)\big] - \nabla\cdot\big[p'\vec{n}\,\delta(f)\big]$$

$$\boxed{\bar\Box^2 p' = -\Big(\frac{M_n}{c}\frac{\partial p'}{\partial t} + \frac{\partial p}{\partial n}\Big)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}\big[M_n p' \delta(f)\big] - \nabla\cdot\big[p'\vec{n}\,\delta(f)\big]}$$

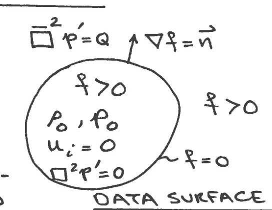
THE SOLUTION OF THIS EQ., <u>THE K EQ.</u> GIVES KIRCHHOFF FORMULA!

LEC. 8/9

THE ACOUSTIC ANALOGY (CONT'D)

DERIVATION OF THE FFOWCS WILLIAMS-HAWKINGS (FW-H) EQ.

WE TAKE THE DATA SURFACE AS A SURFACE OF DISCONTINUITY BY EXTENDING $p' = c^2(\rho - \rho_0)$ TO INSIDE OF THE DATA SURFACE $f = 0$. IT IS CLEAR THAT $p' = 0$ INSIDE $f = 0$ AND, THEREFORE, $\Box^2 p' = 0$ THERE. WE ASSUME THAT INSIDE $f = 0$, FLUID AT CONDITIONS OF UNDISTURBED MEDIUM EXISTS WITH $\vec{u} = 0$, WHERE $\vec{u}$ IS FLUID VELOCITY. WE WANT TO DERIVE AN EQ. OF THE FORM $\bar{\Box}^2 p' = Q$, WHERE $\bar{\Box}^2$ IS THE WAVE EQ. WITH GENERALIZED DERIVATIVES.



DATA SURFACE, PENETRABLE

v_n : LOCAL NORMAL VELOCITY OF DATA SURFACE

MASS CONTINUITY EQ.

$$\frac{\bar{\partial}\rho}{\partial t} + \frac{\bar{\partial}}{\partial x_i}(\rho u_i) = \underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)}_{0} + \Big[(\rho - \rho_0)\underbrace{\frac{\partial f}{\partial t}}_{-v_n} + \rho \underbrace{u_i n_i}_{u_n}\Big]\delta(f)$$

$$= \rho_0 v_n \delta(f) + \rho(u_n - v_n)\delta(f) \equiv Q$$

MOMENTUM EQ. (INVISCID)

$$\frac{\bar{\partial}}{\partial t}(\rho u_i) + \frac{\bar{\partial}}{\partial x_j}(\rho u_i u_j) + \frac{\bar{\partial} p}{\partial x_i} = \underbrace{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial p}{\partial x_i}}_{0}$$

$$+ \Big[\rho u_i \underbrace{\frac{\partial f}{\partial t}}_{-v_n} + \rho u_i \underbrace{u_j n_j}_{u_n} + (p - p_0)n_i\Big]\delta(f) = \big[\rho u_i(u_n - v_n) + (p - p_0)n_i\big]\delta(f)$$

$$\equiv Q_i$$

LEC. 8/10

THE ACOUSTIC ANALOGY (CONT'D)

DERIVATION OF THE FW-H EQ. (CONT'D)

NOW FORM $\frac{\bar{\partial}}{\partial t}$(L.S. OF MASS CON.) $-\frac{\bar{\partial}}{\partial x_i}$(L.S. OF MOM. EQ.) $= \frac{\bar{\partial} Q}{\partial t} - \frac{\bar{\partial} Q_i}{\partial x_i}$

WE GET

$$\frac{\bar{\partial}^2 \rho}{\partial t^2} - \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\underbrace{(\rho u_i u_j + p\delta_{ij})}_{} = \frac{\bar{\partial} Q}{\partial t} - \frac{\bar{\partial} Q_i}{\partial x_i}$$

NOW ADD AND SUBTRACT $\nabla^2 \rho$ TO BOTH SIDES, REPLACE ρ BY $\rho' = \rho - \rho_0$, KEEP ONLY $\bar{\Box}^2(\rho' c^2)$ ON THE LEFT SIDE TO GET

$$\boxed{\begin{aligned} &\bar{\Box}^2(\rho' c^2) = \frac{\bar{\partial} Q}{\partial t} - \frac{\bar{\partial} Q_i}{\partial x_i} + \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\Big[\underbrace{\rho u_i u_j + (p - p_0) - c^2(\rho - \rho_0)}_{T_{ij}}\Big] H(f) \\ &Q = \rho_0 v_n \delta(f) + \rho(u_n - v_n)\delta(f) \\ &Q_i = [\rho u_i(u_n - v_n) + (p - p_0)n_i]\delta(f) \end{aligned}}$$

THIS IS THE FW-H EQ. FOR PENETRABLE DATA SURFACE

$H(f)$: HEAVISIDE FN, $H(f) = \begin{cases} 0 & f < 0 \quad \text{INSIDE DATA SURFACE} \\ 1 & f > 0 \quad \text{OUTSIDE DATA SURFACE} \end{cases}$

WE USE $p' \equiv \rho' c^2$ IN OUR WORK.

THE ACOUSTIC ANALOGY (CONT'D)

COMMENTS OF FW-H EQ.

i) WE HAVE EMBEDDED AN EXTERNAL PROBLEM (FOR $f>0$) INTO AN UNBOUNDED SPACE BECAUSE OF SIMPLICITY OF THE GREEN'S FUNCTION IN UNBOUNDED SPACE. TO CONVINCE YOURSELF THAT THIS IS LEGITIMATE, SEE THE EXAMPLE FOR AN ODE IN MY NASA TR-R450, 1975 (APPENDIX)

ii) IF $f=0$ IS AN IMPENETRABLE SURFACE, i.e. $u_n = v_n$, THEN THE FW-H EQ. BECOMES

$$\boxed{\bar{\Box}^2 p' = \underbrace{\frac{\bar{\partial}}{\partial t}\left[\rho_0 v_n \delta(f)\right]}_{\text{THICKNESS}} - \underbrace{\bar{\nabla}\cdot\left[p\vec{n}\,\delta(f)\right]}_{\text{LOADING}} + \underbrace{\frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right]}_{\text{QUADRUPOLES}}}$$

iii) THICKNESS AND LOADING TERMS OF FW-H EQ. CAN BE USED TO SOLVE THE THICKNESS AND LOADING PROBLEMS OF AERODYNAMICS.

iv) THE NOISE GENERATED BY ALL DISCONTINUITIES IN THE FLOW, SHOCKS, WAKE AND VORTICES ARE INCLUDED IN QUADRUPOLE SOURCE TERM. THE EASIEST WAY TO SEE THIS AND FIND THE SOURCE STRENGTHS IS BY USING GENERALIZED FUNCTIONS!

THE ACOUSTIC ANALOGY (CONT'D)

COMMENTS ON FW-H EQ.

v) TYPICAL SOURCE TERMS IN FW-H & K EQ.

SURFACE TERMS $\frac{\partial}{\partial t}\left[Q\delta(f)\right]$, $\nabla\cdot\left[\vec{Q}\,\delta(f)\right]$

VOLUME TERM (FW-H) $\frac{\partial^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right]$

THE SUBSONIC SOLUTIONS ARE EASY TO GIVE!

vi) WE ALWAYS TRY TO GET THE EXACT SOLUTION FOR EXACT GEOMETRY OF DATA SURFACE $f=0$ AND EXACT KINEMATICS

vii) THERE ARE MANY EQUIVALENT SOLUTIONS OF FW-H EQ. WITH DIFFERENT DEGREES OF COMPLEXITY AND USEFULNESS FOR CODE DEVELOPMENT. IN GENERAL, ONE SHOULD SPEND A LOT OF TIME THINKING ABOUT ALGORITHMS FOR CODE DEVELOPMENT.

The Ffowcs Williams-Hawkings (FW-H) Equation

- Published in 1969 in Philosophical Transactions of the Royal Society
- The derivation is based on Lighthill's acoustic analogy
- Used for helicopter and propeller noise calculations since mid-seventies
- Many computer codes in aircraft industry are based on FW-H equation

$$\boxed{\Box^2 p' = \underbrace{\frac{\partial}{\partial t}[\rho_0 v_n \delta(f)]}_{\text{Thickness}} - \underbrace{\frac{\partial}{\partial x_i}[p n_i \delta(f)]}_{\text{Loading}} + \underbrace{\frac{\partial^2}{\partial x_i \partial x_j}[T_{ij} H(f)]}_{\text{Quadrupole}}}$$

■
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Use of Green's Functions for Discontinuous Solutions

Green's function can be used to find discontinuous solutions if the derivatives in the differential equation are treated as generalized derivatives. This adds to usefulness of Green's function.

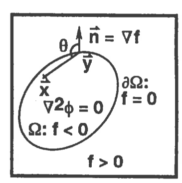
Example: *Green's Identity for Laplace Equation*

Let $\tilde{\phi}(\vec{x}) = \begin{cases} \phi(\vec{x}) & \vec{x} \in \Omega \\ 0 & \vec{x} \notin \Omega \end{cases} \Rightarrow \nabla^2 \tilde{\phi} = 0$ everywhere.

$$\bar{\nabla}\tilde{\phi} = \nabla\tilde{\phi} + \Delta\tilde{\phi}\vec{n}\delta(f) = \nabla\tilde{\phi} - \phi\vec{n}\delta(f)$$

$$\bar{\nabla}^2\tilde{\phi} = \nabla^2\tilde{\phi} - \nabla\phi \cdot \vec{n}\delta(f) - \nabla \cdot [\phi\vec{n}\delta(f)]$$

$$= -\frac{\partial\phi}{\partial n}\delta(f) - \nabla \cdot [\phi\vec{n}\delta(f)]$$



Interior Problem

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Use of Green's Functions for Discontinuous Solutions (Cont'd)

Since this equation is valid in the unbounded space, we can use the Green's function $-\frac{1}{4\pi r}$ to get the Green's identity

$$4\pi\tilde{\phi}(\vec{x}) = \int \frac{1}{r}\frac{\partial\phi}{\partial n}\delta(f)d\vec{y} + \nabla_{\vec{x}}\cdot\int\frac{\phi\vec{n}}{r}\delta(f)d\vec{y}$$

$$= \int_{f=0}\frac{1}{r}\frac{\partial\phi}{\partial n}dS + \nabla_{\vec{x}}\cdot\int_{f=0}\frac{\phi\vec{n}}{r}dS = \int_{f=0}\frac{\phi_n}{r}dS - \int_{f=0}\frac{\phi\cos\theta}{r^2}dS$$

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This method tells us that when $\vec{x}\notin\Omega$, $\tilde{\phi} = 0$ which is not obvious from the classical derivation. The exterior problem is similar.

Note: $r = |\vec{x}-\vec{y}|$ is the only term in the integrands of the above integrals which is a function of $\vec{x}$. We assume that $\vec{x}$ is not located on S and S is piecewise smooth. The justification for the exchange of the divergence and integral operators follows from classical analysis.

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How Does $\delta(f)$ Appear in Applications? (Cont'd)

- In our work the discontinuities in functions are either real (e.g., shock waves) or artificial (e.g., across blade surface in derivation of FW-H eq.).
- **Example**: *Shock surface sources* in Lighthill jet noise theory. Let the shock surfaces be defined by $f(\vec{x}, t) = 0$. We can show that Lighthill's equation is valid in presence of shocks if we interpret the derivatives of the source term as generalized derivatives:

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$$\Box^2 p' = \frac{\bar{\partial}^2 T_{ij}}{\partial x_i \partial x_j}$$

$$= \frac{\bar{\partial}}{\partial x_i}\left[\frac{\partial T_{ij}}{\partial x_j} + \Delta T_{ij}\frac{\partial f}{\partial x_j}\delta(f)\right]$$

$$= \underbrace{\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}}_{\text{Turbulence Source}} + \underbrace{\Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right)\frac{\partial f}{\partial x_i}\delta(f) + \frac{\partial}{\partial x_i}\left[\Delta T_{ij}\frac{\partial f}{\partial x_j}\delta(f)\right]}_{\text{Shock Surface Sources}}$$

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SOLUTION OF FW-H EQUATION

THICKNESS AND LOADING TERMS

WE HAVE TERMS OF THE FOLLOWING TYPES

$$\frac{\partial}{\partial t}\left[Q(\vec{x},t)\delta(f)\right] \quad \text{THICKNESS}, \quad \nabla\cdot\left[\vec{Q}\,\delta(f)\right] \quad \text{LOADING}$$

IF $f=0$ IS THE DATA SURFACE THAT IS IMPENETRABLE, THEN $Q = \rho_0 v_n$ AND $\vec{Q} = p\vec{n}$

1. SOLUTION OF $\square^2 p' = \frac{\partial}{\partial t}\left[Q\delta(f)\right]$

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int \frac{Q(\vec{y},\tau)}{r}\delta(g)\delta(f)\,d\vec{y}\,d\tau$$

ASSUMING $f=0$ IS RIGID (VERY COMMON), FIX AN $\vec{\eta}$-FRAME TO THIS SURFACE SUCH THAT $f(\vec{y}(\vec{\eta},\tau),\tau) \equiv \bar{f}(\vec{\eta})$, i.e. TIME INDEPENDENT. WE HAVE $d\vec{y} = d\vec{\eta}$ BUT, IN GENERAL, $\vec{y} = \vec{y}(\vec{\eta},\tau)$, i.e., WE HAVE $\vec{y}$ NOW A FUNCTION OF SOURCE TIME. WE HAVE

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int \frac{Q[\vec{y}(\vec{\eta},\tau),\tau)]}{r(\vec{\eta},\tau;\vec{x})}\delta(g)\delta(\bar{f})\,d\vec{\eta}\,d\tau$$

NOTE: $\int \equiv \int_{-\infty}^{t}\iiint_{-\infty}^{\infty} \cdots\cdots d\vec{y}\,d\tau$

SOLUTION OF FW-H EQ. (CONT'D)

NEXT LET $\tau \rightarrow g$, WE HAVE $d\tau = \dfrac{dg}{|\partial g/\partial\tau|_{\vec{\eta}}}$

$$g = \tau - t + |\vec{x} - \vec{y}(\vec{\eta},\tau)|/c \leftarrow \underline{\text{NOTE}}\ r = |\vec{x}-\vec{y}(\vec{\eta},\tau)| = r(\vec{\eta},\tau;\vec{x})$$

$$\left.\frac{\partial g}{\partial\tau}\right|_{\vec{\eta}} = 1 - \frac{1}{c}\underbrace{\frac{\partial\vec{y}}{\partial\tau}}_{\vec{v}\ \text{LOCAL SURFACE VELOCITY}}\cdot\hat{r} = 1 - M_r, \quad M_r = \frac{\vec{v}}{c}\cdot\hat{r}$$

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int \left[\frac{Q[\vec{y}(\vec{\eta},\tau),\tau]}{r|1-M_r|}\right]_{g=0}\delta(\bar{f})\,d\vec{\eta}$$

WHAT IS THE MEANING OF $[\cdots\cdots]_{g=0}$? IT MEANS FIND THE EMISSION TIME τ^* BY SOLVING

$$g = \tau^* - t + |\vec{x} - \vec{y}(\vec{\eta},\tau^*)|/c = 0$$

i.e. $\tau^* = \tau^*(\vec{\eta};\vec{x},t)$ [CALLED ALSO RETARDED TIME]

NOW INTEGRATE THE DELTA FUNCTION $\delta(\bar{f})$

$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int_{f=0}\left[\frac{Q[\vec{y}(\vec{\eta},\tau),\tau]}{r|1-M_r|}\right]_{\tau^*} dS}$$

THIS IS THE THICKNESS NOISE PART OF FORMULATION 1.

SOLUTION OF FW-H EQ.

WE WILL WORK MORE ON THIS EQUATION LATER. WE NOW MENTION A FEW FACTS.

i) IN GENERAL, $\partial/\partial t$ IS TAKEN NUMERICALLY

ii) IF ALL PARTS OF THE SURFACE MOVE SUBSONICALLY, EACH POINT ON THE SURFACE HAS ONLY ONE EMISSION TIME τ^*. FOR SUPERSONIC PARTS OF THE SURFACE, EACH POINT ON THE SURFACE AN ODD NUMBER OF EMISSION TIMES, MULTIPLICITY OF THE ROOTS OF $g=0$ INCLUDED.

iii) THE ABOVE EQUATION IS A POOR CANDIDATE FOR PREDICTING THE THICKNESS NOISE OF SUPERSONIC SURFACES.

iv) IT IS COMMON TO WRITE $f=0$ INSTEAD OF $\bar{f}=0$ IN THE ABOVE RESULT, AS WE HAVE DONE. THIS IS DONE TO REDUCE THE NUMBER OF SYMBOLS USED. REMEMBER THE DISTINCTION BETWEEN $f(\vec{y},\tau)$ AND $\bar{f}(\vec{\eta}) = f[\vec{y}(\vec{\eta},\tau),\tau]$.

SOLUTION OF FW-H EQ. (CONT'D

BRINGING $\partial/\partial t$ INTO THE INTEGRAL

REMEMBERING THAT f (ACTUALLY $\bar{f}$!) IS INDEPENDENT OF TIME t, WE HAVE

$$\frac{\partial}{\partial t}\int_{f=0} \frac{Q[\vec{y}(\vec{\eta},\tau^*),\tau^*]}{r(\vec{\eta},\tau;\vec{x})|1-M_r(\vec{\eta},\tau;\vec{x})|}\,dS = \int_{f=0} \frac{\partial}{\partial \tau^*}\left[\frac{Q}{r|1-M_r|}\right]\frac{\partial \tau^*}{\partial t}\,dS$$

FROM $\tau^* - t + |\vec{x} - \vec{y}(\vec{\eta},\tau^*)|/c = 0$, WE GET $\dfrac{\partial \tau^*}{\partial t} = \left[\dfrac{1}{1-M_r}\right]_{\tau^*}$

$$\boxed{4\pi p'(\vec{x},t) = \int_{f=0} \left\{\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\left[\frac{Q[\vec{y}(\vec{\eta},\tau),\tau]}{r|1-M_r|}\right]\right\}_{\tau^*} dS}$$

$$\frac{\partial}{\partial \tau}\bigg|_{\vec{\eta}} \underbrace{Q[\vec{y}(\vec{\eta},\tau),\tau]}_{\bar{Q}(\vec{\eta},\tau)} = \frac{\partial Q}{\partial y_i}\frac{\partial y_i}{\partial \tau} + \frac{\partial Q}{\partial \tau} \equiv \dot{Q}$$

RATE OF CHANGE OF Q AS MEASURED BY AN OBSERVER ON $f=0$

$$\frac{\partial}{\partial \tau} r = \frac{\partial}{\partial \tau}|\vec{x} - \vec{y}(\vec{\eta},\tau)| = -\hat{\vec{r}}\cdot\vec{v} = -v_r$$

SURFACE VELOCITY IN RADIATION DIRECTION

WHEN THESE ARE SUBSTITUTED IN THE ABOVE INTEGRAL, WE GET THE THICKNESS NOISE PART OF FORMULATION 1A USED IN WOPWOP

N

SOLUTION OF FW-H EQ. (CONT'D

2. SOLUTION OF $\Box^2 p' = -\nabla\cdot[\vec{Q}\,\delta(f)]$

FORMAL SOLUTION

$$4\pi p'(\vec{x},t) = -\nabla_x\cdot\int \frac{\vec{Q}(\vec{y},\tau)}{r}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

THE FIRST THOUGHT COMING TO OUR MIND IS TO USE NUMERICAL DIFFERENTIATION. BUT THIS REQUIRES EVALUATION OF A SURFACE INTEGRAL SIX TIMES! NOT VERY EFFICIENT METHOD. CAN WE CONVERT THE SPACE DERIVATIVES TO OBSERVER TIME DERIVATIVE EXACTLY? THE ANSWER IS YES!

$$\nabla_x\cdot\int \vec{Q}(\vec{y},\tau)\,\delta(f)\,\frac{\delta(g)}{r}\,d\vec{y}\,d\tau = \int \vec{Q}\,\delta(f)\cdot\nabla_x\frac{\delta(g)}{r}\,d\vec{y}\,d\tau$$

$$\boxed{\begin{aligned}\nabla_x\frac{\delta(g)}{r} &= \frac{\hat{r}}{c}\,\frac{\delta'(g)}{r} - \hat{r}\,\frac{\delta(g)}{r^2}\\ &= -\frac{\hat{r}}{cr}\frac{\partial}{\partial t}\delta(g) - \hat{r}\,\frac{\delta(g)}{r^2}\end{aligned}}$$

$$\boxed{Q_r = \vec{Q}\cdot\hat{r}}$$

$$= -\int\left[\frac{Q_r}{cr}\,\delta(f)\frac{\partial}{\partial t}\delta(g) + \frac{Q_r}{r^2}\,\delta(f)\,\delta(g)\right]d\vec{y}\,d\tau$$

$$= -\frac{\partial}{\partial t}\int \frac{Q_r}{cr}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau - \int \frac{Q_r}{r^2}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

SOLUTION OF FW-H EQ. (CONT'D)

Cyclic control in fixed frame

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int \frac{Q_r}{cr}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau + \int \frac{Q_r}{r^2}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int_{f=0}\left[\frac{Q_r}{cr|1-M_r|}\right]_{\tau^*}dS + \int_{f=0}\left[\frac{Q_r}{r^2|1-M_r|}\right]_{\tau^*}dS}$$

THIS IS THE LOADING PART OF FORMULATION 1. AGAIN, WE CAN TAKE THE DERIVATIVE WRT t INSIDE THE FIRST INTEGRAL TO GET THE LOADING PART OF FORMULATION 1A

$$\boxed{4\pi p'(\vec{x},t) = \int_{f=0}\left\{\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left[\frac{Q_r}{cr|1-M_r|}\right]\right\}_{\tau^*}dS + \int_{f=0}\left[\frac{Q_r}{r^2|1-M_r|}\right]_{\tau^*}dS}$$

REMEMBER $Q_r = \vec{Q}(\vec{\eta},\tau)\cdot\hat{r}(\vec{\eta},\tau;\vec{x})$ SO THAT

$$\frac{\partial}{\partial\tau}Q_r = \dot{\vec{Q}}\cdot\hat{r} + \vec{Q}\cdot\frac{\partial\hat{r}}{\partial\tau}$$

THIS EXPLAINS WHY FORMULATION 1A HAS SO MANY TERMS!

SOLUTION OF FW-H EQ. (CONT'D)

SUMMARY

THE SOLUTION OF $\Box^2 p' = \frac{\partial}{\partial t}[Q\delta(f)] - \nabla\cdot[\vec{Q}\,\delta(f)]$

WHERE $Q = \rho_o v_n + \rho(u_n - v_n)$, $\vec{Q} = \rho(u_n - v_n)\vec{u} + (p - p_o)\vec{n}$

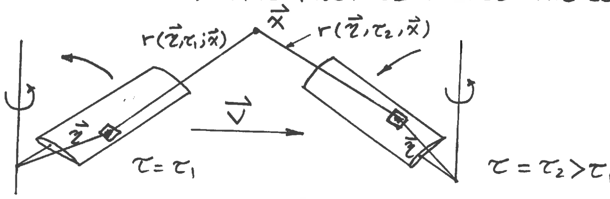
IS:

$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{c\,\partial t}\int_{f=0}\left[\frac{cQ + Q_r}{r(1-M_r)}\right]_{ret} dS + \int_{f=0}\left[\frac{Q_r}{r^2(1-M_r)}\right]_{ret} dS}$$

$Q_r = \vec{Q}\cdot\hat{r}$, $\hat{r} = (\vec{x}-\vec{y})/|\vec{x}-\vec{y}|$, $M_r = \vec{M}\cdot\hat{r}$, $\frac{\partial}{\partial t}$ TAKEN NUMERICALLY

THIS IS FORMULATION 1.

NOTE: $\partial/\partial t = \partial/\partial t|_{\vec{x}}$, THE $\vec{x}$-FRAME IS FIXED TO THE UNDISTURBED MEDIUM. $r = |\vec{x} - \vec{y}(\vec{\eta},\tau)|$, $\vec{\eta}$ IS THE LAGRANGIAN VARIABLE IN A FRAME FIXED TO $S: f=0$. THEREFORE $r = r(\vec{\eta},\tau;\vec{x})$ AND MUST BE INSIDE THE SQ. BRACKETS.



SOLUTION OF FW-H EQ. (CONT'D)

FORMULATION 1A

SOLUTION OF $\Box^2 p'_T = \frac{\partial}{\partial t}[Q\delta(f)]$ (THICKNESS SOURCE)

$$\boxed{4\pi p'_T(\vec{x},t) = \int_{f=0}\left[\frac{\dot{Q}}{r(1-M_r)^2}\right]_{ret} dS + \int_{f=0}\left[\frac{Q(r\dot{M}_r + cM_r - cM^2)}{r^2(1-M_r)^3}\right]_{ret} dS}$$

SOLUTION OF $\Box^2 p'_L = -\nabla\cdot[\vec{Q}\,\delta(f)]$ (LOADING SOURCE)

$$\boxed{\begin{aligned}4\pi p'_L(\vec{x},t) &= \frac{1}{c}\int_{f=0}\left[\frac{\dot{Q}_r}{r(1-M_r)^2}\right]_{ret} dS + \int_{f=0}\left[\frac{Q_r - \vec{M}\cdot\vec{Q}}{r^2(1-M_r)^2}\right]_{ret} dS \\ &+ \frac{1}{c}\int_{f=0}\left[\frac{Q_r(r\dot{M}_r + cM_r - cM^2)}{r^2(1-M_r)^3}\right]_{ret} dS\end{aligned}}$$

DOT $(\dot{\ }) \equiv \frac{\partial}{\partial\tau}\Big|_{\vec{\eta}}$, $\dot{M}_r = \dot{M}_i\cdot\hat{r}_i$, $\dot{Q}_r = \dot{Q}_i\hat{r}_i$, $Q_r = \vec{Q}\cdot\hat{r}$

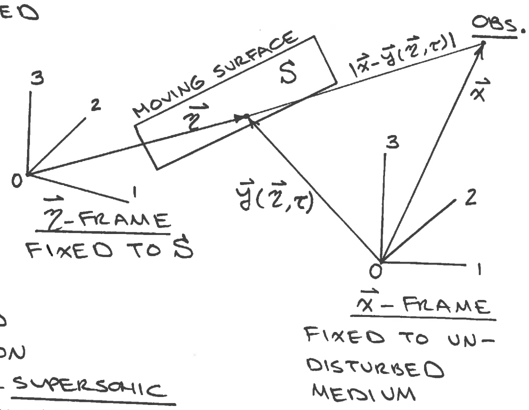
$\frac{\partial}{\partial\tau}\Big|_{\vec{\eta}}$ MEANS RATE OF CHANGE OF A QUANTITY AS MEASURED BY AN INSTRUMENT FIXED TO THE SURFACE $S: f=0$. WE ARE USING THE SUMMATION CONVENTION: $a_i b_i = \sum_{i=1}^{3} a_i b_i$.

SOLUTION OF FW-H EQ. (CONT'D)
TRICKS OF THE TRADE

i) RETARDED TIME IS CALCULATED FROM

$$\tau - t + |\vec{x} - \vec{y}(\vec{\eta},\tau)|/c = 0$$

IN GENERAL THIS IS A TRANSCENTAL (NOT A POLYNOMIAL) EQUATION. USE NUMERICAL METHOD TO FIND τ. NOTE THAT t, $\vec{x}$ AND $\vec{\eta}$ ALWAYS HAVE A NUMERICAL VALUE.



THE SHOOTING TECHNIQUE IS A GOOD METHOD OF SOLVING THIS EQUATION FOR SUBSONIC SURFACES. FOR SUPERSONIC SURFACES, LOTS OF COMPLICATIONS APPEAR. MOST PROBLEMS ARE ASSOCIATED WITH MULTIPLE ROOTS OR NEARLY EQUAL ROOTS. ONE DISCOVERS QUICKLY THAT SMALL ERRORS IN RETARDED TIME CAUSES LARGE ERRORS IN $p'(\vec{x},t)$.

– FOR UNIFORM RECTILINEAR MOTION OF S, WE HAVE ANALYTIC SOLUTION FOR EMISSION TIME – ONE FOR SUBSONIC, TWO OR NONE FOR SUPERSONIC. USE THE GARRICK TRIANGLE TO FIND THE QUADRATIC EQUATION YOU NEED TO SOLVE.

SOLUTION OF FW-H EQ. (CONT'D)
TRICKS OF THE TRADE (CONT'D)

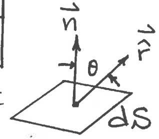
REMEMBER THAT A VECTOR IS A MATHEMATICAL OBJECT THAT HAS A MEANING EVEN WHEN NO FRAME OF REFERENCE IS SPECIFIED. THEREFORE, $\tau - t + |\vec{x} - \vec{y}(\vec{\eta},\tau)|/c = 0$ CAN BE WRITTEN IN EITHER THE FIXED OR MOVING FRAMES. THIS COMMENT ALSO HOLDS FOR ALL DOT PRODUCTS SUCH AS $Q_r = \vec{Q}\cdot\hat{r}$, $M_r = \vec{M}\cdot\hat{r}$, ETC.

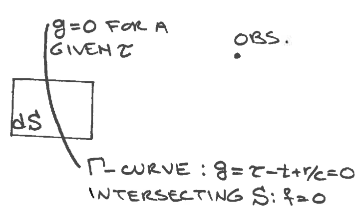
ii) INTEGRATION : USE SMART INTEGRATION METHODS SUCH AS GAUSS-LEGENDRE TECHNIQUE. AVOID RECTANGULAR OR SIMPSON RULES. SMART INTEGRATION METHODS CAN LEAD TO SUBSTANTIAL SAVING IN EXECUTION TIME AND IN INCREASED ACCURACY.

iii) VERY IMPORTANT RELATIONS

$$\boxed{\frac{dS}{|1-M_r|} = \frac{c\,d\Gamma\,d\tau}{\sin\theta} = \frac{d\Sigma}{\Lambda}}$$

$$\boxed{\Lambda^2 = 1 + M_n^2 - 2M_n\cos\theta}$$





$d\Sigma$ IS THE ELEMENT OF THE SURFACE AREA OF $F(\vec{y};\vec{x},t)$ $= f(\vec{y}, t-r/c) = 0$. THIS IS THE LOCUS OF Γ-CURVES FOR $(\vec{x},t)$ FIXED.

SOLUTION OF FW-H EQ. (CONT'D)

TRICKS OF THE TRADE (CONT'D)

TWO OTHER WAYS OF WRITING FORMULATION 1

$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{c\partial t}\int_{-\infty}^{t}\frac{d\tau}{t-\tau}\int_{r=c(t-\tau)}\left[cQ(\vec{y},\tau)+Q_r(\vec{y},\tau)\right]\frac{d\Gamma}{\sin\theta} + \frac{1}{c}\int_{-\infty}^{t}\frac{d\tau}{(t-\tau)^2}\int_{r=c(t-\tau)} Q_r(\vec{y},\tau)\frac{d\Gamma}{\sin\theta}}$$

THE COLLAPSING SPHERE METHOD USED HERE.

– NOTE THAT WE ARE NOT USING LAGRANGIAN VARIABLE HERE. THE VARIABLE $\vec{y}$ IS THE SOURCE POSITION IN A FRAME FIXED TO UNDISTURBED MEDIUM.

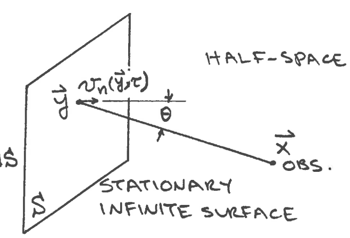
$$\boxed{4\pi p'(\vec{x},t) = \frac{\partial}{c\partial t}\int_{F=0}\frac{1}{r}\left[\frac{cQ+Q_r}{\Lambda}\right]_{ret} d\Sigma + \int_{F=0}\frac{1}{r^2}\left[\frac{Q_r}{\Lambda}\right]_{ret} d\Sigma}$$

– NOTE THAT $r=|\vec{x}-\vec{y}|$ AND SHOULD NOT BE INSIDE THE SQUARE BRACKETS BECAUSE r IS NOT A FUNCTION OF SOURCE TIME. BOTH OF THE ABOVE EQUATIONS ARE VALID FOR SUBSONIC AND SUPERSONIC MOTION OF $S: f=0$. NUMERICAL TIME DIFFERENTIATION CAN CAUSE ERRORS IN $p'(\vec{x},t)$.

SOLUTION OF FW-H EQ. (CONT'D)

RAYLEIGH'S PISTON IN THE WALL REVISITED

WE ARE INTERESTED IN RADIATION INTO THE HALF-SPACE BY VOLUME SOURCES ON THE INFINITE PLANE. WE HAVE SHOWN THAT (RAYLEIGH)



$$2\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int_S \frac{[\rho_0 v_n]_{ret}}{r} dS = \int_S \frac{[\rho_0 \dot{v}_n]_{ret}}{r} dS$$

USING FW-H EQ., WE HAVE

$$4\pi p'(\vec{x},t) = \underbrace{\frac{\partial}{\partial t}\int_S \frac{[\rho_0 v_n]_{ret}}{r} dS}_{\text{RAYLEIGH}:\ 2\pi p'(\vec{x},t)} - \nabla_x\cdot\int_S \frac{[p\vec{n}]_{ret}}{r} dS$$

$\Rightarrow$

$$2\pi p'(\vec{x},t) = -\nabla_x\cdot\int_S \frac{[p\vec{n}]_{ret}}{r} dS$$

TAKE DIVERGENCE INSIDE

$$\boxed{2\pi p'(\vec{x},t) = \int_S\left\{\frac{[\dot{p}]_{ret}\cos\theta}{cr} + \frac{[p]_{ret}\cos\theta}{r^2}\right\} dS}$$

RAYLEIGH'S FIRST INTEGRAL

$\cos\theta = \vec{n}\cdot\hat{r}$

SOLUTION OF FW-H EQ. (CONT'D)

THE CURLE FORMULA : TURBULENT FLOW OVER A STATIONARY SURFACE $S : f(\vec{x},t)=0$

ESSENTIALLY, CURLE GAVE THE SOLUTION OF FW-H EQ. FOR A FLAT SURFACE WITH QUADRUPOLES IN THE VICINITY OF THE SURFACE. THE SOLUTION OF FW-H EQ. (NOT RESTRICTED TO FLAT SURFACES) IN THIS CASE IS :

$$\boxed{4\pi p'(\vec{x},t) = -\nabla_x \cdot \int_S \frac{[p\vec{n}]_{ret}}{r} dS + \frac{\partial^2}{\partial x_i \partial x_j} \int_{f>0} \frac{[T_{ij}]_{ret}}{r} d\vec{y}}$$

IN THE FAR FIELD, WE HAVE THE FOLLOWING SIMPLE RESULT

$$\boxed{4\pi p'(\vec{x},t) = \int_S \frac{[\dot{p}]_{ret}\cos\theta}{cr} dS + \int_{f>0} \frac{[\ddot{T}_{rr}]}{c^2 r} d\vec{y}}$$

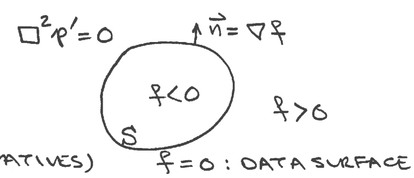
$$T_{rr} = T_{ij}\,\hat{r}_i \cdot \hat{r}_j$$

4/10/01

SOLUTION OF THE FW-H EQ. (CONT'D)

THE KIRCHHOFF FORMULA FOR MOVING SURFACES

THIS RESULT IS MOST SIMPLY OBTAINED BY FIRST EXTENDING p' TO INSIDE THE SURFACE AS FOLLOWS : $\tilde{p}' = \begin{cases} p' & f>0 \\ 0 & f<0 \end{cases}$



$\Rightarrow \square^2 \tilde{p}' = 0$ (ORDINARY DERIVATIVES)

WE NEXT FIND $\bar{\square}^2 p'$ (GENERALIZED DERIVATIVES)

SEE NASA TM-110285 (1996) :

$$\boxed{\begin{aligned}\bar{\square}^2 \tilde{p}' = &-\left(\frac{\partial p'}{\partial n} + \frac{1}{c} M_n \frac{\partial p'}{\partial t}\right)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}\left[M_n p' \delta(f)\right] \\ &- \nabla \cdot \left[p' \vec{n}\, \delta(f)\right]\end{aligned}}$$

NOTE THAT p', $\partial p'/\partial t$ AND $\partial p'/\partial n$ ON THE RIGHT SIDE ARE EVALUATED ON $f = 0_+$, i.e. ON THE EXTERIOR SIDE OF $S: f=0$.

– WE HAVE GIVEN THE SOLUTION OF THE WAVE EQUATION WITH SOURCES OF THE TYPES ABOVE. THE FORMAL SOLUTION OF THIS EQUATION IS THE KIRCHHOFF FORMULA FOR MOVING SURFACES.

LEC. 9/15

SOLUTION OF THE FW-H EQ. (CONT'D)

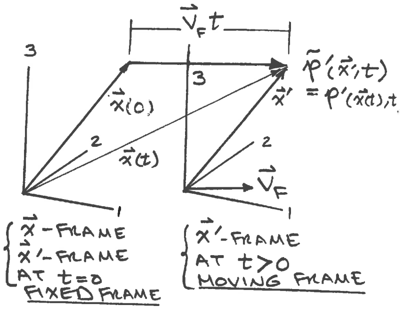
THE KIRCHHOFF FORMULA FOR MOVING SURFACES (CONT'D)

- USE THE KIRCHHOFF FORMULA IF YOU ARE SURE $\Box^2 p' = 0$ IN THE EXTERIOR REGION OF THE DATA SURFACE $f=0$. OTHERWISE USE THE FW-H EQ. WITH PENETRABLE DATA SURFACE.
- NOTE THAT $p'=0$ INSIDE THE SURFACE $f=0$ FOR BOTH THE KIRCHHOFF FORMULA AND THE SOLUTION OF THE FW-H EQUATION. THIS GIVES A FOOL-PROOF METHOD OF TESTING YOUR COMPUTER CODE AND THE ACCURACY OF THE INPUT DATA. IF YOU HAVE DONE EVERYTHING CORRECTLY, THEN YOU SHOULD GET $p'=0$ TO MACHINE ACCURACY INSIDE THE DATA SURFACE $f=0$. TRY THIS FOR MANY POINTS ARBITRARILY CHOSEN
- FROM THE COMPUTATIONAL EXPERIMENTS OF BRENTNER AND FARASSAT (AIAA J., VOL. 36(8), 1998, 1379-1386), IT WAS FOUND THAT IN THE NEAR FIELD OF MOVING SURFACES, THE FW-H EQ. CHANGES AT A SMALLER RATE THAN THE KIRCHHOFF FORMULA AS THE DATA SURFACE IS ENLARGED AND MOVED FARTHER. ALSO THE FW-H EQ. GIVES MUCH CLOSER RESULT THAN THE KIRCHHOFF FORMULA TO MEASURED DATA IN THE NEAR FIELD. MORE WORK ON THIS IS NEEDED.

LEC. 9/16

SOLUTION OF THE FW-H EQ.

MOVING OBSERVER CALCULATIONS



FOR AN OBSERVER MOVING WITH THE AIRCRAFT, THE ACOUSTIC PRESSURE $\tilde{p}'(\vec{x}',t)$ IS CALCULATED AS FOLLOWS. HERE $\vec{x}'$ IS A FRAME FIXED TO THE AIRCRAFT SUCH THAT THE $\vec{x}$ AND $\vec{x}'$ FRAME COINCIDE AT $t=0$. LET THE FLIGHT VELOCITY BE $\vec{V}_F$. NOTE THAT $\vec{x}'$ IS A FIXED OBSERVER POSITION IN THE MOVING FRAME. NOW $\vec{x}$ IS A FUNCTION OF OBSERVER TIME:

$$\vec{x}(t) = \vec{x}(0) + \vec{V}_F t = \vec{x}' + \vec{V}_F(t), \quad \vec{x}(0) = \vec{x}'$$

$$\therefore \quad \boxed{\tilde{p}'(\vec{x}',t) = p'(\vec{x}' + \vec{V}_F t, t)}$$

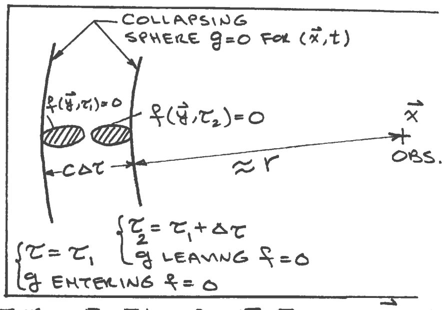
WHERE $p'(\vec{x},t)$ IS THE ACOUSTIC PRESSURE IN THE FRAME FIXED TO THE UNDISTURBED MEDIUM. IN PRACTICE, WE CALCULATE p' AT DISCRETE TIME POINTS WITH TIME GRID Δt. WE USE $\boxed{\tilde{p}'(\vec{x}',t) = p'(\vec{x}' + (n\Delta t)\vec{V}_F, n\Delta t)}$.

SOLUTION OF THE FW-H EQ. (CONT'D)

LOWSON'S FORMULA REVISITED

WE HAVE SEEN THAT LOWSON PROPOSED, BASED ON RESULTS OF LAMB AND LIGHTHILL, THAT THE NOISE FROM A MOVING UNSTEADY FORCE $\vec{F}(t)$ IS THE SOLUTION OF THE WAVE EQUATION

$$\Box^2 p' = -\nabla\cdot\{\vec{F}\,\delta[\vec{x}-\vec{x}_s(t)]\}$$



WHERE $\vec{x}_s(t)$ IS THE POSITION VECTOR OF THE POINT FORCE $\vec{F}(t)$. THIS REQUIRES A PROOF BECAUSE IT IS NOT OBVIOUS. WE START BY ASSUMING THAT THE FORCE IS GENERATED BY A FINITE BODY $f(\vec{y},\tau)=0$. FROM FORMULATION 1

$$4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int_{f=0}\left[\frac{p\cos\theta}{r|1-M_r|}\right]_{ret} dS + \int_{f=0}\left[\frac{p\cos\theta}{r^2|1-M_r|^2}\right]_{ret} dS$$

NOW IF WE ASSUME $\boxed{c\Delta\tau \ll r\ ,\ \Delta\tau \ll T}$ WHERE T IS THE TYPICAL PERIOD OF FLUCTUATION OF p ON THE SURFACE, THEN

$$4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{r}}{r|1-M_r|}\cdot\underbrace{\int_{f=0} p\vec{n}\,dS}_{\vec{F}(\tau)}\right]_{ret} + \left[\frac{\hat{r}}{r^2|1-M_r|^2}\cdot\underbrace{\int_{f=0} p\vec{n}\,dS}_{\vec{F}(\tau)}\right]_{ret}$$

(CONT'D)

SOLUTION OF THE FW-H EQ. (CONT'D)

LOWSON'S FORMULA REVISITED (CONT'D)

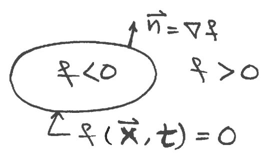
$$\Rightarrow\quad 4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{F_r}{r|1-M_r|}\right]_{ret} + \left[\frac{F_r}{r^2|1-M_r|}\right]_{ret}$$

WHERE $F_r = \vec{F}\cdot\hat{r}$. BUT THIS IS THE SOLUTION OF

$$\Box^2 p' = -\nabla\cdot\{\vec{F}(t)\,\delta[\vec{x}-\vec{x}_s(t)]\}$$

ANOTHER VIEW OF THICKNESS NOISE TERM

LET $H(f)=\begin{cases}1 & f>0\\ 0 & f<0\end{cases}$ BE THE HEAVISIDE FUNCTION. THEN $1-H(f)=\begin{cases}1 & f<0\\ 0 & f>0\end{cases}$.



$$\frac{\partial}{\partial t}[1-H(f)] = -\frac{\partial f}{\partial t}\delta(f) = v_n\,\delta(f)$$

$$\boxed{Q_T = \frac{\partial^2}{\partial t^2}\{\rho_0[1-H(f)]\} = \frac{\partial}{\partial t}[\rho_0 v_n\,\delta(f)]}$$ THICKNESS SOURCE !

$\therefore$ THE SOLUTION OF $\Box^2 p'_T(\vec{x},t) = Q_T$ IS

$$\boxed{\begin{aligned}4\pi p'_T(\vec{x},t) &= \frac{\partial^2}{\partial t^2}\int \rho_0[1-H(f)]\frac{\delta(g)}{r}\,d\vec{y}\,d\tau = \frac{\partial^2}{\partial t^2}\int_{f<0}\left[\frac{\rho_0}{r|1-M_r|}\right]_{ret} d\vec{y}\\ &= \frac{\partial^2}{\partial t^2}\int_{f<0}\frac{\rho_0}{r}\,d\vec{y}\end{aligned}}$$

SOLUTION OF THE FW-H EQ. (CONT'D)

SUCCI'S THICKNESS NOISE FORMULA

SUCCI'S FORMULA GIVES THE THICKNESS NOISE FOR A COMPACT SOURCE. IT IS GENERALLY APPLIED TO A RIGID SURFACE. THE ASSUMPTION OF COMPACTNESS IS $\boxed{c\Delta\tau \ll r}$ HERE.

FROM PREVIOUS PAGE

$$\boxed{\begin{aligned} 4\pi p_T'(\vec{x},t) &= \frac{\partial^2}{\partial t^2}\left[\frac{\rho_0}{r|1-M_r|}\overbrace{\int_{f<0} d\vec{y}}^{\text{VOLUME } V \text{ INSIDE } f=0}\right]_{ret} \\ &= \frac{\partial^2}{\partial t^2}\left[\frac{\rho_0 V}{r|1-M_r|}\right]_{ret} \\ &= \left\{\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left[\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left(\frac{\rho_0 V}{r|1-M_r|}\right)\right]\right\}_{ret} \end{aligned}}$$

TAKE THE TIME DERIVATIVES!

IN APPLICATION TO PROPELLERS, SUCCI DIVIDED THE BLADE INTO SMALL VOLUME ELEMENTS V_i AND SUMMED THE CONTRIBUTION OF ALL VOLUME ELEMENTS TO p_T'. IT WORKS! REMEMBER $r = |\vec{x} - \vec{y}(\vec{\xi},\tau)|$, $\vec{M}$ AND $\hat{r}$ IN $M_r = \vec{M}\cdot\hat{r}$ ARE ALL FUNCTIONS OF TIME. SO THE FINAL RESULT HAS MANY TERMS.

SOLUTION OF THE FW-H EQ. (CONT'D)

ISOM'S THICKNESS NOISE RESULT

WE KNOW THAT $1 - H(f) = \begin{cases} 0 & f>0 \quad \text{OUTSIDE THE SURFACE} \\ 1 & f<0 \quad \text{INSIDE THE SURFACE}\end{cases}$

LET US DEFINE $\phi(\vec{x},t) = \rho_0 c^2[1 - H(f)] = \begin{cases} 0 & f>0 \\ \rho_0 c^2 & f<0 \end{cases}$

$$\bar{\Box}^2\phi = \underbrace{-\frac{\bar{\partial}^2}{\partial t^2}[\rho_0 H(f)]}_{\text{THICKNESS NOISE TERM}} + \underbrace{\bar{\nabla}^2[\rho_0 c^2 H(f)]}_{\nabla\cdot[\rho_0 c^2 \vec{n}\,\delta(f)]}$$

MINUS LOADING TERM WITH $p = \rho_0 c^2$

$$4\pi\phi(\vec{x},t) = \text{THICKNESS NOISE} + \nabla\cdot\int \frac{\rho_0 c^2 \vec{n}}{r}\delta(f)\delta(g)\,d\vec{y}\,d\tau$$

$$= 0 \quad \text{OUTSIDE THE BODY}$$

$$\Rightarrow \text{THICKNESS NOISE} = -\nabla\cdot\int \frac{\rho_0 c^2 \vec{n}}{r}\delta(f)\delta(g)\,d\vec{y}\,d\tau$$

i.e. THICKNESS NOISE IS EQUIVALENT TO THE LOADING NOISE FROM A UNIFORM PRESSURE LOADING OF MAGNITUDE $\rho_0 c^2 \approx 140{,}000$ Pa! THIS IS ISOM'S RESULT. THE ABOVE PROOF IS BY FFOWCS WILLIAMS BUT PUBLISHED BY FARASSAT. ISOM SHOWED THIS RESULT IN THE FAR FIELD NUMERICALLY. FARASSAT PROVED THE RESULT FOR THE FAR FIELD AND FW GAVE THE FULL RESULT.

SOLUTION OF THE FW-H EQ. (CONT'D)

WHAT IS HIDDEN IN THE VOLUME TERM OF FW-H EQUATION

REF.: F. FARASSAT & M.K. MYERS "AN ANALYSIS OF THE QUADRUPOLE NOISE SOURCE OF HIGH SPEED ROTATING BLADES", COMPUTATIONAL ACOUSTICS, VOL. 2, LEE, CAKMAK & VICHNEVETSKY (EDS.) ELSEVIER SCIENCE PUBL., 1990.

WE NOW LOOK AT THE EQUATION $\square^2 p' = \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right]$.

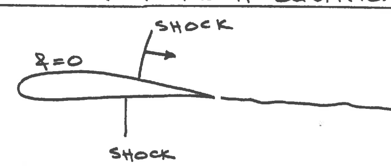
FORMALLY, THE SOLUTION IS

$$4\pi p'(\vec{x},t) = \frac{\bar{\partial}^2}{\partial x_i \partial x_j} \int_{F>0} \frac{[T_{ij}]_{ret}}{r}\, d\vec{y}$$

WHERE $F(\vec{y};\vec{x},t) = f(\vec{y}, t - r/c) = [f(\vec{y},\tau)]_{ret}$. IN THIS FORM, THIS SOLUTION IS NOT OF MUCH USE! THE QUADRUPOLE CAN HAVE MANY REAL OR IDEALIZED DISCONTINUITIES EACH OF WHICH IS A SOURCE OF SOUND! IT IS NOT A GOOD IDEA TO TAKE THE SPACE DERIVATIVES $\bar{\partial}^2/\partial x_i \partial x_j$ INSIDE THE INTEGRAL. THIS CAUSES UNNECESSARY COMPLICATIONS! IT IS BETTER TO WORK WITH THE SOURCE TERM AND USE GENERALIZED FUNCTION THEORY.

SOLUTION OF FW-H EQ. (CONT'D)

WHAT IS HIDDEN IN THE VOLUME TERM OF FW-H EQUATION



$$E = \frac{\bar{\partial}^2}{\partial x_i \partial x_j}[T_{ij}H(f)] = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} H(f)$$ pure quadrupole term

$$+ [2\nabla_2 \cdot \vec{Q}_T + \frac{\partial Q_n}{\partial n} - 4H_f Q_n - Q_G]\,\delta(f) + \hat{Q}_n \delta'(f)$$ blade surface terms

$$+ \left[2\nabla_2 \cdot \vec{q}_T + \Delta\left(\frac{\partial Q'_{n'}}{\partial n}\right) - 4H_k q_{n'} - q_G\right]\delta(k) + \hat{q}_{n'}\delta'(k)$$ shock surface terms

$$+ [Q_\nu]\,\delta(\tilde{f})\delta(f).$$ trailing edge term

LECTURES ON AEROACOUSTICS

BY : F. FARASSAT

FOR : SUMMER STUDENTS
& OTHERS

SUMMER 2010

LANGLEY RESEARCH CENTER
HAMPTON, VIRGINIA

LECTURE 1, JULY 7, 2010 ①

- WAVE EQ. IN 1-D

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = Q(x,t)$$

OR $$\frac{1}{c^2}\phi_{tt} - \phi_{xx} = Q(x,t)$$

INHOMO. WAVE EQ.

$$\frac{1}{c^2}\phi_{tt} - \phi_{xx} = 0 \quad \text{HOMO. WAVE EQ.}$$

1- WE CONSIDER HOMO. CASE FIRST.

$$\phi(x,t) = f(x-ct) + g(x+ct) \quad (*)$$

INFINITE STRING

$$\begin{cases} \phi(x,0) = \phi_0(x) \\ \phi_t(x,0) = \phi_1(x) \end{cases}$$

$$\begin{cases} \phi_0(x) = f(x) + g(x) & (1) \\ \phi_1(x) = -c f'(x) + c g'(x) & (2) \end{cases}$$

FROM (2)

$$\Rightarrow -c f(x) + c g(x) = A + \int_a^x \phi_1(y)\,dy$$

FROM (1) & (2), WE GET

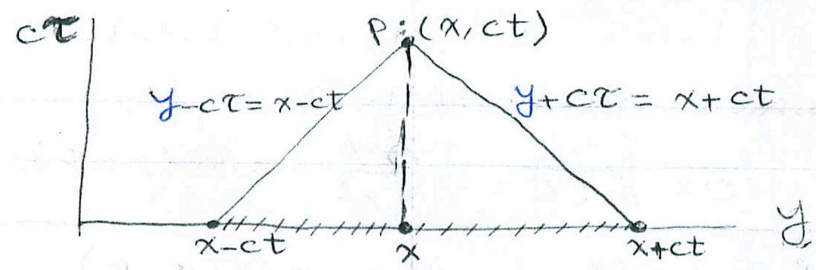
$$\begin{cases} 2c f(x) = c\phi_0(x) - A - \int_a^x \phi_1(y)\,dy \\ 2c g(x) = c\phi_0(x) + A + \int_a^x \phi_1(y)\,dy \end{cases}$$

$$\phi(x,t) = \frac{1}{2}\left[\phi_0(x+ct) + \phi_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\phi_1(y)\,dy$$

$$(*)\begin{cases} f(x-ct) & \text{RIGHT MOVING WAVE} \\ g(x+ct) & \text{LEFT MOVING WAVE} \end{cases}$$

BOTH ARBITRARY FUNCTIONS

(2)

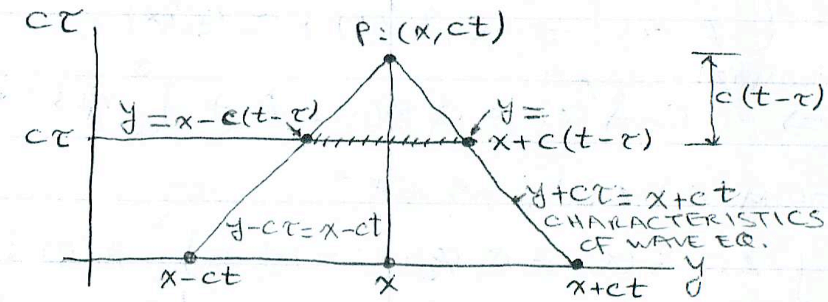


DOMAIN OF DEPENDENCE OF $(x,ct):P$

THE INHOMO. CASE – INFINITE STRING

$$\begin{cases} \frac{1}{c^2}\phi_{tt} - \phi_{xx} = \rho(x,t) & \forall t \in (0,\infty) \\ \phi(x,0) = \phi_0(x) \\ \phi_t(x,0) = \phi_1(x) \end{cases}$$

$$\phi(x,t) = \text{HOMO. SOL.} + \int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} \rho(y,\tau)\, dy$$



DOMAIN OF DEPENDENCE OF (x,ct) AT τ.

WE ARE GETTING SOME SIMILARITY TO 3D WAVE PROPAGATION

(3) FINITE STRING : USE SEPARATION OF VARIABLE

$$\phi(x,t) = X(x)\,T(t)$$

3

$$\Rightarrow \frac{\tau''}{\tau} = c^2 \frac{X''}{X}$$

$$\phi(x,t) = \sum_{\lambda} A_{\lambda} \cos(\lambda x + \epsilon_{\lambda}) \cos(\lambda c t + \delta_{\lambda})$$

WE HAVE AN EIGENVALUE PROBLEM.

THERE ARE MANY APPLICATIONS IN MUSICAL INSTRUMENTS, TRANSDUCERS, ETC. BUT NOT MUCH IN AEROACOUSTICS.

THE WAVE EQ. IN 2D & 3D

$$\Box^2 \phi \stackrel{!}{\equiv} \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = Q(\vec{x}, t)$$

$\vec{x} = (x_1, x_2)$ 2D

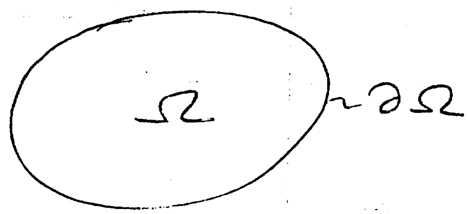
$\vec{x} = (x_1, x_2, x_3)$ 3D

- IN MATH, THE WAVE EQ. IS REFERRED TO AS IN 2+1 & 3+1 DIM. BECAUSE OF THE DIMENSION OF TIME IN PDE.
- THE NATURE OF WAVE PROPAGATION IN 2 & 3 D IS DIFFERENT. THE WORLD IS 3D! WE START WITH 3D AND THEN DERIVE THE SOLUTION FOR 2D BY THE METHOD OF DESCENT.

- MOST PROBLEMS OF AEROACOUSTICS ARE EXTERNAL PROBLEMS OF TWO TYPES:

1 - THE UNBOUNDED 3D DOMAIN (OR 2D)

2 - THE INFINITE REGION EXTERNAL TO A BOUNDED REGION Ω



THE TIME t IS EITHER IN $(-\infty, T]$ OR $[0, T]$. WE ALWAYS ASSUME THAT THE WAVES DIE SUFFICIENTLY FAST AT INFINITY SO THAT OUR SURFACE INTEGRALS OVER A LARGE SURFACE GO TO ZERO AS WE GO TO INFINITY.

- A SOLUTION WITH SPHERICAL SYMMETRY IN 3D, $\phi = \phi(R, t)$

$$0 = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{1}{c^2}\phi_{tt} - \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 \phi_R)$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(R\phi) - \frac{\partial^2}{\partial R^2}(R\phi) = 0$$

OR $$\phi(R,t) = \underbrace{\frac{f(R-ct)}{R}}_{\text{OUTGOING WAVE}} + \underbrace{\frac{g(R+ct)}{R}}_{\text{INCOMING WAVE}}$$

f & g: ARB. FNS

(5)

— FROM THIS SOLUTION, WE CAN MAKE OTHER SOLUTIONS BY SUPERPOSITION METHOD THAT ARE MORE USEFUL:

$$\phi(\vec{x},t) = \int_{\mathbb{R}^3} Q(\vec{y}) \frac{f(r-ct)}{r} \, d\vec{y}$$

$r = |\vec{x} - \vec{y}|$, f ARBITRARY

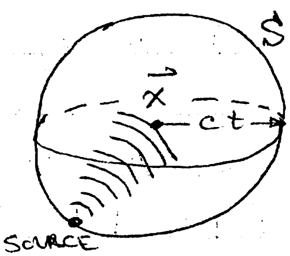
— THIS CAN BE USED TO FIND THE SOLUTION OF THE FOLLOWING <u>INITIAL VALUE PROBLEM</u>:

$$\begin{cases} \Box^2 \phi = 0 \quad \text{IN } \mathbb{R}^3 \\ \phi(\vec{x},0) = \phi_0(\vec{x}) \\ \phi_t(\vec{x},0) = \phi_1(\vec{x}) \end{cases}$$

$$\phi(\vec{x},t) = \frac{1}{4\pi c} \frac{\partial}{\partial t} \int_{r=ct} \frac{\phi_0(\vec{y})}{r} \, dS + \frac{1}{4\pi c} \int_{r=ct} \frac{\phi_1(\vec{y})}{r} \, dS$$

POISON'S SOLUTION

<u>HUYGENS PRINCIPLE</u>: ONLY THE POINTS AT A DISTANCE OF $r=ct$ INFLUENCE $\phi(x,t)$ AT THE TIME t



S SPHERE WITH CENTER AT $\vec{x}$ AND RADIUS ct

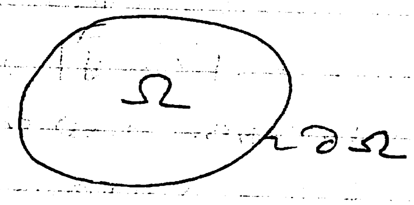
A VERY IMPORTANT RESULT FOR WAVE PROPAGATION IN 3D!

CORRECTED LEC. 1 (6)

— THE UNIQUENESS THM FOR THE WAVE EQ. IN 1, 2 AND 3 D (FINITE REGION)

WE PROVE THE UNIQUENESS THM FOR 2 AND 3D. THE 1 D IS BASED ON THE SAME ENERGY IDENTITY. ASSUME $C=1$

$$\Box^2 \phi = \phi_{tt} - \nabla^2 \phi$$



$$E = \phi_t \phi_{tt} - \phi_t \nabla^2 \phi$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\phi_t^2 - \nabla\cdot(\phi_t \nabla\phi) + \underbrace{\nabla\phi_t\cdot\nabla\phi}_{\frac{1}{2}\frac{\partial}{\partial t}|\nabla\phi|^2} = 0$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\phi_t^2 + |\nabla\phi|^2\right] = \nabla\cdot(\phi_t \nabla\phi)$$

INTEGRATE OVER $\Omega\times[0,T]$, TO GET

$$\frac{1}{2}\int_\Omega \left[\phi_t^2 + |\nabla\phi|^2\right]_0^T d\vec{x} = \int_0^T dt \int_{\partial\Omega} \phi_t\, \phi_n\, dS$$

NOW LET US ASSUME:

$\phi(\vec{x},0)$ <u>AND</u> $\phi_t(\vec{x},0)$ IS GIVEN ON Ω

AND

EITHER ϕ <u>OR</u> ϕ_n IS GIVEN ON $\partial\Omega$

$\forall t \in [0,T]$ $\qquad\qquad$ $\forall$: FOR ALL

(TALK ABOUT COMPATIBILITY OF DATA ON Ω & $\partial\Omega$ AT $t=0$)

THEN THE SOLUTION IS UNIQUE. WE ARE ASSUMING THAT THE DATA ON Ω AND $\partial\Omega$ ARE COMPATIBLE, i.e: $\phi(\vec{x},0)$ FROM INITIAL & BOUNDARY CONDITIONS ARE EQUAL ON $\partial\Omega$.

<u>PROOF</u>: LET $\phi_1(\vec{x},t)$ AND $\phi_2(\vec{x},t)$ BE TWO SOLUTIONS SATISFYING THE INITIAL AND BOUNDARY CONDITIONS. LET $\phi = \phi_1 - \phi_2$

$\Rightarrow$ WE HAVE $\phi(\vec{x},0) = 0$ AND $\phi_t(\vec{x},0) = 0$ ON Ω

AND

$$\phi(\vec{x},t) = 0 \ \underline{\text{OR}} \ \phi_n(\vec{x},t) = 0 \ \text{ON } \partial\Omega$$

$$\forall\, t \in [0,T] \qquad \forall: \text{FOR ALL}$$

$$\therefore \int_\Omega \left[\phi_t^2(\vec{x},T) + |\nabla\phi(\vec{x},T)|^2\right] d\vec{x} = 0$$

$$\Rightarrow \phi_t^2(\vec{x},T) + |\nabla\phi(\vec{x},T)|^2 = 0$$

$$\Rightarrow \phi(\vec{x},T) = \text{CONT} = \phi(\vec{x},0) = 0 \ !$$

$\Rightarrow \phi_1(\vec{x},t) = \phi_2(\vec{x},t)$, $\therefore$ THE SOLUTION IS UNIQUE.

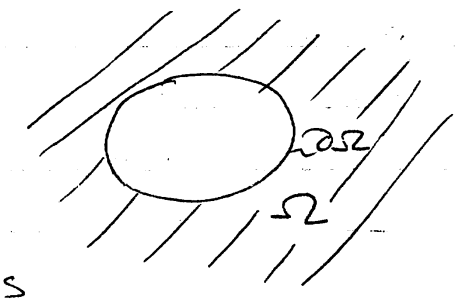
NOTE THAT IN ENGINEERING, UNIQUENESS THEOREMS ARE MORE IMPORTANT THAN EXISTENCE THEOREMS.

<u>TERMINOLOGY</u>: IF ϕ IS SPECIFIED ON $\partial\Omega$, WE SAY WE HAVE A <u>DIRICHLET</u> BC

⑧

IF ϕ_n IS SPECIFIED ON $\partial\Omega$, WE HAVE A NEUMANN PROBLEM

— THE UNIQUENESS RESULT APPLIES TO THE EXTERNAL PROBLEM SHOWN IF THE VOLUME INTEGRAL IS CONVERGENT. THIS VERSION IS WHAT WE NORMALLY USE IN AEROACOUSTICS



LECTURE 2 – JULY 13, 2010

SOLUTIONS OF WAVE EQUATION IN THREE DIMENSIONAL SPACE

- WE PRESENT TWO EQUIVALENT FORMS OF THE SOLUTION OF THE WAVE EQ. IN 3D (UNBOUNDED SPACE $\mathbb{R}^3$)

(1) $$\Box^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = Q(\vec{x}, t)$$

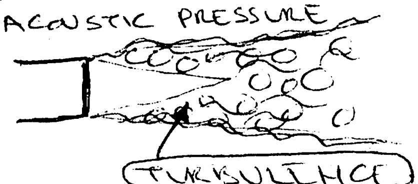
WHERE $Q(\vec{x}, t) \neq 0$ IS CALLED THE INHOMOGENEOUS SOURCE TERM. IT CAN BE NONZERO IN A FINITE REGION OF 3D SPACE

- EXAMPLE FROM AEROACOUSTICS

$$\Box^2 p' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \equiv Q(\vec{x}, t)$$

$T_{ij} \simeq \rho u_i u_j$, p': ACOUSTIC PRESSURE

JET NOISE PROBLEM



THIS IS CALLED LIGHTHILL'S JET NOISE EQ. ρ IS THE FLUID DENSITY WITH VELOCITY u_i. $\rho u_i u_j$ IS CALLED REYNOLDS STRESS. T_{ij} IS CALLED LIGHTHILL'S STRESS TENSOR. IN PRACTICE WE ASSUME T_{ij} IS EITHER MEASURED

OR COMPUTED BY TURBULENCE SIMULATION.

— WE HAVE SHOWN THAT THE FOLLOWING IS A GENERAL SOLUTION OF WAVE EQ. IN 3D (SEE P5, LEC. 1), $r = |\vec{x} - \vec{y}|$

$$\phi(\vec{x}, t) = \int_{\mathbb{R}^3} \frac{\tilde{Q}(\vec{y}) f(r - ct)}{r} \, d\vec{y}$$

$\tilde{Q}(\vec{y})$ AND f ARBITRARY FUNCTIONS.

NOTATION: $\int_{\mathbb{R}^3} \cdots\cdots d\vec{y} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots\cdots dy_1 dy_2 dy_3$

NOW, WE GIVE THE SOLUTION OF EQ. (1) AS FOLLOWS:

$$4\pi\,\phi(\vec{x}, t) = \int_{\mathbb{R}^3} \frac{Q(\vec{y}, t - r/c)}{r} \, d\vec{y}$$

NOTE

THIS IS CALLED THE RETARDED TIME SOLUTION (A BEAUTIFUL RESULT !)

AT THIS STAGE DO NOT CONCERN YOURSELF HOW WE GOT THIS RESULT. YOU MUST LEARN HOW TO INTERPRETE AND USE THIS RESULT.

TERMINOLOGY: $(\vec{x}, t)$: OBSERVER

LEC. 2 ③

SPACE-TIME VARIABLES

$(\vec{y}, \tau)$: SOURCE SPACE-TIME VARIABLES

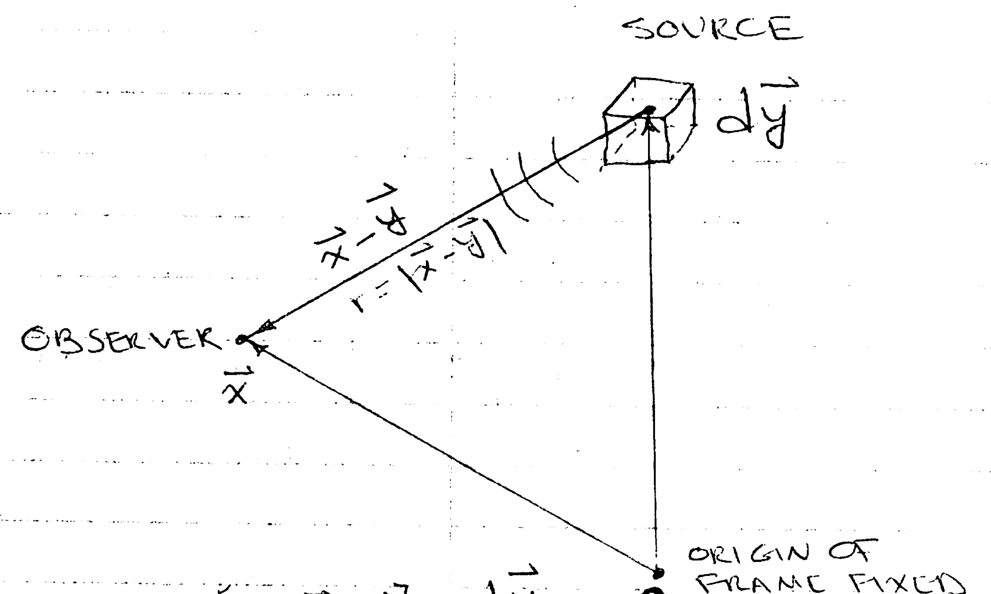
$$Q(\vec{y}, t - r/c) \equiv [Q(\vec{y}, \tau)]_{ret}$$

ret: RETARDED

$\therefore$ WE CAN WRITE THE SOLUTION AS FOLLOWS

$$4\pi\phi(\vec{x}, t) = \int_{\mathbb{R}^3} \frac{[Q(\vec{y}, \tau)]_{ret}}{r}\, d\vec{y}$$

INTERPRETATION



$$d\phi(\vec{x}, t) = \frac{[Q(\vec{y}, \tau)]_{ret}\, d\vec{y}}{4\pi r} = \frac{Q(\vec{y}, t - r/c)\, d\vec{y}}{4\pi r}$$

THE SOURCE AT POINT $\vec{y}$ WITH SOURCE STRENGTH $Q\,d\vec{y}$ SENDS A SIGNAL AT THE RETARDED TIME $t - r/c$ TO ARRIVE AT THE OBSERVER AT THE TIME t. THE SIGNAL IS DIMINISHED (REDUCED) BY SPHERICAL SPREADING <u>INVERSELY</u> PROPORTIONAL TO THE DISTANCE r BETWEEN THE SOURCE AND THE OBSERVER. SO THE INTERPRETATION OF THE SOLUTION IS VERY SIMPLE.

<u>2ND FORM OF THE SOLUTION OF THE WAVE EQUATION (1)</u>

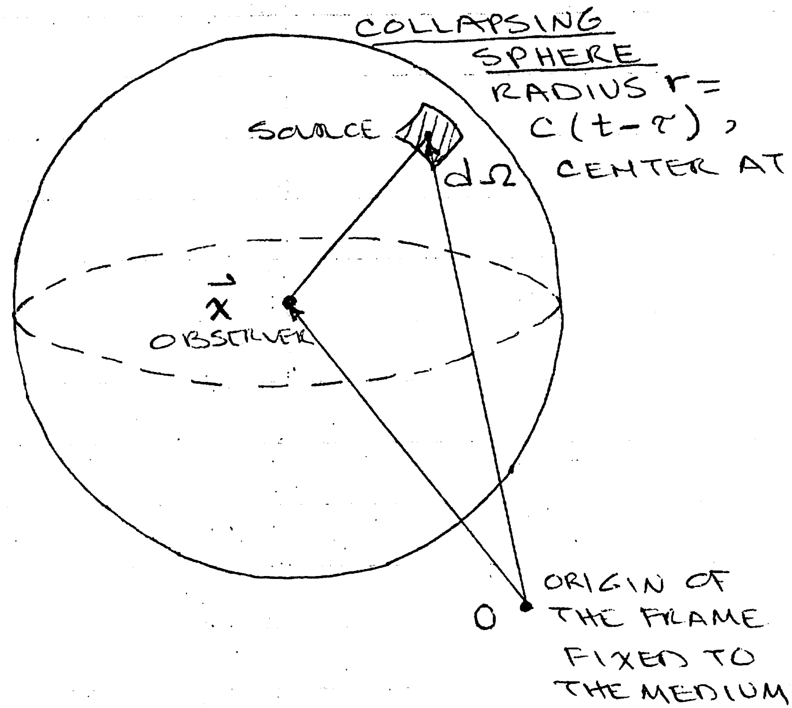
THIS SOLUTION IS LESS WELL-KNOWN BUT <u>VERY</u> USEFUL IN AEROACOUSTICS.

$$4\pi\phi(\vec{x},t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r=c(t-\tau)} Q(\vec{y},\tau)\, d\Omega$$

WHERE $d\Omega$ IS ELEMENT OF SURFACE AREA OF A SPHERE WITH CENTER AT THE OBSERVER AND RADIUS $r = c(t-\tau)$. THIS FORM OF THE SOLUTION OF WAVE

EQ. IS KNOWN AS THE COLLAPSING SPHERE SOLUTION.

INTERPRETATION:



THE COLLAPSING SPHERE IS THE LOCUS OF ALL SOURCES WHOSE SIGNAL ARRIVE SIMULTANEOUSLY AT THE TIME t.

WE HAVE SPHERICAL SPREADING ALSO:

$$\frac{d\tau\, d\Omega}{t-\tau} = \frac{c\, d\tau\, d\Omega}{c(t-\tau)} = \frac{d\vec{y}}{r}$$

SO AGAIN THE INTERPRETION OF THE SOLUTION

IS VERY SIMPLE. AGAIN DO NOT CONCERN YOURSELF RIGHT NOW HOW WE GOT THIS SOLUTION. LEARN HOW TO INTERPRET AND USE THIS SOLUTION IN APPLICATIONS.

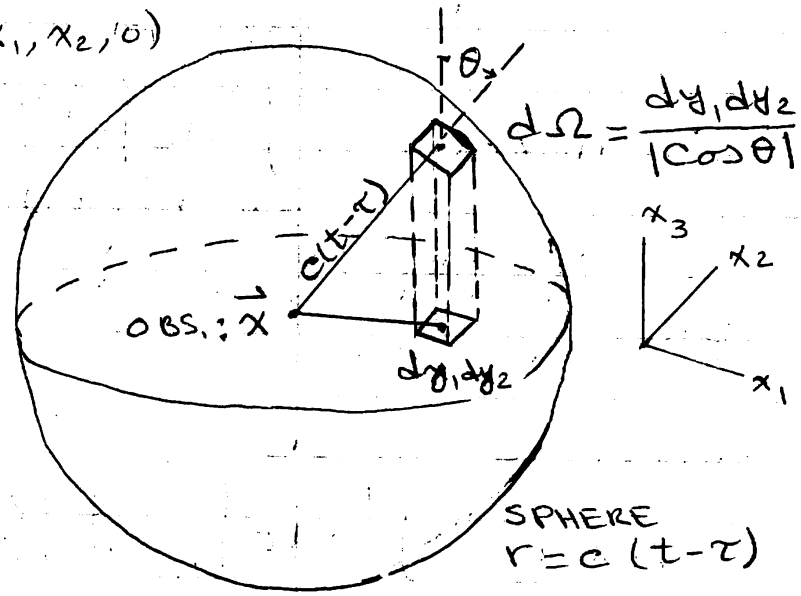
LECTURE 3 - JULY 29, 2010

(1)

HOW TO SOLVE THE WAVE EQ. IN 2D.

$$\Box^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = Q(x_1, x_2, t)$$

$$\vec{x} = (x_1, x_2, 0)$$



RECALL THAT WE HAVE IN 3D

$$4\pi\, \phi(\vec{x}, t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r = c(t-\tau)} Q(\vec{y}, \tau)\, d\Omega$$

IN 2D, WE CAN SET THE PROBLEM IN 3D WITH THE ASSUMPTION THAT Q IS INDEPENDENT OF x_3 (OR y_3). WE ALSO HAVE

$$d\Omega = \frac{dy_1 dy_2}{|\cos\theta|}$$

$$|\cos\theta| = \frac{\left[c^2(t-\tau)^2 - (x_1-y_1)^2 - (x_2-y_2)^2\right]^{1/2}}{c(t-\tau)}$$

IF WE DENOTE $r_2 = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

$$\Rightarrow \cos\theta = \frac{\sqrt{c^2(t-\tau)^2 - r_2^2}}{c(t-\tau)}$$

$$\therefore \; 4\pi\phi(\vec{x},t) = 2\int_{-\infty}^{t} c\,d\tau \int_{r_2=c(t-\tau)} \frac{Q(\vec{y},\tau)\,dy_1 dy_2}{\sqrt{c^2(t-\tau)^2 - r_2^2}}$$

2 ON THE RIGHT SIDE COMES FROM THE FACT THAT WE HAVE TWO HEMISPHERE TO INTEGRATE OVER.

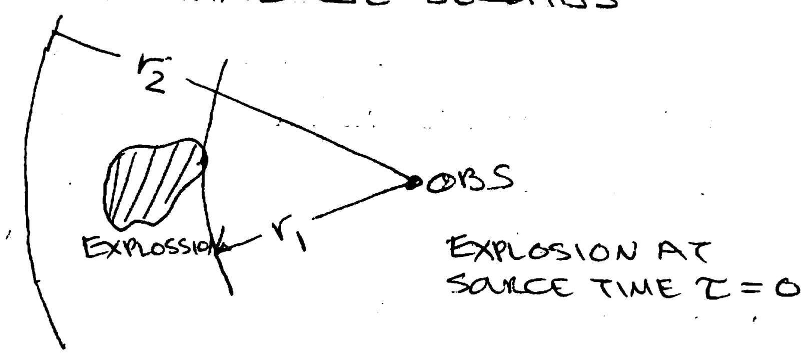
THEREFORE, IN 2D :

$$2\pi\phi(\vec{x},t) = \int_{-\infty}^{t} c\,d\tau \int_{r_2=c(t-\tau)} \frac{Q(\vec{y},\tau)\,dy_1 dy_2}{\sqrt{c^2(t-\tau)^2 - r_2^2}}$$

NOTE THAT $r_2 = c(t-\tau)$ IS A CIRCLE WITH CENTER AT $\vec{x} = (x_1, x_2)$ AND RADIUS $c(t-\tau)$. THIS MEANS THAT WE INTEGRATE ALL THE SOURCES AT THE SOURCE TIME τ OVER THE CIRCLE OF RADIUS $c(t-\tau)$ BUT ATTENUATED <u>INVERSELY</u> BY $\sqrt{c^2(t-\tau) - r_2^2}$ WHERE r_2 IS SIMPLY THE DISTANCE BETWEEN THE OBSERVER AND SOURCE IN 2D.

LEC. 3 ③

ONE MAJOR DIFFERENCE BETWEEN WAVE PROPAGATION IN 2D AND 3D IS THAT FOR AN EXPLOSION IN 2D, WE HAVE A SIGNAL THAT HAS A SHARP BEGINNING BUT ASYMPTOTICALLY GOES TO ZERO AS $t \to \infty$ BUT IN 3D BOTH THE BEGINNING AND END OF THE SIGNAL ARE SHARP. IN 2D, THIS CAN BE SEEN AS FOLLOWS. LET US SAY THAT WE HAVE AN EXPLOSION AT $\tau = 0$ OVER A REGION OF 2D SPACE. LET US ALSO ASSUME THAT THE EXPLOSION TAKES A VERY SMALL TIME $\Delta\tau$ SECONDS



AT $t = \frac{r_1}{c}$, WE HAVE THE BEGINNING OF THE SIGNAL FROM THE EXPLOSION. AT ANY TIME $t > \frac{r_1}{c}$, WE HAVE TO INTEGRATE OVER A CIRCLE $r_2 = ct$, AND THE SIGNAL IS NONZERO BUT DIMINISHES AS $1/t$!

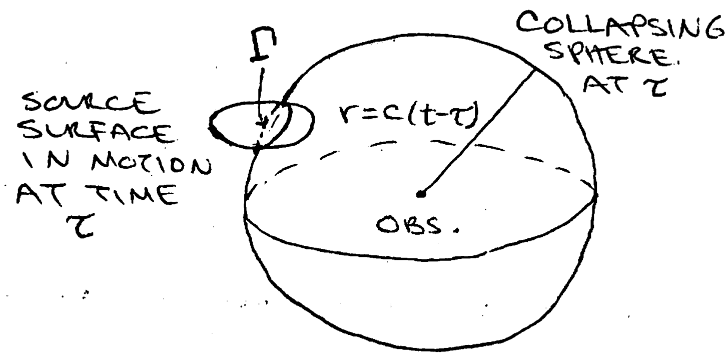
④

SOURCES IN MOTION – AN INTUITIVE APPROACH

WE WANT TO DISCUSS HOW WE CALCULATE THE NOISE FROM MOVING OBJECTS WITHOUT THE USE OF ADVANCED MATHEMATICS. IN PARTICULAR, WE WANT TO CLARIFY THE MEANING OF COMPACT AND NONCOMPACT SOURCES. FROM NOW ON WE CONSIDER THE WAVE EQ. IN 3D

$$\Box^2 p' = Q(\vec{x}, t)$$

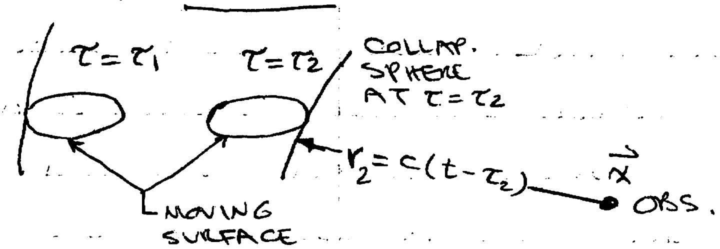
WHERE YOU CAN THINK OF p' AS THE ACOUSTIC PRESSURE. WE DO NOT SAY WHAT Q (THE SOURCE) IS AND HOW WE GET IT MATHEMATICALLY. LET US SAY THAT WE HAVE NOISE GENERATING OBJECT OF SPHEROIDAL SHAPE IN MOTION SENDING SIGNALS OF PERIOD T FROM ITS SURFACE.



RECALL THE COLLAPSING SPHERE SOLUTION OF THE WAVE EQ.

$$4\pi p'(\vec{x},t) = \int_{-\infty}^{t} \frac{d\tau}{t-\tau} \int_{r=c(t-\tau)} Q(\vec{y},\tau)\, d\Omega$$

ON PREVIOUS PAGE, WE HAVE SHOWN THE COLLAPSING SPHERE AND THE MOVING SURFACE AT TIME τ. ONLY THE SOURCES ON THE CURVE OF INTERSECTION Γ OF THE COLLAPSING SPHERE AND THE MOVING SURFACE CONTRIBUTE TO $p'(\vec{x},t)$: REMEMBER THAT $(\vec{x},t)$ ARE KEPT FIXED IN OUR DISCUSSION.



NOW, LET US SAY THAT THE COLLAPSING SPHERE ENTERS AND LEAVES THE MOVING SURFACE AT SOURCE TIMES τ_1 & τ_2, RESPECTIVELY. DEFINE $\Delta\tau = \tau_2 - \tau_1$, AND L = MAX. DIMENSION OF MOVING SURFACE. NOTE THAT $\Delta\tau = \Delta\tau(\vec{x},t)$.

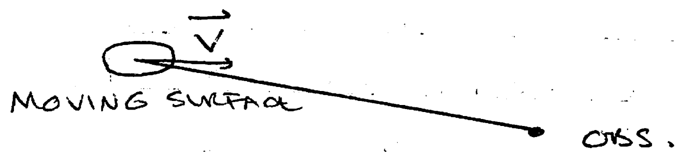
LEC. 3 (6)

LET US SEE WHAT HAPPENS IF

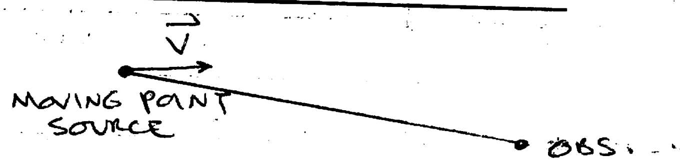
$$\begin{cases} c\,\Delta\tau \ll r_2 \\ \Delta\tau \ll T \end{cases} \qquad (1)$$

r_2 IS THE MINIMUM DISTANCE OF THE OBSERVER FROM THE MOVING SOURCE AS THE COLLAPSING SURFACE INTERSECTS THE SOURCE. T IS THE (TYPICAL) PERIOD OF FLUCTUATION OF SOURCES ON THE MOVING SURFACE. CONDITIONS (1) TELL US THAT $\frac{1}{c(t-\tau)} = \frac{1}{r}$ DOES NOT CHANGE MUCH DURING THE NOISE RADIATION PROCESS FOR OBS. TIME t, AND THAT THE SOURCE TIME FOR THIS PROCESS IS ESSENTIALLY CONSTANT. IN OTHER WORDS, THE MOVING SURFACE IS ACTING AS A <u>POINT SOURCE</u> AT THE OBS. TIME t. WE SAY THAT A MOVING SOURCE IS <u>COMPACT</u> (i.e. ACTS LIKE A POINT SOURCE) IF CONDITIONS (1) HOLD FOR <u>ALL</u> OBSERVER TIME IN AN INTERVAL I. NOTE THAT <u>BOTH</u> CONDITIONS IN (1) ARE NECESSARY FOR COMPACTNESS OF THE SOURCE. COMPACTNESS ASSUMPTION

SIMPLIFIES OUR ACOUSTIC CALCULATIONS CONSIDERABLY.



ACTUAL SITUATION



IDEALIZED SITUATION

NOTE THAT IF THE SOURCE IS NOT IN MOTION, THEN $\Delta\tau \approx \frac{L}{c}$ AND CONDITIONS (1) BECOME

$$(1)\quad \begin{cases} (a)\ L \ll r_2 \text{, MIN. DIST. OF OBS. FROM THE SURFACE} \\ (b)\ \frac{L}{c} \ll T \end{cases}$$

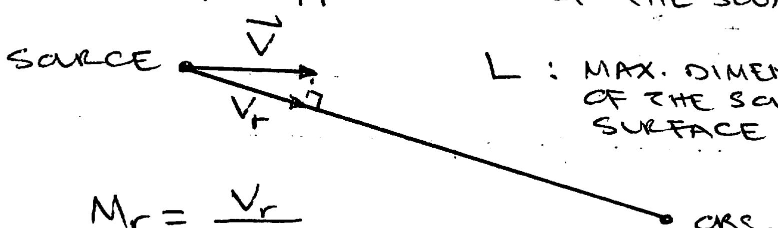
BUT $cT = \lambda$ THE WAVE LENGTH OF SOUND SO CONDITIONS (1) CAN BE WRITTEN AS

$$(1)'\quad \begin{cases} (a)\ L \ll \text{MIN. DIST. OF OBS. FROM THE SURF.} \\ (b)\ L \ll \lambda \end{cases}$$

UNFORTUNATELY, MANY BOOKS ON ACOUSTICS DO NOT EMPHASIZE THAT CONDITION (a) IS ABSOLUTELY ESSENTIAL FOR COMPACTNESS!

FOR A COMPACT SOURCE IN MOTION WITH VELOCITY V, $\vec{M} = \vec{V}/c$, THE FOLLOWING CONDITIONS MUST HOLD FOR <u>ALL</u> THE OBS. TIMES IN AN INTERVAL I:

$$\begin{cases} \text{(a)} & \dfrac{L}{|1-M_r|} \ll \text{MIN. DIST. OF OBS. FROM SOURCE} \\ \text{(b)} & \dfrac{L}{|1-M_r|} \ll \lambda \quad \text{THE WAVELENGTH OF THE SOUND} \end{cases}$$



L : MAX. DIMENSION OF THE SOURCE SURFACE

$$M_r = \frac{V_r}{c}$$

NEXT WE GIVE AN EXAMPLE OF HOW COMPACTNESS ASSUMPTION SIMPLIFIES THE ACOUSTIC ANALYSIS.

LECTURE 3 (9)

LOWSON'S FORMULA

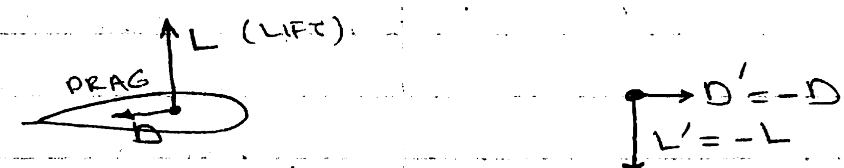
LOWSON HYPOTHESIZED (ON THE BASIS OF THE WORK OF LORD RAYLEIGH AND SIR JAMES LIGHTHILL) THAT THE GOVERNING EQ. FOR NOISE RADIATION FROM A COMPACT FORCE $\vec{F}(t)$ IN MOTION ON THE TRAJECTORY $\vec{x}_s(t)$

$$\Box^2 \rho'(\vec{x},t) = -\nabla \cdot \left[\vec{F}(t)\, \delta[\vec{x}-\vec{x}_s(t)]\right]$$

HERE $\delta(\cdot)$ IS THE DIRAC DELTA FUNCTION WITH THE FOLLOWING PROPERTY

$$\int_{-\infty}^{\infty} \phi(x)\, \delta(x)\, dx = \phi(0)$$

THE DIRAC DELTA FUNCTION IS NOT AN ORDINARY FUNCTION BUT A GENERALIZED FUNCTION.



A BLADE IN MOTION (ACTUAL SITUATION)

FORCE ON THE FLUID IN LOWSON'S FORMULA

NOTE THAT $\vec{F}(t)$ IN LOWSON'S FORMULA IS THE FORCE ON THE FLUID MEDIUM

LEC. 3 ⑩

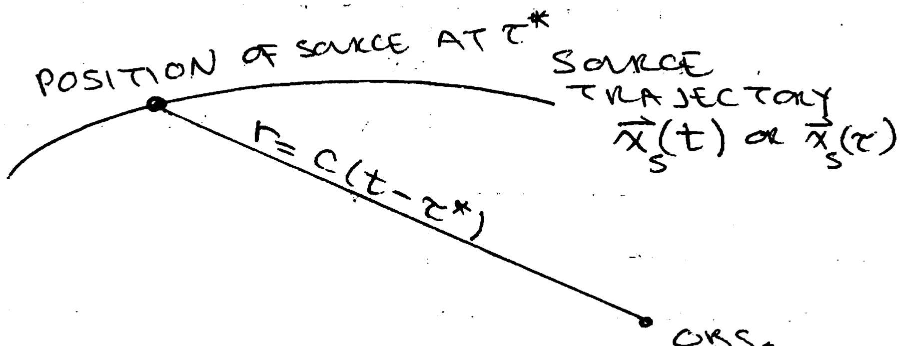
THE SOLUTION OF LOWSON'S EQ.

$$4\pi p'(\vec{x},t) = -\nabla \cdot \left[\frac{\vec{F}(\tau)}{r(1-M_r)} \right]_{\tau^*}$$

$r = |\vec{x} - \vec{x}_s(\tau)|$

WHERE τ^* IS THE SOLUTION OF THE EQ.

$$\tau - t + \frac{|\vec{x} - \vec{x}_s(\tau)|}{c} = 0$$



NOTE THAT τ^* IS SIMPLY THE EMISSION TIME FOR OBS. TIME t.

$\therefore \quad \tau^* = \tau^*(\vec{x},t)$ EMISSION TIME

WE CAN SHOW THAT IF THE SOURCE IS MOVING SUBSONICALLY, WE HAVE A SINGLE EMISSION TIME.

LOWSON'S FORMULA : $i = 1,2,3$

$$4\pi p'(\vec{x},t) = -\frac{\partial}{\partial x_i} \left[\frac{F_i(\tau^*)}{|\vec{x}-\vec{x}_s(\tau^*)|\, M_r(\tau^*)} \right]$$

(SUM ON i)

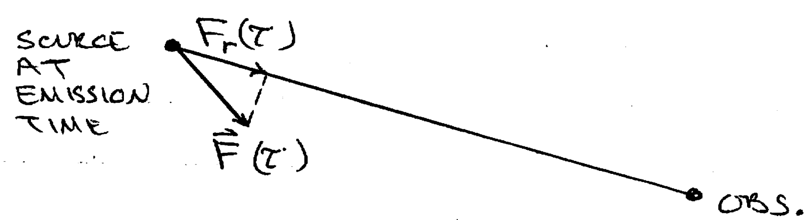
11

NOW USE CALCULUS TO CARRY OUT THE DIFFERENTIATION WRT x_i !. EASY ? NOT REALLY ! THE RESULTING EQUATION (FORMULA) IS VERY LONG AND LOWSON SHOWED THAT HE COULD DERIVE A WELL-KNOWN RESULT FOR PROPELLER NOISE CALCULATION BY GUTIN. SEE LOWSON'S 1965 PAPER IN PROC. OF ROY. SOC. LONDON. WE GIVE THE ESSENCE OF LOWSON'S FORMULA IN THE FAR-FIELD (A RESULT DERIVED BY FARASSAT) :

$$4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{F_r(\tau)}{r(1-M_r)}\right]_{\tau^*} + \left[\frac{F_r(\tau)}{r^2(1-M_r)}\right]_{\tau^*}$$

$$\boxed{r = |\vec{x}-\vec{x}_s(\tau)|}$$

THIS IS AN EXACT RESULT. RECALL THAT $\tau^* = \tau^*(\vec{x},t)$. HERE $F_r(\tau)$ IS FORCE IN THE RADIATION DIRECTION



(12)

WE CAN SHOW THAT

$$4\pi\, p'(\vec{x},t) = \frac{1}{c}\left\{\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\left[\frac{F_r(\tau)}{r(1-M_r)}\right]\right\}_{\tau^*} + \left[\frac{F_r(\tau)}{r^2(1-M_r)}\right]_{\tau^*}$$

THIS IS AGAIN AN <u>EXACT</u> RESULT. IN THE FAR FIELD, WE GET AN AMAZINGLY SIMPLE RESULT THAT WILL EXHIBIT THE ESSENCE OF LAWSON'S FORMULA

$$4\pi\, p'(\vec{x},t) = \left[\frac{\dot{F}_r(\tau)}{c\, r(1-M_r)^2}\right]_{\tau^*} + \left[\frac{\dot{M}_r F_r(\tau)}{c\, r(1-M_r)^3}\right]_{\tau^*}$$

$\boxed{\text{FAR FIELD}}$

WHERE $\dot{F}_r$ AND $\dot{M}_r$ ARE RATES OF CHANGE OF FORCE AND MACH NUMBER IN THE RADIATION DIRECTION AT EMISSION TIME τ^*, RESPECTIVELY. THUS, THE LAWSON'S FORMULA STATES THAT, IN THE FAR FIELD, <u>THE RATE OF FORCE FLUCTUATIONS</u> <u>AND ACCELERATION</u> <u>IN THE RADIATION DIRECTION</u> CONTRIBUTE TO THE NOISE OF A COMPACT FORCE IN MOTION!

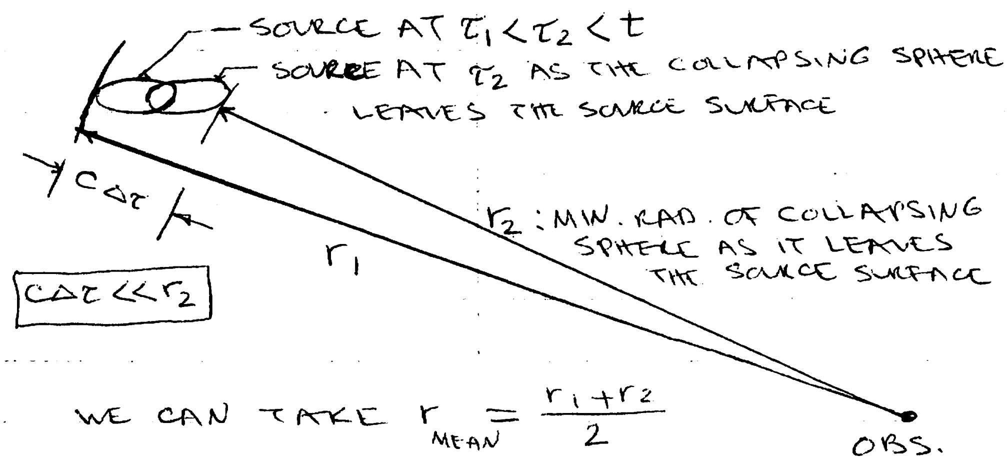
NOTE ADDED AFTER THE LECTURE

MORE ON COMPACT SOURCES

CONDITIONS (1) OF COMPACTNESS (P6 OF LEC.3) ARE

$$\begin{cases} \text{a)} \quad c\Delta\tau \ll r_2 \quad \text{(MIN. RADIUS OF COLL. SPHERE)} \\ \text{b)} \quad \Delta\tau \ll T \end{cases} \qquad (1)$$

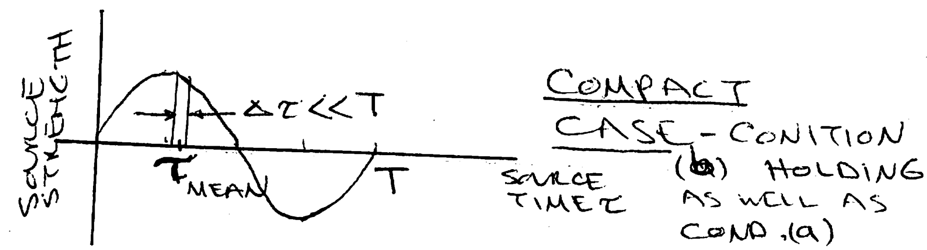
CONDITION (a) TELLS US THAT $r = c(t-\tau)$ DOES NOT CHANGE MUCH AS THE COLLAPSING SPHERE SWEEPS OVER THE SOURCE SURFACE. THIS MEANS THAT r IN THE RADIATION PROCESS CAN BE ASSUMED CONSTANT EQUAL TO A MEAN DISTANCE AS SHOWN BELOW



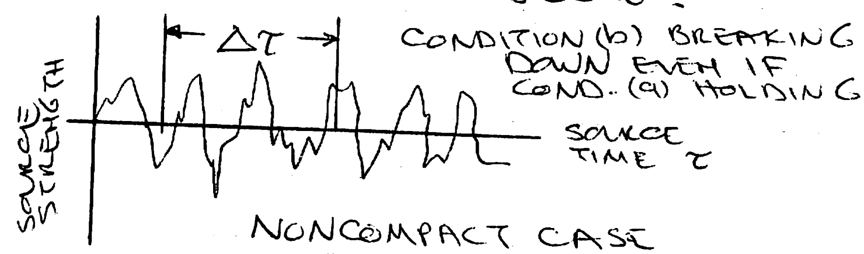
WE CAN TAKE $r_{MEAN} = \dfrac{r_1 + r_2}{2}$

CONDITION (b) TELLS US THAT THE SOURCE STRENGTH $Q(\vec{x}, t)$ ON THE MOVING SURFACE

DOES NOT CHANGE MUCH AS THE COLLAPSING SPHERE, FOR A FIXED $(\vec{x},t)$, CROSSES THE SOURCE SURFACE. THIS



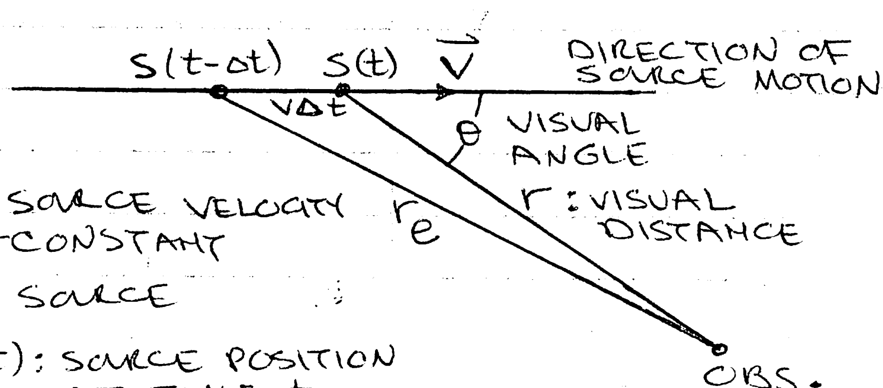
MEANS THAT THE EMISSION TIME CAN BE ASSUMED CONSTANT EQUAL TO τ_{MEAN} AS SHOWN. THIS CONDITION WILL BREAK DOWN IF $\Delta\tau$ IS SO LARGE THAT THE SOURCE GOES THROUGH SEVERAL FLUCTUATING CYCLES OR PERIODS AS SHOWN BELOW:



SO THE SOURCE STRENGTH CANNOT BE CONSIDERED CONSTANT AS THE COLLAPSING SPHERE CROSSES THE SOURCE SURFACE. THIS IS A COMMON SITUATION WHEN THE SOURCE MUST BE CONSIDERED NONCOMPACT EVEN IF CONDITION (a) HOLDS. YOU MAY THINK OF SOURCE STRENGTH AS SURFACE PRESSURE NOW.

LEC. 4 ①

1 – A POINT SOURCE IN UNIFORM RECTILINEAR MOTION AND OBSERVER STATIONARY



$\vec{V}$: SOURCE VELOCITY – CONSTANT

S : SOURCE

S(t) : SOURCE POSITION AT TIME t

S(t–Δt) : SOURCE POSITION AT EMISSION TIME

r_e : EMISSION DISTANCE $= c\Delta t$

USING THE COSINE LAW FOR THE TRIANGLE SHOWN

$$r_e^2 = r^2 + (V\Delta t)^2 + 2rV\Delta t\cos\theta$$

LET $M = V/c$, $V\Delta t = M r_e$

$$(1-M^2)r_e^2 - 2(rM\cos\theta)r_e - r^2 = 0$$

$\Rightarrow$

$$r_e = \frac{r}{1-M^2}\left[M\cos\theta \pm \sqrt{M^2\cos^2\theta + 1 - M^2}\right]$$

$$= \frac{r}{1-M^2}\left[M\cos\theta \pm \sqrt{1 - M^2\sin^2\theta}\right]$$

FOR $M < 1$ WE HAVE ONLY ONE SOLUTION THAT GIVES $r_e > 0$.

IT IS:

$$r_e = \frac{r}{1-M^2}\left[M\cos\theta + \sqrt{1-M^2\sin^2\theta}\right]$$

NOTE THAT THE RIGHT SIDE OF THIS RELATION IS WRITTEN IN TERMS OF VISUAL QUANTITIES r & θ.

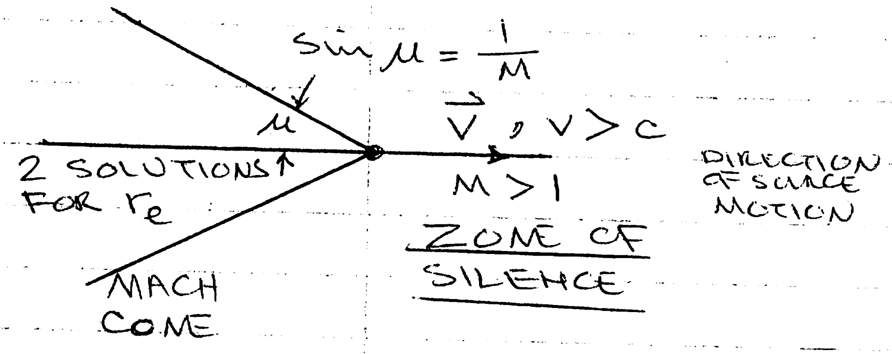
- WE HAVE, THEREFORE, OBTAINED EMISSION TIME $t_e = t - \frac{r_e}{c}$ (RETARDED TIME)
- THIS IS ONE IMPORTANT CASE WHERE WE HAVE EMISSION DISTANCE & TIME ANALYTICALLY.
- THE ABOVE TRIANGLE (WITH VERTICES AT OBS., SOURCE AT t AND $t-\Delta t$) IS CALLED THE GARRICK TRIANGLE.

ED GARRICK WAS A NASA LANGLEY SCIENTIST WHO WORKED ON UNSTEADY AERODYNAMICS AND ACOUSTICS.

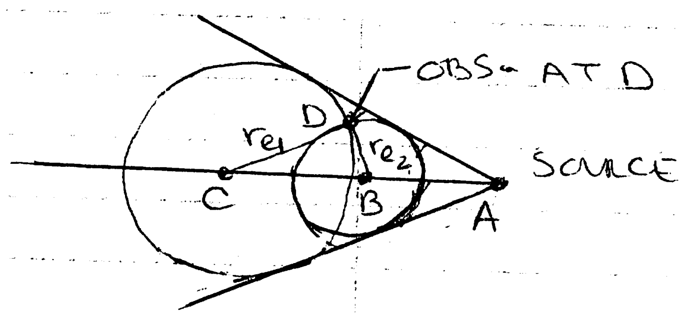
- WHEN $M>1$, WE HAVE NO SOLUTION FOR r_e WHEN $1-M^2\sin^2\theta<0$ OR $\sin^2\theta > \frac{1}{M^2}$. WE ARE THEN

LEC. 4 (3)

IN THE ZONE OF SILENCE.



IF $1 - M^2 \sin^2\theta > 0$, WE ARE INSIDE THE MACH CONE AND WE HAVE TWO SOLUTIONS FOR r_e AS SHOWN BELOW:



A: SOURCE AT TIME t

B: " " " $t - \frac{|AB|}{c}$

C: " " " $t - \frac{|AC|}{c}$

OBS. AT D INSIDE THE MACH CONE GETS TWO SIGNALS FROM SOURCE WHEN IT WAS AT B AND C. BOTH SIGNALS ARE RECEIVED AT TIME t.

LEC. 4

(4)

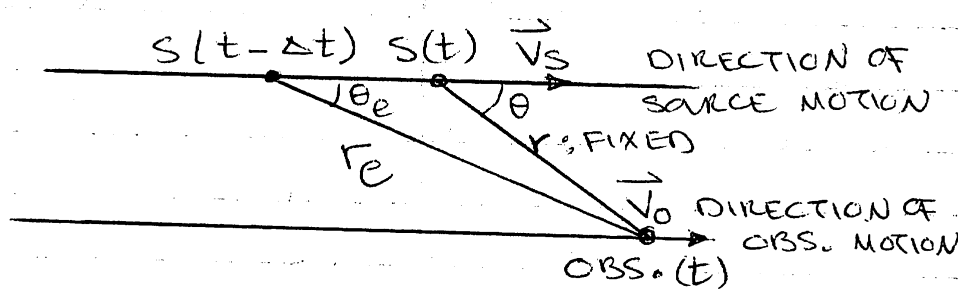
NOTE THAT GARRICK & WATKINS USED THE EMISSION TIME AND DISTANCE RELATION ABOVE FOR $M < 1$ TO EXTEND GUTIN'S PROPELLER NOISE FORMUL (FOR LOADING) TO PROPELLERS IN FORWARD FLIGHT. THIS WAS DONE IN 1950'S.

— THE ABOVE RESULTS SHOW THE CLOSE RELATION OF ACOUSTICS TO LINEAR UNSTEADY AERODYNAMICS.

— WE NOW HAVE THE ANALYTIC FORM OF LOWSON'S FORMULA IN CLOSED FORM FOR A FORCE IN UNIFORM RECTILINEAR MOTION FOR AN OBSERVER STATIONARY OR IN MOTION WITH THE SOURCE. THE FORMULA IS LENGTHY BUT IT IS MORE FUN PUTTING IT ON A COMPUTER!

LEC. 4 (5)

2 — A POINT SOURCE AND OBSERVER IN UNIFORM RECTILINEAR MOTION



— SOURCE & OBS. ARE IN UNIFORM RECTILINEAR MOTION. $\vec{V}_S = \vec{V}_o = \vec{V}$

— THE VISUAL DISTANCE r IS FIXED

USING THE GARRICK TRIANGLE RELATION $\Longrightarrow$ r_e IS FIXED

$$t_e = t - \frac{r_e}{c} \text{ IS FIXED}$$

IF WE ASSUME THAT THE SOURCE HAS $e^{i\omega t}$ BEHAVIOR (i.e. THE SO-CALLED STEADY STATE) $\Longrightarrow$

$$e^{i\omega t_e} = e^{-i r_e/c} \cdot e^{i\omega t}$$

i.e. THE OBSERVER HEARS THE SAME FREQUENCY AS THE SOURCE. HOWEVER, THE EMISSION DISTANCE IS r_e AND NOT r. THIS MEANS THAT IN MOVING FRAME, THE FREQUENCY

(6)

HEARD IS THE SOURCE FREQUENCY BUT THE DIRECTIVITY OF SOURCE CHANGES COMPARED TO STATIONARY SOURCE & OBSERVER.

- NOTE THAT BOTH r_e AND θ_e ARE VELOCITY DEPENDENT. ALSO NOTE THAT THE RELATION FOR r_e WE FOUND IS VALID IN THE NEAR AND FAR FIELDS.
- WE NOW CAN SEE HOW TO EXTEND LAWSON'S FORMULA TO A MOVING FRAME WHEN THE OBSERVER MOVES WITH THE FLUCTUATING FORCE IN UNIFORM RECTILINEAR MOTION.

6 Lectures on the Aeroacoustics of Rotating Blades in the Time Domain

LECTURES ON THE AEROACOUSTICS OF
ROTATING BLADES IN TIME DOMAIN

BY
F. FARASSAT

NASA LANGLEY RESEARCH CENTER AND
GEORGE WASHINGTON UNIVERSITY

LECTURES DELIVERED IN THE DEPARTMENT
OF MECHANICS AND AERONAUTICS, UNIVERSITY
OF ROME "LA SAPIENZA", JULY 1989

BRIEF HISTORY

— IN 1920'S PROPELLER NOISE BECAME IMPORTANT, SOME RESEARCH STARTED, PREDICTION EFFORTS FAILED, SOME GOOD OBSERVATIONS MADE

— IN 1936 GUTIN, USING A RESULT OF LAMB ABOUT A STATIONARY TIME DEPENDENT POINT SOURCE, DERIVED THE CORRECT EXPRESSION FOR DISTRIBUTED (NONCOMPACT) LOADING NOISE. LACKING COMPUTERS, HE APPROXIMATED HIS RESULT FOR A ROTATING POINT FORCE. IT WAS IN FREQ. DOMAIN AND WAS FOR THE FAR FIELD (STATIC PROPELLERS)

— ABOUT THE SAME TIME, DEMING DERIVED A FORMULA FOR THICKNESS NOISE OF NONLIFTING PROPELLERS BASED ON THE RESULT OF RAYLIGH (PISTON IN THE WALL). HE WAS ALSO FORCED TO CALCULATE MOSTLY FOR COMPACT SOURCES. HE DID FIND THAT THICKNESS NOISE IS IMPORTANT BUT THIS RESULT WAS FORGOTTEN FOR MANY YEARS. THE EXPERIMENTAL WORK OF ERNSTHAUSEN IN GERMANY CONFIRMED THE WORK OF DEMING IN LATE 1930'S. (STATIC PROPELLERS)

— IN LATE 40'S NACA STARTED WORKING ON PROPELLER NOISE. IT WAS MAINLY EXPERIMENTAL FINDING PARAMETRIC BEHAVIOR (HUBBARD).

— IN EARLY 50'S GARRICK AND WATKINS EXTENDED GUTIN'S WORK TO PROPELLERS IN FORWARD MOTION. THEY STATED THAT THICKNESS NOISE WAS NOT IMPORTANT. [THE IMPORTANCE OF NONCOMPACTNESS DISCOVERED.]

— IN 50'S HUBBARD & REGIER DID EXPERIMENTAL WORK ON SUPERSONIC PROPELLERS. VERY HIGH NOISE, GUTIN'S THEORY USELESS! NO THICKNESS NOISE INCLUDED IN ACOUSTIC CALCULATIONS.

— ARNOLDI IN 50'S DEVELOPED THICKNESS NOISE FORMULA, COMPACT! HE MADE CHARTS FOR PROPELLER NOISE CALCULATIONS. ONLY FIRST FEW HARMONICS CONSIDERED. ARNOLDI'S WORK DID NOT CONCLUSIVELY SHOW THAT THICKNESS NOISE WAS IMPORTANT BECAUSE OF SOURCE COMPACTNESS.

— IN 60'S HELICOPTER NOISE BECAME IMPORTANT. MORE WORK WAS DONE ON THEORY NOW. TWO WORKS OF PARTICULAR IMPORTANCE WERE BY LOWSON AND WRIGHT: UNSTEADY LOADING NOISE VERY IMPORTANT. BOTH WORKED IN FREQ. DOMAIN, WRIGHT EXCLUSSIVELY AND LOWSON MOSTLY.

— IN 1969 A SIGNIFICANT PAPER BY FFOWCS WILLIAMS AND HAWKINGS APPEARED IN PHILOSOPHICAL TRANSACTIONS: "SOUND GENERATION BY TURBULENCE AND SURFACES IN ARBITRARY MOTION", A264, 321-342

IN THIS PAPER THE NOW FAMOUS FW-H EQUATION WAS DEVELOPED. IT USES ADVANCED MATHEMATICS OF THE TYPE NOT TAUGHT TO THE ENGINEERS.

— ALTHOUGH, AS THE TITLE SHOWS, IT WAS MAINLY ADDRESSED TO TURBULENCE PROBLEMS, ITS IMPACT WAS ON ROTATING MACHINERY MAINLY. IT WAS RECOGNIZED IMMEDIATELY AS A SIGNIFICANT CONTRIBUTION BUT VERY FEW PEOPLE STARTED TO USE IT: FW & H, HAWKINGS & LOWSON (1972), FARASSAT (1973).

— SINCE MIDSEVENTIES THERE HAS BEEN AN EXPLOSION OF RESEARCH ON ROTATING BLADE NOISE BOTH EXPERIMENTAL AND THEORETICAL. PROPFAN RESEARCH

HELP THE DEVELOPMENT OF THEORY AND COMPUTER CODES. VERY FEW PEOPLE APPLIED FW-H EQ. IN THE SPIRIT OF THE ORIGINAL PAPER. EVEN FW & HIS STUDENTS DID NOT ALWAYS FOLLOW THE MATHEMATICAL METHOD OF THE PAPER. MOST PEOPLE REFER TO THE PAPER BECAUSE IT IS FASHIONABLE.

— SINCE 1975, NASA LANGLEY HAS SPENT A LOT OF EFFORT DEVELOPING CODES FOR HELICOPTER ROTORS AND HIGH SPEED PROPELLERS. THIS HAS LED TO DEVELOPMENT OF MANY FORMULATIONS IN TIME DOMAIN SOME OF WHICH WILL BE DISCUSSED HERE.

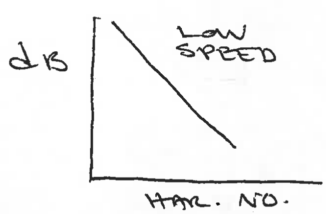
SOURCES OF NOISE IN PROPELLERS AND ROTORS

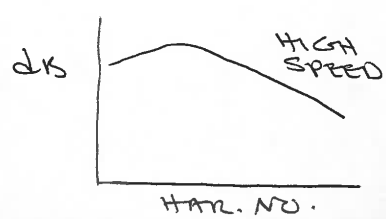
THIN BLADES ∴ LINEAR ACOUSTIC THEORY IS APPLICABLE

— GENERALLY THE PROPELLER MOTION IN SPACE IS REGULAR (i.e. MOSTLY AXISYMMETRIC), THE BLADE MOTION IS HELICOIDAL WITH FIXED PITCH, THE SURFACE PRESSURE VARIATION IS SIMPLE (i.e. AT MOST PERIODIC OR IMPULSIVE CONFINED TO BLADE-WAKE INTERACTION). THESE CHARACTERISTICS MAKE PREDICTION OF THE NOISE OF PROPELLERS SIMPLER THAN HELICOPTER ROTORS. ON THE OTHER HAND, NEW PROPELLERS HAVE SUPERSONIC TIP SPEEDS, ARE HIGHLY TWISTED AND ARE SWEPT BACK CONSIDERABLY. THIS MEANS THAT SIMPLE SOURCE DISTRIBUTION (i.e. PLANE SOURCES) AND SUBSONIC THEORIES WITH DOPPLER SINGULARITY ARE USELESS. THESE FACTS COMPLICATE THE PREDICTION OF PROPELLER NOISE. ESSENTIALLY, THEN, OUR PREDICTION THEORY MUST INCLUDE:

DISCRETE WITH NO BROAD-BAND
- i) THICKNESS NOISE
- ii) LOADING NOISE
  STEADY, PERIODIC, IMPULSIVE
- iii) PART OF QUADRUPOLE NOISE (SHOCKS, VORTICES)

THIS CLASSIFICATION IS FROM THE POINT OF VIEW OF ACOUSTIC ANALOGY (LIGHTHILL, 1954)





SOURCES OF NOISE .....

— HELICOPTERS HAVE TWO ROTORS IN GENERAL — MAIN AND TAIL ROTORS. THE FOLLOWING COMMENTS APPLY MOSTLY TO MAIN ROTORS.

— HELICOPTER ROTOR BLADES ARE THICK (10-12 %), HAVE A VERY COMPLICATED MOTION (PERIODIC OR CYCLIC PITCH CHANGE, FLAPPING, CONING & LEAD-LAG), AND OPERATE IN DISTORTED AND DISTURBED FLOW FIELDS (VORTEX AND WAKE OF PRECEEDING BLADES, TURBULENCE IN THE ATMOSPHERE). THIS MEANS THAT HELICOPTER ROTORS HAVE VERY COMPLICATED UNSTEADY BLADE LOADS AND ARE SUBJECT TO TRANSONIC EFFECTS AND SEPARATION (PARTICULARLY ON THE RETREATING SIDE). ONE OTHER DIFFERENCE WITH PROPELLERS IS THAT HELICOPTER ROTORS OPERATE IN MANY DIFFERENT REGIMES OF FLIGHT EACH OF WHICH CAN PRODUCE NOISE BY A DIFFERENT NOISE GENERATING MECHANISM.

— IN OUR PREDICTION SCHEME, WE MUST THEREFORE INCLUDE, FROM THE POINT OF THE ACOUSTIC ANALOGY, THE FOLLOWING SOURCES OF NOISE FOR HELICOPTER ROTORS

i) THICKNESS NOISE

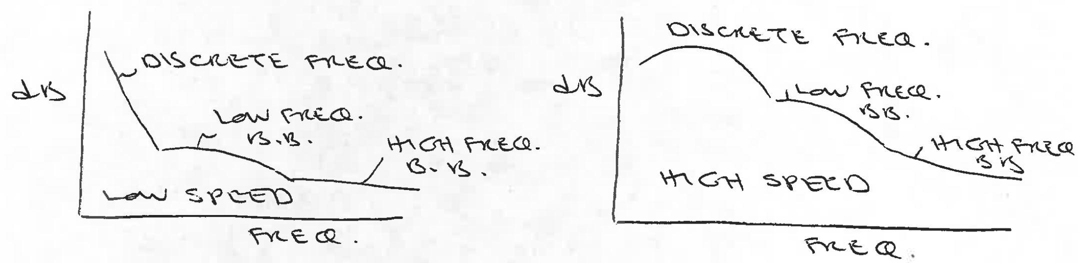
ii) LOADING NOISE

STEADY, PERIODIC, IMPULSIVE, RANDOM

SOURCES OF NOISE ....

iii) QUADRUPOLE SOURCES

LINEAR ANALYSIS ALONE DOES NOT HELP HERE. NON-LINEAR AERODYNAMICS + ACOUSTIC ANALOGY OR KIRCHHOFF FORMULA SHOULD BE USED.



TYPICAL SPECTRUM

— WE WILL DISCUSS DISCRETE FREQ. NOISE PREDICTION. BROADBAND NOISE PREDICTION SO FAR HAS BEEN DEVELOPED IN FREQ. DOMAIN. IT IS BECAUSE TURBULENCE INFORMATION IS USUALLY IN FREQ. DOMAIN (POWER SPECTRA). TIME DOMAIN METHOD MAY ALSO BE BETTER HERE.

— IN HELICOPTER ROTORS, BROADBAND NOISE PREDICTION (BLADE-TURBULENCE, BLADE-WAKE & BL NOISE AND TRAILING EDGE NOISE) MUST BE PERFORMED. THIS IS BECAUSE THE RANGE OF FREQ. OF BB NOISE FALLS IN THE REGION OF EAR SENSITIVITY AND THUS THEY CONTRIBUTE MUCH TO THE PERCEIVED NOISE LEVEL.

— FINALLY, IN RECENT YEARS IT HAS BEEN FOUND

SOURCES OF NOISE ----

THAT FREE ROTOR ANALYSIS IS NOT ENOUGH AND THE EFFECT OF THE NACELLE AND FUSELAGE, BL PROPAGATION MUST BE CONSIDERED IN THE DETERMINATION OF THE ACOUSTIC FIELD OF PROPELLERS AND ROTORS.

POSSIBLE APPROACHES TO NOISE PREDICTION

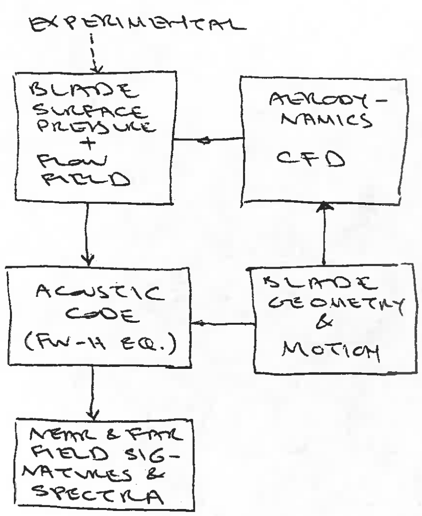
THERE ARE THREE APPROACHES AVAILABLE TO US AS FOLLOWS

i) THE ACOUSTIC ANALOGY

ii) KIRCHHOFF FORMULA FOR MOVING SURFACES

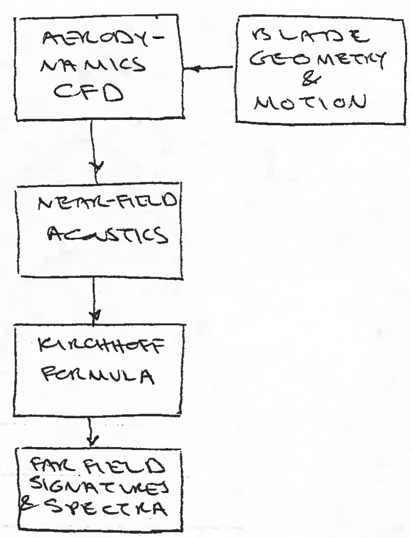
iii) PURELY COMPUTATIONAL APPROACH

THE PURELY COMPUTATIONAL APPROACH IS IN ITS INFANCY AND IS NOT YET DEVELOPED TO INCLUDE PROPELLERS AND HELICOPTER ROTORS. WE WILL THUS CONCENTRATE ON FIRST TWO METHODS.



ACOUSTIC ANALOGY APPROACH

FOR OPEN PROPELLERS & ROTORS



KIRCHHOFF FORMULA APPROACH

FOR OPEN PROPELLERS & ROTORS & DUCTED FANS
(LYRINTZIS & GEORGE, AMES GROUP)

POSSIBLE APPROACHES ----..

— WHAT SHOULD OUR FORMULATIONS AND COMPUTER CODES BE ABLE TO HANDLE? HERE IS OUR WISH LIST:

i) EXACT GEOMETRY AND MOTION OF BLADES
ii) NEAR & FAR FIELD POSITION OF THE OBSERVER
iii) MOTION OF THE OBSERVER
iv) SUBSONIC AND SUPERSONIC MOTION OF THE BLADES (NONCOMPACT SOURCES)
v) EXACT FLOW FIELD AND BLADE SURFACE PRESSURE

THE CURRENT FORMULATIONS BASED ON TIME DOMAIN APPROACH AND CODES DEVELOPED AT LANGLEY ACTUALLY SATISFY THE ABOVE WISH LIST VERY CLOSELY!

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POSSIBLE APPROACHES ----

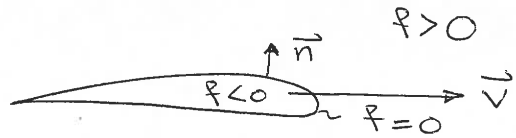
— TIME DOMAIN VS FREQUENCY DOMAIN

IT SEEMS THAT BECAUSE OF THE FOLLOWING SIMPLE RELATION, THESE TWO APPROACHES ARE EQUIVALENT:

$$\text{TIME DOMAIN} \underset{\text{TRANSFORM}}{\overset{\text{FOURIER}}{\longleftrightarrow}} \text{FREQUENCY DOMAIN}.$$

ALTHOUGH IN PRINCIPLE THIS IS SO, IN PRACTICE THEY ARE NOT EQUIVALENT! RATHER THAN BEING TWO COMPETING METHODS, THEY ARE IN FACT COMPLIMENTARY IN MANY RESPECTS. IN GENERAL FREQ. DOMAIN APPROACH IS GOOD FOR QUALITATIVE ANALYSIS GIVING THE EFFECT OF VARIOUS PARAMETERS ON NOISE. IT USES APPROXIMATIONS WHICH TIME DOMAIN METHOD DOES NOT USE. THE LATTER IS THEREFORE PREFERED FOR FINAL ANALYSIS OF THE SOUND FIELD.

THE FFOWCS WILLIAMS-HAWKINGS EQUATION



$$\boxed{\begin{aligned}\Box^2 c^2 \tilde{\rho} &\equiv \Box^2 p' \\ &= \frac{\bar{\partial}}{\partial t}\left[\rho_0 v_n |\nabla f| \delta(f)\right] \\ &\quad - \frac{\bar{\partial}}{\partial x_i}\left[P_{ij} n_j |\nabla f| \delta(f)\right] \\ &\quad + \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right]\end{aligned}}$$

THICKNESS

LOADING

QUADRUPOLE

v_n : LOCAL NORMAL VELOCITY , p' ACOUSTIC PRESSURE IN F.F.

P_{ij} : COMPRESSIVE STRESS TENSOR

$P_{ij} = p\delta_{ij} + E_{ij}$; E_{ij} VISCOUS STRESS TENSOR

$P_{ij} n_j \approx p n_i$, p GAGE PRESSURE

$T_{ij} = \rho u_i u_j + P_{ij} - c^2 \tilde{\rho}\, \delta_{ij}$ LIGHTHILL STRESS TENSOR

THE QUADRUPOLE TERM INCLUDES THE EFFECT OF NONLINEARITIES WHICH PHYSICALLY ARE OF TWO TYPES ; i) THE SPEED OF SOUND IS NOT EQUAL TO C NEAR THE BODY , ii) THE FLUID VELOCITY $|\vec{u}|$ IS NOT SMALL NEAR THE BODY SO THAT THE PROPAGATION SPEED OF SMALL DISTURBANCES NEAR THE BODY IS NOT EQUAL TO THE LOCAL SPEED OF SOUND WRT THE FLUID FIXED FRAME .

THE FW-H EQ.

— <u>NOTE</u>: ALL VELOCITIES IN FW-H EQ. IS WRT THE FRAME FIXED TO THE UNDISTURBED MEDIUM.

— THE ABOVE WAVE EQ. IS VALID IN THE ENTIRE UNBOUNDED SPACE. THE FLUID INSIDE THE BODY IS ASSUMED TO BE AT THE CONDITION OF THE UNDISTURBED MEDIUM: ρ_0, P_0, $T_{ij} = 0$.

— WE CAN ALWAYS DEFINE f SUCH THAT $\nabla f = \vec{n}$ (UNIT OUTWARD NORMAL), i.e. $|\nabla f| = 1$ SO THAT THE THICKNESS AND LOADING TERMS CAN BE WRITTEN AS

$$\frac{\partial}{\partial t}\left[v_n \, \delta(f)\right] \quad ,$$

$$-\frac{\partial}{\partial x_i}\left[P_{ij} \, n_j \, \delta(f)\right] ,$$

RESPECTIVELY.

— THE FW-H EQ. MUST BE VIEWED AS THE FUNDAMENTAL GOVERNING EQ. OF NOISE GENERATION BY MOVING BODIES. AS WE WILL SEE, ONE CAN DERIVE INTEGRAL EQUATIONS FOR AERODYNAMICS ALSO FROM THIS EQUATION.

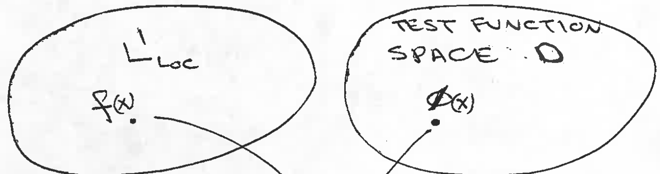
MATHEMATICAL PREPARATION

I) GENERALIZED FUNCTIONS (GEN. FNS.)

REFS : GEL'FAND & SHILOV, VOL. 1 ; GEN. FNS (1964
KANWAL : GEN. FNS – THEORY & TECHNIQUE (198
D.S. JONES : GEN. FNS (1966)
F. FARASSAT : JSV 1977, 55(2), 165–193,
(GEN. DERIVATIVES)

— GEN. FNS. ARE DEFINED AS CONTINUOUS LINEAR FUNCTIONALS ON THE SPACE $\mathcal{D}$ OF C^∞ FUNCTIONS WITH BOUNDED SUPPORT (GEL'FAND & SHILOV). FOR OUR PURPOSE, ONLY SOME CONCEPTS AND FACTS ABOUT GEN. FN THEORY ARE NEEDED WHICH WE WILL GIVE BY EXAMPLES.

— L^1_{loc}: LOCALLY INTEGRABLE FUNCTIONS, i.e. $\int_\Omega |f|\,dx < \infty$ $\forall$ BOUNDED REGION Ω



$$F[\phi] = \int f\phi\,dx$$

WE FIRST CHANGE OUR WAY OF THINKING ABOUT FUNCTIONS BY DESCRIBING THEM BY THEIR ACTIONS ON FUNCTIONS IN SPACE $\mathcal{D}$ THROUGH THE FUNCTIONAL

$$F[\phi] = \int f\phi\,dx \; ; \; F : \mathcal{D} \to \mathbb{R} \text{ or } \mathbb{C} .$$

THIS MEANS THAT WE ABANDON THE POINTWISE DEFINITION OF A FUNCTION AND THINK OF A FUNCTION AS A TABLE OF ITS FUNCTIONAL VALUES ON FUNCTIONS IN SPACE $\mathcal{D}$. WE CAN SHOW THAT IF f AND $g \in L^1_{loc}$

AND $f \neq g$ A.E. $\Longrightarrow F[\phi] = \int f \phi \, dx \neq \int g \phi \, dx = G[\phi]$ FOR SOME $\phi \in D$. THIS MEANS THAT WE ARE ABLE TO DISTINGUISH BETWEEN DIFFERENT FUNCTIONS IN L^1_{loc} USING THE FUNCTIONAL DEFINITION GIVEN ABOVE

WE CAN EASILY SHOWS THAT THE FUNCTIONAL $F[\phi]$ ABOVE IS <u>LINEAR</u>, i.e. $F[\alpha \phi_1 + \beta \phi_2] = \alpha F[\phi_1] + \beta F[\phi_2]$ WE CAN ALSO SHOW THAT IT IS <u>CONTINUOUS</u> IN THE FOLLOWING SENSE: IF $\{\phi_n\}$ IS A SEQUENCE OF FUNCTIONS IN D SUCH THAT $\phi_n \xrightarrow{D} 0 \Longrightarrow F[\phi_n] \longrightarrow 0$. WE MUST DEFINE WHAT $\phi_n \xrightarrow{D} 0$ MEANS. IT MEANS THAT i) THERE EXISTS A BOUNDED REGION $\omega \ni$ SUPPORT $\phi_n \subset \omega$, ii) $\forall$ $|\alpha| = 1, 2, \ldots,$ $D^\alpha \phi_n(x) \longrightarrow 0$ UNIFORMLY IN ω. THIS COMPLICATED DEFINITION OF CONTINUITY ACTUALLY HAS A VERY SIMPLE BUT POWERFUL CONSEQUENCE. WE SAY $\phi_n \xrightarrow{D} \phi$ IF $\phi_n - \phi \xrightarrow{D} 0$ BY THE ABOVE DEFINITION. NOW CONSIDER TWO SEQUENCES OF FUNCTIONS $\{\phi_n\}$ AND $\{\psi_n\} \ni \phi_n \xrightarrow{D} \phi$ AND $\psi_n \xrightarrow{D} \phi$ $\Longrightarrow \forall |\alpha| = 1, 2, \ldots;$ $F[D^\alpha \phi_n] \longrightarrow F[D^\alpha \phi]$ AND $F[D^\alpha \psi_n] \longrightarrow F[D^\alpha \phi]$. THIS IS A VERY DESIRABLE PROPERTY.

WE CAN SHOW THAT THE SPACE L^1_{loc} DOES NOT PRODUCE <u>ALL</u> THE CONTINUOUS LINEAR FUNCTIONALS ON SPACE D. FOR EXAMPLE, WE CAN PROVE $\delta[\phi] = \phi(0)$ IS A CONTINUOUS LINEAR FUNCTIONAL <u>NOT</u> GENERATED BY ANY FUNCTION IN SPACE L^1_{loc}. WE DEFINE

MATH. PREP.

<u>DISTRIBUTIONS</u> OR <u>GENERALIZED FUNCTIONS</u> $\mathcal{D}'$ AS THE SPACE OF CONTINUOUS LINEAR FUNCTIONALS ON SPACE $\mathcal{D}$. SPACE L^1_{loc} OF ORDINARY FUNCTIONS DEFINE <u>REGULAR GEN FNS</u>. OTHER GEN. FNS. ARE CALLED <u>SINGULAR GEN.FN</u> FOR CONVENIENCE OF ALGEBRAIC MANIPULATIONS, WE INTRODUCE SYMBOLIC FUNCTIONS FOR SINGULAR GENERALIZED FUNCTIONS WHICH ONLY HAVE MEANING WHEN THEY APPEAR UNDER AN INTEGRAL SIGN, e.g.

$$\delta[\phi] = \int \overbrace{\delta(x)}^{\text{SYMBOLIC FUNCTION (DIRAC } \delta\text{-FN)}} \phi(x)\, dx \equiv \phi(0)$$

GIVE MEANING BY

IN THIS WAY WE CAN USE DIRAC DELTA FUNCTION IN DIFFERENTIAL OR ALGEBRAIC EQUATION, e.g. THE FUNDAMENTAL SOLUTION OF THE LINEAR D.E. $\ell u = f$ IS $\ell g(x,y) = \delta(x-y)$.

WE WILL NEED ONLY THE CONCEPT OF DIFFERENTIATION OF GEN. FNS. WE MUST DEFINE THE DERIVATIVE IN SUCH A WAY THAT IT CORRESPONDS TO THE USUAL DERIVATIVE OF A FUNCTION WHEN IT EXISTS.

<u>DEFN</u>: THE <u>DERIVATIVE</u> OF GEN. FN $F[\phi]$ IS DEFINED AS $F'[\phi] = -F[\phi']$.

<u>JUSTIFICATION</u>: LET $f \in L^1_{loc} \ni f' \in L^1_{loc}$ ALSO (i.e. f COMPLETELY CONTINUOUS) $\Rightarrow F'[\phi] = \int f'\phi\, dx$
$= -\int f\phi'\, dx = -F[\phi']$.

FROM THIS DEFN, USING MULTI-INDEX NOTATION

$$F^{\alpha}[\phi] = (-1)^{|\alpha|} F[D^{\alpha}\phi]$$

WHERE $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum_1^n \alpha_i$, α_i-NONNEG. INTEGER.

BY THIS DEFINITION, GENERALIZED FUNCTIONS HAVE DERIVATIVES OF <u>ALL</u> ORDERS.

<u>EXAMPLES</u>

i) LET $H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ HEAVISIDE FN

$$H[\phi] \equiv \int_0^{\infty} \phi(x)\,dx \quad , \ \phi \in D$$

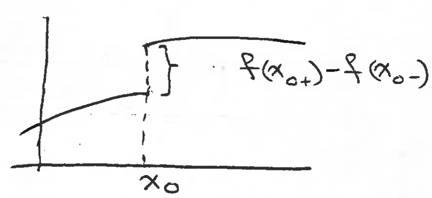
$$H'[\phi] = -H[\phi'] = -\int_0^{\infty} \phi'\,dx = \phi(0) = \delta[\phi]$$

(REMEMBER ϕ HAS COMPACT SUPPORT)

$$\therefore \quad \frac{\bar{d}}{dx} H(x) = \delta(x) \qquad \text{(SYMBOLICALLY)}$$

WHERE $\bar{d}/dx$ STANDS FOR GENERALIZED DERIVATIVE

ii) f HAS A JUMP OF MAGNITUDE $\Delta f = f(x_{0+}) - f(x_{0-})$ AT x_0 AND f' EL...



LET $F[\phi] = \int f\phi\,dx$, $\phi \in D$

$$F'[\phi] = -F[\phi'] = -\int f\phi'\,dx$$

$$= -\int_{-\infty}^{x_{0-}} - \int_{x_{0+}}^{\infty} f\phi'\,dx = \int_{-\infty}^{\infty} f'\phi\,dx + \phi(x_0)\Delta f$$

OR $\dfrac{\bar{d}f}{dx} = \dfrac{df}{dx} + \Delta f\,\delta(x - x_0)$

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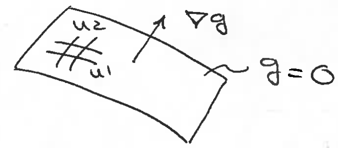
iii) f HAS A JUMP OF MAGNITUDE



$$\Delta f = f(g=0+) - f(g=0-)$$

ACROSS THE SURFACE $g=0$ IN $\mathbb{R}^3$ AND $\frac{\partial f}{\partial x_i} \in L$.
WE USE THE RESULT THE RESULT IN (ii) TO FIND $\overline{\partial f / \partial x_i}$. INTRODUCE $u^3 = g$ AND VARIABLES (u^1, u^2) ON SURFACES $g =$ CONST. $\Longrightarrow$

$$\frac{\overline{\partial f}}{\partial x_i} = \frac{\overline{\partial f}}{\partial u^j} \frac{\partial u^j}{\partial x_i} =$$

$$= \frac{\partial f}{\partial u^1} \frac{\partial u^1}{\partial x_i} + \frac{\partial f}{\partial u^2} \frac{\partial u^2}{\partial x_i} + \left[\frac{\partial f}{\partial u^3} + \Delta f\, \delta(u^3)\right] \frac{\partial u^3}{\partial x_i}$$

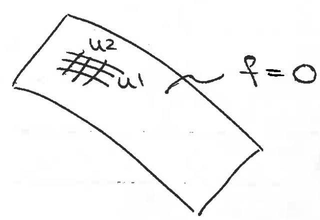
$$= \frac{\partial f}{\partial x_i} + \Delta f \frac{\partial g}{\partial x_i} \delta(g)$$

OR
$$\overline{\nabla} f = \nabla f + \Delta f\, \nabla g\, \delta(g)$$

SIMILARLY

$$\overline{\nabla} \cdot \vec{f} = \nabla \cdot \vec{f} + \Delta \vec{f} \cdot \nabla g\, \delta(g)$$
$$\overline{\nabla} \times \vec{f} = \nabla \times \vec{f} + \Delta \vec{f} \times \nabla g\, \delta(g)$$

iv) WE CAN ALWAYS DEFINE
$f = 0 \ni \nabla f = \vec{n}$ UNIT VECTOR.



WHAT ARE

$$I = \int Q(\vec{x})\, \delta(f)\, d\vec{x},$$

$$I' = \int Q(\vec{x})\, \delta'(f)\, d\vec{x} \quad ?$$

USING $u^3 = f$ AND (u^1, u^2) ON $f =$ CONST., WE CAN SHOW THAT, IF $g_{(3)}$ IS THE DETERMINANT OF THE

COEFFICIENTS OF FIRST FUNDAMENTAL FORM IN $\mathbb{R}^3$ AND $g_{(2)}$ IS THE SAME FUNCTION ON SURFACES $f=$ CONST. $\Longrightarrow$ $g_{(3)} = g_{(2)}(u^1, u^2, u^3)$ AND

$$\frac{\partial \sqrt{g_{(2)}}}{\partial u^3} = -2\sqrt{g_{(2)}}\, H_f$$

WHERE H_f IS THE MEAN CURVATURE OF THE SURFACE $f=$ CONST. NOW WE HAVE

$$\begin{aligned} I &= \int Q\, \delta(u^3)\sqrt{g_{(2)}}\, du^1 du^2 du^2 \\ &= \int_{u^3=0} Q\sqrt{g_{(2)}}\, du^1 du^2 \\ &= \int_{f=0} Q\, dS \end{aligned}$$

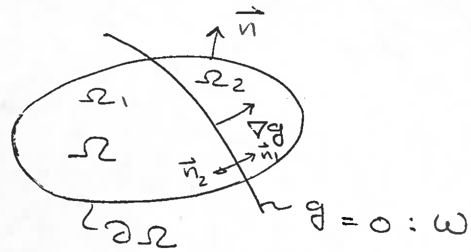
$$\begin{aligned} I' &= \int Q\, \delta'(u^3)\sqrt{g_{(2)}}\, du^1 du^2 du^3 \\ &= -\int \frac{\partial}{\partial u^3}\left[Q\sqrt{g_{(2)}}\right]\delta(u^3)\, du^1 du^2 du^3 \\ &= -\int_{u^3=0}\left[\frac{\partial Q}{\partial u^3} - 2H_f Q\right]\sqrt{g_{(2)}}\, du^1 du^2 \\ &= \int_{f=0}\left[-\frac{\partial Q}{\partial n} + 2H_f Q\right] dS \end{aligned}$$

IF f IS DEFINED SO THAT $\nabla f \neq \vec{n}$, THEN WE CAN SHOW THAT

$$\int Q\, |\nabla f|\, \delta(f)\, d\vec{x} = \int_{f=0} Q\, dS$$

iv) THE DIVERGENCE THEOREM –

$$I = \int_{\partial\Omega} \vec{f}\cdot\vec{n}\, dS = ?$$



$$\Delta\vec{f}\Big|_{g=0} = \vec{f}(g=0+) - \vec{f}(g=0-)$$

$$\Omega = \Omega_1 \cup \Omega_2$$

$$\int_{\partial\Omega_1} \vec{f}\cdot\vec{n}\, dS = \int_{\partial\Omega_1\cap\partial\Omega} \vec{f}\cdot\vec{n}\, dS + \int_{\partial\Omega_1\cap\omega} \vec{f}\cdot\vec{n}_1\, dS = \int_{\Omega_1} \nabla\cdot\vec{f}\, d\vec{x} \qquad (1)$$

$$\int_{\partial\Omega_2} \vec{f}\cdot\vec{n}\, dS = \int_{\partial\Omega_2\cap\partial\Omega} \vec{f}\cdot\vec{n}\, dS + \int_{\partial\Omega_2\cap\omega} \vec{f}\cdot\vec{n}_2\, dS = \int_{\Omega_2} \nabla\cdot\vec{f}\, d\vec{x} \qquad (2)$$

ADDING BOTH SIDES OF (1) AND (2) AND NOTICING THAT $\vec{n}_2 = -\vec{n}_1$, WE HAVE

$$\int_{\partial\Omega} \vec{f}\cdot\vec{n}\, dS - \int_{\partial\Omega_1\cap\omega} \Delta\vec{f}\cdot\vec{n}\, dS = \int_{\Omega} \nabla\cdot\vec{f}\, d\vec{x}$$

WE NOTE ALSO THAT $\partial\Omega_1\cap\omega = \partial\Omega_2\cap\omega$. ALSO USING

$$\int \Delta\vec{f}\cdot\vec{n}\, dS = \int \Delta\vec{f}\cdot\nabla g\,\delta(g)\, d\vec{x}$$

(FROM LAST RESULT PREVIOUS PAGE), WE HAVE

$$\int_{\partial\Omega} \vec{f}\cdot\vec{n}\, dS = \int_{\Omega} \left(\nabla\cdot\vec{f} + \Delta\vec{f}\cdot\nabla g\,\delta(g)\right) d\vec{x}$$

$$= \int_{\Omega} \overline{\nabla}\cdot\vec{f}\, d\vec{x}\,.$$

THIS MEANS THE DIVERGENCE THM IS VALID FOR DISCONTINUOUS VECTOR FIELDS IF GENERALIZED DERIVATIVES IS USED IN PLACE OF ORDINARY DERIVATIVES.

V) <u>CHANGE OF ORDER OF LIMIT PROCESSES</u> — AS LONG AS WE USE GENERALIZED DERIVATIVES WE CAN CHANGE ORDER OF INTEGRATION AND INFINITE SUMS WITH DIFFERENTIATION, i.e.

$$\frac{\bar{\partial}^{|\alpha|}}{\partial x^{\alpha}} \int \cdots = \int \frac{\bar{\partial}^{|\alpha|}}{\partial x^{\alpha}} \cdots\cdots$$

$$\frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \sum_{1}^{\infty} \cdots\cdots = \sum_{1}^{\infty} \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \cdots\cdots$$

IN APPLYING THE FIRST RESULT, TWO SITUATIONS OF INTEREST APPEAR AS FOLLOWS:

1) WE ARE DERIVING A CONSERVATION LAW WHERE WE KNOW THAT THE INTEGRAND IS DISCONTINUOUS, e.g. WE HAVE SHOCKS IN THE FLOW FIELD. HOWEVER, ON PHYSICAL ARGUMENT WE KNOW THAT THE DERIVATIVE OF THE INTEGRAL IS A REGULAR GENERALIZED FUNCTION. THEN THE ORDINARY DERIVATIVE IS THE SAME AS GENERALIZED DERIVATIVE WHICH CAN BE TAKEN INSIDE INTEGRAL ONLY AS GENERALIZED DERIVATIVE, e.g.

$$\frac{\partial}{\partial t}\int_{\Omega} \rho \, d\vec{x} = \frac{\bar{\partial}}{\partial t}\int_{\Omega} \rho \, d\vec{x}$$

$$= \int_{\Omega} \frac{\bar{\partial} \rho}{\partial t} \, d\vec{x} \, .$$

COMBINED WITH THE DIVERGENCE THEOREM, WHEN

WE STUDY THE DERIVATION OF MASS CONTINUITY AND NAVIER-STOKES (OR EULER) EQUATION, THE ABOVE RESULT GIVES THE FOLLOWING IMPORTANT CONCLUSION:

THE EQUATIONS OF MASS CONTINUITY AND MOMENTUM ARE VALID WHEN ORDINARY DERIVATIVES ARE REPLACED BY GENERALIZED DERIVATIVES.

WE WILL SEE THE RAMIFICATIONS OF THIS LATER.

2) LET $I(x)=\int_\Omega Q(x,y)\,dy$ AN IMPROPERLY CONVERGENT INTEGRAL SUCH THAT $\partial^{|\alpha|}I/\partial x^\alpha$ EXISTS AND IS AN ORDINARY FUNCTION. IN GENERAL $\partial^{|\alpha|}Q/\partial x^\alpha$ RESULTS IN A DIVERGENT (OR HYPERSINGULAR) INTEGRAL. WE THEN USE

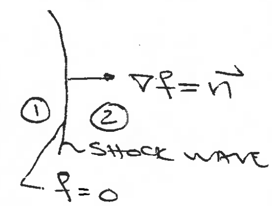
$$\frac{\partial^{|\alpha|}I}{\partial x^\alpha} = \frac{\bar{\partial}^{|\alpha|}I}{\partial x^\alpha} = \int_\Omega \frac{\bar{\partial}^{|\alpha|}}{\partial x^\alpha} Q(x,y)\,dy$$

THIS PROCEDURE IS ENTIRELY EQUIVALENT TO TAKING FINITE PART OF DIVERGENT INTEGRALS AS WILL BE SHOWN LATER. HOWEVER, USING THE PROPERTY OF COMPLETENESS OF SPACE OF GENERALIZED FUNCTIONS, THE ABOVE DEFINITION CAN LEAD TO ENTIRELY NEW (BUT NUMERICALLY EQUIVALENT) EXPRESSIONS WHICH MAY BE MORE USEFUL THAN THE CLASSICAL OR CONVENTIONAL HADAMARD'S RESULT.

VI) <u>TWO APPLICATIONS OF GENERALIZED DERIVATIVES</u>

1) <u>JUMP CONDITIONS ACROSS UNSTEADY SHOCK WAVES</u> –

ASSUME THE SHOCK SURFACE IS GIVEN BY $f(\vec{x},t) = 0$ SUCH THAT $\nabla f = \vec{n}$ THE UNIT OUTWARD NORMAL.



LET Δ STAND FOR JUMP ACROSS THIS SHOCKWAVE DEFINED AS $\Delta = [\ \ ]_{②} - [\ \ ]_{①}$ WHERE REGION ② IS WHERE $\vec{n}$ POINTS INTO.

WE SAID IN SECTION (V) THAT THE MASS CONTINUITY AND MOMENTUM EQS. ARE VALID WHEN ALL THE DERIVATIVES ARE VIEWED AS GEN. DERIVATIVES. LET US SEE WHAT MASS CONTINUITY EQ. GIVES US

$$\frac{\bar{\partial} \rho}{\partial t} = \frac{\partial \rho}{\partial t} + \Delta\rho \frac{\partial f}{\partial t}\,\delta(f) = \frac{\partial \rho}{\partial t} - \Delta\rho\, v_n\, \delta(f)$$

$$\bar{\nabla}\cdot(\rho\vec{u}) = \nabla\cdot(\rho\vec{u}) + \Delta(\rho u)\cdot\vec{n}\ \delta(f)$$
$$= \nabla\cdot(\rho\vec{u}) + \Delta(\rho u_n)\,\delta(f)$$

$$0 = \frac{\bar{\partial}\rho}{\partial t} + \bar{\nabla}\cdot(\rho\vec{u}) = \underbrace{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u})}_{=0} + \underbrace{\left[\Delta(\rho u_n) - \Delta\rho v_n\right]}_{=0}\delta(f)$$

WHERE v_n IS THE LOCAL NORMAL VELOCITY OF THE SHOCK. THEREFORE WE HAVE

$$\Delta(\rho u_n) = \Delta\rho v_n \quad \text{OR} \quad \Delta[\rho(u_n - v_n)] = 0$$

WHICH IS THE UNSTEADY SHOCK JUMP CONDITION.

SIMILARLY THE MOMENTUM EQ. GIVES

MATH. PREP.

$$\frac{\bar\partial}{\partial t}(\rho u_i) = \frac{\partial}{\partial t}(\rho u_i) + \Delta(\rho u_i)\frac{\partial f}{\partial t}\delta(f)$$
$$= \frac{\partial}{\partial t}(\rho u_i) - \Delta(\rho u_i)\, v_n\, \delta(f)$$

$$\frac{\bar\partial}{\partial x_j}[\rho u_i u_j + p\delta_{ij}] = \frac{\partial}{\partial x_j}[\rho u_i u_j + p\delta_{ij}] + \Delta(\rho u_i u_j + p\delta_{ij})\, n_j\, \delta(f)$$
$$= \frac{\partial}{\partial x_j}[\rho u_i u_j + p\delta_{ij}] + \Delta(\rho u_i u_n + p n_i)\,\delta(f)$$

$$0 = \frac{\bar\partial}{\partial t}(\rho u_i) + \frac{\bar\partial}{\partial x_j}[\rho u_i u_j + p\delta_{ij}] = \underbrace{\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}[\rho u_i u_j + p\delta_{ij}]}_{=0}$$
$$+ [\underbrace{\Delta(\rho u_i u_n + p n_i) - \Delta(\rho u_i)\, v_n}_{=0}]\,\delta(f)$$

$\therefore$ IF $\vec{u}_T$ IS THE TANGENTIAL COMPONENT OF THE VELOCITY WE HAVE $\Delta(\rho u_n \vec{u}_T) - v_n \Delta(\rho\vec{u}_T) = 0$

$$\Delta[\rho(u_n - v_n)\vec{u}_T] = 0 \Rightarrow \Delta\vec{u}_T = 0.$$

AND FOR THE NORMAL COMPONENT OF VELOCITY WE HAVE

$$\Delta(\rho u_n^2 + p) - v_n\Delta(\rho u_n) = 0$$

OR

$$\Delta[\rho u_n(v_n - u_n)] = \Delta p$$

$\therefore$

$$m\,\Delta u_n = \Delta p, \quad m = [\rho(v_n - u_n)]_1 = [\rho(v_n - u_n)]_2$$

2) <u>FINITE PART OF DIVERGENT INTEGRALS</u>.

WE USE A SIMPLE EXAMPLE HERE, NAMELY THE CAUCHY PRINCIPAL VALUE:

$$I = \mathrm{PV}\int_a^b \frac{f(x)}{x}\,dx \quad , \quad a < 0 < b \ , \ f \in C^1$$

THIS MUST BE INTERPRETED AS

$$I = \int_a^b f \frac{d}{dx} \ln|x| \, dx \quad , \quad \ln : \text{NAT. LOG}$$

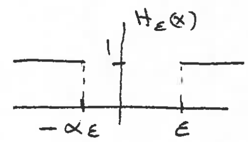
$$= f(b) \ln b - f(a) \ln|a| - \int_a^b f'(x) \ln|x| \, dx$$

IT CAN ALSO BE INTERPRETED AS

$$I = \lim_{\varepsilon \to 0} \int_a^b f(x) \frac{d}{dx} \left[H_\varepsilon(x) \ln|x| \right] dx$$

WHERE

$$H_\varepsilon(x) = \begin{cases} 1 & x < -\alpha\varepsilon \\ 0 & -\alpha\varepsilon < x < \varepsilon \\ 1 & x > \varepsilon \end{cases}$$



FOR SOME $\alpha > 0$. WE HAVE

$$\frac{d}{dx} \left[H_\varepsilon(x) \ln|x| \right] = \ln|x| \left[-\delta(x+\alpha\varepsilon) + \delta(x-\varepsilon) \right] + H_\varepsilon(x) \frac{1}{x}$$

$$= -\ln(\alpha\varepsilon)\, \delta(x+\alpha\varepsilon) + \ln\varepsilon \, \delta(x-\varepsilon) + H_\varepsilon(x) \frac{1}{x}$$

$$\Rightarrow \quad I = \left[-\ln(\alpha\varepsilon) f(-\alpha\varepsilon) + \ln\varepsilon \, f(\varepsilon) + \int_a^b \frac{H_\varepsilon(x) f(x)}{x} dx \right]$$

$$= -f(0) \ln(\alpha) + \lim_{\varepsilon \to 0} \left[\int_a^{-\alpha\varepsilon} + \int_\varepsilon^b \frac{f(x)}{x} dx \right]$$

IF WE TAKE $\alpha = 1$, WE GET THE USUAL DEFINITION OF CAUCHY PRINCIPAL VALUE. BUT IF WE TAKE $\alpha = \frac{1}{2}$, WE GET A DIFFERENT ANALYTIC EXPRESSION BUT <u>NUMERICALLY EQUIVALENT</u> RESULT TO THE USUAL DEFINITION OF CAUCHY PRINCIPAL VALUE.

— INCIDENTALLY $\lim_{\varepsilon \to 0} \frac{d}{dx} \left[H_\varepsilon(x) \ln|x| \right] = \frac{d}{dx} \ln|x|$ FOLLOWS FROM THE THEOREM THAT THE SPACE OF GENERALIZED FUNCTIONS IS COMPLETE.

MATH. PREP.

vii) <u>THE RESTRICTION OF A FUNCTION TO THE SUPPORT OF A DELTA FUNCTION</u>

IN APPLICATIONS, WE OFTEN HAVE TERMS LIKE

$\phi(\vec{x})\,\delta(f)$ & $\nabla\cdot[\vec{Q}(\vec{x})\,\delta(f)]$.

ALGEBRAIC MANIPULATIONS ARE REDUCED IF WE RESTRICT TO THE SURFACE $f=0$ A FUNCTION MULTIPLYING $\delta(f)$. TO SIMPLIFY THE DISCUSSION, LET US CONSIDER $\phi(x)\,\delta(x)$. WE CAN WRITE

$$\phi(x)\,\delta(x) = \phi(0)\,\delta(x)$$

$$\frac{d}{dx}[\phi(x)\,\delta(x)] = \underbrace{\phi'(x)\,\delta(x)}_{\phi'(0)\,\delta(x)} + \phi(x)\,\delta'(x) = [\phi(0)\,\delta(x)]'$$
$$= \phi(0)\,\delta'(x)$$

IT IS OBVIOUS THAT $\phi(0)\,\delta'(x)$ IS SIMPLER THAN $\phi'(0)\,\delta(x) + \phi(x)\,\delta'(x)$. BUT THEY ARE EQUAL:

$$\int[\phi'(0)\,\delta(x) + \phi(x)\,\delta'(x)]\psi(x)\,dx = \phi'(0)\psi(0) - [\phi(x)\psi(x)]'_{x=0}$$
$$= -\phi(0)\psi'(0)$$
$$= \int \phi(0)\,\psi(x)\,\delta'(x)\,dx$$

A SIMILAR RESULT HOLDS FOR MULTIDIMENSIONAL FUNCTIONS. LET US USE $\hat{\phi}(\vec{x}) = \phi(\vec{x})\big|_{f=0}$, THEN $\phi(\vec{x})\,\delta(f) = \hat{\phi}(x)\,\delta(f)$

WE HAVE

$$\nabla[\phi(\vec{x})\,\delta(f)] = \nabla\phi\,\delta(f) + \phi\,\nabla f\,\delta'(f) \qquad (1)$$

$$\nabla[\hat{\phi}(\vec{x})\,\delta(f)] = \nabla_2\hat{\phi}\,\delta(f) + \hat{\phi}\,\nabla f\,\delta'(f) \qquad (2)$$

WHERE ∇_2 IS SURFACE GRADIENT OPERATOR ON $f=0$. THE RIGHT SIDES OF (1) AND (2) ARE ACTUALLY EQUIVALENT. WE MENTION THAT SINCE $\hat{\phi}$ IS RESTRICTED TO $f=0$, WE HAVE $\partial\hat{\phi}/\partial n = 0$.

MATH. PREP.

LET $\vec{Q}$ BE A VECTOR FIELD ($\in D$) AND $\nabla f = \vec{n}$ AS BEFORE. WE WILL INTERPRETE $\vec{Q} \cdot \nabla[\phi\,\delta(f)]$ AND $\vec{Q} \cdot \nabla[\hat{\phi}\,\delta(f)]$ USING EQS (1) AND (2) :

FROM (1) :

$$I_{①} = \int [\vec{Q} \cdot \nabla\phi + \phi\, Q_n\, \delta'(f)]\, d\vec{x} = \int_{f=0} [\vec{Q} \cdot \nabla\phi - \frac{\partial}{\partial n}(\phi Q_n) + 2H_f \phi Q_n]\, dS$$

BUT $\vec{Q} \cdot \nabla\phi - \frac{\partial}{\partial n}(\phi Q_n) = \vec{Q}_T \cdot \nabla_2 \phi + \cancel{\phi_n Q_n} - \cancel{\phi_n Q_n} - \phi \frac{\partial Q_n}{\partial n}$

$$= \vec{Q}_T \cdot \nabla_2 \phi - \phi \frac{\partial Q_n}{\partial n}$$

$\therefore$

$$I_{①} = \int_{f=0} [\vec{Q}_T \cdot \nabla_2\phi - \phi \frac{\partial Q_n}{\partial n} + 2H_f \phi Q_n]\, dS$$

WHERE AS BEFORE $\vec{Q}_T$ IS THE TANGENTIAL COMPONENT OF $\vec{Q}$ TO THE SURFACE $f=0$, $Q_n = \vec{Q} \cdot \vec{n}$ AND H_f IS THE MEAN CURVATURE OF THE SURFACE $f=0$.

FROM (2)

$$I_{(2)} = \int [\vec{Q} \cdot \nabla_2 \hat{\phi} + \hat{\phi}\, Q_n\, \delta'(f)]$$

$$= \int_{f=0} [\vec{Q}_T \cdot \nabla_2 \hat{\phi} - \hat{\phi} \frac{\partial Q_n}{\partial n} + 2H_f \hat{\phi} Q_n]\, dS = I_{(1)}$$

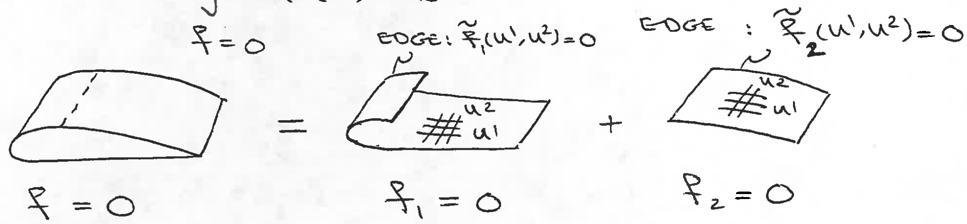
NOTE THAT SINCE $\phi = \hat{\phi}$ ON $f=0$, THE ABOVE TWO RESULTS ARE COMPLETELY EQUIVALENT. BUT USING EQ. (2), WE HAVE LESS ALGEBRAIC MANIPULATIONS.

viii) A RESULT CONCERNING A DELTA FUNCTION WITH SUPPORT ON A SURFACE $f(\vec{x})=0$

CONSIDER THE INTEGRAL

$$I = \int Q(\vec{x})\,\delta(f)\,d\vec{x} \qquad , |\nabla f| = 1$$

$$= \int_{f=0} Q(\vec{x})\,dS$$



LET US BREAK THE SURFACE $f=0$ INTO TWO SURFACES AS SHOWN. FROM LINEARITY OF INTEGRALS, WE SEE THAT

$$I = \int_{f_1=0} Q(\vec{x})\,dS + \int_{f_2=0} Q(\vec{x})\,dS$$

$$= \int Q(\vec{x})\,\delta(f_1)\,dS + \int Q(\vec{x})\,\delta(f_2)\,dS$$

SO THAT

$$\delta(f) = \delta(f_1) + \delta(f_2)$$

IN PRACTICE, WE DEFINE THE EDGES OF THE OPEN SURFACES f_1 AND f_2 BY CURVES $\tilde{f}_1(u^1,u^2)=0$ AND $\tilde{f}_2(u^1,u^2)=0$ SUCH THAT $\tilde{f}_1>0$ ON f_1 AND $\tilde{f}_2>0$ ON f_2 AND THEN

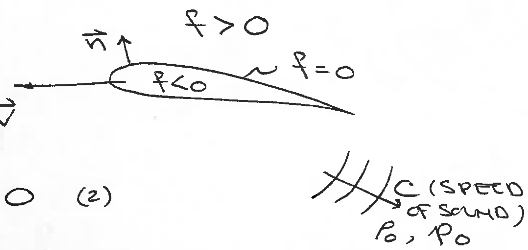
$$f_1(\vec{x}) = H(\tilde{f}_1)\,f(\vec{x})$$

$$f_2(\vec{x}) = H(\tilde{f}_2)\,f(\vec{x})$$

WE USE THIS DECOMPOSITION OF $\delta(f)$ IN FORMULATION 3 AND QUADRUPOLE NOISE ANALYSIS.

iX) DERIVATION OF THE FFOWCS WILLIAMS–HAWKINGS (FW–H) EQUATION

WE GIVE A BRIEF DERIVATION OF THE FW–H EQ. HERE. THE MASS CONTINUITY AND MOMENTUM EQS. ARE



$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x_i}(\rho u_i) = 0 & (1) \\[2ex] \dfrac{\partial}{\partial t}(\rho u_i) + \dfrac{\partial}{\partial x_j}\left[\rho u_i u_j + P_{ij}\right] = 0 & (2) \end{cases}$$

WE ASSUME THAT INSIDE THE BODY THERE IS FLUID WITH THE CONDITIONS OF UNDISTURBED MEDIUM ρ_0, P_0 $(T_{ij}=0)$. WE THUS HAVE A DISCONTINUITY IN THE FLOW FIELD ACROSS $f=0$, $\Delta\rho \equiv \tilde{\rho} = \rho - \rho_0$, $\Delta P = P_a - P_0 \equiv p$ WHERE P_a IS THE PRESSURE ON THE BODY SURFACE AND $\Delta T_{ij} = T_{ij} - 0 = T_{ij}$. WE KNOW THAT THE MASS CONTINUITY AND MOMENTUM EQS. ARE VALID WHEN ALL DERIVATIVES ARE VIEWED AS GENERALIZED DERIVATIVES. LET US SEE WHAT THESE TWO EQS. GIVE US :

$$\begin{cases} \dfrac{\bar{\partial} \rho}{\partial t} = \dfrac{\partial \rho}{\partial t} + \tilde{\rho}\dfrac{\partial f}{\partial t}\delta(f) = \dfrac{\partial \rho}{\partial t} - \tilde{\rho}\, v_n |\nabla f| \delta(f) \\[2ex] \dfrac{\bar{\partial}}{\partial x_i}(\rho u_i) = \dfrac{\partial}{\partial x_i}(\rho u_i) + \rho u_i \dfrac{\partial f}{\partial x_i}\delta(f) = \dfrac{\partial}{\partial x_i}(\rho u_i) + \rho v_n |\nabla f| \delta(f) \end{cases}$$

ADD BOTH SIDES AND USE EQ. (1) ON THE RIGHT SIDE TO GET

$$\frac{\bar{\partial} \rho}{\partial t} + \frac{\bar{\partial}}{\partial x_i}(\rho u_i) = \rho_0 v_n |\nabla f| \delta(f) \qquad (3)$$

NOW, FOR THE MOMENTUM EQ. (2), WE HAVE

$$\begin{cases} \dfrac{\bar{\partial}}{\partial t}(\rho u_i) = \dfrac{\partial}{\partial t}(\rho u_i) + \rho u_i \dfrac{\partial f}{\partial t}\delta(f) = \dfrac{\partial}{\partial t}(\rho u_i) - \rho u_i v_n |\nabla f| \delta(f) \\ \dfrac{\bar{\partial}}{\partial x_j}\left[\rho u_i u_j + P_{ij}\right] = \dfrac{\partial}{\partial x_j}\left[\rho u_i u_j + P_{ij}\right] + \left[\rho u_i u_j + P'_{ij}\right] n_j |\nabla f| \delta(f) \end{cases}$$

$$= \frac{\partial}{\partial x_j}\left[\rho u_i u_j + P_{ij}\right] + \left[\rho u_i v_n + P'_{ij} n_j\right] |\nabla f| \delta(f)$$

WHERE $P'_{ij} = P_{ij} - P_0 \delta_{ij}$. WE NOW DROP PRIME ON P'_{ij} REMEMBRING THAT <u>ALL SURFACE PRESSURES ARE THE GAGE PRESSURE</u> IN FW-H EQUATION. ADDING BOTH SIDES OF THE ABOVE EQS. AND USING EQ. (2) ON THE RIGHT SIDE, WE GET

$$\frac{\bar{\partial}}{\partial t}(\rho u_i) = -\frac{\bar{\partial}}{\partial x_j}(\rho u_i u_j + P_{ij}) + P_{ij} n_j |\nabla f| \delta(f) \qquad (4)$$

NOW TAKE $\frac{\bar{\partial}}{\partial t}$ OF BOTH SIDES OF EQ. (3), $\frac{\bar{\partial}}{\partial x_i}$ OF BOTH SIDES OF EQ. (4) AND SUBTRACT BOTH SIDES OF THE RESULTING EQUATIONS. WE OBTAIN

$$\frac{\bar{\partial}^2}{\partial t^2}\rho = \frac{\bar{\partial}}{\partial t}\left[\rho_0 v_n |\nabla f| \delta(f)\right] - \frac{\bar{\partial}}{\partial x_i}\left[P_{ij} n_j |\nabla f| \delta(f)\right] + \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[\rho u_i u_j + P_{ij}\right] \qquad (5)$$

WE KNOW FROM PHYSICS THAT THE PROPAGATION OF ACOUSTIC PRESSURE FAR FROM THE BODY IS GOVERNED BY THE WAVE EQUATION. REPLACING ρ BY $\tilde{\rho} = \rho - \rho_0$ AND SUBTRACTING $\bar{\nabla}^2 c^2 \tilde{\rho}$ FROM BOTH SIDES OF EQ. (5) GIVES THE FFOWCS WILLIAMS-HAWKINGS EQUATION:

$$\boxed{\begin{aligned}\bar{\Box}^2 c^2 \tilde{\rho} &\equiv \bar{\Box}^2 p' \\ &= \frac{\bar{\partial}}{\partial t}\left[\rho_0 v_n |\nabla f| \delta(f)\right] \\ &- \frac{\bar{\partial}}{\partial x_i}\left[P_{ij} n_j |\nabla f| \delta(f)\right] \\ &+ \frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right]\end{aligned}}$$

THICKNESS SOURCE

LOADING SOURCE

QUADRUPOLE SOURCE

WHERE $p' = c^2 \tilde{\rho}$ IS THE ACOUSTIC PRESSURE IN THE FAR FIELD AND

$$T_{ij} = \rho u_i u_j + P_{ij} - c^2 \tilde{\rho} \delta_{ij}$$

IS THE LIGHTHILL STRESS TENSOR. ALSO, $H(f)$ IS THE HEAVISIDE FUNCTION. IT IS USED HERE AS A REMINDER THAT $T_{ij} = 0$ INSIDE THE BODY.

— WE DO NOT NEED TO INCLUDE $|\nabla f|$ IN THE FW-H EQUATION SINCE WE CAN ALWAYS DEFINE $f(\vec{x}, t)$ SUCH THAT $\nabla f = \vec{n}$ AND THUS $|\nabla f| = 1$ OR USE THE RELATION

$$|\nabla f| \delta(f) = \delta(f / \widehat{|\nabla f|}) \equiv \delta(\tilde{f})$$

WHERE

$$\tilde{f} = \frac{f(\vec{x}, t)}{\widehat{|\nabla f|}} \qquad \left\{ \begin{array}{l} \widehat{|\nabla f|} \text{ RESTRICTION} \\ \text{OF } |\nabla f| \text{ TO } f = 0 \end{array} \right.$$

— NOTE THAT T_{ij} HAS JUMPS ACROSS SHOCK WAVES IN THE FLOW IN ADDITION TO THE JUMP ACROSS THE BODY SURFACE. WE WILL INCLUDE THIS JUMP IN THE QUADRUPOLE NOISE ANALYSIS.

MATH. PREP.

II - SOME RESULTS FROM DIFFERENTIAL GEOMETRY

WE NEED SOME RESULTS FROM DIFF. GEOMETRY WHICH ARE ELEMENTARY BUT USED OFTEN IN OUR WORK.

WE DEFINE THE FIRST FUNDAMENTAL FORM

$$\left.\begin{array}{l}\text{ELEMENT OF}\\ \text{ARC LENGTH}\\ \text{ON A SURFACE}\end{array}\right\} d\ell^2 = g_{ij}\, du^i du^j \qquad i,j = 1-2$$

WHERE THE SURFACE IS GIVEN BY $\vec{r} = \vec{r}(u^1, u^2)$

AND $g_{ij} = \vec{a}_i \cdot \vec{a}_j \;, \quad \vec{a}_i = \dfrac{\partial \vec{r}}{\partial u^i}$.

WE DENOTE BY $g_{(2)}$ THE DETERMINENT OF THE COEFFICIENTS OF THE FIRST FUNDAMENTAL FORM

$$g_{(2)} = \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix} = \left| \frac{\partial \vec{r}}{\partial u^1} \times \frac{\partial \vec{r}}{\partial u^2} \right|^2$$

$$\therefore \left.\begin{array}{l}\text{ELEMENT OF}\\ \text{SURFACE AREA}\end{array}\right\} dS = \sqrt{g_{(2)}}\; du^1 du^2$$

— SECOND FUNDAMENTAL FORM

$$II = b_{ij}\, du^i du^j \qquad i,j = 1-2$$

WHERE

$$b_{ij} = \vec{n} \cdot \frac{\partial^2 \vec{r}}{\partial u^i \partial u^j} = - \frac{\partial \vec{n}}{\partial u^i} \cdot \frac{\partial \vec{r}}{\partial u^j}$$

$$b = \begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{21} \end{vmatrix}$$

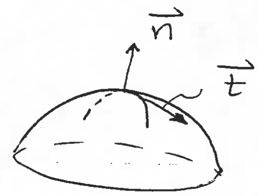
— WEINGARTEN FORMULA

$$\frac{\partial \vec{n}}{\partial u^i} = - b_i^j \, \vec{a}_j$$

WHERE $b_i^j = g^{jk} b_{ki}$ (SUM ON K)

HERE: g^{ij} ARE THE ELEMENTS OF THE INVERSE OF THE MATRIX $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix}$.

MATH. PREP.



— <u>NORMAL CURVATURE</u> κ_t IN DIRECTION $\vec{t}$ (NOT A UNIT VECTOR). LET
$\vec{t} = t^i \vec{a}_i \Rightarrow$

$$b_{ij}\, t^i t^j = \kappa_t |\vec{t}|^2 \;, \quad |\vec{t}|^2 = g_{ij}\, t^i t^j$$

— <u>THE MEAN CURVATURE</u> OF A SURFACE

$$H = \frac{1}{2 g_{(2)}} \left[g_{11} b_{22} - 2 g_{12} b_{12} + g_{22} b_{11} \right] = \frac{1}{2} b_i^i$$

— <u>THE GAUSSIAN CURVATURE</u> OF A SURFACE

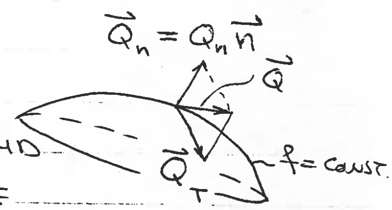
$$K = \frac{b}{g_{(2)}} = b_1^1 b_2^2 - b_1^2 b_2^1$$

— <u>THE PRINCIPAL DIRECTIONS</u> ON A SURFACE ARE GIVEN BY THE TWO SOLUTIONS OF

$$b_2^1 (du^2)^2 + (b_1^1 - b_2^2) du^1 du^2 - b_1^2 (du^1)^2 = 0$$

AND THE PRINCIPLE CURVATURES ARE THE SOLUTIONS OF

$$\kappa^2 - 2H\kappa + K = 0$$



— LET $\vec{Q}$ BE A 3D VECTOR FIELD AND LET $f(\vec{x}) =$ CONST. BE A FAMILY OF SURFACES. WRITE $\vec{Q} = \vec{Q}_T + \vec{Q}_n$ BY DECOMPOSING $\vec{Q}$ TANGENT AND PARALLEL TO THE SURFACES $f =$ CONST. $\Longrightarrow$ $\boxed{\nabla \cdot \vec{Q} = \nabla_2 \cdot \vec{Q}_T + \frac{\partial Q_n}{\partial n} - 2 H_f Q_n}$

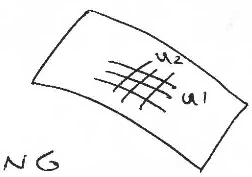
WHERE $\nabla_2 \cdot \vec{Q}_T = \frac{1}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \left[\sqrt{g_{(2)}}\, Q^i \right]$, $i = 1-2$
$\vec{Q}_T = Q^i \vec{a}_i$ AND $g_{(2)}$ IS THE DET. OF COEFF. OF 1ST FUND. FORM ON THE SURFACE $f =$ CONST., H_f ITS MEAN CURVATURE.

MATH. PREP.

— LET $\phi(\vec{x},t)$ BE A FUNCTION MAPPING $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. AND LET $f(\vec{x},t)=0$ BE A SURFACE MOVING IN THE SPACE. WE OFTEN NEED THE FOLLOWING TWO RESULTS:

$$\frac{\partial \hat{\phi}}{\partial x_i} \quad \text{AND} \quad \frac{\partial \hat{\phi}}{\partial t}$$

WHERE, AS BEFORE, $\hat{\phi}$ IS THE RESTRICTION OF ϕ TO $f=0$. WE ASSUME AS BEFORE $|\nabla f|=1$. WE INTRODUCE ON $f=0$ COORDINATE SYSTEM (u^1,u^2) AND TAKE $u^3=f$. WE EXTEND (u^1,u^2) TO $f=\text{CONST.}$ BY ASSUMING (u^1,u^2) IS CONSTANT ALONG NORMALS TO $f=0$. WE CAN EXTEND (u^1,u^2) THE SAME WAY FROM $f(\vec{x},t)=0$ TO $f(\vec{x},t+\delta t)=0$. WE THUS HAVE $\vec{u}=(u^1,u^2,u^3)$ $=\vec{u}(\vec{x},t)$. WITH THIS COORDINATE SYSTEM



$$\hat{\phi}(\vec{x},t) = \hat{\phi}(u^1(\vec{x},t), u^2(\vec{x},t), 0, t)$$

$\therefore$

$$\frac{\partial \hat{\phi}}{\partial x_i} = \frac{\partial \hat{\phi}}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial x^i} \qquad \alpha=1-2$$

$$\frac{\partial \hat{\phi}}{\partial t} = \frac{\partial \hat{\phi}}{\partial u^\alpha}\frac{\partial u^\alpha}{\partial t} + \left.\frac{\partial \hat{\phi}}{\partial t}\right|_{\vec{u}} \qquad \alpha=1-2$$

WE USE THE NOTATION $\dot{\hat{\phi}}$ FOR $\partial\hat{\phi}/\partial t|_{\vec{u}}$. WE NEED TO FIND $\partial u^\alpha/\partial x^i$ OR ∇u^α AND $\partial u^\alpha/\partial t$ FOR $\alpha=1-2$. FROM

$$d u^\alpha|_t = \nabla u^\alpha \cdot d\vec{r}$$

IT IS OBVIOUS THAT $\nabla u^\alpha = \vec{a}^\alpha$ WHERE

MATH. PREP.

$\vec{a}^{\alpha}$ IS DUAL OF THE NATURAL BASIS VECTOR $\vec{a}_{\alpha} = \frac{\partial \vec{x}}{\partial u^{\alpha}}$. THEREFORE

$$\frac{\partial \hat{\phi}}{\partial x_i} \equiv (\nabla \hat{\phi})_i$$

$$= \frac{\partial \hat{\phi}}{\partial u^{\alpha}} (\vec{a}^{\alpha})_i = (\nabla_2 \hat{\phi})_i \quad (\alpha = 1-2)$$

OR $\boxed{\nabla \hat{\phi} = \nabla_2 \hat{\phi}}$ WHERE $\nabla_2 \hat{\phi}$ IS THE SURFACE GRADIENT OF $\hat{\phi}$ WHEN TIME t IS KEPT FIXED.

TO GIVE A MEANING TO $\partial u^{\alpha}/\partial t$, WE NOTE THAT

$$du^{\alpha} = \vec{a}^{\alpha} \cdot d\vec{x} + \frac{\partial u^{\alpha}}{\partial t} dt$$

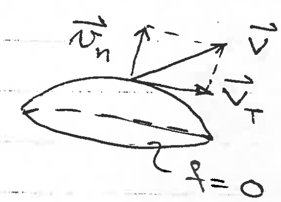
TAKE $\vec{u}$ = FIXED (i.e. TAKE A POINT ON $f = 0$), THEN ITS VELOCITY VECTOR IS $d\vec{x}/dt = \vec{V}$ OR $d\vec{x} = \vec{V} dt$, WE HAVE

$$0 = (\vec{a}^{\alpha} \cdot \vec{V} + \frac{\partial u^{\alpha}}{\partial t}) dt$$

OR $\boxed{\frac{\partial u^{\alpha}}{\partial t} = - v_T^{\alpha}}$ WHERE T STANDS FOR TANGENT COMPONENT OF VELOCITY TO $f = 0$.

WE ALSO HAVE

$$\boxed{\frac{\partial u^3}{\partial t} = - v_n}$$



$$\therefore \quad \boxed{\frac{\partial \hat{\phi}}{\partial t} = - \vec{v}_T \cdot \nabla_2 \hat{\phi} + \dot{\hat{\phi}}}$$

SOLUTIONS OF THE FW-H EQUATION

WE WILL DRIVE SEPARATE RESULTS FOR SUBSONIC AND SUPERSONIC SOURCES. THIS IS BECAUSE SUPERSONIC SOURCE RESULTS ARE MUCH MORE DIFFICULT TO DERIVE THAN SUBSONIC SOURCE FORMULATIONS WE FIRST DERIVE SOME FUNDAMENTAL RESULTS CONCERNING THE SOLUTION OF THE WAVE EQUATIONS

$$\Box^2 \phi = Q(\vec{x},t) \qquad (1)$$

AND

$$\Box^2 \phi = Q(\vec{x},t)\,\delta(f)\,. \qquad (2)$$

WE ASSUME THAT $|\nabla f| = 1$ AND THE SURFACE f IS DEFORMABLE AND IS NOT NECESSARILY A CLOSED SURFACE. IN THE CASE OF AN OPEN SURFACE ONE SHOULD ACTUALLY WRITE EQ. (2) IN THE FORM OF

$$\Box^2 \phi = Q(\vec{x},t)\, H(\tilde{f})\, \delta(f) \qquad (2')$$

WHERE $H(\tilde{f})$ IS THE HEAVISIDE FUNCTION SUCH THAT IT IS EQUAL TO ONE ON THE SURFACE, i.e. $\tilde{f} > 0$ ON THE SURFACE, AND $f = \tilde{f} = 0$ SPECIFIES THE EDGE OF THE OPEN SURFACE.

THE TWO FORMS OF SOLUTION OF EQ. (1)

THE GREEN'S FUNCTION OF THE WAVE EQ. IS $\delta(g)/4\pi r$, $r = |\vec{x} - \vec{y}|$, $g = \tau - t + r/c$ WHERE $(\vec{x},t)$ AND $(\vec{y},\tau)$ ARE THE OBSERVER AND SOURCE COORDINATES, RESPECTIVELY. USING THIS GREEN'S FUNCTION, WE HAVE

SOL. OF FW-H EQ.

$$4\pi\,\phi(\vec{x},t) = \int_{-\infty}^{t}\int_{\mathbb{R}^3} \frac{1}{r} Q(\vec{y},\tau)\,\delta(g)\,d\vec{y}\,d\tau \qquad (3)$$

LET US CHANGE τ TO g, $d\tau = dg$

$$4\pi\,\phi(\vec{x},t) = \int \frac{1}{r} Q(\vec{y}, g+t-r/c)\,\delta(g)\,d\vec{y}\,dg$$

$$= \int_{\mathbb{R}^3} \frac{1}{r} Q(\vec{y}, t-r/c)\,d\vec{y} \qquad (4)$$

(RETARDED TIME SOLUTION)

1ST FORM

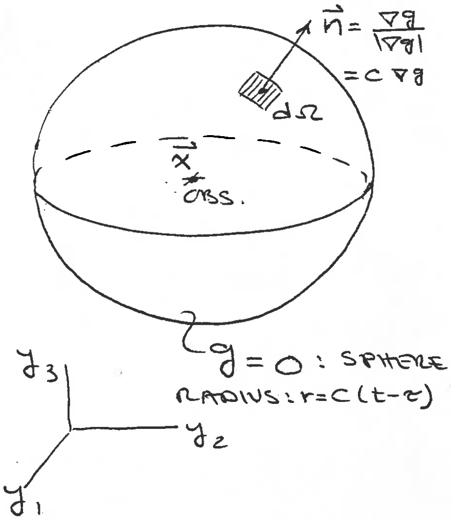
2ND FORM

LET $y_3 \to g$,

$$d\vec{y} = \frac{dy_1\,dy_2\,dg}{\left|\frac{\partial g}{\partial y_3}\right|}$$

$$= \frac{dy_1\,dy_2\,dg}{\underbrace{\left[\left|\frac{\partial g}{\partial y_3}\right| / |\nabla g|\right]}_{n_3} \cdot \underbrace{|\nabla g|}_{\frac{1}{c}}}$$

$$= c\,d\Omega\,dg \qquad (5)$$



WHERE $d\Omega$ IS ELEMENT OF THE SURFACE AREA OF THE SPHERE $g = \tau - t + r/c = 0$, i.e. THE SPHERE WITH CENTER AT THE OBSERVER AND RADIUS $r = c(t-\tau)$. NOTE THAT $\tau \leq t$. AS τ INCREASES FROM $-\infty$ TO t, THE RADIUS OF THIS SPHERE COLLAPSES AT SPEED OF SOUND. IT IS THEREFORE KNOWN AS THE COLLAPSING SPHERE.

SOL. OF FW-H EQ.

SUBSTITUTING EQ. (4) IN EQ. (3) AND INTEGRATING WRT VARIABLE g, WE GET

$$4\pi\phi(\vec{x},t) = \int_{-\infty}^{t} \int_{r=c(t-\tau)} \frac{1}{c(t-\tau)} Q(\vec{y},\tau)\, d\Omega\, d\tau \qquad (6)$$

2ND FORM

(THE COLLAPSING SPHERE SOLUTION)

REMARK : NOTE THAT IF $Q = 0$ INSIDE A MOVING CLOSED SURFACE $f=0$ SUCH THAT $f>0$ OUTSIDE THE SURFACE, THEN WE ARE ACTUALLY SOLVING

$$\Box^2 \phi = Q(\vec{x},t) H(f)$$

THEN EQS. (4) AND (6) BECOME, RESPECTIVELY

$$4\pi\phi(\vec{x},t) = \int_{\mathbb{R}^3\setminus\omega} \frac{1}{r} Q(\vec{y}, t-r/c)\, d\vec{y} \qquad (7)$$

$$4\pi\phi(\vec{x},t) = \int_{-\infty}^{t} \int_{\Omega\setminus\omega} \frac{1}{c(t-\tau)} Q(\vec{y},\tau)\, d\Omega\, d\tau \qquad (8)$$

WHERE ω IS THE REGION INSIDE THE SURFACE $F(\vec{y};\vec{x},t) = f(\vec{y}, t-r/c) = 0$.

THE SURFACE $F=0$ IS GENERATED BY THE INTERSECTION OF THE SURFACES $g=0$ AND $f=0$. IT IS IMPORTANT TO UNDERSTAND THE CONSTRUCTION OF THIS SURFACE. NOTE THE DEPENDENCE ON VARIABLES OF F, i.e. $\vec{y}$, $\vec{x}$ AND t. WE WILL ALWAYS TRY TO VISUALIZE F AS A FUNCTION OF $\vec{y}$ WITH $\vec{x}$ AND t FIXED. THE CONSTRUCTION OF

SOL. OF FW–H EQ.

F IS AS FOLLOWS. WE KNOW THAT FOR FIXED $\vec{x}$ AND t, $g=0$ IS A SPHERE WITH CENTER AT $\vec{x}$ (OBSERVER) AND WHOSE RADIUS COLLAPSES AT SPEED OF SOUND. CONSIDER A TIME τ_0 SUCH THAT THIS SPHERE INTERSECTS THE BODY WHICH IS NOW GIVEN BY EQUATION $f(\vec{y}, \tau_0)=0$. LET THE CURVE OF INTERSECTION OF $g=\tau_0 - t + r/c = 0$ AND $f(\vec{y}, \tau_0) = 0$ BE DENOTED Γ. WE WILL ALWAYS REFER TO THIS CURVE AS THE Γ-CURVE (AT TIME τ_0). AS τ VARIES FROM $-\infty$ TO t, THE LOCUS OF THE Γ-CURVES FORMS THE SURFACE $F(\vec{y}; \vec{x}, t)=0$. WE CALL THIS SURFACE THE Σ-SURFACE FOR $(\vec{x}, t)$. ITS GEOMETRY AND TOPOLOGY IS DEPENDENT ON THE GEOMETRY AND TOPOLOGY OF THE BODY SURFACE $f(\vec{y}, \tau)=0$, ITS MOTION AND THE OBSERVER POSITION AND TIME.

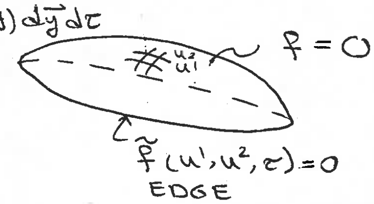
MATHEMATICALLY $f(\vec{y}, \tau)=0$ GENERATES A THREE DIMENSIONAL SURFACE IN FOUR DIMENSIONAL SPACE $(\vec{y}, \tau) \in \mathbb{R}^3 \times (-\infty, t]$. FOR FIXED $(\vec{x}, t)$, $g = \tau - t + |\vec{x} - \vec{y}|/c = 0$ IS THE CHARACTERISTIC CONE OF THE WAVE EQUATION $\Box^2 \phi$ WITH VERTEX AT $(\vec{x}, t)$ AND $(\vec{y}, \tau) \in \mathbb{R}^3 \times (-\infty, t]$. THE INTERSECTION OF THIS CONE AND THE THREE DIMENSIONAL SURFACE $f=0$ GENERATES THE TWO DIMENSIONAL SURFACE $F=0$ (THE Σ-SURFACE) IN THE THREE DIMEN–

SIONAL SPACE $\mathbb{R}^3$. THE Σ-SURFACE IS, THEREFORE THE <u>INFLUENCE SURFACE</u> OF THE POINT $(\vec{x},t)$. IN SUPERSONIC AERODYNAMIC LITERATURE BASED ON LINEAR THEORY (i.e. BASED ON WAVE EQ. IN A MOVING FRAME), $f=0$ IS USUALLY A PLANAR SURFACE (i.e. THE WING PLANFORM) AND THEREFORE ITS Σ-SURFACE IS ALSO A FLAT SURFACE. IT IS CALLED <u>THE ACOUSTIC PLANFORM</u> OF THE WING. FOR THIS REASON, IT IS CUSTOMARY TO CALL THE INFLUENCE SURFACE OF NONPLANAR SURFACE WITH NONZERO VOLUME INSIDE $f=0$ BY THE SAME TERMINOLOGY IN AEROACOUSTICS.

<u>THE VARIOUS FORMS OF SOLUTION OF EQ. (2′)</u>

THE FORMAL SOLUTION OF EQ. (2′)

$$4\pi\,\phi(\vec{x},t) = \int_{-\infty}^{t}\int_{\mathbb{R}^3} \frac{1}{r}\, Q(\vec{y},\tau) H(\tilde{f})\,\delta(f)\,\delta(g)\, d\vec{y}\, d\tau \qquad (9)$$



WHEN WE HAVE THE PRODUCT OF TWO DELTA FUNCTIONS IN AN INTEGRAND, THE RESULT IS AN INTEGRAL OVER THE <u>INTERSECTION</u> OF THE SUPPORTS OF THE DELTA FUNCTIONS. WE ALREADY HAVE CONSTRUCTED THIS SET. IT IS THE Σ-SURFACE! NOTE THAT THE PRESENCE OF $H(\tilde{f})$ IN THE INTEGRAND IS JUST TO INDICATE THAT THE ... IS AN OPEN SURFACE AND DOES NOT AFFECT THE VISUALIZATION AND CONSTRUCTION OF THE Σ-SURFACE

SOL. OF FW-H EQ.

FOR SUBSONIC MOTION OF THE OPEN SURFACE $f=0$, $\tilde{f}>0$, THE CORRESPONDING Σ - SURFACE IS AN OPEN SURFACE ALSO. THIS IS NOT NECESSARILY SO FOR SUPERSONIC SURFACES.

FIRST FORM OF SOLUTION OF EQ. (2')

LET $\tau \to g \implies \frac{\partial g}{\partial \tau} = 1$ AND EQ. (9) BECOMES

$$4\pi \phi(\vec{x},t) = \int \frac{1}{r} Q(\vec{y},\tau) H(\tilde{f}) \delta(f) \delta(g)\, dg\, d\vec{y}$$

$$= \int \frac{1}{r} \left[Q\, H(\tilde{f}) \delta(f) \right]_{g=0} d\vec{y} \qquad (10)$$

NOTE THAT $g = \tau - t + r/c = 0$ MEANS THAT τ IN SQUARE BRACKETS MUST BE REPLACED BY $t - r/c$. LET US DEFINE

$$\tilde{F}(u^1, u^2, t - r/c) \equiv [\tilde{f}]_{ret} \qquad (11\text{-a})$$

$$F(\vec{y}; \vec{x}, t) = [f]_{ret} \qquad (11\text{-b})$$

WHERE ret STANDS FOR THE RETARDED (OR EMISSION) TIME $t - r/c$. THEN WE HAVE

$$4\pi \phi(\vec{x},t) = \int \frac{1}{r} [Q]_{ret} H(\tilde{F}) \delta(F)\, d\vec{y} \qquad (12)$$

WE HAVE ALREADY GIVEN THE INTERPRETATION OF THIS IN THE MATHEMATICAL PREPARATIONS. WE NOTE THAT $|\nabla F| \neq 1$ EVEN IF $|\nabla f| = 1$ AS ASSUMED HERE. WE HAVE

$$4\pi \phi(\vec{x},t) = \int_{\substack{F=0 \\ \tilde{F}>0}} \frac{[Q]_{ret}}{r |\nabla_y F|}\, d\Sigma \qquad (13)$$

SOL. OF FW-H EQ.

THE CALCULATION OF $|\nabla_y F|$ IS SIMPLE SINCE

$$\nabla_y F = \nabla_y f(\vec{y}, t - \tfrac{r}{c})$$
$$= \left[\nabla f(\vec{y}, \tau) + \frac{1}{c}\frac{\partial f(\vec{y}, \tau)}{\partial \tau}\hat{r}\right]_{ret}$$
$$= [\vec{n} - M_n \hat{r}]_{ret} \qquad (14)$$

WHERE $\vec{n} = \nabla f$ IS THE UNIT OUTWARD NORMAL TO THE SURFACE $f = 0$, $M_n = \frac{v_n}{c} = -\frac{1}{c}\frac{\partial f}{\partial \tau}$ IS THE LOCAL NORMAL MACH NUMBER ON THIS SURFACE AND $\hat{r} = (\vec{x} - \vec{y})/|\vec{x} - \vec{y}|$ IS THE UNIT RADIATION VECTOR. NOTE THE DIRECTION OF $\hat{r}$ IS FROM THE SOURCE <u>TO</u> THE OBSERVER. FROM EQ. (14), WE GET

$$|\nabla_y F| = [1 + M_n^2 - 2M_n \cos\theta]_{ret}^{1/2}$$
$$\equiv [\Lambda]_{ret} \qquad (15)$$

WHERE $\cos\theta = \vec{n} \cdot \hat{r}$ IS THEREFORE THE COSINE OF THE ANGLE BETWEEN THE NORMAL TO THE SURFACE $f=0$ AND THE RADIATION DIRECTION.

<u>THE FIRST FORM</u> OF THE SOLUTION OF EQ. (2′) IS THEREFORE:

$$\boxed{4\pi\phi(\vec{x}, t) = \int_{\substack{F=0\\ \tilde{F}>0}} \frac{1}{r}\left[\frac{Q(\vec{y}, \tau)}{\Lambda}\right]_{ret} d\Sigma} \qquad (16)$$

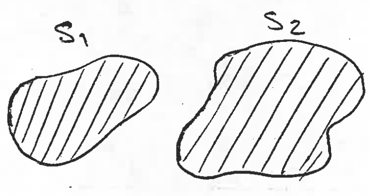
INCIDENTALLY, NOTE THAT THE UNIT NORMAL TO THE Σ-SURFACE, $\vec{N}$, IS GIVEN BY

$$\vec{N} = \frac{\nabla_y F}{|\nabla_y F|}$$
$$= \left[\frac{\vec{n} - M_n \hat{r}}{\Lambda} \right]_{ret} . \qquad (17)$$

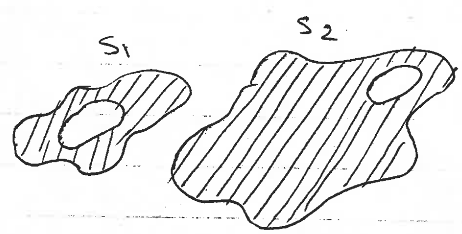
WE WILL NEED THIS RESULT LATER.

<u>THE SECOND FORM OF SOLUTION OF EQ. (2')</u>

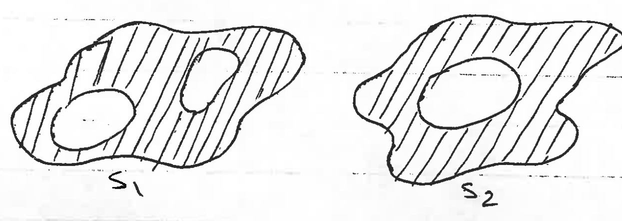
WE NOW ASSUME THAT THE OPEN SURFACE $f=0, \tilde{f}>0$ DOES NOT CHANGE ITS TOPOLOGY, BUT AS BEFORE, IT IS DEFORMABLE AND CHANGES ITS SIZE AS A FUNCTION OF TIME. TWO SURFACES ARE <u>TOPOLOGICALLY EQUIVALENT</u> IF ONE CAN BE DEFORMED CONTINUOUSLY TO THE SECOND SURFACE WITHOUT TEARING. THE FOLLOWING EXAMPLES CLARIFY THIS DEFINITION:



TOPOLOGICALLY EQUIVALLENT

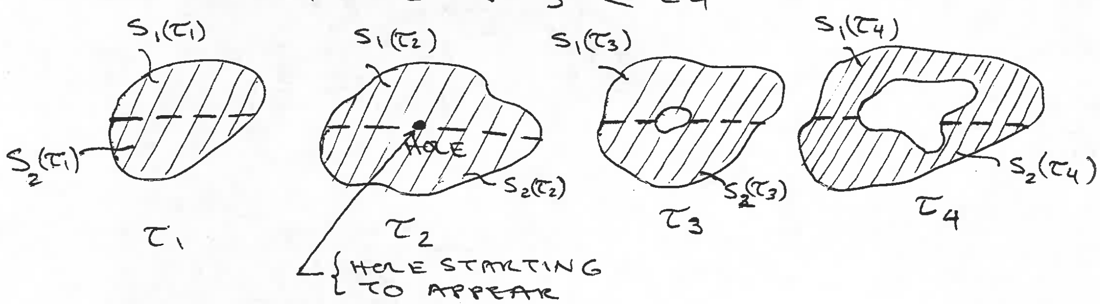


TOPOLOGICALLY EQUIVALENT



<u>NOT</u> TOPOLOGICALLY EQUIVALENT.

WHEN WE SAY THAT THE SURFACE $f=0, \tilde{f}>0$ DOES NOT CHANGE TOPOLOGY, WE MEAN THAT AT ANY TWO TIMES τ_1 AND $\tau_2 \in (-\infty, t]$, THE TWO SURFACES $f(\vec{y}, \tau_1)=0$ $(\tilde{f}>0)$ AND $f(\vec{y}, \tau_2)=0$ $(\tilde{f}>0)$ ARE TOPOLOGICALLY EQUIVALENT. ACTUALLY, THIS DOES NOT CAUSE ANY RESTRICTION IN APPLICATIONS OF AEROACOUSTICS. IF A SURFACE, WHICH WE CAN THINK AS A SHOCK SURFACE, DOES CHANGE ITS TOPOLOGY BY DEVELOPING HOLES OR CLOSING HOLES, THEN IT CAN BE BROKEN INTO SURFACES WITHOUT CHANGING TOPOLOGY IN TIME AS SHOWN BELOW : $\tau_1 < \tau_2 < \tau_3 < \tau_4$

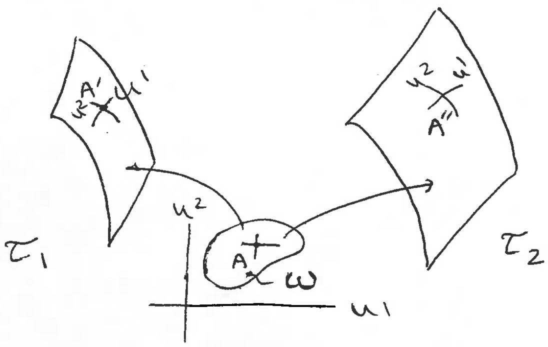


WE SEE THAT S_1 AND S_2 MAINTAIN THEIR TOPOLOGY WHICH IS EQUIVALENT TO A CIRCLE FOR ALL $\tau \in [\tau_1, \tau_4]$.

BECAUSE THE SURFACE $f=0, \tilde{f}>0$ DOES NOT CHANGE ITS TOPOLOGY, WE CAN FIND A <u>TIME INDEPENDENT</u> REGION ω IN (u^1, u^2) SPACE SUCH THAT IT IS MAPPED TO THE OPEN SURFACE. WE USE THIS AS SURFACE COORDINATES. WE USE u^3 AS

SOL. OF FW-H EQ.

THE THIRD COORDINATE ON THE SURFACE SUCH THAT $u^3 = f$ AND THUS $\partial u^3/\partial n = 1$. IN THIS WAY $\tilde{f} = \tilde{f}(u^1, u^2)$ AND TIME DOES NOT APPEAR IN ITS ANALYTIC EXPRESSION.



NOTE THAT $\vec{y} = \vec{y}(\vec{u}, \tau)$, $\vec{u} = (u^1, u^2, u^3)$.
THE METRIC TENSOR g_{ij} IS A FUNCTION OF TIME SINCE THE BASIS VECTORS ARE

$$\vec{a}_i = \frac{\partial \vec{y}}{\partial u^i} \qquad i = 1-2$$

$$\therefore \quad g_{(2)} = g_{(2)}(\vec{u}, \tau)$$

AND

$$g_{(3)} = g_{(2)}(\vec{u}, \tau)$$

LET US USE THE TRANSFORM $\vec{y} \to \vec{u}$. WE HAVE THE JACOBIAN OF TRANSFORMATION $\sqrt{g_{(3)}} = \sqrt{g_{(2)}}$. EQUATION (9) BECOMES

$$4\pi\phi(\vec{x}, t) = \int \frac{1}{r} Q(\vec{y}(\vec{u}, \tau), \tau) H(\tilde{f}) \delta(u^3) \delta(g) \sqrt{g_{(2)}} \, d\vec{u} \, d\tau$$

$$= \int_{-\infty}^{t} \int_{\omega} \left[\frac{Q}{r} \delta(g) \sqrt{g_{(2)}} \right]_{u^3=0} du^1 du^2 \, d\tau \qquad (18)$$

SOL. OF FW-H EQ.

NOTE THAT THE ORDER OF INTEGRATION CAN BE CHANGED SINCE ω IS INDEPENDENT OF τ. NOW USE $\tau \rightarrow g$ AND NOTE THAT THE JACOBIAN OF TRANSFORMATION IS $1/|\partial g/\partial \tau|$. HOWEVER, LOOK CAREFULLY AT g FOR DEPENDENCE ON VARIABLES. FIRST OF ALL

$$\frac{\partial}{\partial \tau} = \left[\frac{\partial}{\partial \tau}\bigg|_{u^1,u^2} \right]_{u^3=0}$$

THAT IS u^1 AND u^2 ARE KEPT FIXED. THEREFOR, HERE IS THE EXACT DEPENDENCE OF VARIABLES OF g:

$$g = \tau - t + \frac{1}{c}\left|\vec{x} - \vec{y}(u^1,u^2,0,\tau)\right| \tag{19}$$

$$\therefore \quad \frac{\partial g}{\partial \tau} = 1 + \frac{1}{c}\frac{\partial r}{\partial y_i}\frac{\partial y_i}{\partial \tau}\bigg|_{u^1,u^2}$$

$$\frac{\partial r}{\partial y_i} = -\hat{r}_i$$

$$\frac{\partial y_i}{\partial \tau}\bigg|_{u^1,u^2} = v_i \quad \left\{ \begin{array}{l} \text{THE VELOCITY OF A POINT} \\ \text{WITH COORDINATES } (u^1,u^2) \\ \text{ON THE SURFACE} \end{array} \right.$$

$$\Rightarrow \quad \frac{\partial g}{\partial \tau} = 1 - \frac{1}{c} v_i \hat{r}_i \equiv 1 - M_r \tag{20}$$

WHERE M_r IS MACH NUMBER IN RADIATION DIRECTION.

THE USE OF $\tau \rightarrow g$ AND INTEGRATION WRT

SOL. OF FW-H EQ.

g OF EQ. (18) GIVES THE SECOND FORM OF SOLUTION

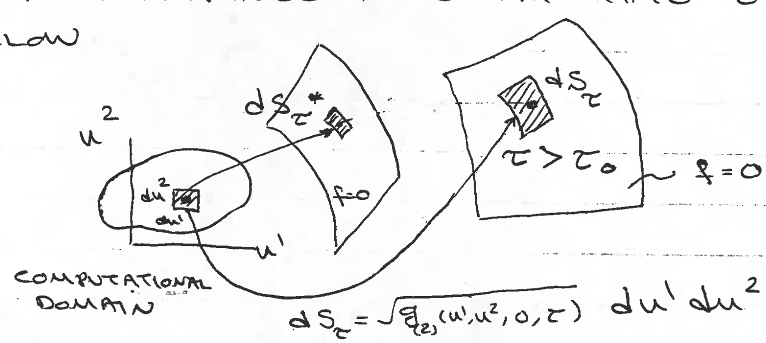
$$\boxed{\begin{aligned} 4\pi\,\phi(\vec{x},t) &= \int_{\omega} \left[\frac{Q\sqrt{g_{(2)}}}{r|1-M_r|}\right]_{\tau^*} du^1 du^2 \\ &= \int_{\omega} \left[\frac{Q(\vec{y},\tau)}{r|1-M_r|}\right]_{\tau^*} dS_{\tau^*} \end{aligned}} \tag{21}$$

WHERE τ^* IS THE SOLUTION OF EQ. (19) KEEPING $u^1, u^2, \vec{x}$ AND t FIXED. WE CALL τ^* THE EMISSION TIME OF THE POINT WITH COORDINATES (u^1, u^2) ON THE MOVING SURFACE. IT SHOULD NOT BE INTERPRETED AS $t - r/c$ SINCE r IS NOW A FUNCTION OF τ THROUGH THE RELATION $r = |\vec{x} - \vec{y}(\vec{u},\tau)|$. IN EQ. (21), WE DEFINE

$$dS_{\tau^*} = \sqrt{g_{(2)}(u^1,u^2,0,\tau^*)}\; du^1 du^2 . \tag{22}$$

EQUATION (21) IS VERY USEFUL IN APPLICATIONS.

— dS_{τ^*} IS INTERPRETED AS FOLLOWS. DIVIDE ω INTO PANELS FOR COMPUTATION PURPOSES. THEN dS_{τ^*} IS THE PHYSICAL SURFACE AREA OF THE MAP OF A PANEL IN ω AT TIME τ^* AS SHOWN BELOW



— NOTE THAT IF $f(\vec{y},\tau)=0$ IS RIGID $\Rightarrow \sqrt{g_{(2)}} =$ CONST. & dS_{τ^*} IS NOT A FUNCTION OF TIME AND WE MAY WRITE $dS_\tau \equiv dS$

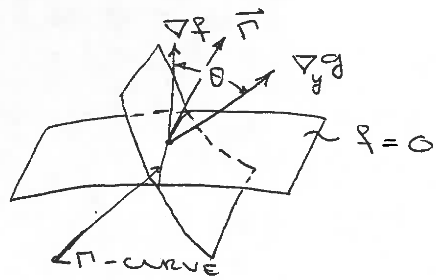
THE THIRD FOR OF SOLUTION OF EQ. (2′)

AGAIN WE START WITH EQ. (9) AND THIS TIME WE TRANSFORM (y_1, y_2) TO (f, g). THE JACOBIAN OF TRANSFORMATION IS $1/|\partial(f,g)/\partial(y_1,y_2)|$ WHERE $\partial(f,g)/\partial(y_1,y_2)$ IS THE JACOBIAN (f,g) WRT VARIABLES (y_1, y_2):

$$\frac{\partial(f,g)}{\partial(y_1,y_2)} = \begin{vmatrix} \frac{\partial f}{\partial y_1} & \frac{\partial f}{\partial y_2} \\ \frac{\partial g}{\partial y_1} & \frac{\partial g}{\partial y_2} \end{vmatrix} = (\nabla f \times \nabla_y g)_3 \qquad (23)$$

WHERE THE SUBSCRIPT 3 STAND FOR THE COMPONENT ALONG y_3- AXIS.

IT IS CLEAR THAT $\nabla f \times \nabla g$ IS TANGENT TO THE Γ- CURVE WHICH IS THE INTERSECTION



OF $f=0$ AND $g=0$. THE ELEMENT OF LENGTH OF THIS CURVE IS

$$d\Gamma = \frac{dy_3}{|\Gamma_3|} \qquad (24)$$

WHERE $\vec{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3)$ IS THE UNIT TANGENT TO THE Γ- CURVE. WE HAVE

$$|\Gamma_3| = \frac{|\nabla f \times \nabla_y g|_3}{|\nabla f \times \nabla_y g|} \qquad (25)$$

$$|\nabla f \times \nabla_y g| = C^{-1} \sin\theta \qquad (26)$$

WHERE θ IS THE ANGLE BETWEEN $\vec{n}$ AND $\vec{F}$.

$$\Rightarrow \quad d\vec{y} = \frac{df\, dg\, dy_3}{|\nabla f \times \nabla_y g|_3}$$

$$= \frac{dy_3}{|\Gamma_3|} \cdot \frac{df\, dg}{|\nabla f \times \nabla_y g|}$$

$$= \frac{c\, df\, dg\, d\Gamma}{\sin\theta} \qquad (27)$$

SUBSTITUTING IN EQ. (9) AND INTEGRATING WRT f AND g GIVES THE THIRD FORM OF THE SOLUTION OF EQ. (2′)

$$\boxed{\begin{aligned} 4\pi\phi(\vec{x},t) &= \int_{-\infty}^{t} \int_{\substack{\Gamma \\ \tilde{f}>0}} \frac{cQ(\vec{y},\tau)}{r\sin\theta}\, d\Gamma\, d\tau \\ &= \int_{-\infty}^{t} \int_{\substack{\Gamma \\ \tilde{f}>0}} \frac{Q(\vec{y},\tau)}{(t-\tau)\sin\theta}\, d\Gamma\, d\tau \end{aligned}} \qquad (28)$$

THIS IS ANOTHER VERY USEFUL FORM OF THE SOLUTION OF EQ. (2′).

WE WILL NOW BUILD UP OUR SOLUTIONS OF THE FW-H EQUATION BASED ON THE ABOVE RESULTS. FROM EQS. (16), (21) AND (28), WE CAN SEE THAT THE FOLLOWING RELATIONS CONNECT THESE EQUATIONS

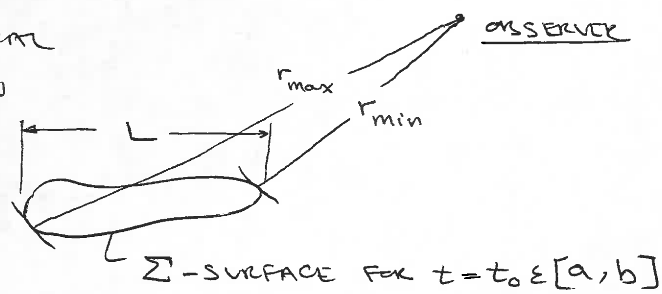
$$\boxed{\frac{d\Sigma}{\Lambda} = \frac{dS_{\tau*}}{|1-M_r|} = \frac{c\, d\Gamma\, d\tau}{\sin\theta}}$$

SOL. OF FW-H EQ.

I- COMPACT SOURCE FORMULATIONS.

WE FIRST EXPLAIN WHAT WE MEAN BY A COMPACT SOURCE. A SOURCE IS CONSIDERED COMPACT IF DIFFERENCES IN RETARDED (OR EMISSION) TIMES ACROSS THE SOURCE IS SMALL COMPARED TO BOTH TYPICAL PERIOD OF FLUCTUATIONS OF THE SOURCE AN PROPAGATION TIME TO THE OBSERVER. WE CAN MAKE THIS PRECISE MATHEMATICALLY BY CONSIDERING THE Σ-SURFACE GENERATED BY THE BOUNDARY OF A FINITE MOVING VOLUME SOURCE OR A SURFACE SOURCE.

LET T BE THE TYPICAL PERIOD OF FLUCTUATION OF THE SOURCE, r_{min} AND r_{max} BE THE MINIMUM AND THE MAXIMUM DISTANCE OF THE OBSERVER FROM THE Σ-SURFACE AND LET L BE THE MAXIMUM DIMENSION OF THE Σ-SURFACE.



THE MAXIMUM DIFFERENCE IN THE RETARDED TIME $(\Delta\tau)_{MAX} = (r_{max} - r_{min})/c$.. WE SAY THE SOURCE IS COMPACT IF BOTH THE FOLLOWING TWO CONDITIONS ARE SATISFIED FOR ALL TIMES t IN SOME INTERVAL $[a,b]$:

$$\boxed{\begin{array}{c}(\Delta\tau)_{MAX} \ll T \\ L \ll r_{min}\end{array}} \qquad (29\text{-}a)$$

IN THIS CASE THE SOURCE CAN BE CONSIDERED AS A POINT SOURCE AND WE CAN USE A MEAN EMISSION

(OR RETARDED) TIME AND THE Σ-SURFACE AN BE APPROXIMATED BY THE ACTUAL SOURCE SURFACE.

a) THE SUCCI THICKNESS NOISE FORMULA:

WE FIRST WRITE THE THICKNESS SOURCE TERM AS A VOLUME SOURCE AS FOLLOWS:

$$\rho_o v_n \delta(f) = \frac{\bar{\partial}}{\partial t}\left[\rho_o\left(1 - H(f)\right)\right]$$

$$\therefore \frac{\partial}{\partial t}\left[\rho_o v_n \delta(f)\right] = \frac{\bar{\partial}^2}{\partial t^2}\left[\rho_o\left(1 - H(f)\right)\right] \tag{30}$$

TAKING $Q = \rho_o(1 - H(f))$ IN EQ. (1), FROM EQ. (4) WE FIND THE SOLUTION OF:

$$\Box^2 p'_T = \frac{\bar{\partial}^2}{\partial t^2}\left[\rho_o\left(1 - H(f)\right)\right]$$

AS

$$4\pi p'_T(\vec{x}, t) = \frac{\bar{\partial}^2}{\partial t^2}\int_{F<0}\left[\frac{\rho_o}{r}\right]_{\tau^*} d\vec{y} \tag{31}$$

USING COMPACTNESS ASSUMPTION, WE GET

$$4\pi p'_\tau(\vec{x}, t) = \frac{\bar{\partial}^2}{\partial t^2}\left[\frac{\rho_o}{r}\right]_{\tau^*}\int_{F<0} d\vec{y}$$

$$= \frac{\bar{\partial}^2}{\partial t^2}\left[\frac{\rho_o V}{r(1 - M_r)}\right]_{\tau^*} \tag{32}$$

WHERE V IS THE VOLUME INSIDE THE SURFACE $f = 0$ WHICH MAY BE ASSUMED DEFORMABLE. FROM EQ. (19), WE FIND

$$\frac{\partial}{\partial t} = \frac{1}{1 - M_r}\frac{\partial}{\partial \tau} \tag{33}$$

USING THIS IN EQ. (32), WE GET

$$4\pi p'_T(\vec{x},t) = \left\{ \frac{1}{1-M_r} \frac{\partial}{\partial \tau} \left[\frac{1}{1-M_r} \frac{\partial}{\partial \tau} \left(\frac{\rho_0 V}{r(1-M_r)} \right) \right] \right\}_{\tau^*} \quad (34)$$

THIS IS SUCCI'S FORMULA WHEN THE DERIVATIVES INSIDE CURLY BRACKETS ARE FULLY TAKEN. IN EQ. (32), NOTE THAT WE HAVE APPROXIMATED THE VOLUME INSIDE F BY $\frac{V}{1-M_r}$ WHICH CAN BE SHOWN BY A LITTLE GEOMETRY.

LAWSON'S FORMULA

WE CAN USE THE FORCE INTENSITY $l_i = P_{ij} n_j$ WHICH IS FORCE / UNIT AREA ACTING ON THE FLUID. WE WANT TO SOLVE

$$\Box^2 p'_L = -\frac{\bar{\partial}}{\partial x_i} \left[P_{ij} n_j \delta(f) \right]$$

$$= -\frac{\bar{\partial}}{\partial x_i} \left[l_i \delta(f) \right] \quad (35)$$

EQUATION (21) GIVES

$$4\pi p'_L(\vec{x},t) = -\frac{\bar{\partial}}{\partial x_i} \int_{f=0} \left[\frac{l_i}{r|1-M_r|} \right]_{\tau^*} dS_{\tau^*}$$

$$= -\frac{\partial}{\partial x_i} \left\{ \left[\frac{1}{r|1-M_r|} \right]_{\tau^*} \underbrace{\int_{f=0} l_i \, dS_{\tau^*}}_{F_i} \right\}$$

$$\boxed{4\pi p'_L(\vec{x},t) = -\frac{\partial}{\partial x_i} \left[\frac{F_i}{r|1-M_r|} \right]_{\tau^*}} \quad (36)$$

WHERE $\vec{F}$ IS THE NET FORCE ON THE FLUID. IT CAN BE SHOWN THAT THIS IS THE SOLUTION OF

THE WAVE EQUATION

$$\Box^2 \rho'_L = -\ \overline{\nabla}\cdot\left[\vec{F}(t)\,\delta(\vec{x}-\vec{S}(t))\right] \quad (37)$$

WHERE $\vec{S}(t)$ IS THE POSITION OF THE POINT SOURCE. BY A METHOD TO BE DISCUSSED NEXT, WE CAN WRITE EQ. (36) EXACTLY AS

$$4\pi\rho'_L(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\left[\frac{F_i\cdot\hat{r}_i}{r|1-M_r|}\right]_{\tau*} + \left[\frac{F_i\cdot\hat{r}_i}{r^2|1-M_r|}\right]_{\tau*}$$

$$\boxed{4\pi\rho'_L(\vec{x},t) = \frac{1}{c}\left\{\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left[\frac{F_i\hat{r}_i}{r|1-M_r|}\right]\right\}_{\tau*} + \left[\frac{F_i\cdot\hat{r}_i}{r^2|1-M_r|}\right]_{\tau*}} \quad (38)$$

WHEN THE DERIVATIVE INSIDE THE CURLY BRACKET IS TAKEN, OR WHEN THE $\partial/\partial x_i$ OF EQ. (36) IS TAKEN INSIDE SQ. BRACKET GIVE LOWSON'S FORMULA

INCIDENTALLY, EQ. (38) EASILY GIVES LOWSON'S FORMULA FOR THE FAR FIELD AS FOLLOWS

$$\boxed{4\pi\rho'_L(\vec{x},t) = \frac{1}{c}\left[\frac{\dot{F}_r}{r|1-M_r|^2}\right]_{\tau*} + \left[\frac{F_r\dot{M}_r}{r|1-M_r|^3}\right]_{\tau*}} \quad (39)$$

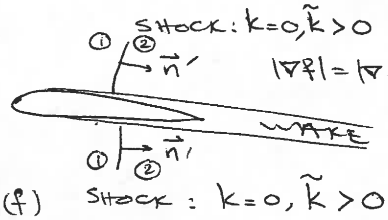
WHERE $\dot{F}_r = \dot{F}_i\cdot\hat{r}_i$, $\dot{M}_r = \dot{M}_i\cdot\hat{r}_i$

SOL. OF FW-H EQ.

II – SUBSONIC BODY AND SHOCK MOTION

IN THIS SECTION, WE DERIVE EQUATIONS USED FOR CALCULATION OF THE NOISE FROM SUBSONIC ROTATING BODIES WITH SHOCK SURFACES ENTIRELY INSIDE THE SONIC SURFACE. FIRST WE WRITE THE QUADRUPOLE TERM IN A FORM SUITABLE FOR APPLICATIONS WHERE THE CONTRIBUTION OF REGIONS OF HIGH GRADIENT CAN BE IDENTIFIED

$$\frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right]$$



$$= \frac{\bar{\partial}}{\partial x_i}\left[\left(\frac{\partial T_{ij}}{\partial x_j} + \Delta T_{ij} n'_j \delta(k)\right) H(f) + T_{ij} n_j \delta(f)\right] \qquad (40)$$

WHERE $k=0$, $\tilde{k}>0$ IS THE OPEN DEFORMABLE SHOCK. WE MAY HAVE MORE THAN ONE SHOCK AS SHOWN IN THE FIGURE. LET US DEFINE

$$(41\text{-a,b}) \quad \begin{cases} Q_i = T_{ij} n_j \\ q_i = \Delta T_{ij} n'_j \,, \quad \Delta T_{ij} = (T_{ij})_2 - (T_{ij})_1 \end{cases}$$

THEN ASSUMING THAT THE SHOCK SURFACE DOES NOT INTERSECT THE BODY SURFACE WE CAN DROP $H(f)$ MULTIPLYING $\delta(k)$ ON THE RIGHT OF EQ. (4). WE ALSO HAVE

$$\frac{\bar{\partial}}{\partial x_i}\left[\frac{\partial T_{ij}}{\partial x_j} H(f)\right] = \left[\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} + \Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right) n'_i \delta(k)\right] H(f) + \frac{\partial T_{ij}}{\partial x_j} n_i \delta(f) \qquad (42)$$

EQUATION (40) CAN THUS BE WRITTEN AS

SOL. OF FW-H EQ.

$$\frac{\bar{\partial}^2}{\partial x_i \partial x_j}\left[T_{ij} H(f)\right] = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} H(f) + \frac{\partial T_{ij}}{\partial x_j} n_i \delta(f)$$

$$+ \Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right) n_i' \delta(k)$$

$$+ \frac{\bar{\partial}}{\partial x_i}\left[Q_i \delta(f)\right] + \frac{\bar{\partial}}{\partial x_i}\left[q_i \delta(k)\right] \tag{43}$$

FOR THE PURPOSE OF THIS SECTION, WE RETAIN THESE TERMS AS THEY ARE. THE FW-H EQUATION BECOMES

$$\boxed{\begin{aligned} \bar{\Box}^2 p' &= \overset{①}{\frac{\partial T_{ij}}{\partial x_j} n_i \delta(f)} + \overset{②}{\Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right) n_i' \delta(k)} \\ &+ \frac{\bar{\partial}}{\partial t}\left[\rho_0 v_n \delta(f)\right] \quad ③ \\ &+ \frac{\bar{\partial}}{\partial x_i}\left[(Q_i - l_i)\delta(f)\right]_{④} + \frac{\bar{\partial}}{\partial x_i}\left[q_i \delta(k)\right]_{⑤} \\ &+ \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} H(f) \quad ⑥ \end{aligned}} \tag{44}$$

WHERE AS BEFORE $l_i = P_{ij} n_j$. WE HAVE

$$\begin{aligned} Q_i - l_i &= (T_{ij} - P_{ij}) n_j \\ &= (\rho u_i u_j - c^2 \tilde{\rho} \delta_{ij}) n_j \\ &= \rho u_i v_n - c^2 \tilde{\rho} n_i \end{aligned} \tag{45}$$

WE NOTE THAT ONLY TERM ⑥ IN EQ. (44) IS A VOLUME TERM. IT IS IDENTICAL TO THE SOURCE TERM OF LIGHTHILL'S JET NOISE THEORY. ALL OTHER TERMS ARE SURFACE TERMS: ①, ③ AND ④ ON THE BODY SURFACE, AND ② AND ⑤ ON THE SHOCK SURFACE. IN GENERAL, THE VOLUME SOURCE IS WEAK EXCEPT IN THE BOUNDARY LAYER

WAKE OR TIP VORTICES. WE WILL GIVE THE EXACT SOLUTION OF EQ. (44) LEAVING THE APPROXIMATIONS TO THE USERS WHICH, NATURALLY, DEPEND ON THE AVAILABILITY OF AERODYNAMIC INPUT DATA.

IN EQ. (44), TERMS 1 AND 2 ARE OF THE TYPE OF EQS. (2) OR (2'). TERM 6 IS OF THE TYPE OF EQ. (1). WE WILL NOW GIVE THE SOLUTION OF TERMS OF THE TYPE OF THE FOLLOWING EQUATIONS:

TERM 3
$$\overline{\square}^2\phi = \frac{\partial}{\partial t}\left[Q(\vec{x},t)H(\tilde{f})\delta(f)\right] \qquad (46)$$

TERMS 4 & 5
$$\overline{\square}^2\phi = \frac{\partial}{\partial x_i}\left[Q_i(\vec{x},t)H(\tilde{f})\delta(f)\right] \qquad (47)$$

WHERE $\tilde{f}=0$, $f=0$ GIVE THE EQUATION OF THE EDGE OF THE DEFORMABLE SURFACE $f=0$, $\tilde{f}>0$.

SOLUTION OF EQUATION (46)

USING EQ. (21) AND THE FACT THAT ω IS NOT A FUNCTION OF TIME, WE GET

$$\boxed{\begin{aligned} 4\pi\phi(\vec{x},t) &= \frac{\partial}{\partial t}\int_{\omega}\left[\frac{Q(\vec{y},\tau)\sqrt{g_{(2)}}}{r\,|1-M_r|}\right]_{\tau^*} du^1\,du^2 \\ &= \int_{\omega}\left\{\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left[\frac{Q(\vec{y},\tau)\sqrt{g_{(2)}}}{r\,|1-M_r|}\right]\right\}_{\tau^*} du^1\,du^2 \end{aligned}} \qquad (48)$$

NOTE CAREFULLY THAT $\partial/\partial\tau \equiv \partial/\partial\tau|_{\vec{u}}$ AND $\partial Q/\partial\tau$ IS VARIATION OF Q WITH TIME AS OBSERVED BY ONE MOVING WITH THE SURFACE. ALSO NOTE THAT WE HAVE NOT ASSUMED THAT $\partial\sqrt{g_{(2)}}/\partial\tau = 0$. WE GIVE THE

SOL. OF FW-H EQ.

FOLLOWING TO FIND THE ANALYTIC EXPRESSION INSIDE THE CURLY BRACKETS:

$$\frac{\partial r}{\partial \tau} = v_r \quad , \quad \frac{\partial \hat{r}_i}{\partial \tau} = \frac{\hat{r}_i v_r - v_i}{r} \qquad (49\text{-a,b})$$

$$\frac{\partial M_r}{\partial \tau} = \frac{1}{cr}\left[r_i \frac{\partial v_i}{\partial \tau} + v_r{}^2 - v^2\right] \qquad (49\text{-c})$$

HERE ALL VELOCITIES REFER TO SURFACE VELOCITY DEFINED AS $\partial \vec{y}/\partial \tau \,|_{\vec{u}}$, AND $v_r = v_i \hat{r}_i$, $v^2 = v_i v_i$.

SOLUTION OF EQUATION (47)

HERE WE USE THE FOLLOWING USEFUL RESULT WHERE $g = \tau - t + r/c$:

$$\boxed{\frac{\partial}{\partial x_i}\left[\frac{\delta(g)}{r}\right] = -\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{r}_i\,\delta(g)}{r}\right] - \frac{\hat{r}_i\,\delta(g)}{r^2}} \qquad (50)$$

THE SOLUTION OF EQ. (47) IS THUS

$$4\pi\phi(\vec{x},t) = \frac{\partial}{\partial x_i}\int \frac{Q_i(\vec{y},\tau)}{r} H(\tilde{f})\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

$$= \int Q_i H(\tilde{f})\,\delta(f)\,\frac{\partial}{\partial x_i}\left[\frac{\delta(g)}{r}\right] d\vec{y}\,d\tau \qquad (51)$$

NOW USE EQ. (50) FOR THE TERM IN SQUARE BRACKETS AND THEN TAKE t OUT OF THE INTEGRAL :

$$4\pi\phi(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\int_{\omega}\left[\frac{Q_i \hat{r}_i \sqrt{g_{(2)}}}{r(1-M_r)}\right]_{\tau *} du^1 du^2$$

$$-\int_{\omega}\left[\frac{Q_i \hat{r}_i \sqrt{g_{(2)}}}{r^2(1-M_r)}\right]_{\tau *} du^1 du^2 \qquad (52)$$

FROM THIS, BY TAKING $\partial/\partial t$ INSIDE THE FIRST INTEGRAL,

WE GET

$$\boxed{\begin{aligned} 4\pi\phi(\vec{x},t) = &-\frac{1}{c}\int_{\omega}\left\{\frac{1}{1-M_r}\frac{\partial}{\partial\tau}\left[\frac{Q_i\hat{r}_i\sqrt{g_{(2)}}}{r|1-M_r|}\right]\right\}_{\tau^*}du^1du^2 \\ &-\int_{\omega}\left[\frac{Q_i\hat{r}_i\sqrt{g_{(2)}}}{r^2|1-M_r|}\right]_{\tau^*}du^1du^2 \end{aligned}} \quad (53)$$

WE THUS HAVE ALL THE ANALYTIC EXPRESSIONS FOR THE SOURCE TERMS OF EQ. (44). WE MENTION THAT THE SOURCE TERMS ③, ④ AND ⑤ RESULT IN EXPRESSIONS OF ORDER $(1-M_r)^3$ IN THE DENOMINATOR WHILE TERMS ① AND ② HAVE $1-M_r$ IN THE DENOMINATOR OF THE ANALYTIC SOLUTION OF EQ. (44). THEREFORE, FOR SURFACES IN MOTION SUPERSONICALLY, THE SOLUTION OF EQ. (44) GIVEN ABOVE WILL HAVE SINGULARITIES WHICH FOR $M_n < 1$ ARE MATHEMATICAL IN NATURE. ALTHOUGH REGULARIZATION OF THESE DIVERGENT INTEGRALS CAN BE DONE, IN GENERAL IMPLIMENTATION ON A COMPUTER WILL BE DIFFICULT AND TIME CONSUMING. WE WILL GIVE BELOW EXPRESSIONS FOR TRANSONIC / SUPERSONIC SOURCES WHICH WILL NOT HAVE SINGULARIES MENTIONED ABOVE AND COMPUTER IMPLIMENTATION IS RELATIVELY SIMPLE.

III – SUPERSONIC MOTION OF BODY AND SHOCKS

WE NOW WRITE THE FW-H EQ. (44) IN A NEW FORM BY TAKING THE DERIVATIVES OF TERMS ③, ④ AND ⑤ REMEMBERING THAT ALL FUNCTIONS MULTIPLYING DELTA FUNCTIONS ARE RESTRICTED TO THE SUPPORT OF THE DELTA FUNCTIONS. WE WRITE THESE TERMS FIRST:

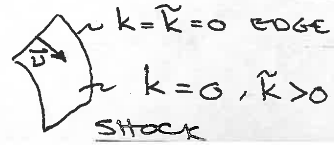
$$E_1 = \frac{\bar{\partial}}{\partial t}\left[\rho_0 \hat{v}_n\, \delta(f)\right] \qquad (54\text{-a})$$

$$E_2 = \frac{\bar{\partial}}{\partial x_i}\left[(\hat{Q}_i - \hat{\ell}_i)\,\delta(f)\right]$$

$$= \bar{\nabla}\cdot\left[(\hat{\vec{Q}} - \hat{\vec{\ell}})\,\delta(f)\right] \qquad (54\text{-b})$$

$$E_3 = \frac{\bar{\partial}}{\partial x_i}\left[\hat{q}_i\,\delta(k)\right] = \bar{\nabla}\cdot\left[\hat{\vec{q}}\,\delta(k)\right] \qquad (54\text{-c})$$

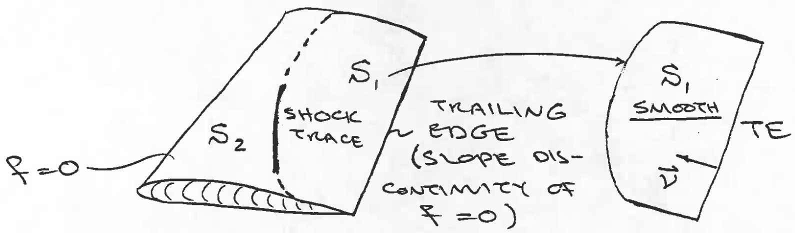
WE HAVE DISCONTINUITIES IN THE FUNCTIONS INSIDE THE SQUARE BRACKETS. THESE DISCONTINUITIES PRODUCE LINE SOURCES. EVEN DISCONTINUITIES IN SLOPE OF $f=0$ CAN PRODUCE EDGE SOURCES AS WILL BE SHOWN BELOW. WE NOTE THAT THE SHOCK SURFACE IS AN OPEN SURFACE WHILE THE BLADE SURFACE IS CLOSED. WE ASSUME THAT THE EDGE CURVE OF THE SHOCK IS GIVEN BY $\tilde{k} = k = 0$ WHERE $\tilde{k} = \tilde{k}(u^1, u^2)$ AS DESCRIBED IN SEC. II AND $\nabla_2 \tilde{k} = \vec{\nu}$ IS THE INWARD UNIT GEODESIC VECTOR, i.e. THE INWARD UNIT VECTOR WHICH IS BOTH TANGENT TO THE SHOCK SURFACE AND NORMAL TO THE EDGE AS SHOWN ON THE RIGHT.



TO INCLUDE SLOPE OR FUNCTION DISCONTINUITIES IN TERMS SUCH AS

SOL. OF FW-H EQ.

E_1 AND E_2, WE BREAK UP THE SURFACE f INTO SMOOTH OPEN SURFACES SUCH THAT PART OR ALL OF THEIR EDGES RUNS ALONG SLOPE OR THE CURVE OF FUNCTION DIS-CONTINUITIES. FOR EXAMPLE THE BLADE BELOW



HAS BEEN CUT INTO TWO SMOOTH OPEN SURFACES S_1 AND S_2 SUCH THAT BOTH FUNCTION DISCONTINUITY (e.g. SURFACE PRESSURE) AND SLOPE DISCONTINUITY OF $f=0$ (e.g. TE) APPEAR ALONG PARTS OF THE EDGES OF S_1 AND S_2. WE CONSIDER ONE OF THESE OPEN SURFACES ONLY AND FOR CLOSED SURFACES SUCH AS A BLADE, WE ADD US THE CONTRIBUTIONS OF ALL THE OPEN SURFACES WHICH FORM THE **BLADE**.

ASSUME THAT $\tilde{f}(u^1,u^2)=0$, $f=0$ DESCRIBE THE EDGE OF AN OPEN SURFACE. THEN WE HAVE

$$
\begin{aligned}
E_1 &= \frac{\bar{\partial}}{\partial t}\left[\rho_0 \hat{v}_n H(\tilde{f})\,\delta(f)\right] \\
&= \rho_0 \frac{\bar{\partial}\hat{v}_n}{\partial t} H(\tilde{f})\,\delta(f) - \rho_0 \hat{v}_n v_\nu\, \delta(\tilde{f})\,\delta(f) \\
&\quad - \rho_0 \hat{v}_n^{\,2} H(\tilde{f})\,\delta'(f) \\
&= \rho_0 (\dot{\hat{v}}_n - \vec{v}_t\cdot\nabla_2 \hat{v}_n) H(\tilde{f})\,\delta(f) - \rho_0 \hat{v}_n v_\nu\, \delta(\tilde{f})\,\delta(f) \\
&\quad - \rho_0 \hat{v}_n^{\,2} H(\tilde{f})\,\delta'(f) \qquad (55)
\end{aligned}
$$

WHERE $v_\nu = -\partial\tilde{f}/\partial t$ IS THE VELOCITY OF THE EDGE IN THE

DIRECTION OF THE INWARD UNIT GEODESIC NORMAL $\vec{\nu}$. WE HAVE ASSUMED HERE THAT $\nabla_2 \tilde{f} = \vec{\nu}$ SO THAT $|\nabla_2 \tilde{f}| = 1$. WE NOTE THAT THE TERM

$$\rho_0 \hat{v}_n v_\nu \delta(\tilde{f}) \delta(f)$$

IS A LINE SOURCE ALONG THE EDGE OF $\tilde{f} = f = 0$, i.e. ALONG THE EDGE OF THE OPEN SURFACE $\tilde{f} > 0, f = 0$. HOWEVER THE LAST TERM IN EQ.(55) ALSO GIVES A LINE SOURCE WHEN WE INTEGRATE THE TERM.

LET US NOW LOOK AT E_3 WHICH IS OF THE SAME TYPE AS E_2. LET $\tilde{k} = 0, k = 0$ BE THE EDGE CURVE OF THE SHOCK AND WE ASSUME $\nabla \tilde{k} = \vec{\nu}'$, THE INWARD UNIT GEODESIC NORMAL OF THE EDGE. THEN USING THE DEFINITION OF DIVERGENCE GIVEN IN THE SECTION ON MATHEMATICAL PREPARATION, i.e. (FOR SURFACE $f = 0$)

$$\nabla \cdot \vec{Q} = \nabla_2 \cdot \vec{Q}_T + \frac{\partial Q_n}{\partial n} - 2 H_f Q_n ,$$

WE HAVE

$$\begin{aligned} E_3 = \nabla \cdot [\hat{\vec{q}} H(\tilde{k}) \delta(k)] &= \nabla_2 \cdot [\hat{\vec{q}}_T H(\tilde{k})] \delta(k) + \frac{\partial}{\partial n'} [\hat{q}_{n'} H(\tilde{k}) \delta(k)] \\ &\quad - 2 H_k H(\tilde{k}) \delta(k) \\ &= \nabla_2 \cdot \hat{\vec{q}}_T H(\tilde{k}) \delta(k) + \hat{\vec{q}}_T \cdot \vec{\nu}' \delta(\tilde{k}) \delta(k) \\ &\quad + \hat{q}_{n'} H(\tilde{k}) \delta'(k) - 2 H_k q_{n'} H(\tilde{k}) \delta(k) \\ &= \nabla_2 \cdot \vec{q}_T H(\tilde{k}) \delta(k) + \cancel{q_{\nu'} \delta(\tilde{k}) \delta(k)} \\ &\quad + \hat{q}_{n'} H(\tilde{k}) \delta'(k) - 2 H_k q_{n'} H(\tilde{k}) \delta(k) \end{aligned} \tag{56}$$

$\boxed{q_{\nu'} = 0 \text{ BY SHOCK JUMP CONDITIONS}}$

WHERE WE HAVE ASSUMED THAT $\vec{n}'$ IS THE UNIT NORMAL TO THE SHOCK SUCH THAT $\nabla k = \vec{n}'$. ALSO WE HAVE DROPPED THE RESTRICTION SYMBOL (^) FROM ...

FUNCTIONS EXCEPT THE THIRD TERM WHICH CONTAINS $\delta'(k)$. THIS IS NEEDED AS A REMINDER IN FINDING THE SOLUTION OF FW-H EQ. WE REMIND THE READERS THAT $\partial \hat{q}_{n'} / \partial n' = 0$. AGAIN THE THIRD TERM IN WHICH $q_{\nu'} = \vec{q}_\tau \cdot \vec{\nu}' = \vec{q} \cdot \vec{\nu}'$ IS A LINE SOURCE. IN A SIMILAR WAY, IF $f=0$, $\tilde{f}>0$ IS AN OPEN SURFACE, THEN

$$E_2 = \overline{\nabla} \cdot [(\hat{\vec{Q}} - \hat{\vec{\ell}}) H(\tilde{f}) \delta(f)]$$
$$= \nabla_2 \cdot (\vec{Q}_\tau - \vec{\ell}_\tau) H(\tilde{f}) \delta(f) + (Q_\nu - \ _\nu) \delta(\tilde{f}) \delta(f)$$
$$+ (\hat{Q}_n - \hat{\ell}_n) H(\tilde{f}) \delta'(f) - 2 H_f (Q_n - \ell_n) H(\tilde{f}) \delta(f)$$
$$(57)$$

ONE MORE STEP IS NEEDED TO WRITE THE FW-H EQ. (44) IN A FORM SUITABLE FOR SUPERSONIC SOURCE MOTION. WE WRITE PART OF TERMS ① AND ② IN EQ. (44) AS

$$\frac{\partial T_{ij}}{\partial x_j} n_i = \frac{\partial}{\partial x_j} [n_i T_{ij}] - \underbrace{T_{ij} \frac{\partial n_i}{\partial x_j}}_{\equiv Q_G}$$
$$= \nabla \cdot \vec{Q} - Q_G$$
$$= \nabla_2 \cdot \vec{Q}_\tau + \frac{\partial Q_n}{\partial n} - 2 H_f Q_n - Q_G \qquad (58\text{-a})$$

$$\Delta \left(\frac{\partial T_{ij}}{\partial x_j} \right) n'_i = \frac{\partial}{\partial x_j} \Delta (T_{ij} n'_i) - \overbrace{\Delta T_{ij} \frac{\partial n'_i}{\partial x_j}}^{\equiv q_G}$$
$$= \nabla_2 \cdot \vec{q}_\tau + \Delta \left(\frac{\partial Q'_{n'}}{\partial n'} \right) - 2 H_k q_{n'} - q_G$$
$$(58\text{-b})$$

WHERE $Q'_{n'} = T_{ij} n'_i n'_j$, $q_{n'} = \Delta Q'_{n'}$.

LET US NOW ASSUME THAT THE BLADE SURFACE IS CUT INTO J SEGMENTS. THEN THE FW-H EQ. CAN BE WRITTEN AS

$$\boxed{\begin{aligned}\bar{\Box}^2 p' = \sum_{\alpha=1}^{J} \Big[& \psi_1(\vec{x},t)\, H(\tilde{f}_\alpha)\delta(f) + \widehat{\psi}_2(\vec{x},t)\, H(\tilde{f}_\alpha)\delta'(f) \\ & + \psi_3(\vec{x},t)\,\delta(\tilde{f}_\alpha)\delta(f)\Big] \qquad \boxed{\text{BLADE SURFACE AND EDGE TERMS}} \\ & + \Big[2\nabla_2\cdot\vec{q}_T - 4H_k q_{n'} + \Delta\Big(\frac{\partial Q_n'}{\partial n'}\Big) - q_G\Big] H(\tilde{k})\delta(k) \\ & + \hat{q}_{n'} H(\tilde{k})\delta'(k) \qquad \boxed{\text{SHOCK SURFACE TERMS}} \\ & + \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} H(f) \qquad \boxed{\text{PURE QUADRUPOLE TERM}} \qquad (59)\end{aligned}}$$

WHERE

$$\psi_1(\vec{x},t) = \rho_0\,(\dot{v}_n - \vec{v}_t\cdot\nabla_2 v_n) + \nabla_2\cdot(2\vec{Q}_T - \vec{\ell}_T) - 2H_f(2Q_n - \ell_n) + \frac{\partial Q_n}{\partial n} - Q_G \qquad (60\text{-a})$$

$$\psi_2(\vec{x},t) = -(\rho_0 v_n^2 + \ell_n) + Q_n \qquad (60\text{-b})$$

$$\psi_3(\vec{x},t = -(\rho_0 v_n v_\nu + \ell_\nu) + Q_\nu \qquad (60\text{-c})$$

AND $f=0$, $\tilde{f}_\alpha > 0$ IDENTIFIES THE αTH OPEN SURFACE COMPOSING THE CLOSED SURFACE $f=0$. WE WILL GIVE THE SOLUTION OF THE WAVE EQUATION

WITH SOURCE TERMS OF THE FORM ON THE RIGHT OF EQ. (59). WE MENTION HERE THAT WE CAN IDENTIFY THICKNESS, LOADING AND QUADRUPOLE CONTRIBUTIONS OF THE CONVENTIONAL (i.e. THE ORIGINAL) FW-H EQ. IN EQS. (60 a–c) AS FOLLOWS :

i) ALL TERMS CONTAINING v_n ARE FROM THICKNESS SOURCE

ii) ALL TERMS CONTAINING $\vec{\ell}_\tau$, ℓ_n AND ℓ_ν ARE FROM LOADING SOURCE

iii) ALL OTHER TERMS IN EQS. 60(a–c) AND (59) ARE FROM QUADRUPOLE SOURCE.

ONLY TWO TYPES OF SOURCE TERMS IN EQ. (59) HAVE NOT BEEN CONSIDERED IN THE SOLUTION OF A WAVE EQUATION. WE MENTION THAT FOR THE SURFACE TERM $\psi_1 H(\tilde{f}_\alpha)\delta(f)$ OR $[\cdots]H(\tilde{k})\delta(k)$, THE SOLUTION OF THE FORM OF EQ. (28) IS USED FOR SUPERSONIC SOURCES IN NASA LANGLEY CODES. WE THUS GIVE THE SOLUTIONS TO THE FOLLOWING TWO WAVE EQUATIONS

$$\bar{\Box}^2\phi = \widehat{\psi_2}(\vec{x},t)\, H(\tilde{f})\, \delta'(f) \qquad (61)$$

$$\bar{\Box}^2\phi = \psi_3(\vec{x},t)\, \delta(\tilde{f})\, \delta(f) \qquad (62)$$

WHERE FOR CONVINIENCE, WE HAVE DROPPED α IN f_α. BELOW WE GIVE THE SOLUTIONS OF THESE EQUATIONS. NOTE THAT THE LAST SHOCK SURFACE TERM IN EQ. (59) IS LIKE THE SOURCE TERM IN EQ. (61).

SOLUTION OF EQUATION (61)

WE HAVE

$$4\pi \phi(\vec{x},t) = \int \frac{1}{r} \hat{\psi}_2(\vec{y},\tau) H(\tilde{f}) \delta'(f) \delta(g) \, d\vec{y} \, d\tau \tag{63}$$

LET $\tau \rightarrow g$, $\Rightarrow |\partial g/\partial \tau| = 1$ AND INTEGRATE THE ABOVE EQUATION WRT g :

$$4\pi \phi(\vec{x},t) = \int \frac{1}{r} \hat{\psi}_2(\vec{y}, t-\tfrac{r}{c}) H(\tilde{f}) \delta'(F) \, d\vec{y} \tag{64}$$

WHERE $F = f(\vec{y}, t - r/c) = [f]_{ret}$ AS BEFORE.

USING THE RESULT GIVEN BEFORE IN MATHEMATICAL PREPARATIONS, WE HAVE :

$$4\pi \phi(\vec{x},t) = \int_{F=0} \left\{ -\frac{1}{\Lambda} \frac{\partial}{\partial N} \left[\frac{\hat{\psi}_2(\vec{y},\tau)}{r\Lambda} H(\tilde{f}) \right] + \frac{2H_F}{r\Lambda^2} H(\tilde{f}) \hat{\psi}_2(\vec{y},\tau) \right\}_{ret} d\Sigma \tag{64}$$

WHERE $\partial/\partial N = \vec{N} \cdot \nabla$, $\vec{N} = (\vec{n} - M_n \hat{r})/\Lambda$ IS THE UNIT NORMAL TO $F=0$. NOTE THAT WE HAVE $|\nabla F| = [\Lambda]_{ret} \neq 1$ AND WE HAVE USED THE FOLLOWING RESULT WHEN $|\nabla f| \neq 1$:

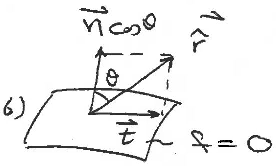
$$I' = \int Q \, \delta'(f) \, d\vec{y}$$

$$= \int_{f=0} \left[-\frac{\partial}{\partial n} \left[\frac{Q}{|\nabla f|} \right] + \frac{2H_f Q}{|\nabla f|} \right] \frac{dS}{|\nabla f|} \tag{65}$$

ALSO NOTE THAT IN EQ. (64), H_F IS THE MEAN

CURVATURE OF THE SURFACE $F=0$. WE MUST TAKE A CAREFUL LOOK AT VARIABLES OF THE EXPRESSION IN THE SQUARE BRACKETS OF EQ.(64). WE KNOW THAT $\frac{\partial}{\partial n}\hat{\psi}_2 = \vec{n}\cdot\nabla\hat{\psi}_2 = 0$. WE WRITE

$$\vec{N} = \frac{\vec{n} - M_n\hat{r}}{\Lambda} = \frac{(1-M_n\cos\theta)\,\vec{n} - M_n\vec{t}}{\Lambda} \qquad (66)$$



WHERE $\vec{t}$ IS THE PROJECTION OF $\hat{r}$ ON THE LOCAL TANGENT PLANE OF $f=0$. NOTE THAT $\vec{t}$ IS NOT A UNIT VECTOR. WE HAVE

$$\begin{aligned}\frac{\partial}{\partial N}\hat{\psi}_2(\vec{y},t-r/c) &= -\frac{M_n}{\Lambda}\vec{t}\cdot\nabla_2\hat{\psi}_2 + \frac{1}{c}\vec{N}\cdot\hat{r}\,\frac{\partial\hat{\psi}_2}{\partial\tau} \\ &= -\frac{M_n}{\Lambda}\sin\theta\,\frac{\partial\hat{\psi}_2}{\partial\sigma_t} + \frac{\cos\theta - M_n}{c\Lambda}\left(\dot{\hat{\psi}}_2 - \vec{v}_t\cdot\nabla_2\hat{\psi}_2\right)\end{aligned} \qquad (67)$$

ALSO ONE OBTAINS

$$\frac{\partial}{\partial N}H(\tilde{f}) = -\frac{M_n}{\Lambda}\vec{t}\cdot\vec{\nu}\,\delta(\tilde{f}) \qquad (68)$$

THIS TERM AGAIN RESULTS IN A LINE INTEGRAL FROM THE FIRST INTEGRAL WHICH WE WRITE AS

$$\equiv \int_{F=0}\left[\frac{\hat{\psi}_2(\vec{y},\tau)\,M_n\,\vec{t}\cdot\vec{\nu}}{r\Lambda^2}\right]_{ret}\delta(\tilde{f})\,d\Sigma = \int \boxed{\frac{\hat{\psi}_2 M_n\vec{t}\cdot\vec{\nu}}{\Lambda}\,\delta(\tilde{f})\delta(f)}\frac{\delta(g)}{r}\,d\vec{y}\,d \qquad (69)$$

THEREFORE, THE TERM IN THE BOX LOOKS LIKE THE

SOURCE TERM OF EQ. (62) WHICH WE WILL DISCUSS BELOW. FINALLY, $\frac{\partial}{\partial N}[1/(\Lambda r)]$ IN FIRST INTEGRAL OF EQ. (64) HAS THE USUAL MEANING SINCE BOTH Λ AND r ARE NOT RESTRICTED TO ANY SURFACE.

IN SUMMARY, THE SOLUTION OF EQUATION (63) IS

$$\boxed{\begin{aligned} 4\pi\,\phi(\vec{x},t) = \int_{\substack{F=0\\ \tilde{f}>0}} \Big\{ -\frac{1}{\Lambda}\frac{\partial}{\partial N}\Big[\frac{\hat{\Psi}_2(\vec{y},\tau)}{r\Lambda}\Big] &+ \frac{2H_F}{r\Lambda^2}\hat{\Psi}_2(\vec{y},\tau)\Big\}_{ret} d\Sigma \\ &+ \int \frac{\hat{\Psi}_2 M_n\, \vec{t}\cdot\vec{\nu}}{r\Lambda}\,\delta(\tilde{f})\,\delta(f)\,\delta(g)\, d\vec{y}\, d\tau \end{aligned}} \quad (70)$$

WHERE $\partial\hat{\Psi}_2/\partial N$ IS GIVEN EXPLICITLY BY EQ. (67). ALSO H_F CAN BE CALCULATED ANALYTICALLY.

SOLUTION OF EQUATION (62)

WE HAVE AGAIN

$$4\pi\,\phi(\vec{x},t) = \int \frac{1}{r}\Psi_3(\vec{y},\tau)\,\delta(\tilde{f})\,\delta(f)\,\delta(g)\, d\vec{y}\, d\tau \quad (71)$$

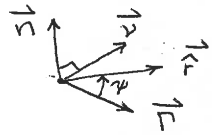
IT IS EASY TO VISUALIZE THE SUBSPACE $\tilde{f}=f=g=0$. IT IS CONSTRUCTED BY THE INTERSECTION OF THE COLLAPSING SPHERE $g=0$ WITH THE EDGE OF THE OPEN SURFACE $\tilde{f}=f=0$. WE CAN ALSO DERIVE THREE FORMS OF THE SOLUTION AS IN THE CASE OF A SOURCE DISTRIBUTION ON A SURFACE.

SOL. OF FW-H EQ.

<u>FIRST FORM</u>

LET $\vec{y} \to (\tilde{f}, f, g)$, WE CAN SEE THAT

$$\left|\frac{\partial(\tilde{f}, f, g)}{\partial(y_1, y_2, y_3)}\right| = |\vec{\nu} \times \vec{n} \cdot \nabla g|$$

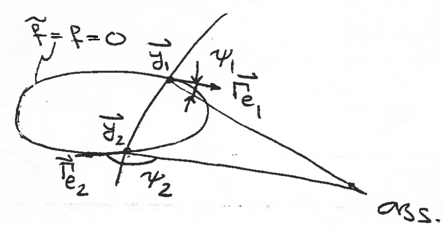


$$= \frac{1}{c}|\vec{\Gamma} \times \vec{r}|$$
$$= \frac{1}{c}|\cos\psi| \qquad (72)$$

WHERE $\vec{\Gamma}_e$ IS THE THE UNIT TANGENT TO THE EDGE OF THE OPEN SURFACE, $\tilde{f} = f = 0$, AND ψ IS THE ANGLE BETWEEN $\vec{\Gamma}_e$ AND $\hat{r}$. WE THUS HAVE

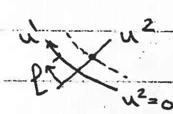
$$\boxed{4\pi\phi(\vec{x}, t) = \int \sum_i \frac{\psi_2(\vec{y}_i, \tau)}{(t-\tau)|\cos\psi_i|}\, d\tau} \qquad (73)$$

THE TIME INTEGRATION IS OVER THE ENTIRE PERIOD OF INTERSECTION OF THE COLLAPSING SPHERE AND THE EDGE CURVE. ALSO MULTIPLE INTERSECTION MUST BE INCLUDED IN THE INTEGRAND AS SHOWN IN THE FIGURE ON THE RIGHT.



<u>SECOND FORM</u>

TAKE $u^2 = \tilde{f}$, $u^3 = f$ AND $u^1 = l$ WHERE l IS THE LENGTH OF THE EDGE CURVE ON $u^2 = 0$ (I.E. EDGE) CURVE FROM A REFERENCE POINT. NOW LET $\vec{y} \to \vec{u}$



$\Rightarrow d\vec{y}\Big|_{\substack{u^2=0\\u^3=0}} = du^1 du^2 dl$ AND THE JACOBIAN OF

TRANSFORMATION 1. WE GET FROM EQ.(71)

$$4\pi\phi(\vec{x},t) = \int \frac{1}{r}\Psi_3(\vec{y},\tau)\,\delta(g)\,d\tau\,d\ell \qquad (74)$$

NOW LET $\tau \to g$ REMEMBERING THAT $\vec{y} = \vec{y}(\vec{u},\tau)$ WHICH GIVES AGAIN $\partial g/\partial\tau = 1 - M_r$. WE GET THE SECOND FORM AS

$$\boxed{4\pi\phi(\vec{x},t) = \int_{\tilde{f}=f=0} \left[\frac{\Psi_3(\vec{y},\tau)}{r\,|1-M_r|}\right]_{\tau^*} d\ell} \qquad (75)$$

WHERE AS BEFORE τ^* IS THE SOLUTION OF THE EQUATION $\tau^* - t + |\vec{x} - \vec{y}(\vec{u},\tau^*)|/c = 0$.

THIRD FORM

THIS TIME WE WANT TO WRITE THE SOLUTION IN TERMS OF THE ELEMENT OF THE LENGTH OF THE EDGE OF THE Σ-SURFACE WHICH WE WILL CALL THE γ-CURVE. THE TANGENT TO THE γ-CURVE WILL BE PARALLEL TO $\nabla\tilde{F}\times\nabla F$ WHERE $\tilde{F} = [\tilde{f}(\vec{y},\tau)]_{ret}$

WE CAN SHOW THAT

$$\nabla\tilde{F} = \vec{\nu} - M_\nu\vec{\hat{r}} \qquad (76\text{-}a)$$

$$\begin{aligned}\therefore \nabla\tilde{F}\times\nabla F &= (\vec{\nu} - M_\nu\vec{\hat{r}})\times(\vec{n} - M_n\vec{\hat{r}})\\ &= \vec{\Gamma}_e - \vec{\hat{r}}\times(M_\nu\vec{n} - M_n\vec{\nu})\end{aligned} \qquad (76\text{-}b)$$

WE WRITE

$$\Lambda_0 = |\nabla\tilde{F}\times\nabla F|, \qquad (77)$$

SO THAT THE UNIT TANGENT TO THE γ-CURVE IS

$$\vec{\gamma} = \frac{\nabla\tilde{F}\times\nabla F}{\Lambda_0}. \qquad (78)$$

WE CAN WRITE EQ.(71), AFTER INTEGRATION WRT g (i.e. FIRST USING $\tau \to g$)

$$4\pi\phi(\vec{x},t) = \int \frac{1}{r}[\Psi_3]_{ret}\,\delta(\tilde{F})\,\delta(F)\,d\vec{y} \qquad (79)$$

NOW LET $(y_1, y_2) \to (\tilde{F}, F)$ AND USE THE FOLLOWING RESULT IN (79)

$$d\vec{y} = \frac{d\tilde{F}\, dF\, dy_3}{|\nabla\tilde{F} \times \nabla F|_3}$$

$$= \frac{d\tilde{F}\, dF}{\Lambda_0} \frac{dy_3}{\gamma_3}$$

$$= \frac{d\tilde{F}\, dF\, d\gamma}{\Lambda_0} \qquad (80)$$

WHERE γ_3 IS THE DIRECTION COSINE OF $\vec{\gamma}$ ALONG y_3-AXIS AND $d\gamma$ IS THE ELEMENT OF LENGTH OF THE γ-CURVE. WE GET THE THIRD FORM

$$\boxed{4\pi\phi(\vec{x}, t) = \int_{\tilde{F}=F=0} \frac{1}{r}\left[\frac{\Psi_3(\vec{y}, \tau)}{\Lambda_0}\right]_{ret} d\gamma} \qquad (81)$$

WE NOTE THAT THE THREE FORMS ARE RELATED THROUGH THE RELATION:

$$\boxed{\frac{c\, d\tau}{|\cos\psi|} = \frac{dl}{|1 - M_r|} = \frac{d\gamma}{\Lambda_0}} \qquad (82)$$

WE HAVE TO USE EITHER FORM 1 OR 3 FOR SUPERSONIC SOURCES. THE POSSIBILITY OF REAL SINGULARITIES IN ACOUSTIC PRESSURE MUST BE CONSIDERED IN APPLICATIONS. IF $M_n < 1$ OVER THE ENTIRE SURFACE $f = 0$, THEN ONLY THE LINE INTEGRALS WILL CONTRIBUTE TO THE SINGULARITIES.

7 Green's Functions (Course)

Lecture Notes on Green's Functions

By

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Lectures delivered to the Aeroacoustics and Structural Acoustics Branches of NASA Langley Research Center October 2005 to March 2006

NOTES ON GREEN'S FUNCTION

WE CONSIDER ODE'S FIRST. WE WORK WITH 2ND ORDER LINEAR ODE'S OF THE TYPE

$$(**) \quad \ell u = A(x)\,u'' + B(x)\,u' + C(x)\,u = f(x) \qquad x \in [0,1]$$

WITH LINEAR HOMOGENEOUS BC'S:

$$\begin{cases} \alpha_1 u(0) + \beta_1 u'(0) + \gamma_1 u(1) + \delta_1 u'(1) = 0 \\ \alpha_2 u(0) + \beta_2 u'(0) + \gamma_2 (u) + \delta_2 u'(1) = 0 \end{cases}$$

ALL $\alpha, \beta, \gamma, \delta$ ARE CONSTANTS. THE TWO BC'S ARE INDEPENDENT. A <u>LINEAR OPERATOR</u> L IS DEFINED AS

$$(*) \quad Lu = f \; : \begin{cases} \ell u = f & , \ell u \text{ LIN. ODE } , x \in [0,1] \\ BC[u] = 0 & BC : \text{LIN. HOMO.} \end{cases}$$

EXAMPLE:

$$Lu = f \; : \begin{cases} u'' = f , & x \in [0,1] \\ \begin{cases} u(0) - u'(0) = 0 \\ u(1) + 2u'(1) = 0 \end{cases} \end{cases}$$

THE <u>GREEN'S FUNCTION</u> OF THE LIN. OP. (*) IS DEFINED AS $G(x,y)$, SUCH THAT

$$u(x) = \int_0^1 f(y)\, G(x,y)\, dy$$

NOTE

THE ORDER OF x AND y IN $G(x,y)$ IS IMPORTANT. WE HAVE SHOWN THAT FOR THE LIN. OP. L (*), ABOVE, $G(x,y)$ HAS THE FOLLOWING PROPERTIES

$$\left.\begin{array}{l} ① \; \ell_x G(x,y) = \delta(x-y) \\ ② \; BC_x[G(x,y)] = 0 \end{array}\right\}$$

THESE CAN BE WRITTEN AS $L_x[G(x,y)] = \delta(x-y)$

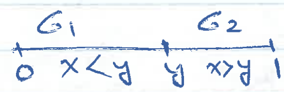
INTERPRETATION OF CONDITIONS (1) AND (2):

$$\ell_x G(x,y) = \begin{cases} 0 & x<y,\ x>y \quad (3) \\ \text{AT } x=y, & G(x,y) \text{ CONTINUOUS} \\ & \Delta\left[\dfrac{\partial G}{\partial x}(x,y)\right] = \dfrac{1}{A(y)} \end{cases} \quad (4)$$

WHERE $\Delta(\cdot)$ = JUMP OF THE FUNCTION $(\cdot)$.
HERE $A(x)$ IS THE COEFFICIENT OF u'' IN ℓu (**, P1). CONDITION (3) MEANS THAT

$G(x,y)$ IS A SOLUTION OF THE HOMOGENEOUS DE $\ell_x G(x,y) = 0$.

$$G(x,y) = \begin{cases} G_1(x,y) & x<y \\ G_2(x,y) & x>y \end{cases}$$



CONDITION (3) MEANS THAT $G_1(y,y) = G_2(y,y)$
CONDITION (4) MEANS THAT

$$\frac{\partial G_2}{\partial x}(y,y) - \frac{\partial G_1}{\partial x}(y,y) = \frac{1}{A(y)}$$

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(5)

SOLUTION OF THE PROBLEM GIVEN IN THE CLASS :

$$L : \begin{cases} u'' = f(x) \\ u(0) - 2u'(1) = 0 \\ u'(0) + u(1) = 0 \end{cases}$$

$$G(x,y) = \begin{cases} A_1 x + B_1 \\ A_2 x + B_2 \end{cases} \text{COND. (1) SATISFIED!}$$

G_1 G_2 ; $x<y$ y $x>y$ 1

USING CONDITION 2

$$G_1(0,y) - 2\frac{\partial G}{\partial x}(1,y) = B_1 - 2A_2 = 0$$

$$\frac{\partial G_1}{\partial x}(0,y) + G_2(1,y) = A_1 + A_2 + B_2 = 0$$

$$\Longrightarrow \quad B_1 = 2A_2 \ , \quad A_1 = -A_2 - B_2$$

$$G(x,y) = \begin{cases} -(A_2+B_2)x + 2A_2 & x<y \\ A_2 x + B_2 & x>y \end{cases}$$

USING CONDITION 3

$$-(A_2+B_2)y + 2A_2 = A_2 y + B_2$$

USING CONDITION 4

$$\frac{\partial G_2}{\partial x}(y,y) - \frac{\partial G_1}{\partial x}(y,y) = A_2 + A_2 + B_2 = 1 \quad (+)$$

$$\begin{cases} 2(1-y)A_2 - (1+y)B_2 = 0 \\ 2A_2 + B_2 = 1 \end{cases}$$

$$A_2 = \frac{\begin{vmatrix} 0 & -(1+y) \\ 1 & 1 \end{vmatrix}}{2\begin{vmatrix} 1-y & -(1+y) \\ 1 & 1 \end{vmatrix}} = \frac{1+y}{4}$$

$$B_2 = \frac{2\begin{vmatrix} 1-y & 0 \\ 1 & 1 \end{vmatrix}}{4} = \frac{2(1-y)}{4}$$

FROM (+)

$$B_1 = 2A_2 = \frac{2(1+y)}{4} \ , \quad A_1 = -A_2 - B_2 = A_2 - 1 = \frac{y-3}{4}$$

⑥

$$\therefore \quad G(x,y) = \begin{cases} \frac{1}{4}\left[(y-3)x + 2(1+y)\right] & x<y \\ \frac{1}{4}\left[(1+y)x + 2(1-y)\right] \end{cases}$$

TEST OF THE BC'S

$$\begin{cases} G_1(0,y) - 2\frac{\partial G_2}{\partial x}(1,y) = \frac{1+y}{2} - \frac{1+y}{4} = 0 \\ \frac{\partial G_1}{\partial x}(0,y) + G_2(1,y) = \frac{y-3}{4} + \frac{3-y}{4} = 0 \end{cases}$$

OKAY!

<u>DEFN</u> : THE <u>ADJOINT</u> L^* OF THE LINEAR OP. L IS DEFINED AS

$$\langle Lu, v\rangle = \langle u, L^* v\rangle$$

WHERE $Lu = \begin{cases} \ell u & \text{LIN. O.D.E.} \quad x\in[0,1] \\ BC[u]=0 & \text{LIN. HOMO.} \end{cases}$

$$L^* v = \begin{cases} \ell^* v & \text{ADJOINT LIN. ODE} \quad x\in[0,1] \\ BC^*[v]=0 & \text{LIN. HOMO.} \end{cases}$$

HERE THE INNER PRODUCT $\langle u,v\rangle$ IS DEFINED AS

$$\langle u,v\rangle = \int_0^1 u(x)v(x)\,dx$$

FOR TWO REAL FUNCTIONS u AND v. NOTE THAT THE INNER PRODUCT HAS THE PROPERTIES

$$\begin{cases} \langle \alpha u, v\rangle = \langle u, \alpha v\rangle = \alpha\langle u,v\rangle \\ \langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle \\ \langle u, v_1 + v_2\rangle = \langle u, v_1\rangle + \langle u, v_2\rangle \\ \langle u,u\rangle = 0 \text{ IF \& ONLY IF } u = 0 \end{cases}$$

FINDING ADJOINT OPERATOR : EXAMPLE 1

$$Lu : \begin{cases} u'' = f(x) \\ u(0) - u'(0) = 0 \\ u(1) + 2u'(1) = 0 \end{cases}$$

$$\langle Lu, v\rangle = \langle u'', v \rangle = \int_0^1 u''v \, dx$$

$$u''v = (u'v)' - u'v'$$
$$= (u'v - uv')' + uv''$$

$$\langle Lu, v\rangle = \int_0^1 u''v \, dx = (u'v - uv')\Big|_0^1 + \int_0^1 u \underbrace{v''}_{\ell^* v} dx = \langle u, L^* v\rangle$$

$$\ell^* v = v''$$

$$0 = (u'v - uv')\Big|_0^1 = u'(1)v(1) - u(1)v'(1)$$
$$- u'(0)v(0) + u(0)v'(0)$$

SUBSTITUTE FOR $u(0)$ & $u(1)$ FROM BC'S

$$u'(1)\underbrace{[v(1) + 2v'(1)]}_{=0 : BC_1[v]} - u'(0)\underbrace{[v(0) - v'(0)]}_{=0 : BC_0[v] = 0}$$

$$\therefore \quad BC^* = \begin{cases} v(0) - 2v'(0) = 0 \\ v(1) + 2v'(1) = 0 \end{cases}$$

NOTE THAT $L^* : \begin{cases} \ell'' = \frac{d^2}{dx^2} \\ BC^* \end{cases}$

SINCE $\ell u = \ell^* u$, $BC[u] = BC^*[u]$
WE SAY THAT $L = L^*$ AND THE OPERATOR
L IS SAID TO BE SELF-ADJOINT.

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FINDING THE ADJOINT OPERATOR

EXAMPLE 2: $Lu : \begin{cases} u'' \\ u(0) - 2u'(1) = 0 \\ u'(0) + 2u(1) = 0 \end{cases}$

$$\langle Lu, v \rangle = \int_0^1 u'' v \, dx = (u'v - uv')\Big|_0^1 + \int_0^1 u v'' dx$$

$$= \langle u, L^* v \rangle$$

$$l^* v = \frac{d^2 v}{dx^2}$$

$$0 = u'(1)v(1) - u(1)v'(1) - \underbrace{u'(0)}_{-2u(1)} v(0) + \underbrace{u(0)}_{2u'(1)} v'(0)$$

$$= u(1)\underbrace{[2v(0) - v'(1)]}_{=0} + u'(1)\underbrace{[2v'(0) + v(1)]}_{=0}$$

$$BC^*[v] = \begin{cases} 2v(0) - v'(1) = 0 \\ 2v'(0) + v(1) = 0 \end{cases}$$

IN THIS CASE $l^* = l$ BUT $BC^*[u] \neq BC[u]$
WE SAY THAT l IS <u>NON-SELFADJOINT</u>.
WE ALSO SAY THAT, SINCE $l = l^*$, L IS
<u>FORMALLY SELFADJOINT</u>.

SHOW THAT THE ADJOINT D.E. FOR
$lu = u'' + 2u'$ IS $l^* v = v'' - 2v'$
SO THAT A LIN. OP. BASED OF lu CANNOT
BE FORMALLY SELFADJOINT.

<u>PROBLEM</u>: FIND THE ADJOINT D.E. FOR
$lu = A(x)u'' + B(x)u + C(x)$

<u>ANS.</u> $l^* v = (A(x)v)'' - (B(x)v)'$
SHOW THIS!

NOTES ON GREEN'S FUNCTION (CONT'D) (9)

WHEN DOES THE GREEN'S FUNCTION NOT EXIST? RATHER THAN GIVE A DETAILED ANSWER, LET US GIVE AN EXAMPLE OF LIN. DIFF. OP. WHICH FAILS TO HAVE A GREEN'S FUNCTION AND THEN STATE OUR MAIN RESULT

$$Lu : \begin{cases} u'' = f(x) & x \in [0,1] \\ \begin{cases} u(0) - 2u'(0) = 0 \\ u(1) - 3u'(1) = 0 \end{cases} \end{cases}$$

HERE WE FIND THAT $u'' = 0$ HAS GENERAL SOLUTION $u = Ax + B$. HOWEVER, WE FIND THAT THE FUNCTION $u = x + 2$ SATISFIES BOTH $u'' = 0$ AND THE BC'S. THIS MEANS THAT WE CANNOT FIND TWO FUNCTIONS $G_1(x,y)$ AND $G_2(x,y)$ FOR $x < y$ AND $x > y$, RESPECTIVELY IN THE PROCESS OF FINDING THE GREEN'S FUNCTION FOR Lu. MATHEMATICALLY, THE EXISTENCE OF u SATISFYING $Lu = 0$, i.e., $\ell u = 0 + BC[u] = 0$, MEANS THAT $\lambda = 0$ IS AN EIGENVALUE OF THE LINEAR OPERATOR. WE HAVE THE FOLLOWING

THEOREM: $Lu : \begin{cases} \ell u = f & \text{LIN. O.D.E.} \\ BC[u] = 0 & \text{LIN. \& HOM.} \end{cases}$

DOES NOT HAVE A GREEN'S FUNCTION IF $\lambda = 0$ IS AN EIGENVALUE OF THE OPERATOR L. THIS MEANS THAT THERE IS A NONTRIVIAL FUNCTION $u(x)$ SUCH THAT

$$(*) \quad \begin{cases} \ell u = 0 \\ BC[u] = 0 \end{cases}$$

"NONTRIVIAL" MEANS NOT IDENTICALLY EQUAL TO ZERO, BECAUSE THE ZERO FUNCTION $u = 0$ SATISFIES $(*)$!

NOTES ON GREEN'S FUNCTION

ANOTHER EXAMPLE OF THE ADJOINT OPERATOR

$$Lu : \begin{cases} u'' + u' - 2u = f(x) & x \in [0,1] \\ u(0) - u'(1) = 0 \\ u'(0) + 2u(1) = 0 \end{cases}$$

$$\langle Lu, v \rangle = \int_0^1 (\ell u) v \, dx$$

$$= \int_0^1 (u'' + u' - 2u) v \, dx$$

$$u''v = (u'v)' - u'v'$$

$$= (u'v - uv')' + uv''$$

$$u'v = (uv)' - uv'$$

$$(\ell u) v = (u'v - uv' + uv)' + u \underbrace{(v'' - v' - 2v)}_{\ell^* v}$$

$$\ell^* v = v'' - v' - 2v$$

$$\langle Lu, v \rangle = \underbrace{(u'v - uv' + uv)\Big|_0^1}_{=0} + \langle u, L^* v \rangle$$

$$u'(1)v(1) - u(1)v'(1) + u(1)v(1) - \underbrace{u'(0)}_{-2u(1)} v(0) + \overbrace{u(0)}^{u'(1)} v'(0)$$

$$- \underbrace{u(0)}_{u'(1)} v(0) = 0$$

$$u'(1)\left[v(1) + v'(0) - v(0)\right] + u(1)\left[-v'(1) + 2v(0) + v(1)\right] = 0$$

$$\therefore \; BC^*[v] : \begin{cases} v(0) - v'(0) - v(1) = 0 \\ 2v(0) + v(1) - v'(1) = 0 \end{cases}$$

SINCE $\ell \neq \ell^*$ & $BC \neq BC^*$, L IS NOT SELF-ADJOINT. WE CAN ALSO SAY L IS NONSELF-ADJOINT

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NOTES ON GREEN'S FUNCTION

- AN IMPORTANT RELATION BETWEEN GREEN'S $G(x,y)$ OF L AND $G^*(x,y)$ OF L^*

$$\langle \underbrace{L_x G(x,y)}_{\delta(x-y)}, G^*(x,z) \rangle_x = \int \ell_x G(x,y) \cdot G^*(x,z)\, dx$$

$$= \langle \delta(x-y), G^*(x,z) \rangle_x = G^*(y,z)$$

$$= \langle G(x,y), \underbrace{L^*_x(x,z)}_{\delta(x-z)} \rangle_x = G(z,y)$$

$$\Rightarrow \quad G^*(x,y) = G(y,x)$$

$$\Rightarrow \quad L_x G(x,y) = L^*_x G^*(x,y) = \delta(x-y)$$
$$= L^*_x G(y,x)$$
$$= L^*_y G(x,y)$$

i.e. IN VARIABLE y

$$\begin{cases} \ell^*_y G(x,y) = \delta(x-y) \\ BC^*_y[G(x,y)] = 0 \end{cases} \Rightarrow L^*_y G(x,y) = \delta(x-y)$$

FOR A SELF-ADJ. OPERATOR $G(x,y) = G(y,x)$

THIS RELATION CAN BE USED TO SIMPLIFY FINDING GREEN'S FUNCTION OF SELF-ADJ. OPERATORS.

FOR SECOND ORDER LIN. ODE'S

$$\ell u = A(x)u'' + B(x)u' + C(x)u \ \& \ BC[u] = 0$$

THERE ARE TWO CONDITIONS THAT WE CAN QUICKLY GUESS THE SELF-ADJOINTNESS OF THE OPERATOR IF ℓ IS FORMALLY SELF-ADJOINT, i.e. $\ell^* = \ell$

LET $x \in [0,1]$

1. IF THE BC'S ARE PERIODIC
2. IF THE BC'S ARE SEPARATED, i.e., IF ONE BC HAS ONLY $u(0)$ & $u'(0)$ AND THE 2ND BC HAS ONLY $u(1)$ AND $u'(1)$.

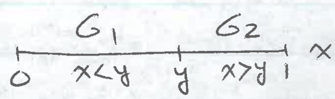
EXAMPLES

1. $$Lu: \begin{cases} \ell u = u'' + u = \ell^* u & x \in [0,1] \\ BC[u]: \begin{cases} u(0) - u(1) = 0 \\ u'(0) - u'(1) = 0 \end{cases} \end{cases} \Rightarrow L \text{ IS SELF-ADJ.}$$

2. $$Lu: \begin{cases} \ell u = u'' = \ell^* u & x \in [0,1] \\ BC[u]: \begin{cases} u(0) - 3u'(0) = 0 \\ 2u(1) + u'(1) = 0 \end{cases} \end{cases} \Rightarrow L \text{ IS SELF-ADJ.}$$

WE FIND THE GREEN'S FUNCTION FOR EX. 2 AND THIS TIME WE USE $G^*(x,y) = G(y,x)$, i.e. G IS SYMMETRIC IN x & y

$$G(x,y) = \begin{cases} G_1(x,y) = A_1 x + B_1 & x < y \\ G_2 = A_2 x + B_2 & x > y \end{cases}$$



G_1 SATISFIES BC_1 IN x: $B_1 - 3A_1 = 0 \Rightarrow B_1 = 3A_1$

G_2 " BC_2 IN x: $2(A_2 + B_2) + A_2 = 3A_2 + 2B_2 = 0$

$$\Rightarrow B_2 = -\tfrac{3}{2} A_2$$

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$$G(x,y) = \begin{cases} A_1(x+3) & x<y \\ A_2(x-\frac{3}{2}) & x>y \end{cases}$$

NOW, WE USE SYMMETRY CONDITION OF $G(x,y)$ AND WRITE G AS FOLLOWS:

$$G(x,y) = \begin{cases} A(y-\frac{3}{2})(x+3) & x<y \\ A(y+3)(x-\frac{3}{2}) & x>y \end{cases}$$

WHERE THE ARROWS SHOW HOW WE CHANGE $x+3$ TO $y+3$, AND $x-\frac{3}{2}$ TO $y-\frac{3}{2}$ TO MAKE G SYMMETRIC IN x AND y. HERE A IS A CONSTANT. NOTE THAT CONTINUITY OF G_1 AND G_2 AT $x=y$ IS AUTOMATICALLY SATISFIED! TO FIND A, WE USE

$$\frac{\partial G_2(y,y)}{\partial x} - \frac{\partial G_1}{\partial x}(y,y) = 1$$

$$A\left[\cancel{y}+3-\cancel{y}+\frac{3}{2}\right) = 1$$

$$A = \frac{2}{9}$$

$$G(x,y) = \begin{cases} \frac{1}{9}(2y-3)(x+3) & x<y \\ \frac{1}{9}(y+3)(2x-3) & x>y \end{cases}$$

HOW DO WE INCLUDE INHOMOGENEOUS BC'S?

WE WILL CONSIDER THE FOLLOWING PROBLEM:

$$lu = u'' = f(x) \qquad x \in [0,1]$$

$$BC: \begin{cases} u(0) - 3u'(0) = 1 \\ 2u(1) + u'(1) = -2 \end{cases}$$

WE ALREADY FOUND THE GREEN'S FN OF THIS

NOTES ON GREEN'S FUNCTIONS

ODE WITH HOMOGENEOUS BC'S, ABOVE. THE EASIEST WAY TO SOLVE THIS PROBLEM IS AS FOLLOWS:

$$u = u_1 + u_2$$

$$\begin{cases} \ell u_1 = f(x) \\ BC[u_1] = 0 \end{cases} \Rightarrow u_1(x) = \int_0^1 f(y)\, G(x,y)\, dy$$

SEE P11

$$\begin{cases} \ell u_2 = 0 \\ BC[u_2] \doteq \begin{cases} u_2(0) - 3u_2'(0) = 1 \\ 2u_2(1) + u_2'(1) = -2 \end{cases} \end{cases}$$

i.e. u_2 IS THE SOLUTION OF HOMOG. D.E. $\ell u_2 = 0$ WITH NON-HOMO. BC'S. IN THIS CASE

$$\ell u_2 = u_2'' = 0 \text{ HAS THE SOLUTION}$$

$$u_2 = Ax + B$$

FROM THE INHOMO. BC'S, WE GET

$$\begin{cases} B - 3A = 1 \\ 2(A+B) + A = -2 \Rightarrow 3A + 2B = -2 \end{cases}$$

$$\Rightarrow A = -\frac{4}{9}, \quad B = -\frac{1}{3}$$

$$\therefore \quad u_2(x) = -\frac{4}{9}x - \frac{1}{3}$$

THIS METHOD WORKS WELL IN PRINCIPLE. SOMETIMES FINDING u_2 MAY BE INCONVENIENT, e.g., WHEN WE WORK WITH PDE'S. THE FOLLOWING METHOD IS MORE ELEGANT BUT MUCH MORE DIFFICULT TO PROVE AND UNDERSTAND. THE IDEA IS THAT $G(x,y)$ ITSELF CAN HELP US FIND u_2 ITSELF. WE DEMONSTRATE THE ALGEBRA BY FIRST SHOWING THAT THE METHOD GIVES WHAT WE ALREADY KNOW WHEN WE HAVE HOMOGENEOUS BC'S. CONSIDER THE LIN. OP.:

$$\boxed{\ell = \frac{d^2}{dx^2}}$$

$$\underbrace{Lu}_{\text{ABOVE EXAMPLE}}: \begin{cases} \ell u = f(x) & x \in [0,1], \text{ LIN. ODE} \\ BC[u] = 0 & \text{LIN. \& HOMO. BC'S} \end{cases}$$

SUPPOSE WE FIND THE GREEN'S FUNCTION $G(x,y)$,

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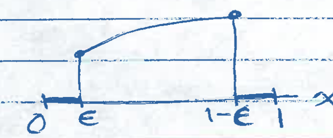
NOTES ON GREEN'S FUNCTIONS

i.e. $u(x) = \int_0^1 f(y)\, G(x,y)\, dy$. WE USE THE MACHINERY OF GENERALIZED FUNCTIONS. I WILL GIVE THE DETAILED DERIVATION FIRST AND THEN GIVE THE ALGEBRA WITHOUT GOING THROUGH THE LIMITING PROCESS OF THE DETAILED DERIVATION. NOTE THAT THE METHOD IS ALSO APPLICABLE TO PDES AND IS A VERY USEFUL TECHNIQUE.

DEFINE u_ϵ AND f_ϵ AS FOLLOWS

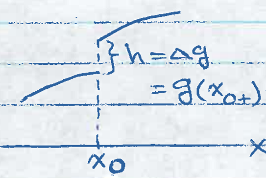


$$u_\epsilon = \begin{cases} u(x) & x \in [\epsilon, 1-\epsilon] \\ 0 & x \in [0,\epsilon) \cup (1-\epsilon, 1] \end{cases}$$

$$f_\epsilon = \begin{cases} f(x) & x \in [\epsilon, 1-\epsilon] \\ 0 & x \in [0,\epsilon) \cup (1-\epsilon, 1] \end{cases} \Rightarrow \boxed{u''_\epsilon = -f_\epsilon(x)}$$

NOW USING THE RULES OF GENERALIZED DIFFERENTIATION, WE FIND $\bar{L} u_\epsilon$, WHERE $\bar{L} = \frac{\bar{d}^2}{dx^2}$ (GENERALIZED 2ND DERIVATIVE).

REMINDER: GENERALIZED DERIVATIVE OF A PIECEWISE SMOOTH FUNCTION $g(x)$ WITH A DISCONTINUITY Δg AT $x = x_0$



$$\frac{\bar{d}g}{dx} \equiv \left.\frac{dg}{dx}\right|_{gen.} = \frac{dg}{dx} + \Delta g\, \delta(x - x_0)$$

ALSO

$$\frac{\bar{d}^2 g}{dx^2} \equiv \left.\frac{d^2 g}{dx^2}\right|_{gen} = \frac{d^2 g}{dx^2} + \Delta\left(\frac{dg}{dx}\right)\delta(x - x_0) + \Delta g\, \delta'(x - x_0)$$

WHERE NOW

$$\Delta\left(\frac{dg}{dx}\right) = \frac{dg}{dx}(x_{0+}) - \frac{dg}{dx}(x_{0-})$$

AND $g(x)$ IS A PIECEWISE C^2 FUNCTION.

(END OF REMINDER)

NOTES ON GREEN'S FUNCTIONS

NOW, WE HAVE

$$\frac{\bar{d}u_\epsilon}{dx} = \frac{du_\epsilon}{dx} + u(\epsilon)\delta(x-\epsilon) - u(1-\epsilon)\delta(x-1+\epsilon)$$

$$\frac{\bar{d}^2u_\epsilon}{dx^2} = \underbrace{\frac{d^2u_\epsilon}{dx^2}}_{f_\epsilon(x)} + u'(\epsilon)\delta(x-\epsilon) - u'(1-\epsilon)\delta(x-1+\epsilon) + u(\epsilon)\delta'(x-\epsilon) - u(1-\epsilon)\delta'(x-1+\epsilon)$$

$$\begin{aligned}
u_\epsilon(x) = & \int_0^1 f_\epsilon(y)\,G(x,y)\,dy \\
& + u'(\epsilon)\int_0^1 \delta(y-\epsilon)G(x,y)\,dy \\
& - u'(1-\epsilon)\int_0^1 \delta(y-1+\epsilon)\,G(x,y)\,dy \\
& + u(\epsilon)\int_0^1 \delta'(y-\epsilon)\,G(x,y)\,dy \\
& - u(1-\epsilon)\int_0^1 \delta'(y-1+\epsilon)\,G(x,y)\,dy \\
= & \int_\epsilon^{1-\epsilon} f(y)\,G(x,y)\,dy \\
& + u'(\epsilon)\,G(x,\epsilon) - u'(1-\epsilon)\,G(x,1-\epsilon) \\
& - u(\epsilon)\frac{\partial G}{\partial y}(x,\epsilon) + u(1-\epsilon)\frac{\partial G}{\partial y}(x,1-\epsilon)
\end{aligned}$$

NOW LET $\epsilon \to 0 \Rightarrow u_\epsilon \to u$ AND

$$u(x) = \lim_{\epsilon\to 0} u_\epsilon(x) = \int_0^1 f(y)\,G(x,y)\,dy$$

$$E\begin{cases} + u'(0)\,G(x,0) - u'(1)\,G(x,1) \\ - u(0)\frac{\partial G}{\partial y}(x,0) + u(1)\frac{\partial G}{\partial y}(x,1) \end{cases}$$

BUT FROM THE BC: $u(0) = 3u'(0)$
$u'(1) = -2u(1)$

$$E = u'(0)\left[G(x,0) - 3\frac{\partial G}{\partial y}(x,0)\right] + u(1)\left[2G(x,1) + \frac{\partial G}{\partial y}(x,1)\right] = 0\,!$$

NOTES ON GREEN'S FUNCTION

THE REASON IS THAT WE HAVE A SELF-ADJOINT OPERATOR SO THAT $BC^* = BC$ AND $G(x,y)$ IN VARIABLE y SATISFIES BC^* AND, THEREFORE, BC:

$$-G(x,0) - 3\frac{\partial G}{\partial y}(x,0) = 0$$
$$2G(x,1) + \frac{\partial G}{\partial y}(x,1) = 0$$

BASED ON THE ABOVE DETAILED ALGEBRA, WE PROPOSE THE FOLLOWING ALGEBRA WHICH IS CORRECT BUT LOOKS AS AD HOC:

JUST ASSUME $u \equiv 0$ WHEN $x \notin [0,1]$ AND APPLY THE RULES OF GENERALIZED DIFFERENTIATION:

$$\frac{\overline{du}}{dx} = \frac{du}{dx} + u(0)\,\delta(x) - u(1)\,\delta(x-1)$$

$$\frac{\overline{d^2u}}{dx^2} = \frac{d^2u}{dx^2} + u'(0)\,\delta(x) - u'(1)\,\delta(x-1) + u(0)\,\delta'(x) - u(1)\,\delta'(x-1)$$
$$= f(x) + u'(0)\,\delta(x) - u'(1)\,\delta(x-1) + u(0)\,\delta'(x) - u(1)\,\delta'(x-1)$$

$$u(x) = \int_0^1 f(y)\,G(x,y)\,dy + u'(0)\,G(x,0) - u'(1)\,G(x,1) - u(0)\frac{\partial G}{\partial y}(x,0) + u(1)\frac{\partial G}{\partial y}(x,1)$$
$$= \int_0^1 f(y)\,G(x,y)\,dy$$

THIS IS NOW SIMILAR TO THE RESULT WE OBTAINED FROM $u(x) = \lim_{\epsilon \to 0} u_\epsilon(x)$ IN PREVIOUS PAGE.

WE MENTION HERE THAT IF $x \notin [0,1]$, THE VALUE OF $u(x) \neq 0$, AND IN FACT, IS NOT EVEN DISCONTINUOUS AT $x=0$ AND $x=1$. HOWEVER, THE ALGEBRA USING THE LIMIT PROCESS IS CORRECT AND RIGOROUS.

WE NOW ADDRESS THE INHOMOGENEOUS BC'S. WE ONLY PRESENT THE AD HOC METHOD BUT YOU SHOULD BE ABLE TO EXPLAIN WHAT WE ARE DOING USING $u(x) = \lim_{\epsilon \to 0} u_\epsilon(x)$ AS ON P13.

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NOTES ON GREEN'S FUNCTIONS

WE WILL ONCE AGAIN SOLVE THE FOLLOWING PROBLEM :

$$\begin{cases} \ell u = u'' = f(x) & x \in [0,1] \\ \begin{cases} u(0) - 3u'(0) = 1 \\ 2u(1) + u'(1) = -2 \end{cases} & \left\{ \begin{array}{l} \text{LIN. INHOMO.} \\ \text{BC'S} \end{array} \right. \end{cases}$$

THE GREEN'S FUNCTION FOR THE LIN. OP.

$$Lu : \begin{cases} \ell u = u'' & x \in [0,1] \\ u(0) - 3u'(0) = 0 \\ 2u(1) + u'(1) = 0 \end{cases}$$

IS

$$G(x,y) = \begin{cases} \frac{1}{9}(2y-3)(x+3) & x < y \\ \frac{1}{9}(y+3)(2x-3) & x > y \end{cases}$$

FOR TAKING GEN. DERIVATIVES OF $u(x)$, ASSUME $u \equiv 0$ WHEN $x \notin [0,1]$, AS BEFORE.

$$\ell u = f + u'(0)\delta(y) - u'(1)\delta(y-1) + u(0)\delta'(y) - u(1)\delta'(y-1)$$

$$\tilde{u}(x) = \int_0^1 f(y)\, G(x,y)\, dy + u'(0)\, G(x,0) - u'(1)\, G(x,1) - u(0)\frac{\partial G}{\partial y}(x,0) + u(1)\frac{\partial G}{\partial y}(x,1)$$

AGAIN $G(x,y)$ SATISFIES THE ADJOINT BC WHICH IS THE SAME AS THE BC

$$\begin{cases} G(x,0) = 3\frac{\partial G}{\partial y}(x,0) \\ \frac{\partial G}{\partial y}(x,1) = -2\,G(x,1) \end{cases}$$ (SEE P 15)

BUT NOW

$$u(0) = 3u'(0) + 1$$
$$u'(1) = -2u(1) - 2$$

NOTE!

SUBSTITUTING THESE RESULTS IN $\tilde{u}(x)$, ABOVE, WE GET

NOTES ON GREEN'S FUNCTION (19)

$$u(x) = \int_0^1 f(y)\,G(x,y)\,dy + 3u'(0)\underbrace{\left[\frac{\partial G}{\partial y}(x,0) - \frac{\partial G}{\partial y}(x,0)\right]}_{=0}$$

$$-2u'(1)\,[\underbrace{G(x,1) - G(x,1)}_{0}]$$

$$-\frac{\partial G}{\partial y}(x,0) + 2\,G(x,1)$$

$$\left[\frac{\partial G_2}{\partial y}(x,0) \qquad\qquad G_1(x,1)\right.$$

$$= \int_0^1 f(y)\,G(x,y)\,dy - \frac{1}{9}(2x-3) - \frac{2}{9}(x+3)$$

$$= \int_0^1 f(y)\,G(x,y)\,dy - \frac{4}{9}x - \frac{1}{3} \quad \text{O.K.!}$$

AGAIN NOTICE THAT $u(x)$ IS NOT DISCONTINUOUS AT $x=0$ AND $x=1$. TO SEE THIS LET $f(y) = \sin y$, USE MATHEMATICA AND PLOT $u(x)$. BASICALLY, TO USE THE GREEN'S FUNCTION SOLUTION OUTSIDE THE RANGE WHERE IT WAS DERIVED IS MEANINGLESS! HOWEVER, SINCE WE SEEM TO USE THE ALGEBRA OF GENERALIZED FUNCTIONS AFTER WE ASSUMED $u \equiv 0$ WHEN $x \notin [0,1]$, THIS QUESTION OF THE MEANING OF u OUTSIDE THE RANGE OF $[0,1]$ ARISES. THE ANSWER IS THAT THE METHOD IS A SHORTCUT TO GETTING THE RIGHT ANSWER BASED ON THE LIMITING PROCESS $u(x) = \lim_{\epsilon\to 0} u_\epsilon(x)$, P.13, AND AS SUCH IT IS AD HOC!

OTHER REMARKS

i) $f(x)$ IN $Lu = f(x)$ CAN BE DISCONTINUOUS. IN FACT, WE USED THIS FACT IN DEFINING $u_\epsilon(x)$ BY TAKING

$$f_\epsilon(x) = \begin{cases} f(x) & x \in [\epsilon, 1-\epsilon], \ \epsilon > 0 \\ 0 & x \in [0,\epsilon) \cup (1-\epsilon, 1] \end{cases}$$

IT IS A GOOD PROBLEM TO CHECK FOR YOURSELF THAT ACTUALLY $f(x)$ CAN BE DISCONTINUOUS.

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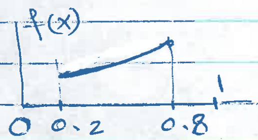
NOTES ON GREEN'S FUNCTIONS.

$$\begin{cases} u'' = f(x) \\ u(0) - 3u'(0) = 0 \\ 2u(1) + u'(1) = 0 \end{cases}$$

$$G(x,y) = \begin{cases} \frac{1}{9}(2y-3)(x+3) & x<y \\ \frac{1}{9}(y+3)(2x-3) & x>y \end{cases}$$

$$f(x) = \begin{cases} e^x & x \in [0.2, 0.8] \\ 0 & \text{OTHERWISE} \end{cases}$$



$$u(x) = \int_0^x \frac{1}{9}(y+3)(2x-3)f(y)\,dy + \int_x^1 \frac{1}{9}(2y-3)(x+3)f(y)\,dy$$

i) $x \in [0, 0.2]$

$$u(x) = \left[\int_{0.2}^{0.8} (2y-3)e^y\,dy\right]\frac{x+3}{9}$$

$$= -0.216487\,(x+3)$$

ii) $x \in [0.2, 0.8]$

$$u(x) = \frac{2x-3}{9}\int_{0.2}^{x} (y+3)e^y\,dy + \frac{x+3}{9}\int_x^{0.8} (2y-3)e^y\,dy$$

$$= -1.62658 + e^x - 1.43789\,x$$

iii) $x \in [0.8, 1]$

$$u(x) = \int_{0.2}^{0.8} \frac{1}{9}(y+3)(2x-3)e^y\,dy$$

$$= \frac{2x-3}{9}\int_{0.2}^{0.8} (y+3)e^y\,dy$$

$$= 0.551356\,(2x-3)$$

NOW SEE THE MATHEMATICA OUTPUT, NEXT PAGES

Example of Discontinuous Forcing Function f(x)

We are studying the differential equation u'' = f (x), where
f (x) = e^x if x ϵ [0.2, 0.8] and
f (x) = 0, otherwise.

The BCs are u (0) - 3 u' (0) = 0
2 u (1) + u' (1) = 0.

The **Green's function** for this problem is:
$G(x, y) = \frac{1}{9}(2y-3)(x+3)$ if x < y and
$= \frac{1}{9}(y+3)(2x-3)$ if x > y

```
f[x_] := Piecewise[{{0, x < 0.2}, {e^x, 0.2 < x < 0.8}, {0, 0.8 < x < 1}}]
```

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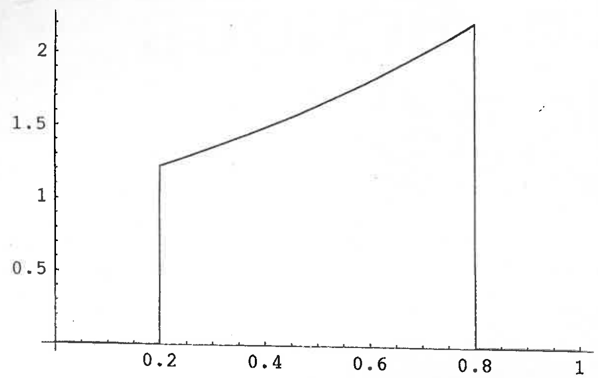
```
Plot[f[x], {x, 0, 1}]
```



- Graphics -

$$\frac{x+3}{9}\int_{0.2}^{0.8}(2\,y-3)\,e^{y}\,dy$$

$-0.216487\,(3+x)$

$$\frac{x+3}{9}\int_{0.2}^{0.8}(2\,y-3)\,e^{y}\,dy\ /.\ x\to 0.2$$

-0.69276

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$$\text{Chop}\left[\text{Simplify}\left[\frac{2x-3}{9}\int_{0.2}^{x}(y+3)\,e^{y}\,dy+\frac{x+3}{9}\int_{x}^{0.8}(2y-3)\,e^{y}\,dy\right]\right]$$

$$-1.62658+1.\,e^{x}-1.43789\,x$$

$$\text{Chop}\left[\text{Simplify}\left[\frac{2x-3}{9}\int_{0.2}^{x}(y+3)\,e^{y}\,dy+\frac{x+3}{9}\int_{x}^{0.8}(2y-3)\,e^{y}\,dy\right]\right]\;/.\;x\to 0.2$$

$$-0.69276$$

$$\text{Chop}\left[\text{Simplify}\left[\frac{2x-3}{9}\int_{0.2}^{x}(y+3)\,e^{y}\,dy+\frac{x+3}{9}\int_{x}^{0.8}(2y-3)\,e^{y}\,dy\right]\right]\;/.\;x\to 0.8$$

$$-0.551356$$

$$\frac{2x-3}{9}\int_{0.2}^{0.8}(y+3)\,e^{y}\,dy$$

$$0.393825\,(-3+2x)$$

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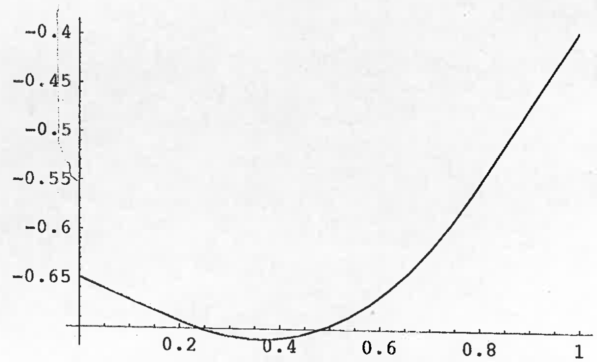
$$\frac{2x-3}{9}\int_{0.2}^{0.8}(y+3)\,e^{y}\,dy \;/.\; x \to 0.8$$

-0.551356

```
u[x_] :=
 Piecewise[{{-0.216487 (x + 3), x < 0.2},
   {-1.62658 + e^x - 1.43789 x, 0.2 < x < 0.8}, {0.393825 (2 x - 3), x > 0.:
```

```
Plot[u[x], {x, 0, 1}, PlotRange → All]
```



- Graphics -

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This is the plot of u (x) for a discontinuous function f (x).
It is seen that the function u (x) and its derivative are **continuous** so that $\bar{u}'' = u''$.

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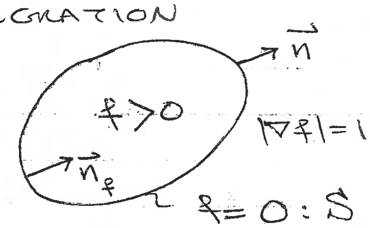
NOTES ON GREEN'S FUNCTION

THE ADJOINT OF A LINEAR DIFFERENTIAL OPERATOR

WE WILL WORK WITH 3-D SPACE HERE. WE DEFINE A SURFACE S IMPLICITLY AS FOLLOWS: $f(\vec{x}) = 0$. INSTEAD OF INTEGRATION BY PARTS, DIVERGENCE THM IS USED IN 3D. YOU MUST LEARN TO WORK WITH THIS RESULT WELL. OUR INNER PRODUCT IS DEFINED BY THE RELATION



$$\langle u, v \rangle = \int_{f>0} u\, v \, d\vec{x}$$

WE DEFINE $\vec{n}$ AS OUTWARD UNIT NORMAL. WE ALSO DEFINE $\vec{n}_f$ AS FOLLOWS:

$$\vec{n}_f = \frac{\nabla f}{|\nabla f|} = \nabla f$$

i.e. $\vec{n}_f$ IS THE UNIT NORMAL TO $f=0$ POINTING INTO THE REGION $f>0$. IN THE ABOVE FIGURE, $\vec{n}_f = -\vec{n}$ EVERYWHERE ON S. NOTE THAT f IS DEFINED SUCH THAT $|\nabla f| = 1$ ON S. THIS CAN ALWAYS BE DONE BECAUSE IF f DOES NOT SATISFY THIS CONDITION, THEN THE NEW IMPLICIT FUNCTION

$$\tilde{f}(\vec{x}) = \frac{f(\vec{x})}{|\nabla f(\vec{x})|}$$

IS SUCH THAT $|\nabla \tilde{f}| = 1$ ON S. PROVE THIS! THIS PROPERTY SIMPLIFIES ALGEBRA IN MANIPULATION

EXAMPLES

1. WE WILL FIRST FIND THE ADJOINT OF

$$L\phi : \begin{cases} l\phi = \nabla^2 \phi & \text{IN } f>0 \\ \phi + \alpha \dfrac{\partial \phi}{\partial n} = 0 & \text{ON } f=0 : S \end{cases}$$

NOTES ON GREEN'S FUNCTION

ii) WE DEFINE A FUNDAMENTAL SOLUTION OF AN ODE ℓu AS ANY SOLUTION $G_F(x,y)$ SUCH THAT

$$\ell_x G_F(x,y) = \delta(x-y)$$

WITHOUT ANY REGARD TO BC'S. IN ORDER TO FIND A FUNDAMENTAL SOLUTION OF A 2ND ORDER ODE:

$$\ell u = A(x)\,u'' + B(x)\,u' + C(x) \qquad x \in [0,1]$$

LET

$$G_F(x,y) = \begin{cases} G_1(x,y) & x<y \\ G_2(x,y) & x>y \end{cases}$$

$$\begin{array}{ccccc} & x<y & & x>y & \\ 0 & G_1(x,y) & y & G_2(x,y) & 1 \end{array}$$

WE MUST SATISFY $\ell G_1(x,y) = \ell G_2(x,y) = 0$

$$G_1(y,y) = G_2(y,y)$$

$$\frac{\partial G_2}{\partial x}(y,y) - \frac{\partial G_1}{\partial x}(y,y) = \frac{1}{A_1(y)}$$

FROM THE FACT THAT FOR THE GREEN'S FUNCTION OF THE ABOVE ODE, WE HAVE

$$L_x G(x,y) = L_y^* G(x,y) = \delta(x-y)$$

WE CAN CONCLUDE:

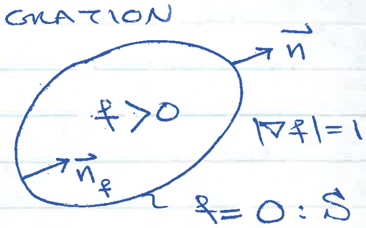
$$\boxed{G(x,y) = G_F(x,y) + \sum_{i,j=1}^{2} A_{ij}\,\phi_i(x)\,\psi_j(y)}$$

WHERE $\phi_i(x)$, $i=1,2$, ARE THE INDEPENDENT SOLUTIONS OF $\ell\,\phi_i(x) = 0$, AND $\psi_j(y)$, $i=1,2$, ARE THE SOLUTIONS OF $\ell^*\psi_j(y) = 0$. HERE A_{ij}, $i,j=1,2$ ARE JUST CONSTANTS. THE ABOVE RESULT EXTENDS DIRECTLY TO PDE'S. IT MEANS THAT ONCE WE FIND A FUNDAMENTAL SOLUTION OF AN ODE OR A PDE, WE ARE FAR CLOSER TO GETTING THE GREEN'S FUNCTION ITSELF.

NOTES ON GREEN'S FUNCTION

THE ADJOINT OF A LINEAR DIFFERENTIAL OPERATOR

WE WILL WORK WITH 3-D SPACE HERE. WE DEFINE A SURFACE S IMPLICITLY AS FOLLOWS: $f(\vec{x}) = 0$. INSTEAD OF INTEGRATION BY PARTS, DIVERGENCE THM IS USED IN 3D. YOU MUST LEARN TO WORK WITH THIS RESULT WELL. OUR



INNER PRODUCT IS DEFINED BY THE RELATION

$$\langle u, v \rangle = \int_{f>0} u\, v\, d\vec{x}$$

WE DEFINE $\vec{n}$ AS OUTWARD UNIT NORMAL. WE ALSO DEFINE $\vec{n}_f$ AS FOLLOWS:

$$\vec{n}_f = \frac{\nabla f}{|\nabla f|} = \nabla f$$

i.e. $\vec{n}_f$ IS THE UNIT NORMAL TO $f=0$ POINTING INTO THE REGION $f>0$. IN THE ABOVE FIGURE, $\vec{n}_f = -\vec{n}$ EVERYWHERE ON S. NOTE THAT f IS DEFINED SUCH THAT $|\nabla f| = 1$ ON S. THIS CAN ALWAYS BE DONE BECAUSE IF f DOES NOT SATISFY THIS CONDITION, THEN THE NEW IMPLICIT FUNCTION

$$\tilde{f}(\vec{x}) = \frac{f(\vec{x})}{|\nabla f(\vec{x})|}$$

IS SUCH THAT $|\nabla \tilde{f}| = 1$ ON S. PROVE THIS!

THIS PROPERTY SIMPLIFIES ALGEBRA IN MANIPULATIONS.

EXAMPLES

1. WE WILL FIRST FIND THE ADJOINT OF

$$L\phi : \begin{cases} l\phi = \nabla^2 \phi & \text{IN } f>0 \\ \phi + \alpha \dfrac{\partial \phi}{\partial n} = 0 & \text{ON } f=0 : S \end{cases}$$

$$\langle \nabla^2\phi, \psi \rangle = \langle \phi, \nabla^{2*}\psi \rangle$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i} \qquad \text{(SUM ON } i\text{)}$$

$$\equiv \phi_{ii}$$

$$\psi\phi_{ii} = (\psi\phi_i)_i - \psi_i\phi_i$$

$$= (\psi\phi_i - \psi_i\phi)_i + \phi\psi_{ii}$$

$$= \nabla\cdot[\psi\nabla\phi - \phi\nabla\psi) + \phi\underbrace{\nabla^2\psi}_{=\nabla^{2*}\psi}$$

$$\therefore \quad \nabla^{2*}\psi = \nabla^2\psi$$

$$\langle \nabla^2\phi, \psi \rangle = \langle \phi, \nabla^2\psi \rangle + \int_{f>0} \nabla\cdot[\psi\nabla\phi - \phi\nabla\psi]\, d\vec{x}$$

$$= \langle \phi, \nabla^2\psi \rangle + \int_{f=0} \left(\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right) dS$$

$$\phi = -\alpha\frac{\partial\phi}{\partial n}$$

$$= \langle \phi, \nabla^2\psi \rangle + \int_{f=0} \frac{\partial\phi}{\partial n}\underbrace{\left(\psi + \alpha\frac{\partial\psi}{\partial n}\right)}_{=0} dS$$

$$BC^* : \psi + \alpha\frac{\partial\psi}{\partial n} = 0$$

$$\therefore \quad L = L^* \quad : \quad \text{SELF-ADJ.}$$

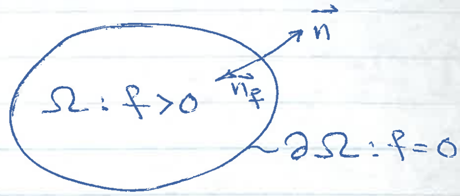
(2) BY EXACTLY THE SAME METHOD, WE CAN SHOW THAT FOR HELMHOLTZ OPERATOR

$$\begin{cases} \mathcal{H}\phi = \nabla^2\phi + k^2\phi & \text{IN } f>0, \ k \text{ REAL} \\ \phi + \alpha\frac{\partial\phi}{\partial n} = 0 & \text{ON } f=0 : S, \ \alpha \text{ REAL} \end{cases}$$

$$\left.\begin{cases} \mathcal{H}^*\psi = \nabla^2\psi + k^2\psi & \text{IN } f>0 \\ \psi + \alpha\frac{\partial\psi}{\partial n} = 0 & \text{ON } f=0 \end{cases}\right\} : \text{SELF-ADJOINT}$$

NOTES ON GREEN'S FUNCTIONS

③ $$\begin{cases} \nabla^2 \phi = g(\vec{x}) & x \in \Omega : f > 0 \\ \phi + \alpha \dfrac{\partial \phi}{\partial n} = \psi(\vec{x}) & \vec{x} \in \partial\Omega : f = 0 \end{cases}$$ INHOM. BC



$$\tilde{\phi} = \begin{cases} \phi & \vec{x} \in \Omega \\ 0 & \vec{x} \notin \Omega \end{cases}$$

$$\tilde{g}(\vec{x}) = \begin{cases} g & \vec{x} \in \Omega \\ 0 & \vec{x} \notin \Omega \end{cases}$$

$$\Longrightarrow \quad \nabla^2 \tilde{\phi} = \tilde{g}(x)$$

$$\langle \underbrace{\bar{\nabla}^2 G(\vec{x},\vec{y})}_{\delta(\vec{x}-\vec{y})}, \tilde{\phi}(\vec{y}) \rangle = \tilde{\phi}(\vec{x})$$

$$= \langle G(\vec{x},\vec{y}), \bar{\nabla}^2 \tilde{\phi}(\vec{y}) \rangle$$

$$\bar{\nabla} \tilde{\phi} = \nabla \tilde{\phi} + \phi(\partial\Omega) \vec{n}_f \, \delta(f)$$

$$\bar{\nabla}^2 \tilde{\phi} = \nabla^2 \tilde{\phi} + \nabla\phi(\partial\Omega) \cdot \vec{n}_f \, \delta(f) + \nabla \cdot [\phi(\partial\Omega) \vec{n}_f \delta(f)]$$

$$= \tilde{g}(\vec{x}) - \frac{\partial \phi}{\partial n} \delta(f) - \nabla \cdot [\phi \vec{n} \, \delta(f)]$$

$$G(\vec{x},\vec{y}) = -\frac{1}{4\pi r} + H(\vec{x},\vec{y}), \quad \nabla_x^2 H(\vec{x},\vec{y}) = 0$$

$$\int G(\vec{x},\vec{y}) \bar{\nabla}^2 \tilde{\phi} \, d\vec{y} = \int_{f>0} g(\vec{y}) G(\vec{x},\vec{y}) \, d\vec{y}$$

$$- \int \frac{\partial \phi}{\partial n}(\vec{y}) \delta(f) G(\vec{x},\vec{y}) \, d\vec{y} - \nabla_x \cdot \int \phi \, \vec{n} \, \delta(f) G(\vec{x},\vec{y}) \, d\vec{y}$$

$$\vec{n} \cdot \nabla_x G(\vec{x},\vec{y}) = \frac{\partial G}{\partial n_x} = -\frac{\partial G}{\partial n_y} \quad : G = G(\vec{x}-\vec{y})\,! \quad \Rightarrow H = H(\vec{x}-\vec{y})$$

$$G(\vec{x},\vec{y}) + \alpha \frac{\partial G}{\partial n_x} = 0 \qquad \vec{x} \text{ ON } \partial\Omega$$

$$G(\vec{x},\vec{y}) + \alpha \frac{\partial G}{\partial n_y} = 0 \qquad y \text{ ON } \partial\Omega$$

NOTES ON GREEN'S FUNCTIONS

$$-\int \phi_n(\vec{y})\,\delta(f)\left[-\alpha \frac{\partial G}{\partial n_y}\right] d\vec{y} + \int \phi(\vec{y})\,\delta(f)\,\frac{\partial G}{\partial n_y}\, d\vec{y}$$

$$= \int \underbrace{\left[\phi(\vec{y}) + \alpha\, \phi_n(\vec{y})\right]}_{\psi(\vec{y})} \frac{\partial G}{\partial n_y}\, \delta(f)\, d\vec{y}$$

$$= \int_{\partial\Omega} \psi(\vec{y})\, \frac{\partial G}{\partial n_y}\, dS$$

$$\therefore \quad \phi(\vec{x}) = \int_{\Omega} g(\vec{y})\, G(\vec{x},\vec{y})\, d\vec{y} + \int_{\partial\Omega} \psi(\vec{y})\, \frac{\partial G}{\partial n_y}(\vec{x},\vec{y})\, dS$$

WHY $G(\vec{x},\vec{y}) = G(\vec{x}-\vec{y})$?

i) $\nabla^2\phi$ IS INVARIANT UNDER TRANSLATION

$\vec{x}' = \vec{x} + \vec{a}$, $\vec{a}$ A CONST. VEC.

$\nabla^2\phi \longrightarrow \nabla'^2\phi'$

$\phi(\vec{x}) = \phi(\vec{x}' - \vec{a}) \equiv \phi'(\vec{x}')$

ii) $$\begin{cases} \nabla_x^2 G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y}) & \text{IN } \Omega \\ G + \dfrac{\partial G}{\partial n_x} = 0 & \text{ON } \partial\Omega \end{cases}$$

LET

$\vec{z} = \vec{x} - \vec{y} \implies$

$$\begin{cases} \nabla_z^2 G = \delta(\vec{z}) & \text{IN } \Omega \\ G + \dfrac{\partial G}{\partial n_z} = 0 & \text{ON } \partial\Omega \end{cases}$$

WE NOTE THAT IN THIS PROBLEM, THERE IS <u>NO</u> DEPENDENCE ON $\vec{y}$, i.e. $G = G(\vec{z}) = G(\vec{x}-\vec{y})$.

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NOTES ON GREEN'S FUNCTIONS

THE GREEN'S FUNCTION OF THE LAPLACE EQ. IN THE UNBOUNDED SPACE (3D)

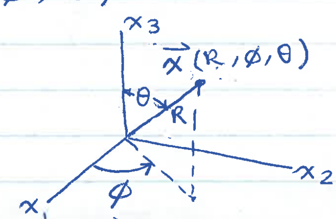
WE SET $\vec{y} = 0$, AND FIND $G(\vec{x})$. WE MUST FIND THE FUNCTION THAT SATISFIES

$$\nabla^2 G = \delta(\vec{x}) \qquad \vec{x} \in \mathbb{R}^3$$

SUBJECT TO THE SOME CONDITION AS $|\vec{x}| \to \infty$.

PRELIMINARIES : THE DELTA FUNCTION IN 3D SPHERICAL COORDINATES (R, ϕ, θ)

i) $\delta(\vec{x} - \vec{x}_0)$, $\vec{x}_0 \neq 0$



LET $\phi(\vec{x})$ BE A TEST FUNCTION

$$\Rightarrow \int_{\mathbb{R}^3} \phi(\vec{x})\, \delta(\vec{x} - \vec{x}_0) = \phi(\vec{x}_0)$$

LET $\phi[\vec{x}(R,\phi,\theta)] = \tilde{\phi}(R,\phi,\theta)$

$$\begin{cases} \vec{x} = (R, \phi, \theta) \\ \vec{x}_0 = (R_0, \phi_0, \theta_0) \end{cases}$$

IN SPHERICAL COORDINATES

$$\delta(\vec{x} - \vec{x}_0) = A\, \delta(R - R_0)\, \delta(\phi - \phi_0)\, \delta(\theta - \theta_0)$$

WHERE A IS UNKNOWN WHICH WE FIND BELOW.

$$\phi(\vec{x}_0) = \tilde{\phi}(R_0, \phi_0, \theta_0)$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{\infty} \tilde{\phi}(R,\phi,\theta)\, A\, \delta(R-R_0)\,\delta(\phi-\phi_0)\,\delta(\theta-\theta_0)\; R^2 \sin\theta \, dR\, d\phi\, d\theta$$

$$= \underbrace{A R_0^2 \sin\theta_0}_{=1}\, \tilde{\phi}(R_0, \phi_0, \theta_0)$$

$$A = \frac{1}{R_0^2 \sin\theta_0}$$

$$\boxed{\begin{aligned} \delta(\vec{x} - \vec{x}_0) &= \frac{1}{R^2 \sin\theta}\, \delta(R-R_0)\,\delta(\phi-\phi_0)\,\delta(\theta-\theta_0) \\ &= \frac{1}{R_0^2 \sin\theta_0}\, \delta(R-R_0)\,\delta(\phi-\phi_0)\,\delta(\theta-\theta_0) \end{aligned}}$$

THE LAST STEP IS BECAUSE $f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$

NOTES ON GREEN'S FUNCTION

ii) $\delta(\vec{x})$

THIS CASE SHOULD BE STUDIED SEPARATELY.
AGAIN, WE GUESS

$$\delta(\vec{x}) = A\,\delta(R)$$

WHERE A IS UNKNOWN. WITH THE NOTATIONS USED IN (i), WE HAVE

$$\phi(0) = \int_{\mathbb{R}^3} \phi(\vec{x})\,\delta(\vec{x})\,d\vec{x}$$

$$= \int_0^\infty \tilde{\phi}(R,\phi,\theta)\,A\,\delta(R)\cdot 4\pi R^2\,dR$$

$$A = \frac{1}{4\pi R^2}$$

$$\boxed{\delta(\vec{x}) = \frac{\delta(R)}{4\pi R^2}}$$

<u>DERIVATION OF THE GREEN'S FUNCTION</u>

NOW WE WORK IN SPHERICAL POLAR COORDINATES

$$\bar{\nabla}^2 G = \delta(\vec{x}) = \frac{\delta(R)}{4\pi R^2}$$

SINCE THE SOURCE IS SPHERICALLY SYMMETRIC, i.e. INDEPENDENT OF ϕ & θ VARIABLES $\Rightarrow G(\vec{x})$ DEPENDS ON R ONLY, i.e., $G = G(R)$. THIS MEANS THAT WE REALLY ARE DEALING WITH AN O.D.E.:

$$\bar{\nabla}^2 G = \frac{\bar{d}^2 G}{dR^2} + \frac{2}{R}\frac{\bar{d}G}{dR} = \frac{\delta(R)}{4\pi R^2}$$

$$4\pi\left(R^2\bar{G}'' + 2R\bar{G}'\right) = \delta(R) \qquad (*)$$

SINCE G HAS NO MEANING FOR $R<0$, WE ASSUME $G=0$ FOR $R<0$. WE NEED TO FIND THE SOLUTIONS OF THE HOMO. ODE

$$R^2 G'' + 2RG' = 0$$

(32)

NOTES ON GREEN'S FUNCTIONS

THE SOLUTION IS GUESSED AS AR^n WHERE n IS UNKNOWN. SUBSTITUTING THIS IN THE ODE, WE GET $n(n-1) + 2n = n(n+1) = 0 \Rightarrow$ $n=0$ OR $n=-1$. FOR $n=0$, WE GET $G = A = $ CONST. BECAUSE $G=0$ FOR $R<0$, THE CONTINUITY PROPERTY OF G AT $R=0$ CAN BE SATISFIED BUT TO GET THE DELTA FUNCTION ON THE RIGHT OF EQ. (*), PREVIOUS PAGE, WE MUST HAVE

$$\Delta\left[4\pi R^2 G'(0_+)\right] = 1$$

WHICH IS IMPOSSIBLE FOR $G=A$ SINCE $G'(0_+)=0$!

<u>DIGRESSION</u>: DISCUSSION OF THE INTERPRETATION OF EQ. (*), LAST PAGE

EQ. (*) MEANS THAT $G(R)$ HAS A FORM

$$G(R) = \begin{cases} G_1(R) & R>0 \\ 0 & R<0 \end{cases}$$

WITH SOME KIND OF DISCONTINUITY AT $R=0$. USING OUR NOTATION $\Delta G = G(0_+) - G(0_-)$ WE HAVE

$$\bar{G}'(R) = G'(R) + \Delta G\,\delta(R)$$

HOWEVER, WE MUST THINK ABOUT THE POSSIBILITY OF $G(R)$ BLOWING UP AT $R=0$ SO THAT ΔG MAY NOT BE DEFINED. ONE NICE WAY TO GET AROUND THIS DIFFICULTY IS THAT WE FIND $G_\varepsilon(R)$ AS THE SOLUTION OF

$$4\pi\left(R^2 \bar{G}_\varepsilon'' + 2R\,\bar{G}_\varepsilon'\right) = \delta(R-\varepsilon)$$

WHERE $\varepsilon>0$ IS A SMALL QUANTITY. WE ASSUME

$$G_\varepsilon(R) = \begin{cases} G_1(R) & R>\varepsilon \\ 0 & R<\varepsilon \end{cases} \Rightarrow G(R) = \lim_{\varepsilon\to 0} G_\varepsilon(R)$$

THEN $\quad \bar{G}_\varepsilon'(R) = G_\varepsilon' + G_1(\varepsilon)\,\delta(R-\varepsilon)$

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NOTES ON GREEN'S FUNCTION

$$\bar{G}_\varepsilon''(R) = G_\varepsilon''(R) + G_1'(\varepsilon)\,\delta(R-\varepsilon) + G_1(\varepsilon)\,\delta'(R-\varepsilon)$$

$$\therefore\quad R^2\bar{G}_\varepsilon'' + 2R\bar{G}_\varepsilon' = \underbrace{R^2 G_\varepsilon'' + 2RG_\varepsilon'}_{=0} + R^2 G_1'(\varepsilon)\,\delta(R-\varepsilon)$$

$$+\, G_1(\varepsilon)\underbrace{\left[R^2\delta'(R-\varepsilon) + 2R\,\delta(R-\varepsilon)\right]}_{(R^2\delta(R-\varepsilon))' = (\varepsilon^2\delta(R-\varepsilon))' = \varepsilon^2\delta'(R-\varepsilon)}$$

$$= \varepsilon^2 G_1'(\varepsilon)\,\delta(R-\varepsilon) + \varepsilon^2 G_1(\varepsilon)\,\delta'(R-\varepsilon)$$

$$\equiv \delta(R-\varepsilon)/4\pi$$

$$\Longrightarrow \lim_{\varepsilon\to 0} \varepsilon^2 G_1(\varepsilon) = 0$$ { WE SHOULD CHECK FOR THIS!

AND $$\lim_{\varepsilon\to 0} 4\pi\varepsilon^2 G_1'(\varepsilon) = 1$$

CONTINUATION OF THE DERIVATION OF THE GREEN'S FUNCTION

THE CASE $n=-1$ GIVES $G_1(R) = \dfrac{A}{R}$

$$\lim_{\varepsilon\to 0} \varepsilon^2\frac{A}{\varepsilon} = 0 \qquad \text{(CHECK!)}$$

$$\lim\ 4\pi\varepsilon^2\left(-\frac{A}{\varepsilon^2}\right) = -4\pi A = 1$$

$$A = -\frac{1}{4\pi}$$

$$\therefore\quad G(R) = \begin{cases} -\dfrac{1}{4\pi R} & R>0 \\ 0 & R<0 \end{cases}$$

OR SIMPLY $$\boxed{G(\vec{x}-\vec{y}) = -\frac{1}{4\pi|\vec{x}-\vec{y}|}}$$

REMINDER: $\nabla_x^2 G = \delta(\vec{x}-\vec{y})$ WE WILL CHECK THIS SOON.

NOTES ON GREEN'S FUNCTIONS

<u>SOLUTION OF LAPLAC'S EQ. IN UNBOUNDED DOMAIN</u>

$$\nabla^2 \phi = \psi(\vec{x}) \qquad \vec{x} \in \mathbb{R}^3$$

$$4\pi \phi(\vec{x}) = -\int_{\mathbb{R}^3} \frac{\psi(\vec{y})}{|\vec{x}-\vec{y}|} \, d\vec{y}$$

IF $\psi(\vec{y})$ IS DEFINED IN A BOUNDED REGION, THEN FOR ALL PRACTICAL $\psi(\vec{y})$, THE INTEGRAL IS CONVERGENT. IF $|\psi(\vec{y})| \approx A|\vec{y}|^n$ FOR LARGE $|\vec{y}|$, $|\vec{y}| > R_L$, THEN FOR $|\vec{x}| << R_L$

$$\left| \int_{|\vec{y}|>R_L} \frac{\psi(\vec{y})}{|\vec{x}-\vec{y}|} d\vec{y} \right| \leqslant \int_{|\vec{y}|>R_L} \frac{|\psi(\vec{y})|}{|\vec{x}-\vec{y}|} d\vec{y}$$

$$\approx \int_{R_L}^{\infty} 4\pi A R^2 \cdot R^{n-1} dR$$

$$= 4\pi A \frac{R^{n+2}}{n+2} \Big|_{R_L}^{\infty}$$

TO HAVE A CONVERGENT INTEGRAL $n+2<0$ OR $\boxed{n<-2}$. THIS IS EASY TO REMEMBER.

<u>CHECKING THAT $\nabla^2 G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y})$</u>

WE SHOULD SHOW THAT FOR THE PROBLEM

$$\nabla^2 \phi = \psi(\vec{x}) \qquad \vec{x} \in \mathbb{R}^3$$

$$-4\pi \psi(\vec{x}) \stackrel{?}{=} \nabla_x^2 \int_{\mathbb{R}^3} \frac{\psi(\vec{y})}{|\vec{x}-\vec{y}|} d\vec{y} = \int_{\mathbb{R}_3} \psi(\vec{y}) \nabla_x^2 \left(\frac{1}{|\vec{x}-\vec{y}|}\right) d\vec{y}$$

$$= \int_{V_\epsilon} \psi(\vec{y}) \nabla_x^2 \left(\frac{1}{|\vec{x}-\vec{y}|}\right) d\vec{y} + \int_{\mathbb{R}^3 \setminus V_\epsilon} \psi(\vec{y}) \nabla_x^2 \left(\frac{1}{|\vec{x}-\vec{y}|}\right) d\vec{y}$$

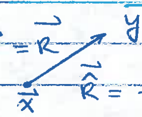
WHERE V_ϵ IS A SMALL SPHERE WITH CENTER AT $\vec{x}$, i.e. $R = |\vec{x}-\vec{y}| \leqslant \epsilon$ AND $\mathbb{R}^3 \setminus V_\epsilon$ IS THE REST OF

NOTES ON GREEN'S FUNCTIONS

THE $\mathbb{R}^3$ SPACE. IN $\mathbb{R}^3 \setminus V_\varepsilon$ WE CAN SHOW THAT $\nabla_x^2(1/|\vec{x}-\vec{y}|) = 0$, THEREFORE,

$$\int_{\mathbb{R}^3 \setminus V_\varepsilon} \psi(\vec{y})\, \nabla_x^2 \left(\frac{1}{|\vec{x}-\vec{y}|} \right) d\vec{y} = 0$$

$\vec{y} - \vec{x} = \vec{R}$



$\hat{R} = \dfrac{\vec{R}}{R}$

FOR THE FIRST INTEGRAL WE WRITE

$$\psi(\vec{y}) = \psi(\vec{x}) + \vec{R} \cdot \nabla_x \psi(\vec{x}) + O(R^2)$$

$$\vec{y} = \vec{x} + (\vec{y} - \vec{x}) = \vec{x} + \vec{R} \;, \text{ WHERE } \vec{R} = \vec{x} - \vec{y}$$

$$\int_{V_\varepsilon} \psi(\vec{y})\, \nabla_x^2 \left(\frac{1}{|\vec{R}|} \right) d\vec{y} = \psi(\vec{x}) \int_{V_\varepsilon} \nabla_y \cdot \nabla_y \left(\frac{1}{R} \right) d\vec{y}$$

$$+ \int_{V_\varepsilon} \vec{R} \cdot \nabla_x \psi(\vec{x})\, \nabla_y^2 \left(\frac{1}{R} \right) d\vec{y}$$

$$+ \; O(\varepsilon^2) \int_{V_\varepsilon} \nabla_y^2 \left(\frac{1}{R} \right) d\vec{y} \qquad (*)$$

$$\int_{V_\varepsilon} \nabla_y \cdot \nabla_y \left(\frac{1}{R} \right) d\vec{y} = \int_{\partial V_\varepsilon} - \frac{\vec{\hat{R}} \cdot \vec{\hat{R}}}{R^2}\, d\vec{S}$$

$$= -4\pi$$

LET $|\nabla \psi(\vec{x})| < M$ WHEN $\vec{x} \in V_\varepsilon \Rightarrow$

$$|\vec{R} \cdot \nabla_x \psi(\vec{x})| \leq |\vec{R}|\, |\nabla_x \psi(\vec{x})| < \varepsilon M$$

WE HAVE:

$$\left| \int_{V_\varepsilon} \vec{R} \cdot \nabla_x \psi(\vec{x})\, \nabla_y^2 \left(\frac{1}{R} \right) d\vec{y} \right| = \left| \int_{\partial V_\varepsilon} \vec{R} \cdot \nabla_x \psi(\vec{x})\, \frac{1}{R^2}\, dS \right|$$

$$\leq \varepsilon M \left| \int_{\partial V_\varepsilon} \frac{1}{R^2}\, dS \right| = 4\pi \varepsilon M \to 0 \text{ AS } \varepsilon \to 0$$

$\left| \int \nabla^2 \left(\frac{1}{R} \right) d\vec{S} \right| = 4\pi \;\therefore$ THE SECOND AND 3RD INTEGRAL IN EQ. (*) GO TO ZERO AS $\varepsilon \to 0$

$$\lim_{\varepsilon \to 0} \int_{V_\varepsilon} \psi(\vec{y})\, \nabla_x^2 \frac{1}{R}\, d\vec{y} = -4\pi \psi(\vec{x}) \qquad \text{CHECK!}$$

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NOTES ON GREEN'S FUNCTIONS

PDES INVARIANT UNDER TRANSLATION OF SPACE AND TIME ORIGINS

i) $\nabla^2\phi$ LAPLACE EQ.

ii) $\nabla^2\phi + k^2\phi$ HELMHOLTZ EQ.

iii) $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi$ THE WAVE EQUATION

iv) $\nabla^2\phi + k\frac{\partial\phi}{\partial t}$ THE HEAT EQ.

IN i) AND ii) $G = G(\vec{x}-\vec{y})$

IN iii) & iv) THE WAVE EQ. IS ALSO INVARIANT UNDER TRANSLATION IN TIME SO THAT

$$G = G(\vec{x}-\vec{y}, t-\tau)$$

WE WILL DISCUSS THIS PROBLEM IN DETAIL LATER. IT IS IMPORTANT TO KNOW THAT THE COMBINATION OF VARIABLES $\vec{x}$ & $\vec{y}$ INTO $\vec{x}-\vec{y}$, AND t & τ INTO $t-\tau$ RESULT IN A LOT OF SIMPLICITY IN DERIVING THE GREEN'S FUNCTIONS OF THE ABOVE PDES. FOR EXAMPLE, FOR LAPLACE'S EQUATION, WE FIND $G(\vec{x})$ BY PUTTING $\vec{y}=0$ IN $G(\vec{x}-\vec{y})$ AND SOLVE FOR $G(\vec{x})$. FOR WAVE EQ., WE TRY TO FIND $G(\vec{x},t)$ BY SETTING $\vec{y}=0$, $\tau=0$. AFTER FINDING $G(\vec{x})$ OR $G(\vec{x},t)$, WE SIMPLY REPLACE $\vec{x}$ BY $\vec{x}-\vec{y}$ AND t BY $t-\tau$ TO GET THE GREEN'S FUNCTION.

A FUNDAMENTAL SOLUTION OF A PDE

A FUNDAMENTAL SOLUTION OF A PDE $\mathcal{L}u$ IS ANY FUNCTION (OR GENERALIZED FUNCTION) $u(\vec{x})$ SUCH THAT $\mathcal{L}u = \delta(\vec{x}-\vec{y})$ WHERE $\mathcal{L}$ IS THE DIFF. EQ. WITH GENERALIZED DIFFERENTIATION. THE FUNDAMENTAL SOLUTION IS NOT UNIQUE SINCE WE CAN ADD ANY SOLUTION OF THE HOMO. PDE $\mathcal{L}u=0$

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NOTES ON GREEN'S FUNCTIONS

TO A FUND. SOL.: TO GET ANOTHER FUND. SOL.

IF WE FIND A FUNDAMENTAL SOLUTION, WE ARE A LONG WAY TOWARD SOLVING A BV PROBLEM.

CONSIDER THE PROBLEM

$$\begin{cases} \nabla^2 \phi = f(\vec{x}) & \text{IN } \Omega \\ \phi + \alpha \dfrac{\partial \phi}{\partial n} = 0 & \text{ON } \partial\Omega \end{cases}$$

LET $h(\vec{x}, \vec{y})$ BE A FUNDAMENTAL SOLUTION $\Rightarrow$ THE GREEN'S FUNCTION

$$G(\vec{x}, \vec{y}) = h(\vec{x}, \vec{y}) + k(\vec{x}, \vec{y})$$

WHERE $k(\vec{x}, \vec{y})$ IS A SOLUTION OF

$$\nabla_x^2 k(\vec{x}, \vec{y}) = 0 \qquad \vec{x} \in \Omega$$

AND $$G(\vec{x}, \vec{y}) + \alpha \frac{\partial G}{\partial n_x} = 0 \qquad \vec{x} \in \partial\Omega$$

$$\Rightarrow \quad k(\vec{x}, \vec{y}) + \alpha \frac{\partial k}{\partial n_x} = -h(\vec{x}, \vec{y}) - \alpha \frac{\partial h}{\partial n_x} \qquad \vec{x} \in \partial\Omega$$

IN THIS CASE THE PROBLEM IS EVEN SIMPLER BECAUSE $h = h(\vec{x} - \vec{y})$, $k = k(\vec{x} - \vec{y})$

NOTE: $\dfrac{\partial}{\partial n_x} = \vec{n} \cdot \nabla_x$

IF IT IS CONVINIENT TO FIND $G(\vec{x} - \vec{y})$ FOR $\vec{y} = \vec{y}_0$, THEN WHAT WE GET IS $G(\vec{x} - \vec{y}_0)$ SO THAT $G(\vec{x} - \vec{y}) = G[(\vec{x} - \vec{y} + \vec{y}_0) - \vec{y}_0]$, i.e., TO GET THE GREEN'S FUNCTION, REPLACE $\vec{x}$ BY $\vec{x} - \vec{y} + \vec{y}_0$.

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NOTES ON GREEN'S FUNCTION

THE GREEN'S FUNCTION OF HELMHOLTZ EQUATION IN UNBOUNDED SPACE – 3 D CASE

WE WILL USE SPHERICAL COORDINATES HERE ALSO WITH $\vec{y} = 0$

$$\nabla^2 G + k^2 G = \frac{\delta(R)}{4\pi R^2} \Rightarrow G = G(R)$$

$$4\pi R^2 \left(\frac{d^2 G}{dR^2} + \frac{2}{R}\frac{dG}{dR} + k^2 G \right) = \delta(R)$$

WE ASSUME A SOLUTION OF THE FORM

$$G(R) = A e^{-ikR} R^n \qquad n: \text{UNKNOWN}$$

TO FIND THE SOLUTION OF THE HOMOGENEOUS ODE

$$R^2 G'' + 2RG' + k^2 R^2 G = 0$$

WE MUST HAVE $(n+1) R^{n+1} + 2ik(n+1) R^n = 0$

$$\Rightarrow n+1 = 0 \text{ OR } \boxed{n = -1}$$

$$\therefore \quad G(R) = A \frac{e^{-ikR}}{R}$$

AGAIN, AS IN THE CASE OF THE LAPLACE EQ., TO FIND A, WE FIND $G_\varepsilon(R)$ IN THE FOLLOWING ODE

$$R^2 G_\varepsilon'' + 2RG_\varepsilon' + k^2 R^2 G_\varepsilon = \frac{\delta(R-\varepsilon)}{4\pi}$$

WITH $G_\varepsilon = 0$ FOR $R < \varepsilon$. WE CAN SHOW

$$-A\cancel{R^2} \cdot \frac{e^{-ikR}}{\cancel{R^2}} \delta(R-\varepsilon) = \frac{\delta(R-\varepsilon)}{4\pi}$$

THIS PART IS SIMILAR TO THE ALGEBRA ON P.27

$$-A e^{-ik\varepsilon} \delta(R-\varepsilon) = \frac{\delta(R-\varepsilon)}{4\pi}$$

$$A = -\frac{e^{ik\varepsilon}}{4\pi}$$

$$G_\varepsilon = -\frac{e^{ik\varepsilon}}{4\pi R} e^{-ikR} \qquad R > \varepsilon$$

$$\boxed{G(R) = -\frac{e^{-ikR}}{4\pi R}}$$

WE HAVE LET $\varepsilon \to 0$ HERE TO GET $G(R)$.

NOTE: $R = |\vec{x} - \vec{y}|$

NOTES ON GREEN'S FUNCTION

REMARKS ON THE DERIVATION

i) WE HAVE ASSUMED $G(R) = A e^{-ikR} R^n$ IN OUR DERIVATION. WE COULD HAVE ASSUMED $G(R) = A e^{ikR} R^n$. OUR CHOICE OF THE SIGN OF THE EXPONENTIAL HAS SOMETHING TO DO WITH OUR CHOICE OF TIME DEPENDENCE OF THE FORM $e^{i\omega t}$ WHEN WE GET THE HELMHOLTZ EQ. FROM THE WAVE EQUATION AND WE WANT OUTGOING WAVES.

ii) WHY DID WE SUDDENLY GET COMPLEX NUMBER e^{ikR} IN OUR GREEN'S FUNCTION? HAVE WE DONE SOMETHING WRONG? STRICTLY SPEAKING YES! WE HAVE

$$-\frac{e^{ikR}}{4\pi R} = \underbrace{-\frac{\cos kR}{4\pi R}}_{G_1(R)} \underbrace{- i\frac{\sin kR}{4\pi R}}_{i\,G_2(R)}$$

WE CAN EASILY SHOW THAT

$$\begin{cases} \nabla_x^2 G_1(R) + k^2 G_1(R) = \delta(R) \\ \nabla_x^2 G_2(R) + k^2 G_2(R) = 0\ ! \end{cases} \quad \text{SHOW THIS}$$

THEREFORE, IF $\psi(\vec{x})$ IS REAL, THEN THE SOLUTION OF

$$\nabla^2\phi + k^2\phi = \psi \qquad \vec{x} \in \mathbb{R}^3$$

WILL BE

$$\phi(\vec{x}) = -\int_{\mathbb{R}^3} \psi(\vec{y})\, G_1(\vec{x}-\vec{y})\, d\vec{y} + i\int_{\mathbb{R}^3} \psi(\vec{y})\, G_2(\vec{x}-\vec{y})\, d\vec{y}$$

WHICH MEANS THAT WE MUST IGNORE THE IMAGINARY PART OF THE ANSWER. BUT WHAT IS THE MEANING OF THE IMAGINARY PART AND HAVE WE, IN SOME

NOTES ON GREEN'S FUNCTIONS

SENSE, FOUND THE CORRECT SOLUTION? LET US STUDY

$$I(\vec{x}) = \int_{\mathbb{R}^3} \psi(\vec{y})\, G_2(\vec{x}-\vec{y})\, d\vec{y}$$

WE HAVE $\nabla^2 I + k^2 I(\vec{x}) = \int_{\mathbb{R}^3} \psi(\vec{y}) \underbrace{\nabla_x^2 G_2}_{=0!} d\vec{y} = 0\,!$

i.e., $I(\vec{x})$ IS A SOLUTION OF THE HOMOGENEOUS PROBLEM $\nabla^2\phi + k^2\phi = 0$. THIS MEANS THAT THE $\phi(\vec{x})$ WE FOUND USING THE GREEN'S FUNCTION

$$G(R) = -\frac{e^{-ikR}}{4\pi R}$$

IS THE CORRECT SOLUTION IN THE <u>UNBOUNDED</u> SPACE SINCE THE SOLUTION IS NOT UNIQUE.

iii) FOR A <u>BOUNDED</u> DOMAIN $G(R)$ IS ONLY A FUNDAMENTAL SOLUTION AND THE GREEN'S FUNCTION $G_B(R)$ IS

$$G_B(R) = G(R) + H(R) \left\{ \begin{array}{l} \text{HERE } H(R) \text{ IS A} \\ \text{SOLUTION OF THE} \\ \text{HOMO. EQ. } \mathcal{H}\phi = 0 \end{array} \right.$$

WHERE

$$\mathcal{H}_x G_B(R) = \mathcal{H}_x[G_1(R) + iG_2(R)] + \mathcal{H}_x H(R)$$

$$= \overbrace{\mathcal{H}_x G_1(R)}^{\delta(R)} + \overbrace{\mathcal{H}_x G_2(R)}^{0} + \overbrace{\mathcal{H}_x H(R)}^{0}$$

HERE $\mathcal{H} = \nabla^2 + k^2$ HELMHOLTZ OPERATOR

$$\therefore \mathcal{H}_x G_B(R) = \delta(R)$$

i.e., THE USE OF $G_2(R)$ IN $G(R)$ DOES <u>NOT</u> MATTER.

NOTES ON GREEN'S FUNCTION

GREEN'S FUNCTION OF THE WAVE EQUATION IN THE UNBOUNDED $\mathbb{R}^3$ SPACE

$$\Box^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi$$

$$G(\vec{x}, t; \vec{y}, \tau) = G(\vec{x}-\vec{y}, t-\tau)$$

WE LET $\vec{y} = 0$, $t = 0$ IN G ...

$$G = G(\vec{x}, t)$$

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \delta(\vec{x})\,\delta(t)$$

WE USE FOURIER TRANSFORM IN t AS FOLLOWS

$$\hat{\phi}(f) = \int_{-\infty}^{\infty} \phi(t)\, e^{-2\pi i f t}\, dt$$

$$\phi(t) = \int_{-\infty}^{\infty} \hat{\phi}(f)\, e^{2\pi i f t}\, df$$

WE HAVE

$$-\nabla^2 \hat{G} - k^2 \hat{G} = \delta(\vec{x})$$

WHERE $k = \frac{2\pi f}{c}$. NOTE THAT $\hat{\delta}(t) = 1$ BY OUR DEFINITION. WE, THEREFORE, HAVE

$$\hat{G}(\vec{x}, f) = \frac{e^{-ik|\vec{x}|}}{4\pi|\vec{x}|}$$

i.e. $\hat{G}$ = −(GREEN'S FN OF HELMHOLTZ EQ.)

$$G(\vec{x}, t) = \frac{1}{4\pi|\vec{x}|}\int_{-\infty}^{\infty} e^{-i2\pi f\left(\frac{|\vec{x}|}{c} - t\right)}\, df$$

$$= \frac{1}{4\pi|\vec{x}|}\,\delta\left(\frac{|\vec{x}|}{c} - t\right)$$

$$\text{or} \quad = \frac{1}{4\pi|\vec{x}|}\,\delta\left(t - \frac{|\vec{x}|}{c}\right)$$

NOTES ON GREEN'S FUNCTIONS

THEREFORE, THE GREEN'S FUNCTION OF THE WAVE EQ. IN THE UNBOUNDED DOMAIN IS

$$\boxed{\begin{aligned} G(\vec{x}-\vec{y}, t-\tau) &= \frac{\delta\left(\tau - t + \frac{|\vec{x}-\vec{y}|}{c}\right)}{4\pi|\vec{x}-\vec{y}|} \\ &= \frac{\delta\left(\tau - t + \frac{r}{c}\right)}{4\pi r} \end{aligned}}$$

WHERE $\vec{r} = \vec{x} - \vec{y}$, $r = |\vec{r}|$. THIS IS THE FORM I AM USED TO THINK ABOUT THE GREEN'S FUNCTION OF THE WAVE EQ. I HAVE TAKEN THIS FROM THE 1969 PAPER OF FFOWCS WILLIAMS & HAWKINGS IN THE PHILOSOPHICAL TRANSACTIONS.

THE RETARDED TIME SOLUTION OF THE HOMOMOGE-NEOUS WAVE EQ. IN THE UNBOUNDED SPACE

$$\Box^2 \phi(\vec{x}, t) = \psi(\vec{x}, t)$$

$$4\pi\phi(\vec{x}, t) = \int_{-\infty}^{t}\int_{\mathbb{R}^3} \frac{\psi(\vec{y}, \tau)}{|\vec{x}-\vec{y}|}\delta\left(\tau - t + \frac{r}{c}\right) d\vec{y}\, d\tau$$

LET $g = \tau - t + r/c \implies \tau = g + t - \frac{r}{c}$

$$d\tau = dg$$

$$4\pi\phi(\vec{x}, t) = \int_{\mathbb{R}^3}\int_{g} \frac{\psi(\vec{y}, g + t - r/c)}{r}\delta(g)\, dg\, d\vec{y}$$

$$= \int_{\mathbb{R}^3} \frac{\psi(\vec{y}, t - r/c)}{r} d\vec{y}$$

THIS IS THE SIMPLEST SOLUTION OF THE WAVE EQUATION. IT IS VERY EASY TO INTERPRET PHYSIC-ALLY.

NOTES ON GREEN'S FUNCTIONS

GREEN'S FUNCTION OF THE HEAT OPERATOR

THE HEAT OPERATOR IS

$$\frac{\partial}{\partial t} - \nabla^2 \qquad \vec{x} \in \mathbb{R}^n,\ n \geq 1\ ,\quad t \in (0,\infty)$$

AND IS A PARABOLIC PDE. WE WILL CONSIDER THE FOLLOWING PROBLEM

$$\begin{cases} \dfrac{\partial \phi}{\partial t} - \nabla^2 \phi = 0 & \vec{x} \in \mathbb{R}^n\ ,\ n \geq 1 \\ \phi(\vec{x},0) = f(\vec{x}) & t \in (0,\infty) \end{cases}$$

STRICTLY SPEAKING, THIS PROBLEM DOES NOT FALL IN THE CATEGORY OF PREVIOUS PROBLEMS FOR WHICH WE FOUND THE GREEN'S FUNCTION. THE ABOVE PROBLEM IS AN INITIAL-VALUE PROBLEM. WE WILL SAY MORE ABOUT THIS LATER IN THIS SECTION. WE USE (GENERALIZED) FOURIER TRANSFORM IN $\mathbb{R}^n$ FOR THE SPACE VARIABLE ONLY.

$$\hat{\phi}(\vec{\xi}, t) = \text{F.T.}\,[\phi(\vec{x},t),\ \vec{x} \to \vec{\xi}\,]$$

$$\hat{\phi}(\vec{\xi},t) = \int_{\mathbb{R}^n} \phi(\vec{x},t)\, e^{\dot{\imath}\,\vec{x}\cdot\vec{\xi}}\, d\vec{x}$$

$\dot{\imath} = 2\pi i$ (MY INVENTION, USED SINCE 1970'S!)

NOTE THAT WITH THIS DEFINITION, THE INVERSE TRANSFORM IS

$$\phi(\vec{x},t) = \int_{\mathbb{R}^n} \hat{\phi}(\vec{\xi},t)\, e^{-\dot{\imath}\,\vec{x}\cdot\vec{\xi}}\, d\vec{\xi}$$

WE NOTE THAT $\widehat{\dfrac{\partial \phi}{\partial t}} = \dfrac{\partial \hat{\phi}}{\partial t}$

SHOW THIS!

$$\widehat{\nabla^2 \phi} = (-\dot{\imath}\,\xi_i)(-\dot{\imath}\,\xi_i)\,\hat{\phi} \qquad (\text{SUM ON } i)$$

$$= -4\pi^2 |\vec{\xi}|^2\, \hat{\phi}(\vec{\xi},t)$$

$$\widehat{\phi(\vec{x},0)} = \hat{\phi}(\vec{\xi},0) = \hat{f}(\vec{\xi})$$

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NOTES ON GREEN'S FUNCTIONS

THE HEAT EQ. + I.C. BECOMES THE FOLLOWING FIRST ORDER ODE WITH THE I.C.

$$\begin{cases} \dfrac{\partial \hat{\phi}}{\partial t} + 4\pi^2 |\vec{\xi}|^2 \hat{\phi} = 0 & t \in (0,\infty) \\ \hat{\phi}(\vec{\xi},0) = \hat{f}(\vec{\xi}) & \text{INITIAL CONDITION} \end{cases}$$

NOTE THAT NOW $\vec{\xi}$ IS JUST A PARAMETER OF THE PROBLEM. THIS IS A SIMPLE ODE! ITS SOLUTION IS

$$\hat{\phi}(\vec{\xi},t) = \hat{f}(\vec{\xi})\, e^{-4\pi^2 |\vec{\xi}|^2 t}$$

NOW IF WE CAN FIND A FUNCTION WHOSE FT IN SPACE IS THE EXPONENTIAL TERM, THEN WE HAVE A PRODUCT OF TWO FT'S FOR WHICH WE KNOW A FAMOUS THEOREM.

LET $\widehat{K(\vec{x},t)} = e^{-4\pi^2 |\vec{\xi}|^2 t}$, THEN

$$\hat{\phi}(\vec{\xi},t) = \hat{f}(\vec{\xi})\, \hat{K}(\vec{\xi},t)$$

$$\Rightarrow \quad \phi(\vec{x},t) = f(\vec{x}) * K(\vec{x},t)$$

$$= \int_{\mathbb{R}^n} f(\vec{y})\, K(\vec{x}-\vec{y},t)\, d\vec{y}$$

$$= \int_{\mathbb{R}^n} f(\vec{x}-\vec{y})\, K(\vec{y},t)\, d\vec{y}$$

THIS IS KNOWN AS <u>THE CONVOLUTION THEOREM</u> OF FOURIER TRANSFORM THEORY. THE FUNCTION $K(\vec{x},t)$ IS SIMPLY THE INVERSE FT OF $\hat{K}(\vec{\xi},t)$

$$K(\vec{x},t) = \int_{\mathbb{R}^n} e^{-4\pi^2 |\vec{\xi}|^2 t - i\vec{x}\cdot\vec{\xi}}\, d\vec{\xi}$$

NOTES ON GREEN'S FUNCTIONS

THIS INTEGRAL CAN BE INTEGRATED EASILY AS FOLLOWS

$$\int_{\mathbb{R}^n} e^{-4\pi^2|\vec{\xi}|^2 t - 2\pi i \vec{x}\cdot\vec{\xi}} d\vec{\xi} = \prod_{j=1}^{n} \int_{\mathbb{R}} e^{-4\pi^2 \xi_j^2 t - 2\pi x_j \xi_j} d\xi_j$$

$$= \frac{1}{(4\pi t)^{n/2}} e^{-|\vec{x}|^2/4t}$$

HERE $\prod_{j=1}^{n}$ IS PRODUCT NOTATION. THE ABOVE INTEGRAL CAN BE FOUND USING MATHEMATICA OR A TABLE OF INTEGRALS (IF YOU ARE OLD-FASHIONED!).

THE FUNCTION $K(\vec{x},t)$ IS CALLED THE <u>HEAT KERNEL</u>. LET US WRITE IT FOR $n=1$, 2 AND 3

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \qquad \text{IN } \mathbb{R}$$

$$K(\vec{x},t) = \frac{1}{4\pi t} e^{-\frac{|\vec{x}|^2}{4t}} \qquad \text{IN } \mathbb{R}^2$$

$$K(\vec{x},t) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|\vec{x}|^2}{4t}} \qquad \text{IN } \mathbb{R}^3$$

WE CALL $K(\vec{x},t)$ ALSO THE GREEN'S FUNCTION OF THE ABOVE I.V. PROBLEM.

WE ARE JUST ONE STEP AWAY FROM SOLVING THE FOLLOWING I.V. PROBLEM

$$\begin{cases} \dfrac{\partial\phi}{\partial t} - \nabla^2\phi = \psi(\vec{x},t) & \vec{x}\in\mathbb{R}^n \\ \phi(\vec{x},0) = f(\vec{x}) \quad \text{I.C.} & t\in(0,\infty) \end{cases}$$

USING $\hat{\phi}(\vec{\xi},t) = FT[\phi(\vec{x},t),\ \vec{x}\to\vec{\xi}]$, WE GET

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NOTES ON GREEN'S FUNCTION

$$\begin{cases} \dfrac{\partial \hat{\phi}}{\partial t} + 4\pi^2 |\vec{\xi}|^2 \hat{\phi} = \hat{\psi}(\vec{\xi}, t) \\ \hat{\phi}(\vec{\xi}, 0) = \hat{f}(\vec{\xi}) \end{cases}$$

THE SOLUTION OF THIS PROBLEM IS

$$\hat{\phi}(\vec{\xi}, t) = \hat{f}(\vec{\xi})\, e^{-4\pi^2 |\vec{\xi}|^2 t} + \int_0^t e^{-4\pi^2 |\vec{\xi}|^2 (t-\tau)} \hat{\psi}(\vec{\xi}, \tau)\, d\tau$$

$$= \hat{f}(\vec{\xi})\, \hat{K}(\vec{\xi}, t) + \int_0^t \hat{\psi}(\vec{\xi}, \tau)\, \hat{K}(\vec{\xi}, t-\tau)\, d\tau$$

$$\Rightarrow \phi(\vec{x}, t) = f(\vec{x}) * K(\vec{x}, \tau) + \int_0^t \psi(\vec{x}, \tau) * K(\vec{x}, t-\tau)\, d\tau$$

$$= \int_{\mathbb{R}^n} f(\vec{y})\, K(\vec{x}-\vec{y}, t)\, d\vec{y} + \int_0^t \int_{\mathbb{R}^n} \psi(\vec{y}, \tau)\, K(\vec{x}-\vec{y}, t-\tau)\, d\vec{y}\, d\tau$$

WE SHOULD CALL $K(\vec{x}-\vec{y}, t-\tau)$ THE GREEN'S FUNCTION OF THIS PROBLEM. WE MANAGED TO GET THE GREEN'S FUNCTION FOR ALL $n \geq 1$!

<u>REMARKS</u>: 1- TO FIND THE GREEN'S FUNCTION FOR $\alpha \dfrac{\partial \phi}{\partial t} - \nabla^2 \phi = \psi$, REPLACE t BY αt IN $K(\vec{x}-\vec{y}, t-\tau)$ FOUND ABOVE.

NOTES ON GREEN'S FUNCTIONS

2 - GREEN'S FUNCTIONS OF LAPLACE EQ. IN n DIMENSIONS — UNBOUNDED SPACE $\mathbb{R}^n$

WE DERIVED THE GREEN'S FN OF ∇^2 IN $\mathbb{R}^3$. IT IS $-1/4\pi r$, $r = |\vec{x} - \vec{y}|$. WE CAN SHOW THAT

$$G(\vec{x}, \vec{y}) = -\frac{1}{2\pi} \ln r \qquad n = 2$$

$$= \frac{-1}{(n-2)\omega_n} \frac{1}{r^{n-2}} \qquad n > 2$$

WHERE ω_n IS THE SURFACE AREA OF UNIT SPHERE IN n DIMENSION. IT CAN BE SHOWN THAT $\omega_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$, WHERE $\Gamma(\cdot)$ IS THE Γ- FUNCTION.

THE BEST PLACE TO READ ABOUT SURFACES AND VOLUMES IN n DIMENSIONS IS R. COURANT, DIFFERENTIAL & INTEGRAL CALCULUS, VOL. 2. A MORE MODERN VERSION OF THIS BOOK IS BY COURANT & JOHN. SOME, MYSELF INCLUDED, CONSIDER THIS BOOK AS THE BEST CALCULUS BOOK OF TWENTIETH CENTURY.

AS WE SAID BEFORE, THE ABOVE GREEN'S FNS ARE ALSO A FUNDAMENTAL SOLUTION IN THE RESPECTIVE SPACES.

NOTES ON GREEN'S FUNCTIONS

GREEN'S FUNCTION OF LAPLACE OPERATOR IN A BOUNDED DOMAIN

WE CONSIDER THE PROBLEM

$$\begin{cases} \nabla^2 \phi = \psi(\vec{x}) & \vec{x} \in D \\ \phi(\vec{x}) = g(\vec{x}) & \vec{x} \in \partial D \end{cases}$$

DIRICHLET B.C.

FIRST FROM EXISTENCE & UNIQUENESS THEOREM, WE KNOW THAT, THIS PROBLEM HAS A UNIQUE SOLUTION FOR ψ AND g SQUARE (LEBESGUE) INTEGRABLE. SEE ANY GOOD BOOK ON PDES.

TO SOLVE THIS PROBLEM, WE USE ...

GREEN'S IDENTITY

$$\int_D (v \nabla^2 u - u \nabla^2 v)\, d\vec{y} = \int_{\partial D} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) dS_y$$

LET $v = \phi$, $u = G(\vec{x}, \vec{y})$

$$\nabla_y^2 G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y})$$

$$\int \Big[\phi(\vec{y}) \underbrace{\nabla_y^2 G(\vec{x}, y)}_{\delta(\vec{x}-\vec{y})} - G(\vec{x}, \vec{y}) \underbrace{\nabla^2 \phi(\vec{y})}_{\psi(\vec{y})}\Big] d\vec{y}$$

$$= \phi(\vec{x}) - \int_D \psi(\vec{y})\, G(\vec{x}, \vec{y})\, d\vec{y}$$

$$= \int_{\partial D} \left[\phi(\vec{y}) \frac{\partial G}{\partial n_y}(\vec{x}, \vec{y}) - G(\vec{x}, \vec{y}) \frac{\partial \phi(\vec{y})}{\partial n_y}\right] dS_y$$

SINCE WE HAVE NO INFORMATION ON $\frac{\partial \phi}{\partial n_y}$, WE IMPOSE THE FOLLOWING CONDITION

$$G(\vec{x}, \vec{y}) = 0 \quad \text{FOR ALL } \vec{y} \in \partial D \text{ AND } \vec{x} \in D$$

$$\therefore\ \phi(\vec{x}) = \int_D \psi(\vec{y})\, G(\vec{x}, \vec{y})\, dy + \int_{\partial D} \phi(\vec{y}) \frac{\partial G}{\partial n_y}(\vec{x}, \vec{y})\, dS_y$$

NOTES ON GREEN'S FUNCTIONS

NOW WE SOLVE THE EIGENVALUE PROBLEM

$$\begin{cases} \nabla^2 u_k(\vec{x}) = \lambda_k u(\vec{x}) & \vec{x} \in D \\ u_k(\vec{x}) = 0 & \vec{x} \in \partial D \end{cases}$$

IT CAN BE SHOWN THAT THIS PROBLEM HAS AN INFINITE NUMBER OF EIGENVALUES AND EIGENFUNCTIONS SUCH THAT $\lambda_n \to \infty$ AS $n \to \infty$. THIS RESULT COMES FROM THE THEORY OF INTEGRAL EQUATIONS. SEE, FOR EXAMPLE, GREEN'S FUNCTIONS BY ROACH. THE COMPLETENESS OF THESE EIGENFUNCTIONS IN SPACE L_2 (SQ. INTEGRABLE FNS) CAN ALSO BE PROVED.

WE CAN FIND $G(\vec{x},\vec{y})$ EASILY AS FOLLOWS.

$$\phi(\vec{x}) = \sum_k d_k u_k(\vec{x})$$

$$\psi(\vec{x}) = \sum_k c_k u_k(\vec{x})$$

WE ASSUME THAT THE EIGENFUNCTIONS ARE NORMALIZED (THEY ARE MUTUALLY ORTHOGONAL)

$$\nabla^2\phi(\vec{x}) = \sum_k d_k \nabla^2 u_k(\vec{x})$$

$$= \sum_k \lambda_k d_k u_k(\vec{x})$$

$$= \sum_k c_k u_k(\vec{x})$$

$$\Rightarrow d_k = \frac{c_k}{\lambda_k}, \quad c_k = \int_D \psi(\vec{y}) u_k(\vec{y}) d\vec{y}$$

$$\phi(\vec{x}) = \sum_k \frac{c_k}{\lambda_k} u_k(\vec{x})$$

$$= \int_D \psi(\vec{y}) \underbrace{\sum_k \frac{u_k(\vec{y}) u_k(\vec{x})}{\lambda_k}}_{G(\vec{x},\vec{y})} d\vec{y}$$

$$\Rightarrow \boxed{G(\vec{x},\vec{y}) = \sum_{k=1}^{\infty} \frac{u_k(\vec{x}) u_k(\vec{y})}{\lambda_k}}$$

SEE NOTE 1 ON P53

THE PROBLEM WE ARE CONSIDERS IS SELF-ADJOINT.

THEREFORE, $G(\vec{x},\vec{y}) = G(\vec{y},\vec{x})$. WE SEE THAT THE INFINITE SERIES FOR $G(\vec{x},\vec{y})$ SATISFIES THIS SYMMETRY PROPERTY.

FOR THE CONVERGENCE PROPERTY OF OUR RESULT, WE MENTION THAT THE RESULT HAS THE FOLLOWING PROPERTY. LET

$$\phi_n(\vec{x}) = \int_D \psi(\vec{y}) \sum_{k=1}^{n} \frac{u_k(\vec{x})\,u_k(\vec{y})}{\lambda_k}\, d\vec{y}$$

$$\Longrightarrow \lim_{n\to\infty} \|\phi(\vec{x}) - \phi_n(\vec{x})\|^2 \equiv$$

$$\lim_{n\to\infty} \int_D |\phi(\vec{x}) - \phi_n(\vec{x})|^2\, d\vec{x} = 0$$

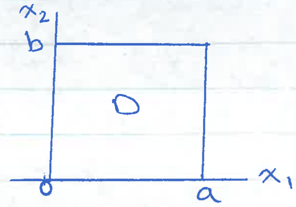
WE CAN FIND THE GREEN'S FUNCTION FOR THE LAPLACE OPERATOR WITH NEUMANN BC IN A SIMILAR WAY (SEE ROACH); ALSO FOR HELMHOLTZ AND WAVE OPERATORS.

WE MENTION THAT FOR SIMPLE DOMAINS D, WE CAN SOLVE FOR EIGENVALUES AND EIGENFUNCTIONS ANALYTICALLY. FINDING THESE NUMERICALLY FOR LARGE ORDERS (LARGE K) IS HOPLESS. UNFORTUNATELY, FOR MOST PROBLEMS OF ACOUSTICS, FINDING EIGENVALUES AND EIGENFUNCTIONS ANALYTICALLY IS IMPOSSIBLE. WE WILL SAY SOME MORE ABOUT THIS LATER. FIRST WE PRESENT ONE EXAMPLE.

NOTES ON GREEN'S FUNCTION

EXAMPLE –

$$\begin{cases} \nabla^2 \phi = \psi(\vec{x}) & \vec{x} \in D \\ \phi(\vec{x}) = g(\vec{x}) & \text{ON } \partial D \end{cases}$$



THIS IS A 2-D PROBLEM. TO FIND THE GREEN'S FN WE MUST FIND EIGENFNS $u_k(x)$ SUCH THAT

$$\begin{cases} \nabla^2 u_k = \lambda_k u_k & \vec{x} \text{ IN } D \\ u_k(\vec{x}) = 0 & \vec{x} \in \partial D \end{cases}$$

THE EIGENVALUES AND EIGENFUNCTIONS CAN BE FOUND BY SEPARATION OF VARIABLES. BECAUSE D IS SO SIMPLE, WE CAN ACTUALLY GUESS THE EIGENFNS:

$$u_{k\ell}(\vec{x}) = A_{k\ell} \sin\left(\frac{k\pi x_1}{a}\right) \sin\left(\frac{\ell\pi x_2}{b}\right)$$

WHERE $A_{k\ell}$ IS FOUND SUCH THAT $\|u_{k\ell}(\vec{x})\|_2 = \int_0^a \int_0^b u_{k\ell}^2(\vec{x})\, dx_2\, dx_1 = 1$

USING MATHEMATICA, WE FIND THAT

$$A_{k\ell} = \frac{2}{\sqrt{ab}}$$

THE CORRESPONDING EIGENVALUES ARE

$$\lambda_{k\ell} = -\pi^2 \left[\left(\frac{k}{a}\right)^2 + \left(\frac{\ell}{b}\right)^2\right]$$

$$\therefore G(\vec{x}, \vec{y}) = -\frac{4}{\pi^2 ab} \sum_{k,\ell=1}^{\infty} \frac{\sin\left(\frac{k\pi x_1}{a}\right)\sin\left(\frac{\ell\pi x_2}{b}\right)\sin\left(\frac{k\pi y_1}{a}\right)\sin\left(\frac{\ell\pi y_2}{b}\right)}{(k/a)^2 + (\ell/b)^2}$$

IN THIS CASE THE DETERMINATION OF THE EIGENVALUES AND EIGENFNS WERE VERY EASY. IT IS NOT SO ALWAYS!

NOTES: 1 - FROM THE RESULT ON P50, WE HAVE

$$G(\vec{x},\vec{y}) = \sum_{k=1}^{\infty} \frac{u_k(\vec{x})\, u_k(\vec{y})}{\lambda_k}$$

$$\Rightarrow \nabla_x^2 G(\vec{x},\vec{y}) = \sum_{k=1}^{\infty} \frac{u_k(\vec{y})}{\lambda_k} \nabla_x^2 u_k(\vec{x})$$

$$= \sum_{k=1}^{\infty} u_k(\vec{x})\, u_k(\vec{y})$$

THIS IS AN IMPORTANT RESULT. BECAUSE WE HAVE A SELF-ADJ. PROBLEM, WE HAVE $G^*(\vec{x},\vec{y}) = G(\vec{y},\vec{x}) = G(\vec{x},\vec{y})$

$$\Rightarrow \nabla_x^2 G(\vec{x},\vec{y}) = \nabla_y^2 G(\vec{x},\vec{y}) = \sum_{k=1}^{\infty} u_k(\vec{x})\, u_k(\vec{y})$$

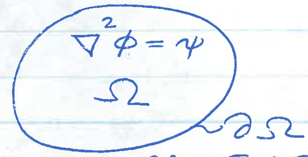
FOR THE EXAMPLE OF PREVIOUS PAGE, WE HAVE USED MATHEMATICA 5.1 TO SHOW THAT WE DO INDEED GET δ-FUNCTION BEHAVIOR FROM THE SUM ABOVE. THE MATHEMATICA OUTPUT IS GIVEN IN THE NEXT FEW PAGES.

2 - THE ABOVE GREEN'S FUNCTION IN THE FORM OF AN INFINITE SERIES HAS A VERY GOOD CONVERGENCE PROPERTY BECAUSE $\lambda_{k\ell} \to \infty$ VERY FAST AS $k + \ell \to \infty$.

NOTES ON GREEN'S FUNCTIONS

UNIQUENESS THEOREMS FOR LAPLACIAN OPERATOR ON BOUNDED DOMAINS

IN THE PREVIOUS SECTION, WE ASSUMED THAT THE DIRICHLET PROBLEM



$$\begin{cases} \nabla^2 \phi = \psi & \vec{x} \in \Omega \\ \phi(\vec{x}) = f(\vec{x}) & \vec{x} \in \partial\Omega \end{cases}$$

DIRICHLET BC

HAS A UNIQUE SOLUTION. SINCE THE PROOF IS VERY SIMPLE, WE GIVE IT HERE. LET US SAY THAT WE FIND TWO DIFFERENT SOLUTIONS $\phi_1(\vec{x})$ AND $\phi_2(\vec{x})$ SATISFYING THE BC ON $\partial\Omega$. THEN $\gamma(\vec{x}) = \phi_1(\vec{x}) - \phi_2(\vec{x})$ SATISFIES

$$\begin{cases} \nabla^2 \gamma = 0 & \vec{x} \in \Omega \\ \gamma(\vec{x}) = 0 & \vec{x} \in \partial\Omega \end{cases}$$

WE MUST SHOW THAT $\gamma(\vec{x}) = 0$ FOR $\vec{x} \in \Omega$ FOR UNIQUENESS OF THE SOLUTION OF LAPLACIAN WITH DIRICHLET BC. HERE IS THE PROOF:

$$0 = \int_\Omega \gamma \nabla^2 \gamma \, d\vec{x} = \int_\Omega [\nabla \cdot (\gamma \nabla \gamma) - |\nabla \gamma|^2] d\vec{x}$$

$$= \underbrace{\int_{\partial\Omega} \gamma \frac{\partial \gamma}{\partial n} dS}_{\gamma = 0 \text{ ON } \partial\Omega} - \int_\Omega |\nabla \gamma|^2 d\vec{x}$$

$$= - \int_\Omega |\nabla \gamma|^2 d\vec{x}$$

NOTE: THIS PROOF APPLIES TO ANY DIMENSION

SINCE $|\nabla \gamma|^2 \geq 0 \Rightarrow |\nabla \gamma| = 0 \Rightarrow \gamma = \text{CONST.}$

$\Rightarrow$ CONST. $= 0$ BECAUSE $\gamma = 0$ ON $\partial\Omega$!

WE NOW CONSIDER THE NEUMANN PROBLEM:

$$\begin{cases} \nabla^2 \phi = \psi(\vec{x}) & \vec{x} \in \Omega \\ \dfrac{\partial \phi}{\partial n} = g(\vec{x}) & \vec{x} \in \partial\Omega \end{cases}$$

NEUMANN BC

HERE THE SITUATION IS SOMEWHAT DIFFERENT.

NOTES ON GREEN'S FUNCTIONS

AS IN THE CASE OF DIRICHLET PROBLEM, WE SOLVE FIRST THE FOLLOWING PROBLEM:

$$\begin{cases} \nabla^2 \phi = \psi & \vec{x} \in \Omega \\ \frac{\partial \phi}{\partial n} = 0 & \vec{x} \in \partial\Omega \end{cases}$$

LET US FIRST LOOK AT THE FOLLOWING RESULT

$$\int_\Omega \psi \, d\vec{x} = \int_\Omega \nabla^2 \phi \, d\vec{x} = \int_\Omega \nabla \cdot \nabla \phi \, d\vec{x}$$

$$= \int_{\partial\Omega} \frac{\partial \phi}{\partial n} \, d\vec{x} = 0$$

THIS IS A NECESSARY CONDITION FOR THE EXISTENCE OF THE SOLUTION. LET US NEXT AT THE FOLLOWING PROBLEM

$$\begin{cases} \nabla^2 \phi = 0 & \vec{x} \in \Omega \\ \frac{\partial \phi}{\partial n} = 0 & \vec{x} \in \partial\Omega \end{cases}$$

i.e. WE ARE ASKING THE QUESTION "DOES THE HOMO. PDE WITH HOMO. BC. HAVE A NONTRIVIAL (NONZERO) SOLUTION?" TO ANSWER THIS QUESTION, WE FOLLOW A PROCEDURE SIMILAR TO THAT WE USED TO ESTABLISH UNIQUENESS OF THE SOLUTION TO DIRICHLET PROBLEM:

$$0 = \int_\Omega \phi \underbrace{\nabla^2 \phi}_{=0} \, d\vec{x} = \int_{\partial\Omega} \phi \underbrace{\frac{\partial \phi}{\partial n}}_{=0} dS - \int_\Omega |\nabla \phi|^2 \, d\vec{x}$$

$$= - \int_\Omega |\nabla \phi|^2 \, d\vec{x}$$

$$\Rightarrow |\nabla \phi| = 0, \quad \text{i.e.} \quad \phi = \text{CONST.} = C$$

THIS TIME, HOWEVER, THE CONSTANT CANNOT BE DETERMINED. NOW LOOKING AT THE EIGENVALUE PROBLEM:

$$\nabla^2 u_i = \lambda_i u_i \quad \vec{x} \in \Omega, \quad \frac{\partial u_i}{\partial n} = 0 \quad \vec{x} \in \partial\Omega$$

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NOTES ON GREEN'S FUNCTIONS

WE SEE THAT, FOR $\lambda_1 = 0$, $u_1(\vec{x}) = C$ IS AN EIGENFUNCTIONS. THE CONDITION

$$\int_\Omega \psi(\vec{x})\, d\vec{x} = 0 = \int C\, \psi(\vec{x})\, d\vec{x}$$

MEANS THAT ψ IS ORTHOGONAL TO $u_1(\vec{x})$. HERE WE USE THE FOLLOWING INNERPRODUCT DEFINITION:

$$\langle u, v \rangle = \int_\Omega u(\vec{x})\, v(\vec{x})\, d\vec{x}$$

THE ABOVE RESULTS SO FAR TELL US THAT

i) $\phi(\vec{x})$ CAN ONLY BE DETERMINED UP TO A CONSTANT. IF ϕ IS A SOLUTION $\Rightarrow$ ϕ + CONST. IS ALSO A SOLUTION.

ii) ψ SHOULD BE ORTHOGONAL TO THE CONSTANT FUNCTION, i.e. $\int_\Omega \psi\, d\vec{x} = 0$.

$\therefore$ PROVIDED THAT THE EIGENVALUE PROBLEM

$$\nabla^2 u_i = \lambda_i u_i \quad \vec{x} \in \Omega, \quad \frac{\partial u_i}{\partial n} = 0 \quad \vec{x} \in \partial\Omega$$

HAS A COMPLETE SET OF EIGENFUNCTIONS, WE SET

$$\phi(\vec{x}) = \sum_{i=2}^{\infty} d_i\, u_i(\vec{x})$$

$$\psi(\vec{x}) = \sum_{i=2}^{\infty} c_i\, u_i(\vec{x})$$

HERE EIGENFNS ARE ORTHONORMAL (* SEE P 65 (NOTE))

$$c_i = \langle \psi, u_i \rangle = \int_\Omega \psi(\vec{y})\, u_i(\vec{y})\, d\vec{y}$$

$$\nabla^2(\vec{x}) = \sum_{i=2}^{\infty} d_i \lambda_i\, u_i(\vec{x}) = \sum_{i=2}^{\infty} c_i\, u_i(\vec{x})$$

$$\Rightarrow d_i = \frac{c_i}{\lambda_i}$$

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NOTES ON GREEN'S FUNCTIONS

$$\Rightarrow \quad \phi(\vec{x}) = \sum_{i=2}^{\infty} \frac{c_i u_i(\vec{x})}{\lambda_i} + \text{CONST.}$$

$$= \int_{\Omega} \left[\sum_{i=2}^{\infty} \frac{u_i(\vec{x}) u_i(\vec{y})}{\lambda_i} \right]$$

$$\therefore \quad G(\vec{x}, \vec{y}) = \sum_{i=2}^{\infty} \frac{u_i(\vec{x}) u_i(\vec{y})}{\lambda_i} \quad \text{GREEN'S FUNCTION}$$

NOW WE CONSIDER THE INHOMO. PDE WITH INHOMO. NEUMANN BC.

$$\begin{cases} \nabla^2 \phi = \psi(\vec{x}) & \vec{x} \in \Omega \\ \dfrac{\partial \phi}{\partial n} = g(\vec{x}) & \vec{x} \in \partial\Omega \end{cases}$$

LET US FIRST DRIVE A <u>NECESSARY</u> CONDITION FOR EXISTENCE OF SOLUTION :

$$\int_{\Omega} \psi(\vec{x})\, d\vec{x} = \int_{\Omega} \nabla^2 \phi \, d\vec{x} = \int_{\partial\Omega} \frac{\partial \phi}{\partial n} dS = \int_{\partial S} g \, dS$$

WE WILL SEE WHAT THIS MEANS LATER. WE KNOW THAT THE HOMO. PDE WITH HOMO. BC HAS A NONTRIVIAL SOLUTION $u_1 = \text{CONST}$. AGAIN WE USE ORTHONORMAL EIGENFUNCTIONS FOR EXPANSION OF THE SOLUTION USING THE GREEN'S FUNCTION ABOVE. THE GREEN'S THM (IDENTITY) IS

$$\int_{\Omega} (u \nabla^2 v - v \nabla^2 u)\, d\vec{x} = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS$$

TAKE $v = G(\vec{x}, \vec{y}) \Rightarrow \nabla^2 G = \delta(\vec{x} - \vec{y})$

$$\frac{\partial G}{\partial n_y} = \sum_{i=2}^{\infty} \frac{u_i(\vec{x})}{\lambda_i} \underbrace{\frac{\partial u_i}{\partial n_y}}_{=0} = 0$$

ALSO TAKE: $u = \phi(\vec{y}) \Rightarrow$

NOTES ON GREEN'S FUNCTIONS

$$\int_{\Omega} \phi(\vec{y})\,\delta(\vec{x}-\vec{y})\,d\vec{y} - \int_{\Omega} G(\vec{x},\vec{y})\,\psi(\vec{y})\,d\vec{y} = -\int_{\partial\Omega} G(\vec{x},\vec{y})\underbrace{\frac{\partial\phi}{\partial n_y}}_{g(\vec{y})}\,dS$$

WHERE $\frac{\partial}{\partial n_y} = \vec{n}\cdot\nabla_y$

$$\therefore\quad \phi(\vec{x}) = \int_{\Omega} G(\vec{x},\vec{y})\,\psi(\vec{y})\,d\vec{y} - \int_{\partial\Omega} G(\vec{x},\vec{y})\,g(\vec{y})\,dS + \text{CONST.}$$

- WHAT IS THE MEANING OF THE NECESSARY CONDITION? LET US DEFINE TWO KINDS OF INNER PRODUCTS:

$$\langle u, v\rangle_1 = \int_{\Omega} u\,v\,d\vec{x}$$

$$\langle u, v\rangle_2 = \int_{\partial\Omega} u\,v\,dS$$

$\Rightarrow$ USING THE DEFINITION OF $G(\vec{x},\vec{y})$ AS AN INFINITE SERIES, WE HAVE

$$\phi(\vec{x}) = \sum_{i=2}^{\infty} \frac{u_i(\vec{x})}{\lambda_i}\left[\langle \psi, u_i\rangle_1 - \langle g, u_i\rangle_2\right] + \text{CONST.}$$

THE NECESSARY CONDITION CAN BE WRITTEN AS $\langle \psi, u_1\rangle - \langle g, u_1\rangle = 0$

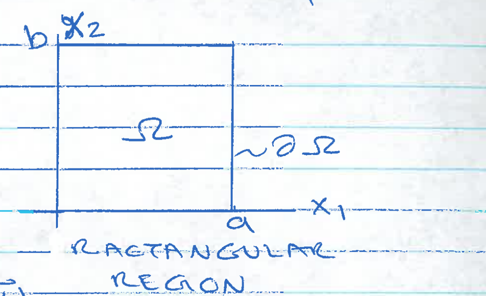
THIS MEANS THAT THE EIGENVALUE $\lambda_1 = 0$ BECOMES IRRELEVANT TO THIS PROBLEM. BUT THE SOLUTION IS OBTAINED TO WITHIN A CONSTANT. WE MENTION HERE THAT THE ABOVE ARE USED OFTEN IN SOLVING ODES AND PDES AND ARE STUDIED IN LINEAR OPERATOR THEORY. IN PARTICULAR, LINEAR ALGEBRA IS PART OF THE LIN. OP. THEORY IN FINITE DIMENSIONS.

NOTES ON GREEN'S FUNCTION

EXAMPLE

GREEN'S FUNCTION

FOR



$$\begin{cases} \nabla^2 \phi = \psi(\vec{x}) & \vec{x} \in \Omega \\ \dfrac{\partial \phi}{\partial n} = g \end{cases}$$

$$G(\vec{x}, \vec{y}) = \sum_{i=2}^{\infty} \frac{u_i(\vec{x})\, u_i(\vec{y})}{\lambda_i}$$

WE USE AGAIN SEPARATION OF VARIABLES TO FIND THE EIGENVALUES AND EIGENFUNCTIONS CORRESPONDING TO $\lambda_i \neq 0$. WE HAVE TO USE DOUBLE INDICES i AND j NOW

$$u_{ij}(\vec{x}) = \frac{2\sqrt{\gamma_{ij}}}{\sqrt{ab}} \cos\left(\frac{\pi x_1}{a}\right) \cos\left(\frac{\pi x_2}{a}\right)$$

WHERE $\gamma_{00} = 0, \quad \gamma_{i0} = \gamma_{0j} = \frac{1}{2}, \quad \gamma_{ij} = 1 \begin{cases} i \geq 1 \\ j \geq 1 \end{cases}$

$$\lambda_{ij} = -\pi^2 \left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2 \right] \qquad i \geq 0, \; j \geq 0$$

$$G(\vec{x}, \vec{y}) = -\frac{4}{ab} \sum_{i,j=0}^{\infty} \gamma_{ij} \frac{\cos\left(\frac{\pi i x_1}{a}\right) \cos\left(\frac{\pi j x_2}{b}\right) \cos\left(\frac{\pi i y_1}{a}\right) \cos\left(\frac{\pi j y_2}{b}\right)}{\pi^2 \left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2 \right]}$$

NOTE: THE DIRICHLET & NEUMANN PROBLEMS ARE SELF-ADJOINT. WE CAN SHOW THAT

i) THE EIGENVALUES ARE ALL REAL

ii) THE EIGENFUNCTIONS ARE MUTUALLY ORTHOGONAL

THE PROOF OF "COMPLETENESS" IS USUALLY COMPLICATED. ONE PROOF REQUIRES THAT WE SHOW, FOR SELF-ADJOINT PROBLEM,

$\boxed{u_i\text{'s ORTHO-NORMAL}}$

$$\sum_{i=1}^{\infty} u_i(\vec{x})\, u_i(\vec{y}) = \delta(\vec{x} - \vec{y})$$

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NOTES ON GREEN'S FUNCTIONS

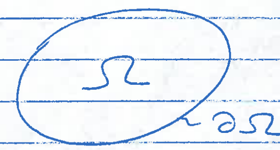
SOME USEFUL RESULTS

- GREEN'S THEOREMS – OR GREEN'S IDENTITIES

1ST THM : $u \in C^2$, i.e. 2ND DER. OF u CONTINUOUS

$$u\nabla^2 u = \nabla\cdot(u\nabla u) - |\nabla u|^2$$

$$\int_\Omega u\nabla^2 u\, d\vec{x} = \int_{\partial\Omega} u\frac{\partial u}{\partial n}\, ds - \int_\Omega |\nabla u|^2 d\vec{x}$$



WE HAVE USED THIS RESULT TO PROVE UNIQUENESS OF DIRICHLET AND NEUMANN PROBLEMS FOR LAPLACIAN OPERATOR

2ND THM : $u \in C^2$

$$u\nabla^2 v = \nabla\cdot(u\nabla v) - \nabla u\cdot\nabla v$$

$$v\nabla^2 u = \nabla\cdot(v\nabla u) - \nabla u\cdot\nabla v$$

$$u\nabla^2 v - v\nabla^2 u = \nabla\cdot(u\nabla v - v\nabla u)$$

$$\int_\Omega (u\nabla^2 v - v\nabla^2 u)\, d\vec{x} = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\vec{s}$$

WE HAVE USED THIS TO FIND THE SOLUTION OF $\nabla^2\phi = \psi \quad \vec{x}\in\Omega, \quad \phi(\vec{x}) = f(\vec{x}), \vec{x}\in\partial\Omega$

- THE SOLUTION OF

$$\begin{cases} \nabla^2\phi - k\phi = \psi & \vec{x}\in\Omega \\ \phi(\vec{x}) = f(\vec{x}) & \vec{x}\in\partial\Omega \end{cases}$$

NOTE: WE FIRST SOLVE: $\nabla^2\phi - k\phi = \psi$, $\vec{x}\in\Omega$; $\phi(\vec{x}) = 0$, $\vec{x}\in\partial\Omega$

SUPPOSE WE SOLVE THE EIGENVALUE PROBLEM

$$\begin{cases} \nabla^2 u_i = \lambda_i u_i \\ u_i(\vec{x}) = 0 \end{cases}$$

$$\psi(\vec{x}) = \sum_{i=1}^{\infty} c_i u_i(\vec{x}) \qquad u_i \text{ : ORTHONORMAL}$$

$$\phi(\vec{x}) = \sum_{i=1}^{\infty} d_i u_i(\vec{x})$$

$$\nabla^2\phi - k\phi = \sum_{i=1}^{\infty} \underbrace{d_i(\lambda_i - k)}_{=c_i} u_i = \psi(\vec{x})$$

NOTES ON GREEN'S FUNCTIONS

$$\Rightarrow \quad c_i = (\lambda_i - k) d_i$$

$$d_i = \frac{c_i}{\lambda_i - k}$$

$$\phi(\vec{x}) = \sum_{i=1}^{\infty} \frac{c_i}{\lambda_i - k} u_i(\vec{x})$$

$$= \int_\Omega \left[\sum_{i=1}^{\infty} \frac{u_i(\vec{x}) u_i(\vec{y})}{\lambda_i - k} \right] (\vec{y}) \, d\vec{y}$$

$$\therefore \quad G(\vec{x}, \vec{y}) = \sum_{i=1}^{\infty} \frac{u_i(\vec{x}) u_i(\vec{y})}{\lambda_i - k}$$

NECECESSARY CONDITION: $k \neq \lambda_i$ FOR $i \geq 1$

THIS GREEN'S FUNCTION SOLVES

$$\begin{cases} \nabla^2 \phi - k\phi = \psi & \vec{x} \in \Omega \\ \phi(\vec{x}) = 0 & \vec{x} \in \partial\Omega \end{cases}$$

TO SOLVE PROBLEM WITH BC $\phi(\vec{x}) = f(\vec{x})$ $\vec{x} \in \partial\Omega$, WE USE GREEN'S 2ND IDENTITY TAKING $u = \phi$, $v = G(\vec{x}, \vec{y})$, $\nabla_y^2 G - k G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y})$

$$\int_\Omega [\phi(\vec{y}) \underbrace{\nabla_y^2 G}_{kG(\vec{x},\vec{y}) + \delta(\vec{x}-\vec{y})} - \underbrace{\nabla^2 \phi(\vec{y})}_{\psi(\vec{y}) + k\phi(\vec{y})} \cdot G(\vec{x}, \vec{y})] \, d\vec{y}$$

$$= \int_\Omega [\phi \, \delta(\vec{x} - \vec{y}) - G(\vec{x}, \vec{y}) \psi(\vec{y})] \, d\vec{y}$$

$$= \phi(\vec{x}) - \int_\Omega G(\vec{x}, \vec{y}) \psi(\vec{y}) \, d\vec{y}$$

$$= \int_{\partial\Omega} \left[\phi(\vec{y}) \frac{\partial G}{\partial n_y} - \underbrace{G(\vec{x}, \vec{y})}_{=0} \frac{\partial \phi}{\partial n_y} \right] dS$$

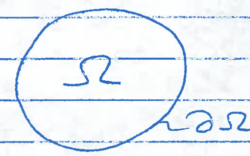
$$= \int_{\partial\Omega} f(\vec{y}) \frac{\partial G}{\partial n_y} \, dS$$

NOTES ON GREEN'S FUNCTIONS

THE HEAT EQUATION IN BOUNDED DOMAIN

WE WANT TO SOLVE

$$\begin{cases} -\nabla^2\phi + \dfrac{\partial \phi}{\partial t} = \psi(\vec{x}) & \vec{x}\in\Omega,\ t\geq 0 \\ \text{I.C.}\quad \phi(\vec{x},0) = f(\vec{x}) & \vec{x}\in D \\ \text{B.C.}\quad \phi(\vec{x},t) = g(\vec{x},t) & \vec{x}\in\partial D,\ t\geq 0 \end{cases}$$



PHYSICALLY, THIS IS A HEAT CONDUCTION PROBLEM WHERE WE ARE GIVEN THE INITIAL CONDITION AND THE BC (e.g., WE DROP THE SOLID WITH BOUNDARY $\partial\Omega$ IN A CONST. TEMP. BATH). INTUITIVELY, THEREFORE, THE SOLUTION EXISTS AND IS UNIQUE. THIS CAN ALSO BE PROVED ANALYTICALLY. HERE WE SOLVE THE ABOVE PROBLEM BY FINDING THE GREEN'S FUNCTION.

WE START BY SOLVING THE FOLLOWING PROBLEM FIRST

$$\begin{cases} \text{HOMO. D.E.}\quad -\nabla^2\phi + \dfrac{\partial\phi}{\partial t} = 0 & \vec{x}\in\Omega,\ t\geq 0 \\ \text{HOMO. BC}\quad \phi(\vec{x},t) = 0 & \vec{x}\in\partial\Omega,\ t\geq 0 \\ \text{I.C.}\quad \phi(\vec{x},0) = f(\vec{x}) & \vec{x}\in\Omega \end{cases}$$

LET US DEFINE THE EIGENVALUE PROBLEM

$$\begin{cases} \nabla^2 u_i + \lambda_i u_i(\vec{x}) = 0 & \vec{x}\in\Omega \\ u_i(\vec{x}) = 0 & \vec{x}\in\partial\Omega \end{cases}$$

ASSUME EIGENFUNCTIONS ARE ORTHONORMALIZED.

DEFINE

$$\phi(\vec{x},t) = \sum_{i=1}^{\infty} c_i(t)\, u_i(\vec{x})$$

THIS FUNCTION SATISFIES THE B.C.

$$-\nabla^2\phi + \frac{\partial\phi}{\partial t} = \sum_{i=1}^{\infty}\big[\underbrace{\lambda_i c_i + \dot{c}_i}_{=0}\big]\, u_i(\vec{x}) = 0$$

$$\therefore\ \dot{c}_i + \lambda_i c_i = 0 \Rightarrow c_i(t) = \alpha_i e^{-\lambda_i t}$$

NOTES ON GREEN'S FUNCTIONS

WHERE α_i IS A CONSTANT.

$$\phi(\vec{x},t) = \sum_{i=1}^{\infty} \alpha_i e^{-\lambda_i t} u_i(\vec{x})$$

$$\phi(\vec{x},0) = \sum_{i=1}^{\infty} \alpha_i u_i(\vec{x}) = f(\vec{x})$$

$$\therefore \quad \alpha_i = \langle u_i, f \rangle = \int_{\Omega} f(\vec{y})\, u_i(\vec{y})\, d\vec{y}$$

$$\phi(\vec{x},t) = \int_{\Omega} f(\vec{y}) \left[\sum_{i=1}^{\infty} e^{-\lambda_i t} u_i(\vec{y}) u_i(\vec{x}) \right] d\vec{y}$$

WE WRITE

$$G(\vec{x},\vec{y},t) = H(t) \sum_{i=1}^{\infty} e^{-\lambda_i t} u_i(\vec{x}) u_i(\vec{y})$$

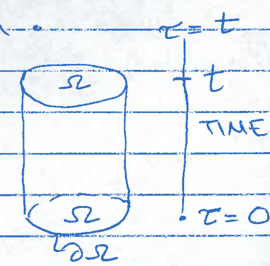
THIS IS THE GREEN'S FUNCTION FOR OUR PROBLEM. HERE $H(t)$ IS THE HEAVISIDE FUNCTION. NOTE THAT

$$-\nabla_x^2 G + \frac{\partial G}{\partial t} = \delta(t) \sum_{i=1}^{\infty} u_i(\vec{x}) u_i(\vec{y})$$

$$= \delta(\vec{x}-\vec{y})\,\delta(t) \qquad (*) \text{ PROVE THIS!}$$

NOW WE SOLVE THE FIRST PROBLEM.

$$L\phi = \begin{cases} \dfrac{\partial \phi}{\partial t} - \nabla^2 \phi = \psi(\vec{x}) & \vec{x} \in \Omega,\ t \geq 0 \\ \phi(\vec{x},t) = g(\vec{x},t) & \vec{x} \in \partial\Omega,\ t \geq 0 \\ \phi(\vec{x},0) = f(\vec{x}) & \vec{x} \in \Omega \end{cases}$$



WE NOW USE A GREEN-LIKE IDENTITY TO SOLVE:

$*$ IN DERIVING THIS RESULT YOU WILL FIND OUT THAT THE HEAVISIDE FUNCTION $H(t)$ IS ESSENTIAL IN THE APPEARANCE OF $\delta(t)$ ON THE RIGHT OF THIS RELATION.

NOTES ON GREEN'S FUNCTION

THIS PROBLEM. WE NOTE THAT

$$L^* \phi = -\frac{\partial \phi}{\partial t} - \nabla^2 \phi$$

ALSO LET US DEFINE

$$G(\vec{x}, \vec{y}, t-\tau) = H(t-\tau) \sum_{i=1}^{\infty} e^{-\lambda_i (t-\tau)} u_i(\vec{x}) u_i(\vec{y})$$

$$\Rightarrow L_x G = \delta(\vec{x}-\vec{y})\, \delta(t-\tau)$$

$$L^*_y G = \delta(\vec{x}-\vec{y})\, \delta(t-\tau)$$

NOW WE DERIVE THE FOLLOWING GREEN'S IDENTITY FOR HEAT EQUATION : $u(\vec{y},\tau)$; $v(\vec{y},\tau)$

$$u L^* v - v L u = -\left(u \frac{\partial v}{\partial \tau} + v \frac{\partial u}{\partial \tau}\right) + v \nabla^2 u - u \nabla^2 v$$

$$= -\frac{\partial}{\partial \tau}(uv) + \nabla \cdot (v \nabla u - u \nabla v)$$

$$\therefore \int_0^t \int_\Omega (u L^* v - v L u)\, d\vec{y}\, d\tau = -\int_\Omega \left[uv\right]_0^t d\vec{y} + \int_0^t \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) dS\, d\tau$$

NOW LET

$$v = G(\vec{x}, \vec{y}, t-\tau) = H(t-\tau) \sum_{i=1}^{\infty} e^{-\lambda_i (t-\tau)} u_i(\vec{x}) u_i(\vec{y})$$

AND $u = \phi(\vec{y}, \tau)$, $L\phi = \psi(\vec{y}, \tau)$

$$\int_{0+}^{t-} \int_\Omega \phi(\vec{y},\tau) \underbrace{L^*_{y,\tau} G}_{\delta(\vec{x}-\vec{y})\delta(t-\tau)}\, d\vec{y}\, d\tau - \int_{0+}^{t-} \int_\Omega \psi(\vec{y},\tau)\, G\, d\vec{y}\, d\tau$$

(SEE NOTE PAGE 84)

$$-\int_\Omega \left[\phi(\vec{y},t)\, G(\vec{x},\vec{y}, t-t_-) - \phi(\vec{y},0)\, G(\vec{x},\vec{y}, t-0_+)\right] d\vec{y}$$

$$+ \int_{0+}^{t-} \int_{\partial\Omega} \left(G \frac{\partial \phi}{\partial n_y} - \phi \frac{\partial G}{\partial n_y}\right) dS_y\, d\tau \qquad (*)$$

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NOTES ON GREEN'S FUNCTION

NOW WE NOTE:

$$G(\vec{x},\vec{y},t-t_-) = \underbrace{H(t-t_-)}_{\longrightarrow 0} \sum_i e^{-\lambda_i(t-t_-)} u_i(\vec{x})\, u_i(\vec{y})$$

$$\lim_{t_-\to t} G(\vec{x},\vec{y},t-t_-) = \sum_{i=1}^{\infty} u_i(\vec{x})\, u_i(\vec{y}) = \delta(\vec{x}-\vec{y})$$

$$\lim_{\tau\to 0+} G(\vec{x},\vec{y},t-\tau) = G(\vec{x},\vec{y},t)$$

$$\int_{0+}^{t_-}\int_{\Omega} \phi(\vec{y},\tau)\,\delta(\vec{x}-\vec{y})\,\underbrace{\delta(t-\tau)}_{=0!} = 0\ !$$

SEE NOTE P84

$G \equiv 0$ ON $\partial\Omega$ FOR ALL $\tau \geq 0$

$$\therefore \int_{\Omega} \phi(\vec{y},t_-)\,G(\vec{x},\vec{y},t-t_-)\,d\vec{y} \rightarrow$$

$$\int_{\Omega} \phi(\vec{y},t)\,\delta(\vec{x}-\vec{y})\,d\vec{y} = \phi(\vec{x},t)$$

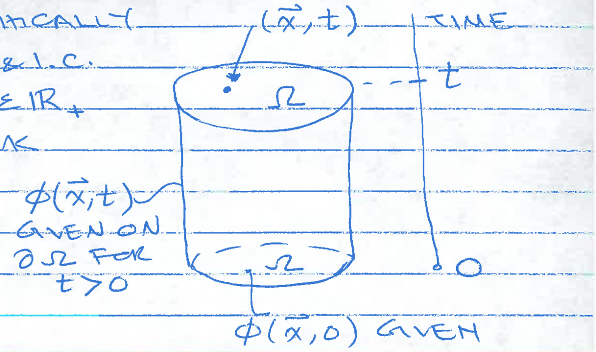
REARRANGING EQ. (*), P70, WE GET

$$\boxed{\begin{aligned}\phi(\vec{x},t) = &\int_0^t\int_{\Omega} \psi(\vec{y},\tau)\,G(\vec{x},\vec{y},t-\tau)\,d\vec{y}\,d\tau \\ &+ \int_{\Omega} f(\vec{y})\,G(\vec{x},\vec{y},t)\,d\vec{y} \\ &- \int_0^t\int_{\partial\Omega} g(\vec{y},\tau)\,\frac{\partial G}{\partial n_y}(\vec{x},\vec{y},t-\tau)\,dS_y\,d\tau\end{aligned}}$$

$$\frac{\partial}{\partial n_y} = \vec{n}\cdot\nabla_y$$

THIS PROBLEM IS UNUSUAL BECAUSE, WE DO NOT GET $\phi(\vec{x},t)$ FROM $L_y^* G(\vec{x},\vec{y},t-\tau) = \delta(\vec{x}-\vec{y})\,\delta(t-\tau)$. WE KNOW WE HAVE THE

RIGHT RESULT BECAUSE WE CAN PROVE THE PROBLEM WE HAVE POSED HAS A UNIQUE SOLUTION. WE CAN GRAPHICALLY ILLUSTRATE THE BC. & I.C. IN $\Omega \times \mathbb{R}_+$, $\vec{x} \in \Omega$, $t \in \mathbb{R}_+$ AS SHOWN. IF WE THINK



OF THIS AS A CONTAINER, THE I.C. IS GIVEN BY THE BOTTOM SURFACE AND THE BC IS GIVEN BY THE SIDE OF THE CONTAINER. THE POINT $(\vec{x},t)$ IS ON THE TOP SURFACE OF THE CONTAINER. THE THEORY OF PARABOLIC PDE'S CAN SHOW THE UNIQUENESS OF THE SOLUTION AT $(\vec{x},t)$. SEE M.G. SMITH "INTRODUCTION TO THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS", VAN NOSTRAND, 1967 (OUT OF PRINT). THERE IS AN EXAMPLE OF CONSTRUCTION OF EIGENFUNCTIONS IN THIS BOOK. ALSO SEE TIKHONOV & SAMARSKI "PDES OF MATHEMATICAL PHYSICS", CHAP. 6, P 422-464, DOVER BOOKS.

NOTES ON GREEN'S FUNCTIONS

NOTE ON GREEN'S FUNCTION OF THE WAVE EQUATION IN UNBOUNDED DOMAIN

WE DERIVED THE GREEN'S FUNCTION OF THE WAVE EQ. IN $\mathbb{R}^3$ BY CONVERTING THE WAVE OPERATOR TO THE HELMHOLZ EQ. WE USED A FOURIER TRANSFORM IN TIME. WE DID THIS BECAUSE WE ALREADY HAD THE GREEN'S FUNCTION OF HELMHOLTZ EQ. BUT FROM THE METHOD WE USED FOR HEAT EQUATION, WE HAVE ANOTHER GOOD METHOD.

$$\square^2 G(\vec{x},t) = \delta(\vec{x})\delta(t)$$

$$\hat{G}(\vec{\xi},t) = FT\left[G(\vec{x},t), \vec{x}\to\vec{\xi}\right]$$

$$\widehat{\square^2 G(\vec{x},t)} = (-\imath)^2|\vec{\xi}|^2\hat{G} - \frac{1}{c^2}\frac{\partial^2\hat{G}}{\partial t^2}$$

$$= -4\pi^2|\vec{\xi}|^2\hat{G} - \frac{1}{c^2}\frac{\partial^2\hat{G}}{\partial t^2} = \delta(t)$$

$$\begin{cases} n: \text{DIMENSION OF THE SPACE} \\ \imath = 2\pi i, \quad |\vec{\xi}|^2 = \xi_1^2 + \xi_2^2 + \dots + \xi_n^2 \end{cases}$$

NOW, REMEMBERING THAT CAUSALITY REQUIRES THAT $\hat{G}(\vec{\xi},t) = 0$ FOR $t<0$, WE CAN SOLVE THE ABOVE 2ND ORDER (ODE) GREEN'S FN PROBLEM IN t. INVERTING THE FT GIVES $G(\vec{x},t)$

$$G(\vec{x},t) = \int_{\mathbb{R}^n} \hat{G}(\vec{\xi},t)\, e^{-\imath\vec{\xi}\cdot\vec{x}}\, d\vec{\xi}$$

THIS REQUIRES SOME DEXTERITY IN ALGEBRAIC MANIPULATIONS... OR JUST USE MATHEMATICA!

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NOTES ON GREEN'S FUNCTIONS

SOME IMPORTANT APPLICATIONS OF GREEN'S FUNCTIONS

BY IMBEDDING A PROBLEM IN ANOTHER SPACE FOR WHICH WE HAVE A GREEN'S FUNCTION, WE CAN SOLVE OR OBTAIN AN INTEGRAL EQUATION WHICH HELPS IN SOLVING AN I.V. OR B.V. PROBLEM. WE GIVE SOME EXAMPLES HERE.

1. THE INITIAL VALUE PROBLEM FOR THE WAVE EQUATION IN $\mathbb{R}^3$ (POISSON'S SOLUTION)

WE ARE DEALING WITH THE UNBOUNDED 3D SPACE ($\mathbb{R}^3$). WE KNOW THAT THE GREEN'S FUNCTION (FUND. SOL.) FOR THIS PROBLEM IS

$$G(\vec{x}-\vec{y}, t-\tau) = \frac{\delta(g)}{4\pi r} \qquad \tau \leq t$$
$$= 0 \qquad \tau > t$$
$$g = \tau - t + r/c$$

TO SOLVE

$$\begin{cases} \Box^2 \phi = \psi(\vec{x}, t) & \vec{x} \in \mathbb{R}^3,\ t \geq 0 \\ \phi(\vec{x}, 0) = f(\vec{x}) & \vec{x} \in \mathbb{R}^3 \\ \dfrac{\partial \phi}{\partial t}(\vec{x}, 0) = h(\vec{x}) & \vec{x} \in \mathbb{R}^3 \end{cases}$$

WE IMBED THIS PROBLEM INTO ONE FOR $\vec{x} \in \mathbb{R}^3$, $t \in (-\infty, \infty)$ AS FOLLOWS. LET US DEFINE

$$\tilde{\phi}(\vec{x}, t) = \begin{cases} \phi(\vec{x}, t) & t \geq 0,\ \vec{x} \in \mathbb{R}^3 \\ 0 & t < 0,\ \vec{x} \in \mathbb{R}^3 \end{cases}$$

NOTE: HERE ALL DERIV. ARE ORD. DERIVAT. $\Longrightarrow \quad \Box^2 \tilde{\phi} = 0 \qquad$ FOR $t < 0,\ \vec{x} \in \mathbb{R}^3$

$\therefore$ DEFINE ALSO $\tilde{\psi} = \begin{cases} \psi & t \geq 0 \quad \vec{x} \in \mathbb{R}^3 \\ 0 & t < 0,\ \vec{x} \in \mathbb{R}^3 \end{cases}$

$$\Longrightarrow \quad \Box^2 \tilde{\phi} = \tilde{\psi}(\vec{x}, t) \qquad \vec{x} \in \mathbb{R}^3, \quad t \in (-\infty, \infty)$$

LET US NOW SET THIS PROBLEM IN GENERALIZED

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NOTES ON GREEN'S FUNCTIONS

FUNCTION SPACE. WE NEED TO FIND $\overline{\Box}^2 \tilde{\varphi}$ WHERE $\overline{\Box}^2$ IS THE WAVE OPERATOR WITH GENERALIZED DERIVATIVES. WE ASSUME BELOW THAT ϕ, f AND $g \in C^2$ (FNS WITH TWO CONTINUOUS DERIVATIVES). THIS IS NOT NECESSARY BUT SIMPLIFIES OUR WORK. SEE MY NASA PAPERS ON GENERALIZED FUNCTIONS. WE HAVE

$$\frac{\bar{\partial}\tilde{\phi}}{\partial t} = \frac{\partial\tilde{\phi}}{\partial t} + \Delta\phi\,\delta(t)$$

$$= \frac{\partial\tilde{\phi}}{\partial t} + \phi(\vec{x},0)\,\delta(t)$$

$$= \frac{\partial\tilde{\phi}}{\partial t} + f(\vec{x})\,\delta(t)$$

$$\frac{\bar{\partial}^2\tilde{\phi}}{\partial t^2} = \frac{\partial^2\tilde{\phi}}{\partial t^2} + h(\vec{x})\,\delta(t) + f(\vec{x})\,\delta'(t)$$

$$\overline{\nabla}^2\tilde{\phi} = \nabla^2\tilde{\phi}$$

$$\therefore\quad \overline{\Box}^2\tilde{\phi} = \Box^2\tilde{\phi} + \frac{1}{c^2}h(\vec{x})\,\delta(t) + \frac{1}{c^2}f(\vec{x})\,\delta'(t)$$

$$= \tilde{\psi}(\vec{x},t) + \frac{1}{c^2}h(\vec{x})\,\delta(t) + \frac{1}{c^2}f(\vec{x})\,\delta'(t)$$

$$\therefore\quad 4\pi\,\tilde{\phi}(\vec{x},t) = \int_{-\infty}^{t}\int_{\mathbb{R}^3} \tilde{\psi}(\vec{y},\tau)\,\frac{\delta(g)}{r}\,d\vec{y}\,d\tau$$

$$+ \frac{1}{c^2}\int_{-\infty}^{t}\int_{\mathbb{R}^3} h(\vec{y})\,\delta(\tau)\,\frac{\delta(g)}{r}\,d\vec{y}\,d\tau$$

$$+ \frac{1}{c^2}\int_{-\infty}^{t}\int_{\mathbb{R}^3} f(\vec{y})\,\delta'(\tau)\,\frac{\delta(g)}{r}\,d\vec{y}\,d\tau$$

SHOW THIS!

$$= \int_{0}^{t}\int_{\mathbb{R}^3} \frac{\psi(\vec{y},t-r/c)}{r}\,d\vec{y}$$

$$+ t\,M[h] + \frac{\partial}{\partial t}\left(t\,M[f]\right)$$

(POISSON'S FORMULA)

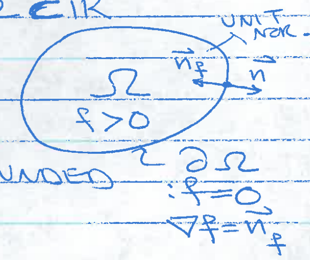
NOTES ON GREEN'S FUNCTIONS

WHERE $M[h]$ AND $M[f]$ ARE THE MEAN OF THE FUNCTIONS h AND f ON AN SPHERE WITH CENTER AT $\vec{x}$ AND RADIUS EQUAL TO ct.

2. LET $\nabla^2 \phi = \psi(\vec{x}) \quad \vec{x} \in \Omega, \; \Omega \in \mathbb{R}^3$

EITHER ϕ OR $\frac{\partial \phi}{\partial n}$ IS GIVEN ON $\partial\Omega$

$\phi = g(\vec{x})$ OR $\frac{\partial \phi}{\partial n} = h(\vec{x}) \quad \vec{x} \in \partial\Omega$



WE IMBED THIS PROBLEM IN UNBOUNDED $\mathbb{R}^3$. LET

$$\tilde{\phi} = \begin{cases} \phi & \vec{x} \in \Omega \\ 0 & \vec{x} \in \mathbb{R}^3 \setminus \Omega \end{cases}$$ (i.e. THE REST OF $\mathbb{R}^3$, EXTERNAL TO Ω)

$$\tilde{\psi} = \begin{cases} \psi & \vec{x} \in \Omega \\ 0 & \vec{x} \in \mathbb{R}^3 \setminus \Omega \end{cases}$$

$$\nabla^2 \tilde{\phi} = \tilde{\psi}(\vec{x}) \quad \vec{x} \in \mathbb{R}^3 \,!$$

$$\bar{\nabla} \tilde{\phi} = \nabla \tilde{\phi} + g(\vec{x}) \vec{n}_f \delta(f)$$

$$= \nabla \tilde{\phi} - g(\vec{x}) \vec{n} \, \delta(f)$$

$$\bar{\nabla}^2 \tilde{\phi} = \nabla^2 \tilde{\phi} - h(\vec{x}) \delta(f) - \nabla \cdot [g(\vec{x}) \vec{n} \delta(f)]$$

$$= \tilde{\psi}(\vec{x}) - h(\vec{x}) \delta(f) - \nabla \cdot [g(\vec{x}) \vec{n} \delta(f)]$$

$$4\pi \tilde{\phi}(\vec{x}) = -\int_{\mathbb{R}^3} \frac{\tilde{\psi}(\vec{y})}{r} d\vec{y} + \int_{\mathbb{R}^3} \frac{h(\vec{y}) \delta(f)}{r} d\vec{y}$$

$$+ \nabla_x \cdot \int_{\mathbb{R}^3} \frac{\vec{n} g(\vec{y}) \delta(f)}{r} d\vec{y}$$

$$= -\int_{\Omega} \frac{\psi(\vec{y})}{r} d\vec{y} + \int_{\partial\Omega} \frac{h(\vec{y})}{r} dS_y$$

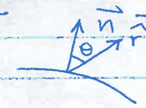
$$+ \nabla_x \cdot \int_{\partial\Omega} \frac{\vec{n} g(\vec{y})}{r} dS_y$$

HERE $\vec{n} = \vec{n}(\vec{y})$ ON $\partial\Omega$. WE CAN NOW BRING

∇_x INSIDE THE LAST INTEGRAL USING

$$\nabla_x r = \hat{\vec{r}}, \quad \hat{\vec{r}} = \frac{\vec{x}-\vec{y}}{r}$$

$$\cos\theta = \vec{n}\cdot\hat{\vec{r}}$$



$$4\pi\,\tilde{\phi}(\vec{x}) = -\int_{\Omega} \frac{\psi(\vec{y})}{r}\,d\vec{y} + \int_{\partial\Omega} \frac{h(\vec{y})}{r}\,dS_y - \int_{\partial\Omega} \frac{\cos\theta\; g(\vec{y})}{r^2}\,dS_y$$

NOTE THAT FOR NEUMANN BC, WE ARE GIVEN $h(\vec{y})$. THE FUNCTION g MUST BE FOUND BY BRINGING $\vec{x}$ ONTO $\partial\Omega$ AND SOLVING THE RESULTING INTEGRAL EQ. FOR $g(\vec{y})$. THERE ARE SOME SUBTLETIES THAT YOU NEED TO KNOW IN GETTING THE INTEGRAL EQUATION. YOU WILL LEARN THESE SUBTLETIES IN A COURSE ON BOUNDARY ELEMENT METHOD. THERE IS THE POSSIBILITY OF HAVING MORE THAN ONE CHOICE OF INTEGRAL EQUATION. FOR THIS PROBLEM IF $\vec{x}$ IS BROUGHT ONTO $\partial\Omega$ FROM <u>OUTSIDE</u> THE SURFACE, WE GET THE FOLLOWING INTEGRAL EQUATION

$$-\int_{\Omega} \frac{\psi(\vec{y})}{r}\,d\vec{y} + \int_{\partial\Omega} \frac{h(\vec{y})}{r}\,dS_y - 2\pi g(\vec{x}) - \int_{\partial\Omega} \frac{\cos\theta\; g(\vec{y})}{r^2}\,dS_y = 0$$

WHICH IS AN INTEGRAL EQ. OF THE FIRST TYPE. IF $\vec{x}$ IS BROUGHT ONTO $\partial\Omega$ FROM INSIDE, WE GET

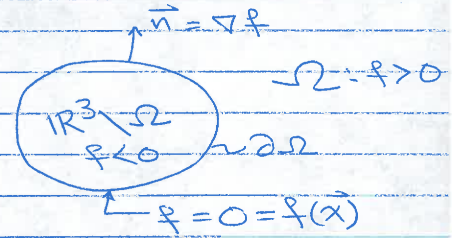
$$4\pi\, g(\vec{x}) = -\int_{\Omega} \frac{\psi(\vec{y})}{r}\,d\vec{y} + \int_{\partial\Omega} \frac{h(\vec{y})}{r}\,dS_y + 2\pi g(\vec{x}) - \int_{\partial\Omega} \frac{\cos\theta\; g(\vec{y})}{r^2}\,dS_y$$

WHICH IS EQUIVALENT TO THE FIRST INT. EQ.

NOTES ON GREEN'S FUNCTIONS (78)

FOR $\vec{x} \in \partial\Omega$, IT CAN BE SHOWN THAT ALL THE ABOVE INTEGRALS ARE CONVERGENT. FOR DIRICHLET B.C. WHERE $g(\vec{x})$ IS GIVEN, THE ABOVE INTEGRAL EQ. FOR $h(\vec{y})$ OF 2ND KIND WITH DIFFERENT PROPERTIES THAN INT. EQ. OF THE FIRST KIND.

3. THE KIRCHHOFF FORMULA FOR A STATIONARY SURFACE



$$\tilde{\phi} = \begin{cases} \phi & \vec{x} \in \Omega \\ 0 & \vec{x} \in \mathbb{R}^3 \setminus \Omega \end{cases}$$

$$\Box^2 \tilde{\phi} = 0 \qquad \vec{x} \in \mathbb{R}^3$$

$\Box^2 \phi = 0$ FOR $\vec{x} \in \Omega$

$$\frac{\partial^2 \tilde{\phi}}{\partial t} = \frac{\partial^2 \tilde{\phi}}{\partial t}$$

$$\tilde{\nabla} \tilde{\phi} = \nabla \tilde{\phi} + \phi(\vec{x}) \vec{n}\, \delta(f)$$

$$\tilde{\nabla}^2 \tilde{\phi} = \nabla^2 \tilde{\phi} + \frac{\partial \phi}{\partial n} \delta(f) + \nabla \cdot [\phi \vec{n}\, \delta(f)]$$

$$\bar{\Box}^2 \tilde{\phi} = \underbrace{\Box^2 \tilde{\phi}}_{=0} - \underbrace{\frac{\partial \phi}{\partial n}}_{\equiv \phi_n} \delta(f) - \nabla \cdot [\phi \vec{n}\, \delta(f)]$$

$$4\pi \tilde{\phi}(\vec{x}, t) = -\int_{\partial\Omega} \frac{[\phi_n]_{ret}}{r} dS - \nabla_x \cdot \int_{\partial\Omega} \frac{[\phi \vec{n}]_{ret}}{r} dS$$

$$= -\int \frac{[\phi_n]_{ret}}{r} dS + \int \left\{ \frac{[\dot{\phi} \cos\theta]_{ret}}{cr} + \frac{[\phi \cos\theta]_{ret}}{r^2} \right\} dS$$

BECAUSE $\nabla_x \cdot [\vec{n}\, \phi(\vec{y}, t - r/c)] = -\frac{\vec{n} \cdot \hat{r}}{c} [\dot{\phi}]_{ret}$

$$= -\frac{1}{c} \cos\theta\, [\dot{\phi}]_{ret}$$

NOTES ON GREEN'S FUNCTIONS (79)

THIS IS THE SIMPLEST METHOD OF FINDING THE KIRCHHOFF FORMULA FOR A STATIONARY SURFACE DERIVED IN 19TH CENTURY BY KIRCHHOFF. FOR THE KIRCHHOFF FORMULA FOR A MOVING SURFACE, SEE MY NASA TM-110285, 1996. THE FORMULA WAS ORIGINALLY DERIVED BY W. R. MORGANS IN 1932 BY CLASSICAL ANALYSIS. THERE WAS A QUESTION ABOUT ITS CORRECTNESS WHICH WAS VERIFIED BY USING GENERALIZED FUNCTION THEORY.

NOTES ON GREEN'S FUNCTIONS

THE UNIQUENESS THEOREM FOR WAVE EQUATION

WE CONSIDER THE WAVE EQ. WITH $c = 1$

$$\Box^2 \phi = \phi_{tt} - \nabla^2 \phi$$

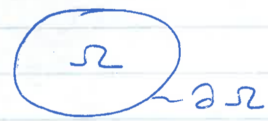
$$\phi_t \Box^2 \phi = \phi_t \phi_{tt} - \phi_t \nabla^2 \phi$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2}\phi_t^2\right) - \nabla \cdot [\phi_t \nabla \phi] - \frac{\partial}{\partial t}\left[\frac{1}{2}|\nabla \phi|^2\right]$$

NOW LET US CONSIDER THE FOLLOWING TWO PROBLEMS :

$$\Box^2 \phi = \psi \quad \begin{cases} x \in \Omega \\ t \in [0,\infty) \end{cases}$$

$$\begin{cases} \phi(\vec{x},0) = f(\vec{x}) & \vec{x} \in \Omega \\ \dfrac{\partial \phi}{\partial t}(\vec{x},0) = g(\vec{x}) & \vec{x} \in \Omega \end{cases}$$



① EITHER $\phi(\vec{x},t) = h(\vec{x},t) \quad \vec{x} \in \partial\Omega, t \in (0,\infty)$

② OR $\dfrac{\partial \phi}{\partial n}(\vec{x},t) = h(\vec{x},t)$ " "

CONSISTENCY OF BC & I.C. :

FOR PROB. ①, WE REQUIRE $h(\vec{x},0_+) = f(\vec{x})$, $\vec{x} \in \partial\Omega$

FOR PROB. ②, WE REQUIRE $\frac{\partial h}{\partial t}(\vec{x},0_+) = \frac{\partial g}{\partial n}(\vec{x})$, $\vec{x} \in \partial\Omega$

WE WILL SEE THAT THESE CONSISTENCY CONDITIONS ARE REQUIRED FOR THE UNIQUENESS THEOREM.

UNIQUENESS THM : THE SOLUTIONS OF PROBLEMS ① AND ② ARE UNIQUE.

PROOF : LET ϕ_1 AND ϕ_2 BE TWO SOLUTIONS.
TAKE $\phi = \phi_1 - \phi_2 \Rightarrow \phi$ SATISFIES

$$\Box^2 \phi = 0 \quad \vec{x} \in \Omega, \; t \in [0,\infty)$$

$$\phi(\vec{x},0) = 0 \quad \vec{x} \in \Omega$$

$$\frac{\partial \phi}{\partial t}(\vec{x},0) = 0 \quad \vec{x} \in \Omega$$

EITHER $\phi(\vec{x},t) = 0$ OR $\dfrac{\partial \phi}{\partial n}(\vec{x},t) = 0$, $\vec{x} \in \partial\Omega, t \in [0,\infty)$

NOTES ON GREEN'S FUNCTIONS

WE HAVE

$$\phi_t \Box^2 \phi = 0 = \frac{\partial}{\partial t}\left\{\frac{1}{2}\phi_t^2 + \frac{1}{2}|\nabla\phi|^2\right] - \nabla\cdot[\phi_t \nabla\phi]$$

$$\Rightarrow \int_0^T \int_\Omega \frac{\partial}{\partial t}\left[\frac{1}{2}\phi_t^2 + \frac{1}{2}|\nabla\phi|^2\right] d\vec{x}\, dt$$

$$= \int_0^T \int_{\partial\Omega} \phi_t \frac{\partial \phi}{\partial n}\, dS\, dt$$

$= 0$ BY CONDITIONS ON $\partial\Omega$

$$\Rightarrow \int_\Omega \left[\frac{1}{2}\phi_t^2 + \frac{1}{2}|\nabla\phi|^2\right]_{t=T} =$$

$$\int_\Omega \left[\frac{1}{2}\phi_t^2 + \frac{1}{2}|\nabla\phi|^2\right]_{t=0}$$

NOW FROM INITIAL CONDITIONS $\left[\frac{1}{2}\phi_t^2 + \frac{1}{2}|\nabla\phi|^2\right]_{t=0} = 0 \Rightarrow$

$$\left[\phi_t^2 + |\nabla\phi|^2\right]_{t=T} = 0 \quad \vec{x}\in\Omega$$

$$\Rightarrow \phi(\vec{x},T) = \text{CONST.} = \phi(\vec{x},0) = 0$$

$\therefore\ \phi_1 = \phi_2$ i.e. THE SOLUTION IS UNIQUE.

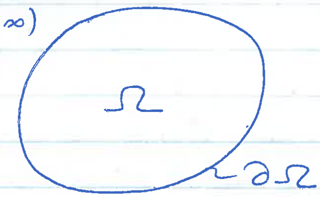
WHERE HAVE WE USED THE CONDITION $h(\vec{x},0+) = f(\vec{x})$? THIS CONDITION MEANS THAT $\phi(\vec{x},0) \in C$ ON $\partial\Omega$ SO THAT $|\nabla\phi|$ DOES NOT BLOW UP ON $\partial\Omega$. SO THIS IS A NECESSARY CONDITION. WE NEED TO ASSUME ADDITIONALLY THAT h & $f \in C^2$.

BASICALLY, THE METHOD WORKS IF $\frac{\partial\phi}{\partial n}$ OR $\frac{\partial\phi}{\partial n} + \alpha(\vec{x},t)\phi$ $(\alpha \geq 0)$ ON $\partial\Omega$ FOR $t\in[0,\infty)$.

NOTES ON GREEN'S FUNCTIONS

GREEN'S FUNCTION FOR WAVE EQUATION IN A BOUNDED DOMAIN

$$\begin{cases} \Box^2 \phi = \psi(\vec{x},t) & \vec{x} \in \Omega,\ t \in [0,\infty) \\ \phi(\vec{x},0) = f(\vec{x}) & \vec{x} \in \Omega \\ \phi_t(\vec{x},0) = g(\vec{x}) & \vec{x} \in \Omega \\ \phi(\vec{x},t) = h(\vec{x},t) & \vec{x} \in \partial\Omega,\ t \in [0,\infty) \\ h(x,0_+) = f(\vec{x}) & \end{cases}$$



G IS A FUNCTION OF $t-\tau$. WE LET $\tau = 0$ FIRST.

SOLVE $\begin{cases} -\nabla^2 u_n = \lambda_n u_n & \vec{x} \in \Omega \\ u_n(\vec{x}) = 0 & \vec{x} \in \partial\Omega \end{cases}$

$$G(\vec{x},\vec{y},t) = \sum_n C_n(t)\, u_n(\vec{x})\, u_n(\vec{y})$$

$$\Box^2_{(\vec{x},t)} G = \delta(\vec{x}-\vec{y})\,\delta(t)$$

$$= \sum_n \left[c^{-2} C_n'' + \lambda_n C_n\right] u_n(\vec{x})\, u_n(\vec{y})$$

IF WE MAKE $\frac{1}{c^2} C_n'' + \lambda_n C_n = \delta(t)$,

THEN BECAUSE

$$\sum_n u_n(\vec{x})\, u_n(\vec{y}) = \delta(\vec{x}-\vec{y})$$

WE HAVE FOUND THE GREEN'S FUNCTION!

WE HAVE $C_n(t) = 0$ FOR $t < 0$. THIS FOLLOWS FROM CAUSALITY. (SEE P85 FOR λ_n)

$$C_n(t) = A_n \sin c\sqrt{\lambda_n}\, t = A_n \sin c k_n t$$

WHERE $k_n = \sqrt{\lambda_n}$. WHAT IS A_n? C_n' HAS A DISCONTINUITY AT $t = 0$:

$$\frac{1}{c^2} \cdot c k_n A_n = \frac{k_n A_n}{c} = 1 \implies A_n = \frac{c}{k_n}$$

$$G(\vec{x},\vec{y},t-\tau) = \sum_{n=1}^{\infty} \frac{c}{k_n} \sin c k_n (t-\tau)\, u_n(\vec{x})\, u_n(\vec{y})$$

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NOTES ON GREEN'S FUNCTION

WE DEFINE

$$\langle u, v \rangle = \int_0^{t_+} \int_{\Omega} u\, v \, d\vec{y}\, d\tau$$

THIS PROBLEM IS SELF-ADJ. (REMEMBER THAT THE LIN. OP. IS $\Box^2$, & $\phi = 0$ ON $\partial\Omega$). $\therefore$ $\Box^{2*} = \Box^2$

$$\langle \Box^{2*}_{(y,\tau)} G(\vec{x},\vec{y},t-\tau), \phi(\vec{y},\tau) \rangle = \langle \delta(\vec{x}-\vec{y})\delta(t-\tau), \phi(\vec{y},\tau) \rangle$$

$$= \phi(\vec{x},t) = \langle G(\vec{x},\vec{y},t-\tau), \overbrace{\Box^2_{(\vec{y},\tau)} \phi(\vec{y},\tau)}^{\psi(\vec{y},\tau)} \rangle + \int_0^{t_+}\int_{\Omega} E(G,\phi)\, d\vec{y}\, d\tau$$

WE NEXT FIND $E(G,\phi)$ BY INTEGRATION BY PARTS:

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial \tau^2} \cdot \phi(\vec{y},\tau) = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial \tau^2} \cdot G + (G_\tau \phi - G\phi_t)_\tau$$

$$\nabla_y^2 G \cdot \phi(\vec{y},\tau) = \nabla^2 \phi \cdot G + \nabla_y \cdot (\phi \nabla G - G\nabla\phi)$$

$$\Rightarrow \quad \Box^2_{(y,\tau)} G \cdot \phi = G\, \Box^2_{(y,\tau)}\phi + (G_\tau\phi - G\phi_t)_\tau / c^2$$
$$- \nabla_y \cdot (\phi \nabla_y G - G\nabla_y \phi)$$

$$\therefore \quad E(G,\phi) = \frac{1}{c^2}(G_\tau \phi - G\phi_\tau) - \nabla_y \cdot (\phi \nabla G - G\nabla\phi)$$

$$\int_0^{t_+}\int_{\Omega} E(G,\phi)\, d\vec{y}\, d\tau = \frac{1}{c^2}\int_{\Omega} \left[G_\tau \phi - G\phi_\tau\right]_0^{t_+} d\vec{y}$$
$$- \int_0^{t_+}\int_{\partial\Omega} \left(\phi \frac{\partial G}{\partial n_y} - G\frac{\partial \phi}{\partial n_y}\right) dS\, d\tau$$

BY CAUSALITY:

$$G(\vec{x},\vec{y},\underbrace{t-t_+}_{<0}) = 0$$

$$\frac{\partial G}{\partial \tau}(\vec{x},\vec{y},t-t_+) = 0$$

NOTES ON GREEN'S FUNCTIONS

ALSO SINCE $u_n = 0$ ON $\partial\Omega \Rightarrow G = 0$ ON $\partial\Omega$

$\therefore$ USING THE IC'S AND THE BC

$$\boxed{\begin{aligned}\phi(\vec{x},t) = &\int_0^t \int_\Omega G(\vec{x},\vec{y},t-\tau)\,\psi(\vec{y},\tau)\,d\vec{y}\,d\tau \\ &- \frac{1}{c^2}\int_\Omega \Big[f(\vec{y})\,\frac{\partial G}{\partial \tau} - G\, g(\vec{y})\Big]\,d\vec{y} \quad G(\vec{x},\vec{y},t) \\ &- \int_0^t \int_{\partial\Omega} h(\vec{y})\,\frac{\partial G}{\partial n_y}\,ds\,d\tau\end{aligned}}$$

SEE P 71 ALSO NOTE: HAD WE TAKEN $\langle u, v\rangle = \int_0^{t_-}\int_\Omega u v\, d\vec{y}\, d\tau$

THEN $\langle \delta(\vec{x}-\vec{y})\,\delta(t-\tau), \phi(\vec{y},\tau)\rangle = 0$ BECAUSE $\delta(t-t_-) = 0$ AND THEN FROM THE PROPERTY OF G, WE HAVE

$$\begin{cases} G(\vec{x},\vec{y},t-t_-) = 0 & (\text{CONT. OF } G \text{ AT } t=0) \\ \dfrac{\partial G}{\partial \tau}(\vec{x},\vec{y},t-t_-) = -c^2\,\delta(\vec{x}-\vec{y}) \end{cases}$$

(THE SECOND RESULT COMES FROM G ON P82).

NOW

$$\langle \square^{2*}_{(y,\tau)} G, \phi(\vec{y},\tau)\rangle = \langle \delta(\vec{x}-\vec{y})\,\delta(t-\tau), \phi(\vec{y},\tau)\rangle$$

$$= 0 = \langle G, \square^2_{(y,\tau)}\phi(\vec{y},\tau)\rangle + \int_0^{t_-}\int_\Omega E(G,\phi)\,d\vec{y}\,d\tau$$

$$= \langle G, \psi(\vec{y},\tau)\rangle + \int_\Omega \frac{1}{c^2}\big[G_\tau\phi - G\phi_\tau\big]_0^{t_-} d\vec{y}$$

$$- \int_0^t \int_{\partial\Omega} \Big(\phi\frac{\partial G}{\partial n_y} - G\frac{\partial \phi}{\partial n_y}\Big)\,ds\,d\tau$$

$$= \langle G, \psi(\vec{y},\tau)\rangle - \int_\Omega \phi(\vec{y},t)\,\delta(\vec{x}-\vec{y})\,d\vec{y}$$

$$- \frac{1}{c^2}\int_\Omega (G_\tau\phi - G\phi_\tau)\Big|_{\tau=0} d\vec{y} - \int_0^t\int_{\partial\Omega}\Big(\phi\frac{\partial G}{\partial n_y} - G\frac{\partial\phi}{\partial n_y}\Big)ds\,d\tau$$

NOTES ON GREEN'S FUNCTIONS

REARRANGING, WE GET THE SAME RESULT AS ON THE TOP OF PREVIOUS PAGE. THE FIRST METHOD IS THE SAME AS MORSE & FESHBACH USED AND THE 2ND METHOD IS USED BY ROACH. I LIKE THE FIRST METHOD BECAUSE IT IS CONSISTENT WITH WHAT WE HAVE ALWAYS USED:

$$\langle \delta(x), f(x) \rangle = f(0)$$

THE SECOND METHOD WAS ALSO USED BY ROACH FOR THE HEAT EQ. (P 71 OF OUR NOTES). THAT SOLUTION METHOD SHOULD ALSO BE CHANGED TO TO THE FIRST METHOD HERE WHEN I REWRITE THE NOTES.

IS λ_n REAL AND POSITIVE?

WE FOUND λ_n FROM $-\nabla^2 u_i = \lambda_i u_i$. WE HAVE

$$u_i \nabla^2 u_i = \nabla \cdot (u_i \nabla u_i) - |\nabla u_i|^2 = -\lambda_i u_i^2$$

$$\int_\Omega u \, \nabla^2 u \, d\vec{y} = \int_{\partial\Omega} u_i \underbrace{\frac{\partial u}{\partial n}}_{=0 \text{ ON } \partial\Omega} d\vec{y} - \int_\Omega |\nabla u_i|^2 d\vec{y}$$

$$= -\int_\Omega |\nabla u_i|^2 d\vec{y} = -\lambda_i \int u_i^2 d\vec{y}$$

$$\Rightarrow \lambda_i = \frac{\int_\Omega |\nabla u_i|^2 d\vec{y}}{\int_\Omega u_i^2 d\vec{y}} \quad \text{REAL \& POSITIVE!}$$

FOR OTHER IMPORTANT FACTS ABOUT EIGENVALUES AND EIGENFUNCTIONS SEE COURANT & HILBERT, NAIMARK, TITCHMARSH (2 VOL), STAKGOLD, 2ND VOLUME, HELWIG, ETC.

NOTES ON GREEN'S FUNCTIONS (A1)

APPENDIX A

A GENERAL METHOD OF FINDING THE ADJOINT BC'S FOR 2ND ORDER LIN. ODE'S

$$Lu: \begin{cases} lu = f & x \in [0,1] \\ \begin{cases} \alpha_1 u(0) + \alpha_2 u'(0) + \alpha_3 u(1) + \alpha_4 u'(1) = 0 \\ \beta_1 u(0) + \beta_2 u'(0) + \beta_3 u(1) + \beta_4 u'(1) = 0 \end{cases} \end{cases}$$

ALL α'S AND β'S CONSTANT. THE INTEGRATION BY PARTS OF THE LEFTSIDE OF THE FOLLOWING GIVES

$$\langle lu, v \rangle = \langle u, l^* v \rangle + \left. \begin{matrix} \gamma_1 u(0) + \gamma_2 u'(0) \\ + \gamma_3 u(1) + \gamma_4 u'(1) \end{matrix} \right\} = 0$$

WHERE γ'S ARE LINEAR FUNCTIONS OF $v(0), v'(0), v(1), v'(1)$. IN MATRIX FORM, WE HAVE 3 LINEAR EQS AS FOLLOWS

$$\mathbb{M} \begin{bmatrix} u_0 \\ u'_0 \\ u_1 \\ u'_1 \end{bmatrix} \equiv \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{bmatrix} \begin{bmatrix} u_0 \\ u'_0 \\ u_1 \\ u'_1 \end{bmatrix} = 0$$

WHERE $u_0 \equiv u(0)$, $u'_0 = u'(0)$, ETC. NOW $BC^*[v]$ IS FOUND BY SETTING THE DETERMINANT OF ANY 3X3 MATRIX FROM THE 3X4 MATRIX OF COEFFICIENTS TO ZERO. WE CAN ONLY FORM 4 SUCH MATRICES. THE REASON IS THAT THE NULL SPACE OF $BC[u]$ IS 2-DIMENSIONAL $\Rightarrow$ RANK OF THE ABOVE MATRIX IS 2.

EXAMPLES

① $$\begin{cases} lu = u'' \\ BC[u]: \begin{cases} u(0) - 2u'(0) = 0 \\ u(1) - 3u'(1) = 0 \end{cases} \end{cases}$$

$$\langle u'', v \rangle = \langle u, v'' \rangle + (u'v - v'u)\Big|_0^1$$

NOTES ON GREEN'S FUNCTION

A2

- $$(u'v - v'u)\Big|_0^1 = u'(1)v(1) - u(1)v'(1) - u'(0)v(0) + u(0)v'(0)$$

$$\gamma_1 = v'(0) \equiv v_0'$$

$$\gamma_2 = -v_0$$

$$\gamma_3 = -v_1'$$

$$\gamma_4 = v_1$$

$$\Rightarrow \quad \mathbb{M} = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ v_0' & -v_0 & -v_1' & v_1 \end{bmatrix}$$

WE HAVE FOUR 3x3 MATRICES WHOSE DETERMINANTS ARE :

- $$\begin{vmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ v_0' & -v_0 & -v_0' \end{vmatrix} = v_0 + 2v_0' \equiv v(0) - 2v'(0) = 0$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ v_0' & -v_1' & v_1 \end{vmatrix} = v_1 + 3v_1' \equiv v(1) - 3v'(1) = 0$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 0 & 1 & -3 \\ -v_0 & -v_1 & v_1 \end{vmatrix} = v(1) - 3v'(1)$$

$$\begin{vmatrix} 1 & -2 & 0 \\ 0 & 0 & -3 \\ v_0' & -v_0 & v_1 \end{vmatrix} = -3\left[v(0) - 2v'(0)\right]$$

SO WE ONLY HAVE TWO INDEP. BC'S :

- $$BC^*[v] \begin{cases} v(0) - 2v'(0) = 0 \\ v(1) - 3v'(1) = 0 \end{cases} = BC[v] \qquad L: \text{SELF-ADJ.}$$

A3

NOTES ON GREEN'S FUNCTION

(2) $L : \begin{cases} lu = u'' + u' - 2u \\ \begin{cases} u(0) - u'(1) = 0 \\ u'(0) + 2\,u(1) = 0 \end{cases} \end{cases}$

$$\langle lu, v \rangle = \langle u, l^* v \rangle + (u'v - uv' + uv)\Big|_0^1$$

$$l^* v = v'' - v' - 2u$$

$$(u'v - uv' + uv)\Big|_0^1 = u_1' v_1 - u_1 v_1' + u_1 v_1 - u_0' v_0 + u_0 v_0' - u_0 v_0$$

$$\gamma_1 = -v_0 + v_0'$$

$$\gamma_2 = -v_0$$

$$\gamma_3 = v_1 - v_1'$$

$$\gamma_4 = v_1$$

$$\mathbb{M} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 0 \\ -v_0 + v_0' & -v_0 & v_1 - v_1' & v_1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ -v_0 + v_0' & -v_0 & v_1 - v_1' \end{vmatrix} = 2v_0 + v_1 - v_1' = 0$$

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -v_0 + v_0' & -v_0 & v_1 \end{vmatrix} = -v_0 + v_0' + v_1 = 0$$

THESE ARE THE SAME AS WHAT WE FOUND ON PAGE 8 OF OUR NOTES.

(A4)

GREEN'S FUNCTION

- COMMENTS ABOUT THE FIRST CHAPTER OF ROACH'S BOOK "GREEN'S FUNCTION" 1982

1. NOTE THE NICE "DERIVATION" OF THE GREEN'S FUNCTION OF THE LIN. OP.

$$Lu: \begin{cases} u'' + k^2 u = -f(x) & x \in [0, l] \\ u(0) = u(l) = 0 \end{cases}$$

BOOK EQ (1.1) (1.2)

THE GUESS FOR THE SOLUTION

$$u(x) = A(x)\cos kx + B(x)\sin kx \qquad \text{BOOK EQ } (1.3)$$

IS FAMILIAR TO PEOPLE WHO HAVE TAKEN A COURSE IN ODE.

2. ROACH DEFINES THE DIFFERENTIAL EQUATION AS $lu = -f(x)$

i.e. HE USES A CONVENTION HERE WHICH IS DIFFERENT THAN OTHER AUTHORS. HOWEVER, HE ALSO DEFINES

$$L_x G(x,y) = -\delta(x-y)$$

SO THAT

$$u(x) = \int_0^l f(y)\, G(x,y)\, dy \qquad \text{BOOK EQ. } (1.11)$$

I HAVE NO IDEA WHY ROACH DOES THIS. NOT BEING USED TO THESE TWO CHANGES OF NOTATION, I GET CONFUSED READING ROACH'S BOOK. HOWEVER, ROACH'S BOOK IS AN EXCELLENT BOOK THAT YOU SHOULD READ FOR ADVANCED CONCEPTS. HERE ARE THE THREE BOOKS THAT I STRONGLY RECOMMEND:

1. G.F. ROACH: GREEN'S FUNCTIONS, 2ND ED.,

A5

NOTES ON GREEN'S FUNCTIONS
CAMBRIDGE UNIV. PRESS, 1982

2. G. BARTON : ELEMENTS OF GREEN'S FUNCTIONS AND PROPAGATION – POTENTIALS, DIFFUSION AND WAVES, NEW ED. 1989 (CORRECTED REPRINT 1991), OXFORD UNIVERSITY PRESS
BARTON IS A PHYSICIST AND THIS BOOK IS VERY GOOD WITH NICE APPLICATIONS PARTICULARLY FOR ENGINEERS & SCIENTISTS

3. C. LANCZOS : LINEAR DIFFERERENTIAL OPERATORS, DOVER BOOKS, 1997 (ORIGINAL 1960)

LANCZOS IS ONE OF MY FAVORITE AUTHORS. HE WRITES BEAUTIFULLY AND CLEARLY. READ HOW LANCZOS SHOWS ONE CAN "GUESS" THE EXISTENCE OF GREEN'S FUNCTION OF A LIN. DIFF. OP. IF ONE USES FINITE DIFFERENCE APPROACH TO THE SOLUTION OF THE OPERATOR. MANY MATHEMATICIANS DO NOT REFER TO THIS BOOK BECAUSE IT DOES NOT MEET THEIR STANDARD OF RIGOR BUT MANY MATHEMATICIANS CAN NOT EXPLAIN MATH AS CLEARLY AS LANCZOS DOES! WE NEED MORE AUTHORS LIKE LANCZOS.

8 The Mathematics of Near Field Acoustical Holography (Course)

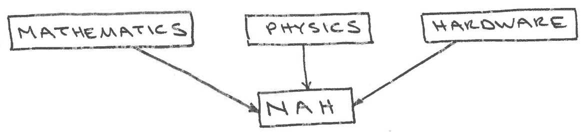
F. FARASSAT

THE MATHEMATICS OF NEAR FIELD ACOUSTICAL HOLOGRAPHY (NAH)

F. FARASSAT

PRESENTED AT LANGLEY RESEARCH CENTER - START DATE AUGUST 23, 2000

- ACOUSTICAL HOLOGRAPHY: CONSTRUCTION OF THE ACOUSTIC FIELD OF A SOURCE FROM MEASUREMENTS ON A SURFACE (A PLANE, CYLINDER, ETC.)
- APPEARED MID 1960'S, HAS LIMITED RESOLUTION
- NAH (WILLIAMS & MAYNARD, 1980) HAS HIGH RESOLUTION. MEASUREMENTS ARE PERFORMED IN NEARFIELD. GIVES ACOUSTIC PRESSURE AND INTENSITY VECTOR.
- NAH IS ESSENTIALLY AN INVERSE PROBLEM (A HOT TOPIC!)



- WE WILL COVER MATH & PHYSICS IN THIS COURSE.

MATHEMATICAL CONCEPTS

1. FOURIER TRANSFORM

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \qquad \text{1-D}$$

$$F(\vec{k}) = \int_{-\infty}^{\infty} f(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x} \qquad \text{3-D}$$

$$d\vec{x} = dx_1\, dx_2\, dx_3 \quad \text{OR} \quad d\vec{x} = dx\, dy\, dz \quad \text{FOR 3D}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \qquad \text{1-D}$$

$$f(\vec{x}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} F(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k} \qquad \text{3-D}$$

$$d\vec{k} = dk_1\, dk_2\, dk_3 \quad \text{or} \quad d\vec{k} = dk_x\, dk_y\, dk_z$$

- FOR TIME DEPENDENT FUNCTIONS, WE DEFINE:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt \;;\; f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega$$

NOTE SIGN OF EXPONENTIALS HERE

THE REASON FOR REVERSAL OF THE SIGN OF THE ARGUMENT OF THE EXPONENTIAL IS TO MAINTAIN THE FORM OF PLANE WAVES IN THE INVERSE TRANSFORM:

$$f(\vec{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} F(\vec{k}, \omega)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}\, d\vec{k}\, d\omega$$

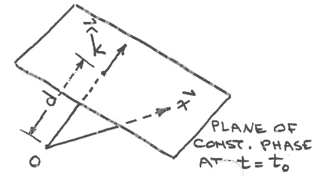
$e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ IS A PLANE WAVE BECAUSE IF WE FREEZE TIME, SAY $t = t_0$, THE SURFACES OF CONSTANT PHASE ARE PLANES. LET $\vec{k}\cdot\vec{x} - \omega t_0 = \theta_0$ (CONST.) $\Longrightarrow$

$$\vec{k}\cdot\vec{x} = \omega t_0 + \theta_0 = \text{CONST.}$$

$$d = \frac{\vec{k}}{k}\cdot\vec{x} = \hat{k}\cdot\vec{x} = \frac{\omega t_0 + \theta_0}{k} = \text{CONST.},\ k = |\vec{k}|$$

THIS IS A PLANE!

— WE WILL STUDY PLANE WAVES FURTHER IN LATER LECTURES.



— SOME PROPERTIES OF F.T.

i) $$F[f'(x)] = \int_{-\infty}^{\infty} f'(x)\, e^{-ikx}\, dx \qquad \text{(INTEGRATE BY PARTS)}$$

$$= ik \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx = ik\,F(k)$$

$$F[\partial_j f(\vec{x})] = \int_{-\infty}^{\infty} \frac{\partial f}{\partial x_j}\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{x}$$

$$= i k_j F(\vec{k})$$

— IN PARTICULAR $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$

$$F[\nabla^2 f(\vec{x})] = -k^2 F(\vec{k}), \quad k = |\vec{k}|$$

SIMILARLY $$F\left(\frac{\partial f}{\partial t}\right) = \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}\, e^{i\omega t}\, dt = -i\omega F(\omega)$$ (NOTE SIGN)

$$F\left(\frac{\partial^2 f}{\partial t^2}\right) = -\omega^2 F(\omega)$$

ii) SHIFT THEOREM

$E_\alpha f = f(x-\alpha)$ SHIFT TO THE RIGHT BY α

$$F[E_\alpha f] = \int_{-\infty}^{\infty} \underbrace{f(x-\alpha)}_{y} e^{-ikx} dx = \int_{-\infty}^{\infty} f(y) e^{-ik(y+\alpha)} dy$$

$$= e^{-ik\alpha} F(k)$$

$$E_\alpha f(\vec{x}) = f(\vec{x}-\vec{\alpha}) \Rightarrow F[E_\alpha f] = e^{-i\vec{k}\cdot\vec{\alpha}} F(\vec{k})$$

- NOTE THAT FOR SHIFTS IN TIME, WE HAVE A CHANGE OF SIGN OF EXPONENTIAL

$$F[E_\alpha f(t)] = F[f(t-\alpha)]$$
$$= e^{i\omega\alpha} F(\omega)$$

iii) CONVOLUTION THEOREM

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy$$
$$F[f * g(x)] = F(k) G(k)$$

- 3-D CASE

$$\int_{-\infty}^{\infty} f(\vec{x}-\vec{y}) g(\vec{y}) d\vec{y} = f * g(\vec{x})$$

$$F[f * g(\vec{x})] = F(\vec{k}) G(\vec{k})$$

2. THE DIRAC DELTA FUNCTION

i) $$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0) \; ; \; \int_{-\infty}^{\infty} \delta(x-x_0) f(x) dx = f(x_0)$$

$$\int \delta(\vec{x}-\vec{x}_0) f(\vec{x}) d\vec{x} = f(\vec{x}_0)$$

THIS IS CALLED THE SIFTING PROPERTY.

ii) $$F[\delta(x-x_0)] = \int_{-\infty}^{\infty} \delta(x-x_0) e^{-ikx} dx = e^{-ikx_0}$$

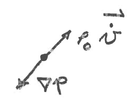
$$\therefore \; \delta(x-x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)} dk \quad \left\{ \begin{array}{l} \text{A VERY} \\ \text{IMPORTANT} \\ \text{RESULT} \end{array} \right.$$

iii) $$\int_{-\infty}^{\infty} \delta^{(n)}(x-x_0) f(x) dx = (-1)^n f^{(n)}(x_0)$$

PLANE WAVES

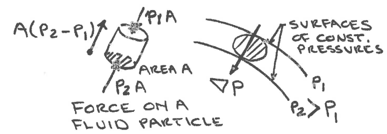
– LINEARIZED EULER EQUATION

$$\rho_0 \dot{\vec{v}} = \rho_0 \frac{\partial \vec{v}}{\partial t} = -\nabla p$$



NOTE THAT ∇p SHOWS THE DIRECTION OF MAXIMUM INCREASE OF PRESSURE. THE FORCE / UNIT VOLUME OF FLUID IS $-\nabla p$.

– IN THIS FIGURE THE NET FORCE $A(P_2 - P_1)$ IS IN THE DIRECTION OF $-\nabla p$



– ACOUSTIC INTENSITY

$$\vec{I} = p(\vec{x},t)\,\vec{v}(\vec{x},t) \qquad \text{INSTANTANEOUS ACOUSTIC INTENSITY}$$

INST. AC. INT. IS ACOUSTIC POWER / UNIT AREA CROSSING A REGION $\perp \vec{I}$ AT ANY MOMENT. WE HAVE THE RELATION

$$\boxed{\frac{\partial e}{\partial t} = -\nabla\cdot\vec{I}}\ , \quad e = \frac{1}{2}\rho_0 v^2 + \frac{1}{2}\frac{p^2}{\rho_0 c^2}$$

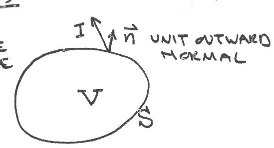
e IS CALLED THE ACOUSTIC ENERGY DENSITY

PLANE WAVES-(CONT'D)

$$\underbrace{\frac{\partial}{\partial t}\int_V e\,dV}_{\text{RATE OF INCREASE OF ENERGY IN } V} = -\int_V \nabla\cdot\vec{I}\,dV \quad \left\{\text{NOW USE THE DIVERGENCE THM}\right.$$

$$= -\int_S \vec{I}\cdot\vec{n}\,dS$$

$$= -\underbrace{\int_S I_n\,dS}_{\text{ACOUST. POWER LEAVING } V} \qquad I_n = \vec{I}\cdot\vec{n}$$



– HARMONIC TIME DEPENDENT (STEADY STATE)

WE ASSUME $p(\vec{x},t) = P(\vec{x})\,e^{-i\omega t}$, $\vec{v}(\vec{x},t) = \vec{V}(\vec{x})\,e^{-i\omega t}$

THEN THE EULER EQ. GIVES $\boxed{i\omega\rho_0\vec{V} = \nabla P}$ P & V COMPLEX

WE NOW USE LOWER CASE LETTERS p & v FOR P & V.

– AVERAGED ACOUSTIC INTENSITY :

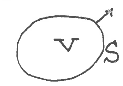
LET $\omega = 2\pi f$, f : FREQUENCY , $T = \frac{1}{f}$ PERIOD OF SOUND

$$\vec{I}(\omega) = \frac{1}{T}\int_0^T \vec{I}\,dt = \frac{1}{T}\int_0^T \mathrm{Re}\,p\ \mathrm{Re}\,\vec{v}\,dt$$

$$= \frac{1}{2}\mathrm{Re}\big[P\,\underbrace{\vec{V}^*}_{\text{COMP. CONJ. OF } \vec{V}}\big] \quad \text{(REMEMBER } P \text{ \& } \vec{V} \text{ ARE COMPLEX)}$$

BECAUSE $\frac{1}{T}\int_0^T \frac{\partial e}{\partial t}\,dt = \frac{1}{T}\left[e(T)-e(0)\right] = 0$, WE HAVE

$$\boxed{\nabla\cdot\vec{I}(\omega) = 0} \Rightarrow \int_S I_n\, dS = 0$$



- WE SHOULD PROPERLY WRITE $I(\vec{x},\omega)$ BUT WE DROP $\vec{x}$!
- WE CAN SHOW THAT THE ACOUSTIC PRESSURE SATISFIES THE WAVE EQUATION:

$$\boxed{\square^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0}$$

THIS CAN BE OBTAINED FROM

$$\begin{array}{ll} \text{TAKE } \frac{\partial}{\partial t} & \left\{ \begin{array}{ll} \frac{1}{c^2}\frac{\partial p}{\partial t} + \rho_0 \nabla\cdot\vec{v} = 0 & \text{LINEARIZED MASS CONT. EQ.} \\ \nabla p + \rho_0 \frac{\partial \vec{v}}{\partial t} = 0 & \text{LINEARIZED MOM. EQ.} \end{array} \right. \\ \text{TAKE } \nabla\cdot & \end{array}$$

SUBTRACT TO GET $\square^2 p = 0$

FOR STEADY STATE $\boxed{H P = \nabla^2 P + k^2 P = 0}$, $k = \frac{\omega}{c}$

THIS IS THE HELMHOLTZ EQUATION.

- MORE ON PLANE WAVES

ANY ACOUSTIC FIELD CAN BE DECOMPOSED INTO PLANE WAVES BY FOURIER TRANSFORM:

$$p(\vec{x},t) = \frac{1}{(2\pi)^4}\int P(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\, d\vec{k}\, d\omega$$

WHERE

$$P(\vec{k},\omega) = \int p(\vec{x},t)\, e^{-i(\vec{k}\cdot\vec{x}-\omega t)}\, d\vec{x}\, dt$$

IT APPEARS THAT THE WAVE NUMBER VECTOR $\vec{k}$ MUST BE REAL. BUT THIS IS NOT SO. THE IMPORTANT CASE OF EVANESCENT WAVES CAN ONLY BE CONSIDERED WHEN $\vec{k}$ IS COMPLEX. WE, THEREFORE, CONSIDER TWO SEPARATE CASES FOR STEADY STATE:

i) PROPAGATING UNDAMPED WAVES

ii) EVANESCENT (DAMPED) WAVES

PLANE WAVES - (CONT'D) LEC. 2/5

i) PROPAGATING UNDAMPED WAVES (STEADY STATE)

$$p = A\, e^{i(\vec{k}\cdot\vec{x} - \omega t)} \quad , \; A \text{ COMP. CONST.}$$

TO SATISFY THE WAVE EQ., WE MUST HAVE

$$|k|^2 = k_1^2 + k_2^2 + k_3^2 = \frac{\omega^2}{c^2} \equiv k^2$$

THIS MEANS THAT THE TIP OF VECTORS $\vec{k}$ IN THE WAVE NUMBER SPACE ARE ON THE SPHERE OF RADIUS k.

SOME FACTS TO REMEMBER

- $\omega = 2\pi f$, f : FREQUENCY , PERIOD $T = \frac{1}{f}$
- WE HAVE ALREADY SHOWN (LEC. 1, P3) THE THE SURFACES OF CONSTANT PHASE ARE PLANES
- SPEED OF PROPAGATION IN THE DIRECTION OF $\vec{k}$

$$\vec{k}\cdot\dot{\vec{x}} - \omega = \vec{k}\cdot(v_p \hat{\vec{k}}) - \omega = k v_p - \omega = 0$$

$$v_p = \frac{\omega}{k} = c \quad \text{SPEED OF SOUND !}$$

$$\hat{\vec{k}} = \vec{k}/k \quad \text{(UNIT LENGTH)}$$

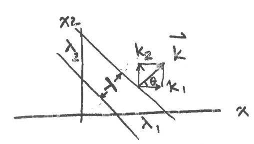
PLANE W ES - (CONT'D) LEC /6

- WAVE LENGTH $\lambda = \frac{2\pi}{k} = cT$
- WAVE LENGTH ON THE AXES

$$\lambda_1 = \frac{2\pi}{k_1} \; , \; \lambda_2 = \frac{2\pi}{k_2} \; , \; \lambda_3 = \frac{2\pi}{k_3}$$

NOTE THAT $\lambda_i \geqslant \lambda$, $i = 1, 2, 3$

- A 2D EXAMPLE



$$k_1 = k\cos\theta$$

$$k_2 = k\sin\theta$$

$$\lambda_1 = \frac{2\pi}{k\cos\theta} = \frac{\lambda}{\cos\theta} \quad , \quad \lambda_2 = \frac{2\pi}{k\sin\theta} = \frac{\lambda}{\sin\theta}$$

- IN 3D, IF $\hat{\vec{k}} = (\hat{k}_1, \hat{k}_2, \hat{k}_3)$, THEN

$$\lambda_i = \frac{\lambda}{\hat{k}_i} \quad (i = 1, 2, 3)$$

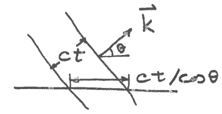
PLANE WAVES - (CONT'D) LEC. 2/7

- TRACE SPEED ALONG THE AXES

$$\vec{k}\cdot\vec{x} - \omega t = k_1 x_1 + k_2 x_2 + k_3 x_3 - \omega t$$

$$\dot{x}_1 = \frac{\omega}{k_1} = \frac{\omega/k}{k_1/k} = \frac{c}{\hat{k}_1} \qquad \text{FROM } \frac{d}{dt}(k_1 x_1 - \omega t) = 0$$

$$\dot{x}_2 = \frac{c}{\hat{k}_2}, \quad \dot{x}_3 = \frac{c}{\hat{k}_3}$$



$\therefore$ TRACE SPEED $\geqslant c$

- AVERAGED INTENSITY

$$i\omega\rho_0\vec{V} = \nabla p = \nabla A e^{i\vec{k}\cdot\vec{x}} = iA\vec{k}e^{i\vec{k}\cdot\vec{x}}$$

$$\boxed{\vec{V} = \frac{p}{\rho_0 c}\hat{k}}$$

$$\boxed{\vec{I} = \frac{|p|^2}{2\rho_0 c}\hat{k} = \frac{|A|^2}{2\rho_0 c}\hat{k}}$$

PLANE WAVES - (CONT'D) LE 2/8

ii) EVANESCENT (DAMPED) WAVES (STEADY STATE)

$$p = A e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{k} = \vec{k}_R + i\vec{k}_I \qquad \text{COMPLEX}$$

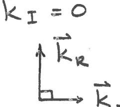
WE MUST HAVE

$$k_1^2 + k_2^2 + k_3^2 = k^2 \equiv \left(\frac{\omega}{c}\right)^2 \text{ REAL}$$

$$k_1^2 + k_2^2 + k_3^2 = \underbrace{k_R^2 - k_I^2}_{\text{REAL}} + \underbrace{2i\vec{k}_R\cdot\vec{k}_I}_{\text{IMAGINARY}} = \underset{\text{REAL}}{k^2}$$

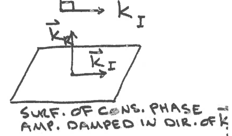
$\therefore$ WE MUST HAVE $\vec{k}_R \perp \vec{k}_I$ SO THAT $\vec{k}_R\cdot\vec{k}_I = 0$

- $k_I^2 = k_R^2 - k^2 \geqslant 0 \quad \therefore \boxed{k_R^2 \geqslant k^2}$



- $p = A e^{-\vec{k}_I\cdot\vec{x}} e^{i(\vec{k}_R\cdot\vec{x} - \omega t)}$

$\vec{k}_I\cdot\vec{x} > 0$ TO HAVE DAMPING

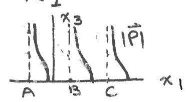


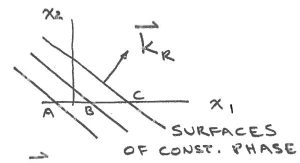
- WE HAVE A DAMPED PROPAGATING PLANE WAVE

- A USEFUL EXAMPLE

TAKE $\vec{k}_R = (k_1, k_2, 0)$, $\vec{k}_I = (0, 0, k_3)$

$\Rightarrow \quad \vec{k}_R \cdot \vec{k}_I = 0$





$\vec{k}_I$: OUT OF THE PLANE

- AVERAGED INTENSITY

$$\vec{I} = \frac{|A|^2}{2\rho_0 c} e^{-2\vec{k}_I \cdot \vec{x}} \hat{\vec{k}}_R$$

$$\hat{\vec{k}}_R = \frac{\vec{k}_R}{k_R}$$

- PROPAGATION SPEED OF EVANESCENT WAVES

$$\vec{k}_R \cdot \dot{\vec{x}} - \omega = 0 \ , \quad \dot{\vec{x}} = v_p \hat{k}_R$$

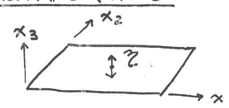
$$k_R v_p - \omega = 0 \ , \quad v_p = \frac{\omega}{k_R} \leqslant \frac{\omega}{k} = c$$

$\therefore \quad v_p \leqslant c \quad$ i.e. EVANESCENT WAVES TRAVEL AT SUBSONIC SPEED

STANDING WAVES ON AN INFINITE VIBRATING PLANE

— ASSUME NORMAL VELOCITY OF STANDING WAVE $\eta(x_1, x_2) = \eta_0 \cos(k_{01}x_1)\cos(k_{02}x_2)$

NOTE THAT THIS IS THE AMPLITUDE DISTRIBUTION OF THE VELOCITY.



WE GUESS THAT THE ACOUSTIC PRESSURE IS GIVEN BY

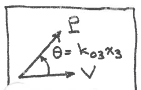
$$P = P_0\, e^{i k_{03} x_3} \cos(k_{01}x_1)\cos(k_{02}x_2)$$

THIS SHOULD SATISFY THE HELMHOLTZ EQ., THEREFORE

$$k_{03} = \pm\sqrt{k^2 - k_{01}^2 - k_{02}^2}\ , \quad k = \frac{\omega}{c}$$

HERE ω IS ANG. FREQ. OF OSCILLATION. WE REJECT THE NEGATIVE SIGN BECAUSE IT WOULD GIVE US INCOMING WAVES. THE EULER EQ. $i\rho_0\omega\vec{V} = \nabla P$ GIVES (LOOKING AT 3RD COMPONENT):

$$i\rho_0\omega\eta_0 = i P_0 k_{03}\ , \quad \boxed{P_0 = \frac{\eta_0\rho_0\omega}{k_{03}}}$$



PROPAGATING WAVE

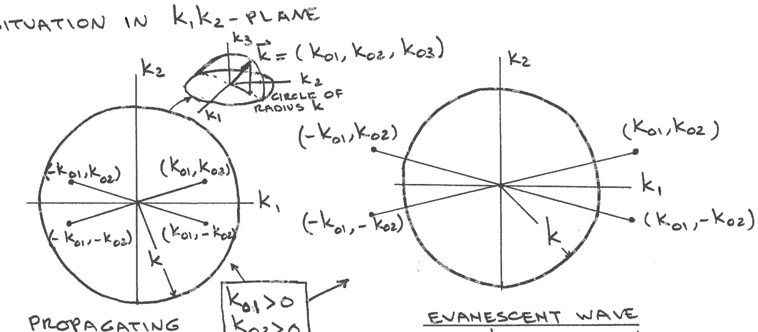
$$\boxed{P(\vec{x}) = \frac{\eta_0\rho_0\omega}{k_{03}}\, e^{i k_{03}x_3}\cos(k_{01}x_1)\cos(k_{02}x_2)}$$

INTERPRETATION IN WAVENUMBER SPACE (k-SPACE)

USING $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, WE HAVE

$$\cos k_{01}x_1 \cos k_{02}x_2 = \frac{1}{4}(e^{ik_{01}x_1} + e^{-ik_{01}x_1})(e^{ik_{02}x_2} + e^{-ik_{02}x_2})$$

MULTIPLYING, WE GET FOUR WAVES OF EQUAL AMPLITUDES OF THE TYPE $e^{\pm i k_{01}x_1 \pm i k_{02}x_2}$. WE HAVE THE FOLLOWING SITUATION IN k_1k_2-PLANE



PROPAGATING WAVE: $k_{03} > 0$ REAL

EVANESCENT WAVE k_3 IMAGINARY

EVANESCENT WAVES

HERE $k_{01}^2 + k_{02}^2 > k^2$ AND $k_{03} = \sqrt{k^2 - k_{01}^2 - k_{02}^2}$

$= \pm i \sqrt{k_{01}^2 + k_{02}^2 - k^2} \equiv \pm i\, k'_{03}$, WE TAKE +VE SIGN TO HAVE DAMPED WAVE. WE HAVE

$$P(\vec{x}) = -i \frac{\eta_0 \rho_0 \omega}{k'_{03}} e^{-k'_{03} x_3} \cos(k_{01}x_1)\cos(k_{02}x_2)$$

TRUE EVERY-WHERE!

- WE NOTE THAT P LAGS V BY $\pi/2$
  ON THE SURFACE $x_3=0$: $I_3(x_1,x_2,0) = \frac{1}{2} Re(\eta P^*) = 0$

- PARTICLE VELOCITY IN THE FIELD

$$\boxed{\vec{V} = -i \frac{1}{\rho_0 \omega} \nabla P} \text{ EASY TO REMEMBER!}$$

$$= \frac{\eta_0}{k'_{03}} e^{-k'_{03}x_3} \big(k_{01} \sin(k_{01}x_1)\cos(k_{02}x_2),\; k_{02}\cos(k_{01}x_1)\sin(k_{02}x_2),\; k'_{03}\cos(k_{01}x_1)\cos(k_{02}x_2) \big)$$

- NOTE V_1 & $V_2 \neq 0$ ON PL[illegible] $x_3=0$

MORE ON PROPAGATING AND EVANESCENT WAVES

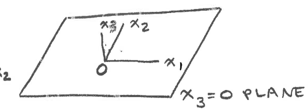
- FOR PROPAGATING WAVES, THE TRACE VELOCITY OF WAVES ON THE VIBRATING SURFACE IS

$$v_T = \frac{\omega}{\sqrt{k_{01}^2 + k_{02}^2}} > \frac{\omega}{k} = c \quad \text{(SUPERSONIC)}$$

- WE HAVE ALREADY SHOWN THAT THE TRACE VELOCITY OF EVANESCENT WAVES IS SUBSONIC.
- P AND η ARE IN PHASE FOR PROPAGATING WAVES ON THE VIBRATING SURFACE. AS $k_{03} = \sqrt{k^2 - k_{01}^2 - k_{02}^2} \longrightarrow 0$ IT APPEARS THAT $P \to \infty$. HOWEVER, IN THIS CASE $\vec{k} = (\pm k_{01}, \pm k_{02}, 0)$, i.e. $V_3 = 0$! THIS MEANS THAT WE CANNOT SATISFY THE BC $\eta = V_3$ ON THE VIBRATING SURFACE. AS $k_{03} \to 0$, $|\vec{V}| \to \infty$ ALSO SO THAT AN INFINITE AMOUNT OF ENERGY IS REQUIRED TO MAINTAIN VIBRATION. IF WE LET $k_{03} \to 0$ FOR AN EVANESCENT WAVE, THE ACOUSTIC ENERGY DENSITY NEAR THE PLATE GROWS TO INFINITY — A CONDITION WHICH IS PHYSICALLY IMPOSSIBLE.

THE ANGULAR SPECTRUM – FOURIER ACOUSTICS

$$\hat{P}(k_1,k_2) = \text{F.T.}\left[P(x_1,x_2,0)\right] = \iint_{-\infty}^{\infty} P(x_1,x_2,0)\, e^{-i(k_1x_1+k_2x_2)}\, dx_1\, dx_2$$



$$k_3 = \sqrt{k^2 - k_1^2 - k_2^2} \quad \text{(CAN BE COMPLEX!)}$$

$$P(\vec{x}) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \hat{P}(k_1,k_2)\, e^{i\vec{k}\cdot\vec{x}}\, dk_1\, dk_2 \qquad (*)$$

$$\vec{k} = (k_1, k_2, k_3(k_1,k_2))$$

THIS MEANS THAT $\boxed{\text{F.T.}\left[P(\vec{x}),\ (x_1,x_2)\to(k_1,k_2)\right] = \hat{P}(k_1,k_2)\, e^{ik_3x_3}}$ VERY IMPORTANT

– $\hat{P}(k_1,k_2)$ IS CALLED THE ANGULAR SPECTRUM OF $P(x_1,x_2,0)$.

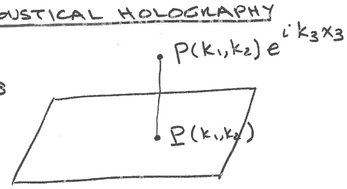
(*) THIS RESULT IS GENERALIZATION OF THE RESULT FOR WAVES ON INFINITE PLATES USING SUPERPOSITION PRINCIPLE.

– FOR EVANESCENT WAVES WE HAVE TO REPLACE $e^{ik_3x_3}$ BY $e^{-|k_3|x_3}$

– THE ABOVE RESULT RELATES $[\text{F.T. } P]_{x_3>0}$ TO $[\text{F.T. } P]_{x_3=0}$

THIS IS THE BASIS OF ACOUSTICAL HOLOGRAPHY

$$\hat{P}(k_1,k_2,x_3) \equiv \hat{P}(k_1,k_2)\, e^{ik_3x_3}$$



– LET $x_3 > x_3'$, THEN

$$\hat{P}(k_1,k_2,x_3) = \hat{P}(k_1,k_2)\, e^{ik_3x_3} = \hat{P}(k_1,k_2)\, e^{ik_3x_3'} \cdot e^{ik_3(x_3-x_3')}$$
$$= \hat{P}(k_1,k_2,x_3')\, e^{ik_3(x_3-x_3')}$$

THIS IS EXTRAPOLATION OF F.T. OF P FROM PLANE $x_3' = \text{CONST.}$ TO $x_3 > x_3'$. AGAIN THIS IS AN IMPORTANT RESULT.

— F.T. OF VELOCITY PHASOR $\vec{V} \Rightarrow \hat{\vec{V}}$: $(x_1, x_2) \to (k_1, k_2)$

$$\hat{\vec{V}} = -\frac{i}{\rho_0 \omega} \widehat{\nabla P}$$ (USE THE RULE FOR F.T. OF DERIVATIVE OF A FUNC'N)

$$\Rightarrow \quad \boxed{\hat{\vec{V}} = \frac{1}{\rho_0 \omega} \vec{k}\, \hat{P}(k_1, k_2, x_3)}$$

THIS MEANS THAT $\hat{\vec{V}}$ IS IN THE DIRECTION OF $\vec{k} = (k_1, k_2, k_3(k_1, k_2))$ IN THE SPACE.

— IF $x_3 > x_3'$, AGAIN WE HAVE <u>EXTRAPOLATION RULE</u>

$$\boxed{\hat{\vec{V}}(k_1, k_2, x_3) = \hat{\vec{V}}(k_1, k_2, x_3')\, e^{i k_3 (x_3 - x_3')}}$$

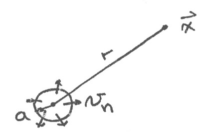
IN PARTICULAR $\quad \hat{V}_3(k_1, k_2, x_3) = \hat{V}_3(k_1, k_2, x_3')\, e^{i k_3 (x_3 - x_3')}$

$$= \frac{k_3}{\rho_0 \omega} \hat{P}(k_1, k_2, x_3')\, e^{i k_3 (x_3 - x_3')}$$

<u>RADIATION FROM SIMPLE SOURCES</u>

i) <u>MONOPOLES</u>

Q : RATE OF MASS INJECTION



$$p(\vec{x}, t) = \frac{1}{4\pi} \frac{\partial}{\partial t} \int_S \frac{\rho_0 [v_n]_{ret}}{r}\, dS$$

$$= \frac{1}{4\pi} \frac{\partial}{\partial t} \frac{[\rho_0 v_n S]_{ret}}{r}$$

$Q = 4\pi a^2 \rho_0 v_n$

$= \rho_0 S v_n$

S AREA OF THE SPHERE

$\boxed{[v_n]_{ret} = v_n(t - r/c)}$

$$\boxed{p(\vec{x}, t) = \frac{1}{4\pi} \frac{[\dot{Q}]_{ret}}{r}}$$

$\boxed{\dot{Q} = \rho_0 S \underbrace{\dot{v}_n}_{\text{ACCELERATION IN RADIAL DIRECTION}}}$

IF $v_n = V_n e^{-i\omega t}$, $p = P e^{-i\omega t}$

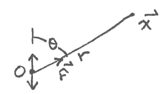
$$\boxed{P = -\frac{i\omega \rho_0 S}{4\pi} \frac{e^{ikr}}{r} V_n} \qquad k = \frac{\omega}{c}$$

$p(\vec{x}, t)$ IS THE SOLUTION OF $\boxed{\Box^2 p = \dot{Q}(t)\, \delta(\vec{x})}$ A MONOPOLE SOURCE AT THE ORIGIN.

ii) <u>DIPOLES</u>

$$p(\vec{x},t) = -\frac{1}{4\pi}\frac{\partial}{\partial x_i}\left\{\frac{[f_i]_{ret}}{r}\right\}$$

$$= \frac{1}{4\pi}\left\{\underbrace{\frac{[\dot{f}_r]_{ret}}{cr}}_{\text{FARFIELD}} + \underbrace{\frac{[f_r]_{ret}}{r^2}}_{\text{NEAR-FIELD}}\right\}$$



OSCILLATING FORCE $\vec{f}(t)$ AT ORIGIN

$$\boxed{\Box^2 p = -\frac{\partial}{\partial x_i}[f_i \, \delta(\vec{x})]}$$

A STATIONARY DIPOLE

$$f_r = \vec{f}\cdot\frac{\vec{r}}{r} = \vec{f}\cdot\hat{r}$$

IF $\vec{f} = \vec{F}\, e^{-i\omega t}$, $p = P(\vec{x})\, e^{-i\omega t}$

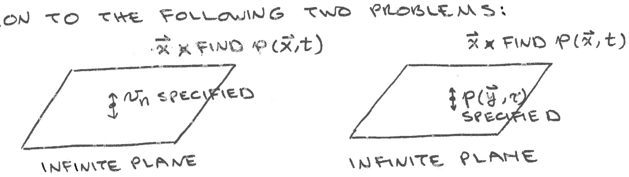
$$P(\vec{x}) = -\frac{F_i}{4\pi}\frac{\partial}{\partial x_i}\left[\frac{e^{ikr}}{r}\right] \qquad r = |\vec{x}|, \; k = \frac{\omega}{c}$$

IF THE DIPOLE IS AT $\vec{y}$, LET $r = |\vec{x} - \vec{y}| \Rightarrow$

$$\boxed{P(\vec{x}) = -\frac{F_i}{4\pi}\frac{\partial}{\partial x_i}\left[\frac{e^{ikr}}{r}\right] = \frac{F_i}{4\pi}\frac{\partial}{\partial y_i}\left[\frac{e^{ikr}}{r}\right]}$$

<u>RAYLEIGH'S INTEGRALS</u>

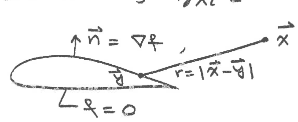
THESE INTEGRALS ARE MORE EASILY OBTAINED IF WE USE TIME DOMAIN METHOD. THE TWO RAYLEIGH INTEGRALS GIVE THE SOLUTION TO THE FOLLOWING TWO PROBLEMS:



THE FFOWCS WILLIAMS-HAWKINGS EQUATION IS (LINEAR CASE!):

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{\partial}{\partial t}\left[\rho_0 v_n \delta(f)\right] - \frac{\partial}{\partial x_i}\left[p n_i \delta(f)\right]$$

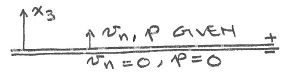
$$\boxed{[v_n]_{ret} \equiv v_n(\vec{y}, t - \tfrac{r}{c})}$$



<u>SOLUTION</u> : $4\pi p(\vec{x},t) = \dfrac{\partial}{\partial t}\displaystyle\int_{f=0} \frac{\rho_0 [v_n]_{ret}\, dS}{r} - \frac{\partial}{\partial x_i}\int \frac{[p n_i]_{ret}\, dS}{r}$ <u>STATIONARY SURFACE</u>

NOW LET $f=0$ BECOME INFINITELY THIN, BUT STILL HAVE TWO SIDES. ASSUME ALSO IT IS AN INFINITE PLANE. LOOKING AT THE PLANE EDGEWISE.

WE ARE INTERESTED IN p IN THE REGION $x_3>0$.



THEN WE CAN ASSUME $v_{n_-}=0$, $p_-=0$ AS SHOWN AND SINCE $\vec{n}=(0,0,1)$, WE HAVE

$$4\pi p(\vec{x},t) = \frac{\partial}{\partial t}\iint_{-\infty}^{\infty} \frac{\rho_0[v_n]_{ret}}{r}\,dy_1\,dy_2 - \frac{\partial}{\partial x_3}\iint_{-\infty}^{\infty} \frac{[p]_{ret}}{r}\,dy_1\,dy_2 \qquad \boxed{\text{CURLE'S FORMULA}}$$

WE CAN WRITE

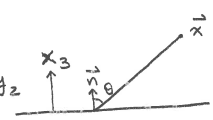
$$\frac{\partial}{\partial t}\iint_{-\infty}^{\infty} \frac{\rho_0[v_n]_{ret}}{r}\,dy_1\,dy_2 = \iint_{-\infty}^{\infty} \frac{\rho_0[\dot{v}_n]_{ret}}{r}\,dy_1\,dy_2$$

$\dot{v}_n$ IS THE LOCAL ACCELERATION OF THE PLANE IN x_3 DIRECTION

$$\text{LET}\quad I = -\frac{\partial}{\partial x_3}\iint_{-\infty}^{\infty} \frac{[p]_{ret}}{r}\,dy_1\,dy_2$$

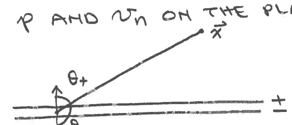
$$= -\iint_{-\infty}^{\infty} \frac{\partial}{\partial x_3}\left[\frac{p(y_1,y_2,t-\frac{r}{c})}{r}\right]dy_1\,dy_2$$

$$= \iint_{-\infty}^{\infty}\left[\underbrace{\frac{\cos\theta\,[\dot{p}]_{ret}}{cr}}_{\text{FAR-FIELD}} + \underbrace{\frac{\cos\theta\,[p]_{ret}}{r^2}}_{\text{NEAR-FIELD}}\right]dy_1\,dy_2$$



NOTE THAT $\frac{\partial r}{\partial x_3}=\cos\theta$. NOTE THAT CURLE'S FORMULA REQUIRES THAT WE KNOW BOTH p AND v_n ON THE PLANE.

NOW ASSUME THE SITUATION SHOWN ON THE RIGHT. WE SEE THAT $\cos\theta_- = -\cos\theta_+$ AND THE CONTRIBUTION OF THE PRESSURE INTEGRALS IS ZERO, i.e. THE PRESSURE INTEGRALS FROM + & – SURFACES CANCEL. WE GET



$$\left.\begin{aligned} v_{n_+} &= v_{n_-}\\ p_{n_+} &= p_{n_-}\end{aligned}\right\}\ \text{ASSUME!}$$

$$4\pi p(\vec{x},t) = 2\iint_{-\infty} \frac{\rho_0[\dot{v}_n]_{ret}}{r}\,dy_1\,dy_2$$

OR

$$\boxed{p(\vec{x},t) = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{\rho_0[\dot{v}_n]_{ret}}{r}\,dy_1 dy_2}\quad \boxed{\text{RAYLEIGH'S FORMULA}}$$

COMPARING THIS TO CURLE'S FORMULA, WE SEE THAT

$$\boxed{\begin{aligned} p(\vec{x},t) &= -\frac{1}{2\pi}\frac{\partial}{\partial x_3}\iint_{-\infty}^{\infty}\frac{[p]_{ret}}{r}\,dy_1 dy_2 \\ &= \frac{1}{2\pi}\iint_{-\infty}^{\infty}\left[\frac{\cos\theta[\dot{p}]_{ret}}{cr} + \frac{\cos\theta[p]_{ret}}{r^2}\right]dy_1 dy_2 \end{aligned}}$$

FORM 1A OF FERI

THIS SHOULD ALSO BE CALLED RAYLEIGH'S FORMULA BUT IT IS NOT. IT IS ALSO <u>NOT</u> KIRCHHOFF FORMULA BECAUSE THERE IS NO $\partial p/\partial n = p_n$ HERE. NOW LET US FIND THE RAYLEIGH INTEGRALS RELEVANT TO ACOUSTICAL HOLOGRAPHY.

LET $v_n = V_n(\vec{x})\,e^{-i\omega t}$, $p = P(\vec{x})\,e^{-i\omega t}$ WHERE V_n AND P ARE COMPLEX AMPLITUDES. THEN

$$[\dot{v}_n]_{ret} = -i\omega V_n e^{-i\omega(t-r/c)} = -i\omega V_n e^{ikr} e^{-i\omega t}$$

$$[p]_{ret} = P\,e^{-i\omega(t-r/c)} = P e^{ikr} e^{-i\omega t} \qquad \boxed{k=\frac{\omega}{c}}$$

THE FORMULA FOR $p(\vec{x},t)$ IN TERMS OF p, FOR AMPLITUDES BECOMES

$$\begin{aligned} P(\vec{x}) &= -\frac{1}{2\pi}\frac{\partial}{\partial x_3}\iint_{-\infty}^{\infty}\frac{P(y_1,y_2,0)\,e^{ikr}}{r}\,dy_1 dy_2 \\ &= -\frac{1}{2\pi}\iint_{-\infty}^{\infty}P(y_1,y_2,0)\frac{\partial}{\partial x_3}\left[\frac{e^{ikr}}{r}\right]dy_1 dy_2 \\ &= \frac{1}{2\pi}\iint_{-\infty}^{\infty}P(y_1,y_2,0)\frac{\partial}{\partial y_3}\left[\frac{e^{ikr}}{r}\right]dy_1 dy_2 \end{aligned}$$

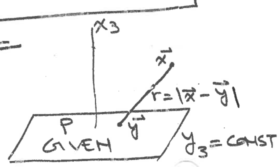
$$\boxed{\frac{\partial r}{\partial x_3} = -\frac{\partial r}{\partial y_3}} \qquad \boxed{x_3 \geqslant 0}$$

FOR $x_3 \geqslant y_3 \neq 0$, WE HAVE

$$\boxed{P(\vec{x}) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}P(\vec{y})\frac{\partial}{\partial y_3}\left[\frac{e^{ikr}}{r}\right]dy_1 dy_2}$$

<u>RAYLEIGH'S FIRST INTEGRAL</u>

$r = |\vec{x}-\vec{y}|$



THE SIGN IN WILLIAM'S BOOK IS WRONG!
[WILLIAMS AGREES THAT THIS IS A MISPRINT].

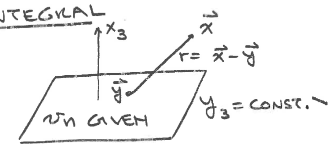
RAYLEIGH'S FORMULA GIVES

$$P(\vec{x}) = -\frac{i\omega\rho_0}{2\pi}\iint_{-\infty}^{\infty} \frac{V_n(y_1,y_2,0)}{r} e^{ikr}\, dy_1\, dy_2 \qquad \boxed{x_3 \geq 0}$$

FOR $x_3 \geq y_3 \neq 0$, WE HAVE

$$\boxed{P(\vec{x}) = -\frac{i\omega\rho_0}{2\pi}\iint_{-\infty}^{\infty} V_n(\vec{y})\, \frac{e^{ikr}}{r}\, dy_1\, dy_2} \qquad x_3 \geq y_3$$

<u>RAYLEIGH'S 2ND INTEGRAL</u>



$P_v = \dfrac{e^{ikr}}{r}$ IS THE <u>PROPAGATOR</u> FOR v_n

$P_p = \dfrac{\partial}{\partial y_3}\left[\dfrac{e^{ikr}}{r}\right]$ IS THE <u>PROPAGATOR</u> FOR p

THE CONCEPT OF PROPAGATOR IS VERY IMPORTANT IN ACOUSTICAL HOLOGRAPHY.

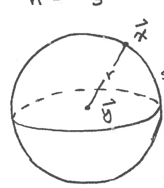
<u>PROPAGATOR CONCEPT</u>

WE HAVE SHOWN THAT (RAYLEIGH'S 2ND INTEGRAL)

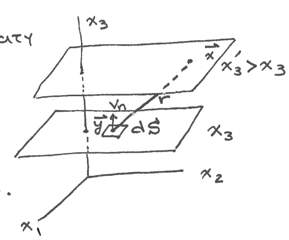
$$P(\vec{x}) = -\frac{i\omega\rho_0}{2\pi}\iint_{-\infty}^{\infty} V_n(\vec{y})\, \frac{e^{ikr}}{r}\, dy_1\, dy_2$$

$$= -\frac{i\omega\rho_0}{2\pi}\int_S V_n(\vec{y})\, \overbrace{P_v(\vec{x},\vec{y})}^{\text{PROPAGATOR}}\, dS$$

$V_n \equiv V_3$ AMP. OF NORMAL VELOCITY



SURFACE OF CONSTANT PHASE OF PROPAGATOR P_v, PHASE $= kr$.



PROPAGATOR IS ESSENTIALLY THE GREEN'S FUNCTION OF THE PROBLEM OF WAVE PROPAGATION IN HALF-SPACE WHEN v_n (OR V_n) IS SPECIFIED ON A SURFACE. P_v IS SPHERICALLY SYMMETRIC. NOTE $P_v(\vec{x},\vec{y}) = P_v(r)$, $r = |\vec{x}-\vec{y}|$.

PLANE WAVES -(CONT'D) LEC. 5/2

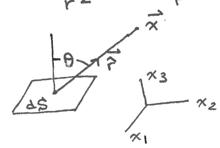
FROM RAYLEIGH'S 1ST INTEGRAL, WE HAVE

$$P(\vec{x}) = \frac{1}{2\pi}\int_S P(\vec{y})\, P_p(\vec{x},\vec{y})\, dS$$

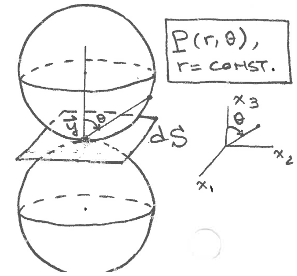
$$P_p(\vec{x},y) = \frac{\partial}{\partial y_3}\left(\frac{e^{ikr}}{r}\right) = \overbrace{-\frac{ik\hat{r}_3\, e^{ikr}}{r}}^{\text{FAR FIELD}} + \overbrace{\frac{\hat{r}_3\, e^{ikr}}{r^2}}^{\text{NEAR FIELD}} \equiv P_p(r,\theta)$$

$$\hat{r}_3 = \frac{\partial r}{\partial x_3} = -\frac{\partial r}{\partial y_3} = \cos\theta$$

$$\hat{\vec{r}} = \frac{\vec{r}}{r} \quad \text{UNIT RADIATION VECTOR}$$



<u>ON A SPHERE OF RADIUS r</u> WITH CENTER AT $\vec{y}$, BOTH THE NEAR AND FAR FIELD TERMS HAVE THE DIRECTIVITY SHOWN ON THE RIGHT. THESE TWO SPHERES ARE ALSO THE SURFACES OF CONSTANT PHASE. THE PHASE FOR FAR FIELD TERM IS $kr-\pi/2$ AND FOR NEAR FIELD TERM IS kr. MAXIMUM INFLUENCE IS FOR $\theta = 0, \pi$. FOR $\theta = \pi/2$, $P_p = 0$.



PLANE WAVES -(CONT'D) LEC 5/3

<u>EWALD SPHERE CONSTRUCTION</u>

LET $|\vec{y}| \ll |\vec{x}|$, THEN WE CAN WRITE

$$r \approx |\vec{x}|\left(1 - \frac{1}{|\vec{x}|}\,\vec{x}\cdot\vec{y}\right) \quad \left\{ \begin{array}{l}\text{FAR FIELD}\\ \text{APPROXIMATION}\end{array}\right.$$

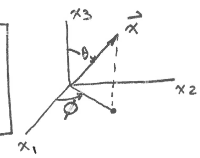
$$P_p(\vec{x},\vec{y}) = \frac{e^{ikr}}{r} \approx \frac{e^{ik|\vec{x}|}}{|\vec{x}|}\; e^{-ik\vec{x}\cdot\vec{y}/|\vec{x}|}$$

WE HAVE SHOWN EARLIER THAT THE DIRECTION OF PROPAGATION OF THE WAVE WITH WAVE NUMBER (k_{01}, k_{02}) IS GIVEN BY $\left(k_{01}, k_{02}, \sqrt{k^2 - k_{01}^2 - k_{02}^2}\right) = \vec{k}$. SO IF THE OBSERVER IS AT $\vec{x}$, ONLY THE WAVE NUMBER $\boxed{\vec{k}/k = \vec{x}/|\vec{x}|}$ WILL CONTRIBUTE TO PRESSURE AT $\vec{x}$. USING RAYLEIGH'S 2ND INTEGRAL, WE GET

$$P(\vec{x}) = -i\rho_0\omega\,\frac{e^{ik|\vec{x}|}}{2\pi|\vec{x}|}\underbrace{\int_S V_n(y_1,y_2)\,e^{-i\vec{k}\cdot\vec{y}}\,dS}_{\hat{V}_n(k_1,k_2)}$$

$$\boxed{P(\vec{x}) = -\frac{i\rho_0\omega}{2\pi}\,\frac{e^{ik|\vec{x}|}}{|\vec{x}|}\,\hat{V}_n(k_1,k_2)} \qquad \boxed{\begin{array}{l}\text{2 DIM'L}\\ \text{FOURIER}\\ \text{TRASFORM OF}\\ V_n(y_1,y_2)\end{array}}$$

THIS RESULT IMPLIES THAT THE DIRECTIVITY FUNCTION $D(\theta,\phi)$ FOR THE FAR FIELD IS

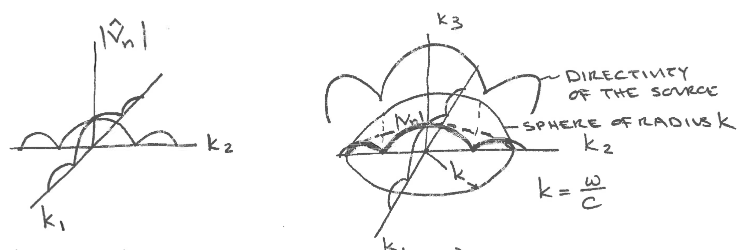


$$\boxed{\begin{aligned} D(\theta,\phi) &= -\frac{i\rho_0\omega}{2\pi}\,\hat{V}_n(k_1,k_2) \\ (k_1,k_2) &= \left(\frac{x_1}{|\vec{x}|},\ \frac{x_2}{|\vec{x}|}\right) \end{aligned}}$$

$$\boxed{p(\vec{x}) = \frac{e^{ik|\vec{x}|}}{|\vec{x}|}\,D(\theta,\phi)}$$

A SIMPLE RESULT TO REMEMBER

<u>THE EWALD SPHERE CONSTRUCTION</u> (FROM X-RAY DIFFRACTION THEORY) IS A METHOD OF VISUALIZING THE FAR FIELD DIRECTIVITY OF SOURCES ON A PLANE. GIVEN $V_n(y_1,y_2)$, FIRST LET US FIND THE 2 DIM'L FOURIER TRANSFORM $\hat{V}_n(k_1,k_2)$. THEN PLOT (USE DENSITY PLOT OF MATHEMATICA) $|\hat{V}_n(k_1,k_2)|$ ON 3-D $\vec{k}$-SPACE. THE DENSITY PLOT IS ON $k_3=0$ PLANE. GIVEN $k=\omega/c$, DRAW SPHERE OF RADIUS k WITH CENTER AT ORIGIN. THEN FOLLOW THE FOLLOWING F URE



$(k_1, k_2, |\hat{V}_n|)$-SPACE $\qquad$ $\vec{k}$ – SPACE

AS YOU SEE, THIS IS AN INGENIUS METHOD OF VISUALIZING THE DIRECTIVITY PATTERN OF PLANAR SOURCES.

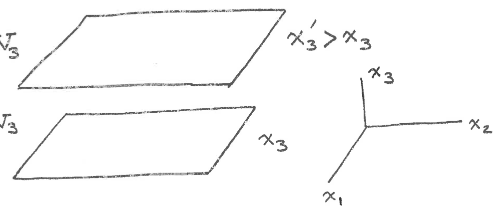
SEE FIGS. 2.14 & 2.15 OF WILLIAMS' BOOK.

PLANAR NAH

THE PROBLEM STATEMENT:

MEASUREMENT PLANE, $\underline{P}$ OR V_3

FIND $\underline{P}$ AND V_3 HERE



WE KNOW THAT GIVEN $\underline{P}$ OR V_n ON x_3-PLANE, THE RAYLEIGH INTEGRALS GIVE $\underline{P}$ (OR V_n) ON $x_3' > x_3$. NOW WE WANT TO RECONSTRUCT $\underline{P}$ OR V_n ON x_3-PLANE FROM MEASUREMENTS ON x_3'-PLANE.

— WE MUST INVERT AN INTEGRAL EQUATION OF FIRST KIND WHICH IS KNOWN TO GENERATE ILL-POSED PROBLEMS, i.e., WE CAN GET WRONG RESULTS IF WE ARE NOT CAREFUL!

WILLIAMS PREFERS TO SOLVE THE PROBLEM WORKING WITH F.T. OF $\underline{P}(\vec{x})$ AND $V_n(\vec{x})$. THIS IS PROBABLY DICTATED BY INSTRUMENTATION AND AVAILABILITY OF F.F.T.

WE HAVE SHOWN FOR $x_3' > x_3$

$$\hat{\underline{P}}(k_1,k_2,x_3') = \hat{\underline{P}}(k_1,k_2,x_3)\, e^{i k_3 (x_3'-x_3)}$$

$$k_3 = \sqrt{k^2-k_1^2-k_2^2}\ ,\quad k^2 \geq k_1^2+k_2^2$$

$$= i\,|k_3| \qquad k^2 < k_1^2+k_2^2$$

$$\hat{V}_n(k_1,k_2,x_3') = \frac{k_3}{\rho_0\omega}\, e^{i k_3 (x_3'-x_3)}\, \hat{\underline{P}}(k_1,k_2,x_3)$$

$$\equiv G(k_1,k_2,x_3'-x_3)\, \underbrace{\hat{\underline{P}}(k_1,k_2,x_3)}_{\mathcal{F}[\underline{P}(x_1,x_2,x_3)]}$$

VERY IMPORTANT $\therefore$ $\boxed{V_n(x_1,x_2,x_3') = \mathcal{F}^{-1}\left[G(k_1,k_2,x_3'-x_3)\,\mathcal{F}\left[\underline{P}(x_1,x_2,x_3)\right]\right]}$

ALL FOURIER TRANSFORMS & INVERSE ARE 2D $(x_1,x_2) \Longrightarrow (k_1,k_2)$.

WE HAVE SHOWN THAT

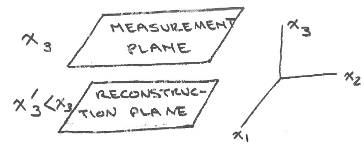
$$V_n(x_1, x_2, x_3') = \mathcal{F}^{-1}\left[G(k_1,k_2,x_3'-x)\,\mathcal{F}[\rho(x_1,x_2,x_3)]\right]$$

FROM THIS WE HAVE, FROM OUR FIRST LECTURE

$$\boxed{V_n(x_1,x_2,x_3') = \mathcal{F}^{-1}[G(k_1,k_2,x_3'-x_3)] * \rho(x_1,x_2,x_3)}$$

THE CONVOLUTION IS 2 DIM'L IN (x_1, x_2) HERE.

NOTE $x_3' - x_3 < 0$ SO THAT WE HAVE $e^{-|k_3|(x_3'-x_3)}$ IS EXPONENTIALLY GROWING FOR EVANESCENT WAVES.



THIS CAUSES ILL-POSEDNESS OF THE PROBLEM : ERRORS OF MEASUREMENTS AT x_3 GROW AT x_3' TO SUCH AN EXTENT THAT $V_n(x_1, x_2, x_3')$ CAN BECOME MEANINGLESS.

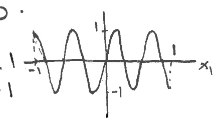
— SEE SEC. 3.4 OF WILLIAMS FOR AN ERROR ANALYSIS OF ONE DIM'L WAVE PROPAGATION.

$$G(k_1,k_2,x_3) = \frac{k_3}{\omega}\, e^{i k_3 x_3}, \quad k_3 = k_3(k_1,k_2)$$

IN GENERAL, $v_n(\vec{x},t) = V_n(\vec{x})\,e^{-i\omega t}$ ON A SURFACE HAS FOURIER COMPONENTS WITH $|\vec{k}| > \frac{\omega}{c}$ WHICH PRODUCE EVANESCENT WAVES WHICH DECAY FOR $x_3 > 0$.

EXAMPLE : $V_3 = \sin \frac{7\pi x_1}{2} \qquad |x_1| < 1$

$\qquad = 0 \qquad |x_1| > 1$



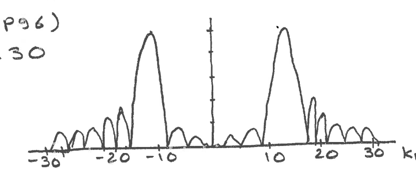
$$\hat{V}_3(\vec{k}) = \int_{-\infty}^{\infty}\int_{-1}^{1} \sin 3.5\pi x_1 \; e^{-i(k_1x_1+k_2x_2)}\,dx_1\,dx_2$$

$$= \frac{2\pi}{2i}\,\delta(k_2)\int_{-1}^{1}\left[e^{i(3.5\pi-k_1)x_1} - e^{-i(3.5\pi+k_1)x_1}\right]dx_1$$

$\boxed{\operatorname{sinc} x = \frac{\sin x}{x}}$

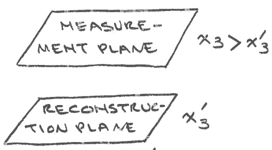
$$= \pi\,\delta(k_2)\left[\operatorname{sinc}(3.5-k_1) + \operatorname{sinc}(3.5\pi+k_1)\right]$$

THE FUNCTION IN SQ. BRACKETS, IN ABSOLUTE VALUE, LOOKS LIKE (FIG. 3.2, WILLIAMS, P96)
IT IS FAIRLY LARGE FOR $k_3 = 30$ ALSO.



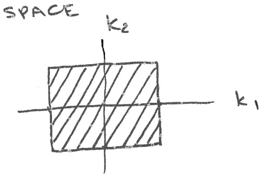
<u>ONE REMEDY FOR ILL-POSEDNESS</u>

WE NOTE THAT FOR HIGH $|\vec{k}| > \frac{\omega}{c}$, $e^{k_3'(x_3 - x_3')}$ IS SO SMALL THAT IT IS OF THE ORDER OF THE ERROR IN MEASUREMENT OF $\underline{P}(\vec{x})$ AT x_3.



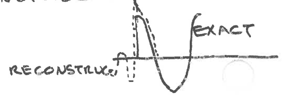
SO <u>LIMIT $|\vec{k}|$ TO FINITE VALUES</u> SUCH THAT $k_3' = |k_3|$ DOES NOT GET VERY BIG. THIS IS EQUIVALENT TO PUTTING A RECTANGULAR WINDOW IN $k_1 k_2$-SPACE

THIS IS <u>A REASONABLY GOOD METHOD</u>. SEE FIG. 3.3 (P96) OF WILLIAMS.



k_1 & $k_2 = 0$ <u>OUTSIDE</u> THE RECTANGLE

<u>PROBLEM</u>: WE CAN GET GIBBS PHENOMENON WHERE THERE ARE DISCONTINUITIES IN V_n.



<u>OTHER REMEDIES FOR ILL-POSEDNESS</u>

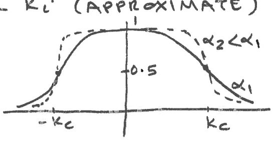
THE METHOD OF PREVIOUS SLIDE IS EQUIVALENT TO REPLACING $\mathcal{F}^{-1}[G(k_1, k_2, x_3' - x_3)] = \check{G}(x_1, x_2, x_3' - x_3)$ BY $\mathcal{F}^{-1}[G(k_1, k_2, x_3' - x_3)\, H(k_1, k_2)]$ WHERE $H(k_1, k_2)$ IS A HEAVISIDE FUNCTION WHICH IS ONE INSIDE THE RECTANGLE AND ZERO OUTSIDE IT. WE HAVE MANY OTHER CHOICES OF WINDOWS. ONE IS

$$\Pi(k_i) = \begin{cases} 1 - \frac{1}{2}\exp\left[-(1 - |k_i|/k_c)/\alpha\right] & |k_i| \leq k_c \\ \frac{1}{2}\exp\left[(1 - |k_i|/k_c)/\alpha\right] & |k_i| > k_c \end{cases}$$

$\boxed{i = 1 \text{ or } 2}$

k_c IS A CUT-OFF VALUE FOR k_i (APPROXIMATE)

SEE FIG. 3.5, WILLIAMS



— AS $\alpha \to 0$, THE WINDOW BECOMES MORE RECTANGULAR.

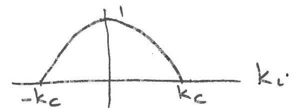
OTHER REMEDIES (CONT'D)

ANOTHER GOOD CHOICE FOR WINDOW, PARTICULARLY FOR CONTROLLING GIBBS PHENOMENON, IS LANCZOS WINDOW

$$\Pi(k_i) = \begin{cases} \cos(\pi |k_i| / 2k_c), & k_i \leq k_c \\ 0 & |k_i| > k_c \end{cases} \quad \boxed{i = 1 \text{ or } 2}$$

WHERE k_c IS THE EXACT CUT-OFF FOR k_i

THIS SIMPLE WINDOW HAS VERY NICE PROPERTIES. IT SHOULD BE OUR FIRST CHOICE!

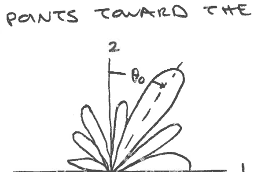


TRAVELING WAVE ON A BAFFLED SQUARE PLATE

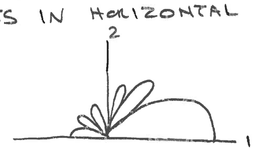
$$V_3(x_1, x_2, 0) = \begin{cases} V_{30}\, e^{i k_{01} x} & |x| < L/2,\ |y| < L/2 \\ 0 & \text{ELSEWHERE} \end{cases}$$

$$\hat{V}_3(k_1, k_2) = V_{30} L^2 \operatorname{sinc}\left(\frac{k_2 L}{2}\right) \operatorname{sinc}\frac{(k_1 - k_{10})L}{2}$$

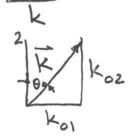
THERE IS NO CHANGE OF DIRECTIVITY IN X_2 DIRECTION. THE MAIN LOBE IN x_1-DIRECTION POINTS TOWARD THE DIRECTION OF TRAVELING WAVE



WE SEE THAT IF $k_{01} = k$, THEN $\theta_0 = \frac{\pi}{2}$, THE MAIN LOBE POINTS IN HORIZONTAL DIRECTION

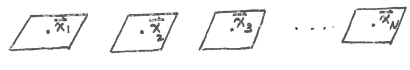


$$\sin\theta_0 = \frac{k_{01}}{k}$$



FIRST PRODUCT THEOREM FOR ARRAYS

$\left\{\begin{array}{l}\text{N IDENTICAL}\\ \text{RADIATORS}\\ \text{NOT NECESSARILY}\\ \text{IN PHASE}\end{array}\right\}$



LET $W_n(x_1, x_2) = Q_n W(\vec{x} - \vec{x}_n) \qquad n = 1, 2, \dots, N$

Q_n <u>COMPLEX</u>, $\vec{x} = (x_1, x_2)$, $\vec{x}_n = (x_{n1}, x_{n2})$

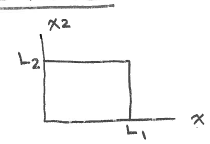
$$\hat{W}_n = Q_n \widehat{W(\vec{x} - \vec{x}_n)} = Q_n \widehat{W(\vec{\alpha})}\, e^{-i\vec{k}\cdot\vec{x}_n}$$

$$\boxed{D(\theta, \phi) = -\frac{i\rho_0\omega}{2\pi}\, \hat{W} \sum_{n=1}^{N} Q_n e^{-i\vec{k}\cdot\vec{x}_n}}$$

THIS MEANS THE DIRECTIVITY PATTERN OF N IDENTICAL RADIATORS IS THE PRODUCT OF THE DIRECTIVITY PATTERN OF ONE RADIATOR AND THE SUM OF N POINT SOURCES LOCATED AT $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N$ WITH PHASES $Q_1, Q_2, \dots, Q_n$

NOTE: F.T. $Q_n \delta(\vec{x} - \vec{x}_n) = Q_n e^{-i\vec{k}\cdot\vec{x}_n}$

<u>RADIATION FROM FINITE SIMPLY SUPPORTED PLATE</u>



$$\Psi W = \nabla^4 W(\vec{x}, \omega) - k_f^4 W(\vec{x}, \omega) = 0$$

$$\nabla^4 = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^2, \quad k_f = \left(\frac{\rho_s h \omega}{D}\right)^{1/4}$$

$$D = \frac{E h^3}{12(1-\nu^2)} \quad \text{FLEXTURAL RIGIDITY}$$

E = YOUNG MODULUS, h = THICKNESS OF PLATE

ν = POISSON RATIO, ρ_s = DENSITY kg/m^2 OF PLATE

k_f = BENDING WAVE NO.

BC'S FOR <u>SIMPLY SUPPORTED PLATES</u>

$$\begin{cases} W = 0, \ \dfrac{\partial^2 W}{\partial x_1^2} = 0 & x_1 = 0, \ x_1 = L_1 \\[2ex] W = 0, \ \dfrac{\partial^2 W}{\partial x_2^2} = 0 & x_2 = 0, \ x_2 = L_2 \end{cases}$$

ONCE WE FIND W, WE USE RAYLEIGH'S 2ND INTEGRAL TO FIND p.
WE ASSUME THAT FLUID LOADING IS NEGLIGIBLE.

<u>SOLUTIONS OF $\Psi W = 0$ SATISFYING BC'S</u>

LET $k_1 = \frac{2\pi}{2L_1} = \frac{\pi}{L_1}$, $k_2 = \frac{2\pi}{2L_2} = \frac{\pi}{L_2}$

$$\Phi_{mn}(\vec{x}) = \frac{2}{\sqrt{L_1 L_2}} \sin(m k_1 x_1) \sin(n k_2 x_2) \qquad m, n = 1, 2, \ldots$$

SATISFY $\boxed{\Psi \Phi_{mn} = 0}$ PROVIDED THAT

(*) $\boxed{m^2 k_1^2 + n^2 k_2^2 = k_f^2}$ (SQ. OF BENDING WAVE NO.)

Φ_{mn} ALSO SATISFY BC'S FOR SIMPLY SUPPORTED PLATES.

IN ADDITION WE HAVE ORTHOGONALITY CONDITION

$$\langle \Phi_{mn}, \Phi_{pq} \rangle = \int_0^{L_2} \int_0^{L_1} \Phi_{mn}(\vec{x}) \Phi_{pq}(\vec{x})\, dx_1 dx_2 = \delta_{mp} \delta_{nq}$$

$$= \begin{cases} 0 & \text{IF } m \neq p \text{ \underline{OR} } n \neq q \\ 1 & \text{IF } m = p \text{ \underline{AND} } n = q \end{cases}$$

NOTE THAT $k_f = \sqrt{\omega}/\alpha$, $\alpha = \left(\frac{D}{\rho_s h}\right)^{1/4} = \left[\frac{E h^2}{12 \rho_s (1-\nu^2)}\right]^{1/4}$ SKUDRZYK CONSTANT

(*) USING $\underbrace{k_f = \sqrt{\omega}/\alpha}_{\text{DISPERSION REL.}} \Rightarrow \boxed{\omega = \alpha^2 (m^2 k_1^2 + n^2 k_2^2) \equiv \omega_{mn} \text{ EIGENFREQ.}}$, SEE NEXT PAGE!

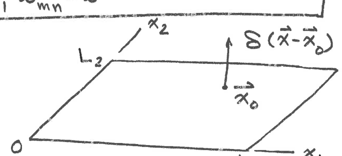
FOR EACH MODE (m, n), THERE IS AN <u>EIGENFREQUENCY</u> $\omega_{mn} = \alpha^2 [m^2 k_1^2 + n^2 k_2^2]$. NOTE THAT FOR A GIVEN PLATE (i.e. FIXED α), ω_{mn} IS DETERMINED BY GEOMETRY (i.e. L_1 & L_2). WE CAN CALCULATE ω_{mn} IN ADVANCE.

— NOTE : $\nabla^4 \Phi_{mn} = \frac{\rho_s h}{D} \omega_{mn}^2 \Phi_{mn} = \frac{\omega_{mn}^2}{\alpha^4} \Phi_{mn}$

THE SOLUTION OF $\Psi W = \delta(\vec{x} - \vec{x}_0)$ IS

$$\boxed{W(\vec{x}, \vec{x}_0, \omega) = \alpha^4 \sum_{m,n=1}^{\infty} \frac{1}{\omega_{mn}^2 - \omega^2} \Phi_{mn}(\vec{x}) \Phi_{mn}(\vec{x}_0)}$$

THIS IS <u>THE GREEN'S FUNCTION</u> OF THE OPERATOR Ψ WHICH IS A <u>SELF-ADJOINT</u> OPERATOR.



$$\langle \Psi W_1, W_2 \rangle = \langle W_1, \Psi W_2 \rangle$$

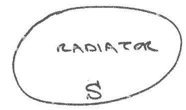
WE MUST USE BC'S OF SIMPLY SUPPORTED PLATE HERE.

— FOR A GIVEN ω, <u>ALL MODES</u> OF THE PLATE ARE <u>EXCITED</u>! BUT <u>NOT ALL MODES</u> CONTRIBUTE TO <u>RADIATION</u> TO FAR-FIELD.

RADIATED POWER FROM PLANAR RADIATORS

$$\Pi(\omega) = \frac{1}{2}\int_S \mathrm{Re}\left[p(\vec{x})V_3^*(\vec{x})\right]dS \qquad \vec{x} = (x_1, x_2, 0)$$

WE CAN FIND AN EXPRESSION IN FOURIER (WAVE NUMBER) SPACE AS FOLLOWS



$$p(\vec{x}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \hat{p}(k_1,k_2)e^{i\vec{k}\cdot\vec{x}}\,d\vec{k}$$

$$V_3^*(\vec{x}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \hat{V}_3^*(k_1',k_2')\,e^{-i\vec{k}'\cdot\vec{x}}\,d\vec{k}'$$

$$\Pi(\omega) = \frac{1}{32\pi^4}\,\mathrm{Re}\iint_{\vec{k}\,\vec{k}'} \hat{p}(\vec{k})\,\hat{V}_3^*(\vec{k}')\Big[\underbrace{\int_{-\infty}^{\infty} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\,d\vec{x}}_{4\pi^2\delta(\vec{k}-\vec{k}')}\Big]\,d\vec{k}\,d\vec{k}'$$

$$= \frac{1}{8\pi^2}\,\mathrm{Re}\int_{\vec{k}'} \frac{\rho_0\omega}{k_3}\,|\hat{V}_3(\vec{k})|^2\,d\vec{k}$$

HERE WE USED $\hat{p}(\vec{k}) = \frac{\rho_0\omega}{k_z}\hat{V}_3$, $k_z = \sqrt{k^2-k_1^2-k_2^2}$

k_z IS REAL INSIDE CIRCLE $k^2 = k_1^2+k_2^2$: S_r, r: RADIATION

$$\boxed{\Pi(\omega) = \frac{\rho_0\omega}{8\pi^2}\int_{S_r}\frac{|\hat{V}_3(\vec{k})|^2}{\sqrt{k^2-k_1^2-k_2^2}}\,dk_1\,dk_2} \qquad \vec{k} = (k_1,k_2)$$

RADIATED POWER FROM PLANAR RADIATORS (CONT'D)

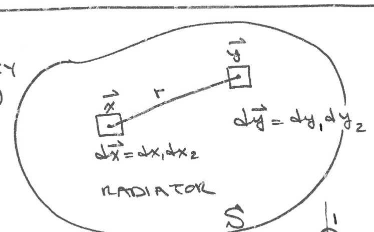
BOUWKAMP'S RESULT

$$\boxed{\Pi(\omega) = \frac{\rho_0\omega}{4\pi}\int_S\int_S V_3(\vec{x})\,V_3^*(\vec{y})\,\frac{\sin(kr)}{r}\,d\vec{x}\,d\vec{y}}$$

$r = |\vec{x}-\vec{y}|$

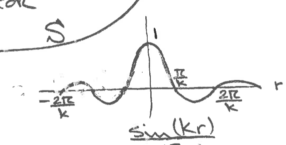
THIS IS A REAL QUANTITY BECAUSE $\Pi^*(\omega) = \Pi(\omega)$

THIS IS A FOUR-DIM'L INTEGRAL.



NOTE THAT WE CAN WRITE

$$K(\vec{x}) = \int_S V_3^*(\vec{y})\,\frac{\sin(kr)}{r}\,d\vec{y}$$

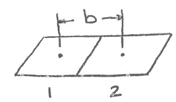


$$\Longrightarrow \Pi(\omega) = \frac{\rho_0\omega}{4\pi}\int_S K(\vec{x})\,V_3(\vec{x})\,d\vec{x}$$

$K(\vec{x})$ HAS TO BE $A(\vec{x})V^*(\vec{x})$, WHERE $A(\vec{x})$ REAL & POSITIVE!

EDGE AND CORNER MODES

RADIATION FROM TWO IDENTICAL PANELS (POWER RADIATED):



$Q_n = \int_S v_3 \, dS = 1$ (ASSUME) FOR EACH PANEL

$$\boxed{\Pi(\omega) = 2\,\Pi_1(\omega)\left(1 \pm \frac{\sin(kb)}{kb}\right)}$$

$$\Pi_1(\omega) = \frac{\rho_0 c k^2}{4\pi}$$

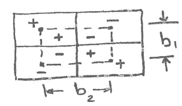
WE USED FIRST ARRAY THM TO GET THIS RESULT.

USE + SIGN IF TWO PANELS ARE IN PHASE AND – SIGN IF OUT OF PHASE

$kb = \frac{2\pi b}{\lambda}$, λ ACOUSTIC WAVE LENGTH

$\frac{\sin(kb)}{kb} \approx 1$ IF $kb \ll 1$ OR $b \ll \lambda$

THIS MEANS THAT FOR TWO PANELS VIBRATING OUT OF WITH $b \ll \lambda$, THERE IS ALMOST COMPLETE CANCELLATION OF RADIATION. IN THE FIGURE ON THE RIGHT, THE FOUR SMALL PANELS IN THE CENTER ARE INEFFICIENT RADIATORS ASSUMING $b_1 < b_2 \ll \lambda$.



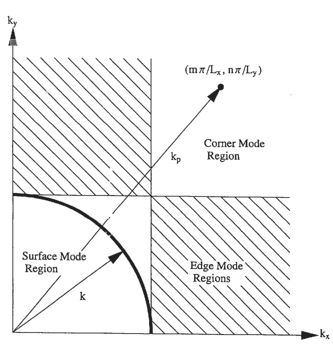


Figure 2.29: Radiation classification of normal modes of a simply supported plate.

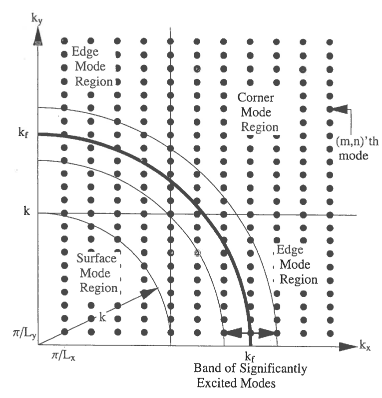


Figure 2.34: *k*-space diagram with eigenmodes displayed as dots. The circular ring is a region in which the amplitudes of the excited modes are the largest. The plate is excited at the frequency ω, $k = \omega/c$ and $k < k_f$.

PLANE WAVES – (CONT'D) LEC. 7/10

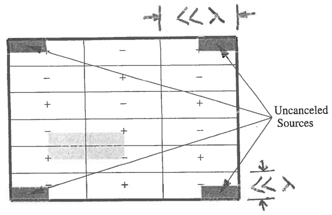


Figure 2.31: Example of a corner mode. All adjacent regions (one shown in light gray) cancel. Only the four corner regions are left uncancelled.

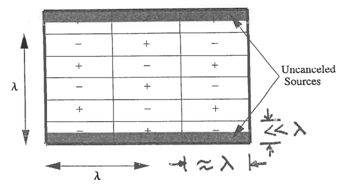


Figure 2.32: Example of an edge mode. Adjacent vertical regions cancel one another, but the horizontal regions on the edge no longer cancel leaving two strips uncancelled.

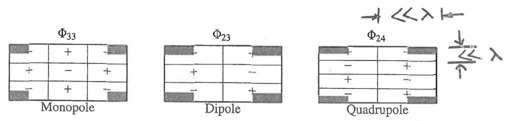


Figure 2.33: Low frequency limits for the radiation classification of normal modes.

TE/1

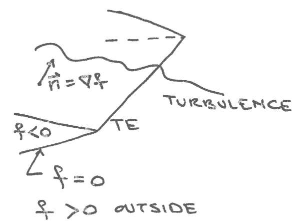
TE NOISE PREDICTION

ANALYTIC RESULTS

F. FARASSAT

THE GOVERNING EQUATION –

$$\Box^2 p' = \underbrace{\frac{\bar{\partial}^2}{\partial x_i \partial x_j}}_{\text{GEN. DERIVATIVES}} \left[T_{ij} H(f) \right]$$



$H(f)$: HEAVISIDE FN

THE RHS OF THE WAVE EQ. IS THE QUADRUPOLE TERM OF THE FW-H EQ.

$T_{ij} = \rho u_i u_j + (p' - c^2 \rho') \delta_{ij}$

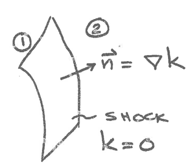
THE SOLUTION : $\boxed{4\pi p(\vec{x}, t) = \dfrac{\partial^2}{\partial x_i \partial x_j} \displaystyle\int_{f>0} \dfrac{[T_{ij} H(f)]_{ret}}{r} d\vec{y}}$

THIS IS NOT VERY USEFUL BECAUSE IT WILL NOT TELL US WHAT REGION OF FLOW CONTRIBUTES MOST TO THE NOISE. $r = |\vec{x} - \vec{y}|$, $(\vec{x}, t)$ & $(\vec{y}, \tau)$ OBS. & FIELD SPACE-TIME VARIABLES, ret : RETARDED TIME $\tau = t - r/c$.

THE BASIC IDEA : REGIONS OF HIGH GRADIENTS (SHOCKS) AND ABRUPT CHANGES OF BC'S (TE) CAN BE DOMINANT NOISE PRODUCERS.

HOW TO FIND THE SOURCE STRENGTH : USE GENERALIZED DIFFERENTIATION OF QUADRUPOLE TERM

EXAMPLE : SHOCK NOISE SOURCES



$$\bar{\partial}_{ij} T_{ij} = \bar{\partial}_i \left\{ \partial_j T_{ij} + \Delta T_{ij} n_j \delta(k) \right\}$$

GEN. DERIV.

$$= \partial_{ij} T_{ij} + \boxed{\Delta(\partial_j T_{ij}) n_i \delta(k) + \partial_i \left[\Delta T_{ij} n_j \delta(k) \right]}$$

QUADRUPOLE (INEFFICIENT)

MONOPOLE

DIPOLE

$\boxed{\text{EFFICIENT}}$

$\Delta = [\ \ ]_2 - [\ \ ]_1$

SOURCE STRENGTH OF TE NOISE

$$\bar{\partial}_{ij}\left[T_{ij}\,H(f)\right] = \bar{\partial}_i\left[\partial_j T_{ij}\,H(f) + \underbrace{T_{ij}n_j}_{T_{in}}\delta(f)\right]$$

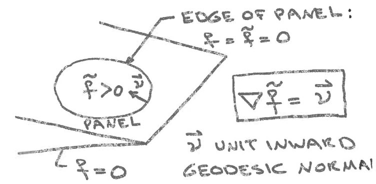
$$= \partial_{ij}T_{ij}\,H(f) + \boxed{(\partial_j T_{ij})n_i\,\delta(f)}$$

$$\boxed{+\ \bar{\partial}_i\left[T_{in}\,\delta(f)\right]}$$

SURFACE SOURCE TERMS

$$T_{in} = (\vec{T})_i$$

TO FIND THE CONTRIBUTION OF THE EDGE, WE MANIPULATE THE LAST SOURCE TERM FOR A PANEL :



$$\bar{\partial}_i\left[T_{in}\,H(\tilde{f})\,\delta(f)\right]$$

$$= \bar{\nabla}\cdot\left[\vec{\underset{\sim}{T}}\,H(\tilde{f})\,\delta(f)\right]$$

$\vec{\underset{\sim}{T}}$ IS THE RESTRICTION OF $\vec{T}$ TO $f = 0$, *i.e.* $\dfrac{\partial\vec{\underset{\sim}{T}}}{\partial n} = 0$.

SOURCE STRENGTH OF TE NOISE (CONT'D)

$$\bar{\nabla}\cdot\left[\vec{\underset{\sim}{T}}\,H(\tilde{f})\,\delta(f)\right] = \nabla_2\cdot\left[\vec{T}_\tau\,H(\tilde{f})\right]\delta(f) \quad \text{SURFACE DIVERGENCE}$$

$$+\frac{\partial}{\partial n}\left[\underset{\sim}{T}_n\,H(\tilde{f})\,\delta(f)\right] - 2H_f\,T_n\,H(\tilde{f})\,\delta(f)$$

$$= \nabla_2\cdot\vec{T}_\tau\,H(\tilde{f})\,\delta(f) + \underbrace{\underset{\sim}{T}_n\,H(\tilde{f})\,\delta'(f)}_{\text{HAS A LINE SOURCE HIDDEN HERE}}$$

$$+\underbrace{\vec{T}_\tau\cdot\vec{\nu}\;\delta(\tilde{f})\,\delta(f)}_{\text{SOURCE ON EDGE OF THE PANEL}}$$

$$\vec{T} = \vec{T}_\tau + T_n\,\vec{n}$$

TANGENTIAL COMPONENT — NORMAL COMPONENT OF $\vec{T}$

$$\vec{T}_\tau\cdot\vec{\nu} = T_{n\nu}\,, \quad T_n = T_{nn} = T_{ij}\,n_i\,n_j$$

$$= T_{ij}\,n_i\,\nu_j$$

TE/6

<u>SOLUTION OF $\quad \Box^2 p' = T_{n\nu}\,\delta(\tilde{f})\,\delta(f)$</u>

FROM NASA TP 3428 (FARASSAT), PAGE 39

$$4\pi p_1'(\vec{x},t) = \int\limits_{\substack{F=0\\ \tilde{F}=0}} \frac{1}{r}\left[\frac{T_{n\nu}}{\Lambda_0}\right]_{ret} dL$$

$$F = [f]_{ret}, \quad \tilde{F} = [\tilde{f}]_{ret}, \quad \Lambda_0 = |\nabla F \times \nabla \tilde{F}|$$

THE <u>LINE INTEGRAL</u> PART OF THE SOLUTION OF

$\Box^2 p' = \underset{\sim}{T}_{nn} H(\tilde{f})\,\delta'(f)$ IS

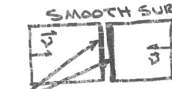
$$4\pi p_2'(\vec{x},t) = -\int\limits_{\substack{F=0\\ \tilde{F}=0}} \frac{[T_{nn}\cot\theta']_{ret}}{r\Lambda^2}\, dL$$

$\Lambda = |\nabla F|$, θ' IS THE ANGLE BETWEEN ∇F & $\nabla \tilde{F}$

TE/7

<u>TE NOISE FORMULA</u>

SMOOTH SURFACE



THE LINE INTEGRALS ALONG THE <u>COMMON</u> <u>EDGE</u> LYING ON A SMOOTH SURFACE CANCEL!

$\Lambda_0 = \Lambda \tilde{\Lambda} \sin\theta'$

$\tilde{\Lambda} = |\nabla \tilde{F}|$

$$\boxed{4\pi p_{TE}' = \int\limits_{\underbrace{\substack{F=0\\ \tilde{F}=0}}_{TE}} \frac{1}{r}\left\{\frac{1}{\Lambda \sin\theta'}\left(\frac{T_{n\nu}}{\tilde{\Lambda}} - \frac{T_{nn}\cos\theta'}{\Lambda}\right)\right\}_{ret} dL}$$

THE CONTRIBUTIONS OF THE UPPER & LOWER SURFACES MUST BE ADDED SEPARATELY

ALL VELOCITIES IN T_{ij} ARE WRT A FRAME FIXED TO UNDISTURBED MEDIUM.

9 Workshop on Kirchhoff Formulas

WORKSHOP ON KIRCHHOFF FORMULAS

BY

F. FARASSAT

AEROACOUSTICS BRANCH

FLUID MECHANICS AND ACOUSTICS DIVISION

NASA LANGLEY RESEARCH CENTER

HAMPTON, VIRGINIA

HELD FEBRUARY 15, 1995, LANGLEY RESEARCH CENTER

1

AVAILABLE METHODS OF NOISE PREDICTION IN AEROACOUSTICS

TODAY WE HAVE THREE METHODS AVAILABLE. THESE ARE:

i) THE ACOUSTIC ANALOGY BASED ON FFOWCS WILLIAMS - HAWKINGS (FW-H) EQUATION (1969). IT IS THE MOST DEVELOPED METHOD AND IS WIDELY IN USE IN THE AIRCRAFT INDUSTRY.

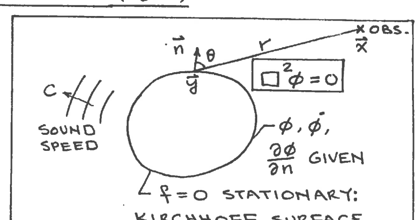
ii) THE KIRCHHOFF FORMULA BASED METHOD. ORIGINALLY SUGGESTED BY HAWKINGS (1979), THIS METHOD IS CURRENTLY UNDER DEVELOPMENT. AVAILABILITY OF HIGH RESOLUTION AERODYNAMICS AND POWERFUL COMPUTERS MAY MAKE THIS APPROACH VERY POPULAR IN THE FUTURE.

iii) THE CFD BASED CAA (COMPUTATIONAL AEROACOUSTICS). THIS METHOD IS UNDER DEVELOPMENT AND IS THE LEAST MATURE OF THE THREE METHODS. IT MAY BE APPROPRIATE FOR SOME PROBLEMS. COMPUTATIONAL TECHNIQUES DEVELOPED HERE WILL ALSO HELP THE ABOVE TWO METHODS.

2

CLASSICAL KIRCHHOFF FORMULA (1882)

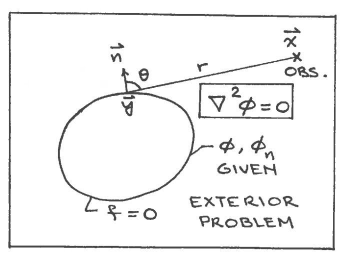
$$4\pi \phi(\vec{x}, t) = \int_{f=0} \frac{[-\phi_n + c^{-1}\dot{\phi}\cos\theta]_{ret}}{r}\, dS + \int_{f=0} \frac{[\phi\cos\theta]_{ret}}{r^2}\, dS$$



ret : RETARDED TIME , $\phi_n = \frac{\partial\phi}{\partial n}$, $r = |\vec{x} - \vec{y}|$

— GIVES ϕ IN TERMS OF VALUES OF ϕ, $\dot{\phi}$ AND ϕ_n ON THE KIRCHHOFF SURFACE $f = 0$. IT IS GREEN'S IDENTITY FOR THE WAVE EQUATION. COMPARE WITH THE FOLLOWING IDENTITY FOR LAPLACE EQUATION:

$$4\pi\,\phi(\vec{x}) = -\int_{f=0} \frac{\phi_n}{r}\, dS + \int_{f=0} \frac{\phi\cos\theta}{r^2}\, dS$$



— WE DERIVE BOTH ABOVE RESULTS BY THE SAME METHOD USING GF THEORY.

CLASSICAL KIRCHHOFF FORMULA (CONT'D)

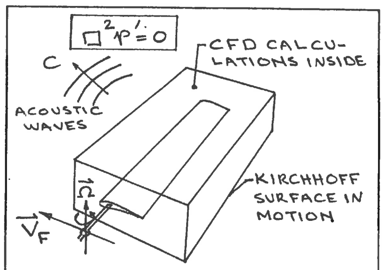
— DERIVED IN 1882 BY G. KIRCHHOFF

— SEE CLASSICAL DERIVATION BY D.S. JONES "THE THEORY OF ELECTROMAGNETISM", PERGAMON PRESS, 1964, SEC. 1.17, P40. ALSO SEE M. BORN AND E. WOLF "PRINCIPLES OF OPTICS", PERGAMON PRESS, 1970, SEC. 8.3, P 375 (GOOD APPLICATIONS HERE).

— APPLICATIONS IN OPTICS, ELECTROMAGNETISM AND ACOUSTICS ARE VERY EXTENSIVE. UNTIL RECENTLY THE CLASSICAL KIRCHHOFF FORMULA HAS BEEN USED EITHER AS APPROXIMATION OR FOR QUALITATIVE UNDERSTANDING OF FIELDS GOVERNED BY THE WAVE EQUATION. THE AVAILABILITY OF HIGH SPEED DIGITAL COMPUTERS HAS CHANGED THIS PICTURE. SIMULATION OF THE WAVE FIELD IS POSSIBLE AND REWARDING! EXTENSION TO MOVING SURFACES HAS OPENED NEW APPLICATIONS.

— SEE ALSO A.D. PIERCE "ACOUSTICS", ACOUST. SOC. AM., 1989, P180.

WHY ARE KIRCHHOFF FORMULAS IMPORTANT IN ACOUSTICS?



KIRCHHOFF DATA p', $\dot{p}'$, p_n' OBTAINED FROM CFD ON KIRCHHOFF SURFACE

— ACCURATE PREDICTION OF THE NOISE OF HELICOPTER ROTORS, PROPELLERS AND DUCTED FANS, PARTICULARLY AT DESIGN STAGE, IS NEEDED TO CONTROL THE PASSENGER AND PUBLIC ANNOYANCE AND TO MEET NOISE STANDARDS.

— LOW NOISE AIRCRAFT AND PROPULSION SYSTEMS SELL BETTER IN THE INTERNATIONAL MARKET. THEREFORE, NOISE PREDICTION TOOLS TO MEET U.S. AIRCRAFT AND ENGINE INDUSTRY NEEDS MUST BE DEVELOPED.

— KIRCHHOFF FORMULAS FOR MOVING SURFACES COUPLED TO ADVANCED CFD CODES SUPPLY AN EFFICIENT AND POWERFUL TOOL FOR NOISE PREDICTION. SEE BOX ABOVE.

WHAT IS THIS WORKSHOP ABOUT?

- OUR PRIMARY PURPOSE IN THIS WORKSHOP IS THE DERIVATION OF TWO KIRCHHOFF FORMULAS FOR SUBSONIC AND SUPERSONIC SURFACES.
- WHEN WORKING WITH INHOMOGENEOUS WAVE EQUATION FOR MOVING SOURCES USING CLASSICAL METHODS, WE NOTICE THAT THE ALGEBRAIC MANIPULATIONS QUICKLY BECOME COMPLICATED. WE LOSE TRACK OF CANCELLATIONS AND SIMPLIFICATIONS. WE NEED SPECIAL TOOLS FROM MATHEMATICS WHICH GIVE US SIMPLE AND DIRECT METHOD OF DERIVATION.
- THE SECONDARY PURPOSE OF THIS WORKSHOP IS TO GIVE ALL THE NECESSARY TOOLS FROM GENERALIZED FUNCTION (GF) THEORY, P.D.E.'S AND DIFFERENTIAL GEOMETRY.
- OUR RESULTS ARE FOR APPLICATION. OUR KIRCHHOFF FORMULAS ARE APPLICABLE TO OPEN SURFACES (e.g., PANELS).

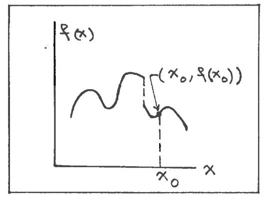
METHOD OF DERIVING KIRCHHOFF FORMULAS

WE REDUCE THE DERIVATION OF THE THREE KIRCHHOFF FORMULAS HERE TO THE SOLUTION OF WAVE EQUATION

$$\Box^2 \phi = Q$$

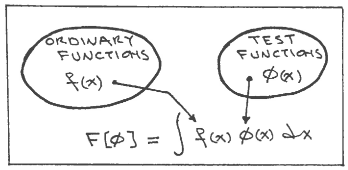
WHERE Q IS A GENERALIZED FUNCTION (SUCH AS $q\,\delta(f)$). THIS METHOD AVOIDS THE USE OF GREEN'S IDENTITY IN FOUR DIMENSIONS. ONE MUST, THEREFORE, LEARN SOME GENERALIZED FUNCTION THEORY. THE SOURCE DISTRIBUTIONS ARE ON MOVING SURFACES AND INVARIABLY THE GEOMETRY OF THESE SURFACES ENTER THE DERIVATION. WITHOUT THE KNOWLEGE OF DIFFERENTIAL GEOMETRY OF SURFACES, WE CANNOT IDENTIFY SURFACE CURVATURE TERMS AND OTHER GEOMETRIC QUANTITIES RESULTING IN A LARGE NUMBER OF MEANINGLESS TERMS IN THE KIRCHHOFF FORMULA. A FORMULA IN THIS FORM IS NOT VERY USEFUL IN APPLICATIONS.

MODELS OF FUNCTIONS



OLD (CONVENTIONAL) MODEL : WE THINK OF A FUNCTION AS A TABLE OF ORDERED PAIRS $(x, f(x))$ WHERE FOR EACH x, $f(x)$ IS UNIQUE. THIS TABLE CAN BE GRAPHED AS SHOWN AND USUALLY HAS AN UNCOUNTABLE NUMBER OF ORDERED PAIRS.

NEW MODEL : WE THINK OF A FUNCTION f BY ITS ACTION (FUNCTIAL VALUES) ON A GIVEN SPACE OF ORDINARY FUNCTIONS CALLED TEST FUNCTION SPACE. THIS ACTION FOR ORDINARY FUNCTIONS IS DEFINED BY $F[\phi] = \int f(x)\,\phi(x)\,dx$.



THE FUNCTION f IS NOW DEFINED (IDENTIFIED, THOUGHT OF) BY THE NEW TABLE $\{F[\phi],\ \phi \in$ TEST FUNCTION SPACE$\}$. THIS VIEW OF ORDINARY FUNCTIONS NOW ALLOWS US TO INCORPORATE $\delta(x)$ INTO MATHEMATICS RIGOROUSLY.

A FAMILIAR EXAMPLE OF THINKING ABOUT FUNCTIONS BY NEW MODEL

CONSIDER SPACE OF PERIODIC FUNCTIONS WITH PERIOD 2π. TAKE THE TEST FUNCTION SPACE TO BE THE SPACE FORMED BY FUNCTIONS $\phi_n = \exp(inx)$, $n = 0, \pm 1, \pm 2, \ldots$. LET f BE PERIODIC WITH PERIOD 2π. THE FOURIER COEFFICIENTS OF f CAN BE VIEWED AS FUNCTIONALS ON TEST FUNCTION SPACE BY THE RELATION

$$F[\phi_n] = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{inx}\, dx$$

FROM THE THEORY OF FOURIER ANALYSIS, WE KNOW THAT THE FOLLOWING TABLE OF FOURIER COEFFICIENTS (*i.e.* FUNCTIONAL VALUES OF f ON TEST FUNCTION SPACE) CONTAINS THE SAME INFORMATION AS $f(x)$: $\{F[\phi_n],\ n = 0, \pm 1, \pm 2, \ldots\}$.

— NOTE THAT IF $f(x) \neq g(x)$, WHERE $g(x)$ IS ANOTHER PERIODIC FUNCTION WITH PERIOD 2π, THEN $F[\phi_n] \neq G[\phi_n] = \frac{1}{2\pi}\int_0^{2\pi} g(x)\, e^{inx}\, dx$ FOR SOME n, *i.e.* THE NEW TABLE UNIQUELY DEFINES FUNCTIONS.

ELEMENTARY GENERALIZED FUNCTION THEORY

THE MAIN REASON TO DEVELOP GF THEORY IS TO INCLUDE MATHEMATICAL OBJECTS SUCH AS THE DIRAC DELTA "FUNCTION" $\delta(x)$. THIS FUNCTION HAS THE SIFTING PROPERTY

$$\int_{-a}^{a} \phi(x)\,\delta(x)\,dx = \phi(0)$$

TO INCLUDE THESE OBJECTS IN MATHEMATICS, WE NEED TO CHANGE OUR THINKING ABOUT FUNCTIONS. THE REASON WE MUST CHANGE OUR THINKING ABOUT FUNCTIONS IS THAT NO ORDINARY FUNCTION CAN HAVE THE SIFTING PROPERTY. WE MUST THEREFORE ENLARGE THE SPACE OF FUNCTIONS BY A PROCESS FAMILIAR IN MATHEMATICS: DEFINE (LOOK AT, VIEW) FUNCTIONS IN A NEW WAY WHICH INCLUDES ALL ORDINAY FUNCTIONS AS WELL AS OBJECTS LIKE THE DIRAC DELTA FUNCTION. THIS IS A CHANGE OF PRADIGM FAMILIAR TO US WHEN WE LEARNED FRACTIONS, NEGATIVE NUMBERS AND COMPLEX NUMBERS.

DEFINITION OF GENERALIZED FUNCTIONS

- A FUNCTIONAL ON A SPACE OF FUNCTIONS Ω IS A MAPPING (A RULE) OF Ω INTO SCALARS (REAL OR COMPLEX NUMBERS).

  EXAMPLES: TAKE Ω AS SPACE OF DIFFERENTIABLE FUNCTIONS. THE FOLLOWING ARE FUNCTIONALS ON Ω, $\phi \in \Omega$

  i) $F[\phi] = \phi'(0) + 2\phi(1)$, ii) $F[\phi] = \int_0^1 \phi^2(x)\,dx$

  iii) $F[\phi] = \sin[\phi(0)]$ iv) $F[\phi] = 2\phi(1) + \int_{-1}^{1} \phi(x)\,dx$

- IN GF THEORY THE FUNCTIONALS ACT ON VARIOUS TEST FUNCTION SPACES DEPENDING ON THE PROBLEM. WE DEFINE GENERALIZED FUNCTIONS ON THE FOLLOWING TEST FUNCTION SPACE:

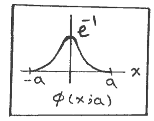
  SPACE D OF TEST FUNCTIONS: INFINITELY DIFFERENTIABLE FUNCTIONS WITH BOUNDED SUPPORT.

- THE SUPPORT OF A FUNCTION ϕ IS THE CLOSURE OF THE SET ON WHICH $\phi \neq 0$. WE USE $\operatorname{supp} \phi$ FOR SUPPORT OF ϕ.

DEFINITION OF GENERALIZED FUNCTIONS (CONT'D)

– EXAMPLES OF FUNCTIONS IN D.

i) LET $\phi(x;a) = \begin{cases} \exp\left[\frac{a^2}{x^2-a^2}\right] & |x|<a \\ 0 & |x|\geq a \end{cases} \Rightarrow \phi(x;a) \in D$



ii) LET $g(x)$ BE ANY CONTINUOUS FUNCTION, THEN

$\psi(x) = \int_b^c g(y)\phi(x-y;a)\,dy$, WHERE $[b,c]$ IS A FINITE INTERVAL, BELONGS TO D. WE CAN SHOW THAT $\operatorname{supp}\psi(x) = [b-a, c+a]$.

– EXAMPLE (ii), ABOVE, SHOWS THAT SPACE D IS POPULATED WITH AN UNCOUNTABLY INFINITE NUMBER OF FUNCTIONS. THIS MEANS THAT THE TABLE OF FUNCTIONAL VALUES ON D IN OUR NEW MODEL OF FUNCTIONS HAS AN UNCOUNTABLY INFINITE NUMBER OF MEMBERS.

– BY AN ORDINARY FUNCTION WE MEAN A LOCALLY (LEBESGUE) INTEGRABLE FUNCTION.

– A REMINDER: IN OUR NEW MODEL OF THINKING ABOUT FUNCTIONS, WE IDENTIFY AN ORDINARY FUNCTION $f(x)$ BY TABLE $\{F[\phi] = \int f\phi\,dx, \phi \in D\}$.

a real valued fn f defined on $\mathbb{R}$ is Lebesgue integrable if there exists a sequence of step fns $\{f_n\}$ ∋ $\sum_{n=1}^{\infty}\int|f_n| < \infty$ and $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for every $x \in \mathbb{R}$ ∋ $\sum_{n=1}^{\infty}\int|f_n| < \infty$

DEFINITION OF GENERALIZED FUNCTIONS (CONT'D)

THE FUNCTIONAL $F[\phi] = \int f\phi\,dx$ IS LINEAR AND CONTINUOUS

– A FUNCTIONAL ON D IS LINEAR IF $F[\alpha\phi_1 + \beta\phi_2] = \alpha F[\phi_1] + \beta F[\phi_2]$ FOR ALL ϕ_1 AND ϕ_2 IN D

– EXAMPLES i) $F[\phi] = \phi(0)$ IS LINEAR, ii) $F[\phi] = 2\phi'(1) - \int f\phi\,dx$ f AN ORDINARY FUNCTION IS LINEAR. iii) $F[\phi] = \phi^2(0)$ IS NONLINEAR

– A FUNCTIONAL ON D IS CONTINUOUS IF $F[\phi_n] \to 0$ IF $\phi_n \xrightarrow{D} 0$.

– A SEQUENCE OF FUNCTIONS $\{\phi_n\}$ IN D CONVERGES TO ZERO IN D, WRITTEN AS $\phi_n \xrightarrow{D} 0$, IF ϕ_n AND ALL ITS DERIVATIVES CONVERGE UNIFORMLY TO ZERO AND $\operatorname{supp}\phi_n \subset I$ FOR ALL n WHERE I IS A FIXED BOUNDED INTERVAL.

– THIS DEFINITION SEEMS VERY STRANGE BUT GIVES GENERALIZED FUNCTIONS SOME OF ITS NICEST PROPERTIES.

– EXAMPLE. i) LET $\phi_n = \frac{1}{n}\phi(x;a)$, WHERE $\phi(x;a)$ IS DEFINED ON P ,
$\Rightarrow \phi_n \xrightarrow{D} 0$; ii) $\phi_n = \frac{1}{n}\phi(x;an) \Rightarrow \phi_n \stackrel{D}{\not\to} 0$.

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DEFINITION OF GENERALIZED FUNCTIONS (CONT'D)

– EXAMPLES: *i*) ALL LINEAR FUNCTIONALS DEFINED ON PREVIOUS PAGE ARE CONTINUOUS.

ii) $\delta[\phi] = \phi(0)$ IS CONTINUOUS. (IT IS ALSO LINEAR.)

– LET $\phi \in D$, ANY ORDINARY FUNCTION f DEFINES A CONTINUOUS LINEAR FUNCTIONAL ON D BY THE RELATION $F[\phi] = \int f\phi\,dx$. BUT ORDINARY FUNCTIONS DO NOT EXHAUST ALL CONTINUOUS LINEAR FUNCTIONALS ON D.

– DEFINITION OF GENERALIZED FUNCTIONS : A CONTINUOUS LINEAR FUNCTIONAL ON SPACE D DEFINES A GENERALIZED FUNCTION. THE SPACE OF ALL GENERALIZED FUNCTIONS IS DENOTED D'.

EXAMPLES *i*) $\delta[\phi] = \phi(0)$ DEFINES A GENERALIZED FUNCTION. WE CAN SHOW THAT THERE IS NO ORDINARY FUNCTION $f(x)$ SUCH THAT $\int f(x)\,\phi(x)\,dx = \phi(0)$. THIS MEANS THAT D' IS LARGER THAN THE SPACE OF ORDINARY FUNCTIONS.

ii) $G[\phi] = 2\phi'(1) + 3\int f\phi\,dx$, f ORD. FN., DEFINES A GEN. FN.

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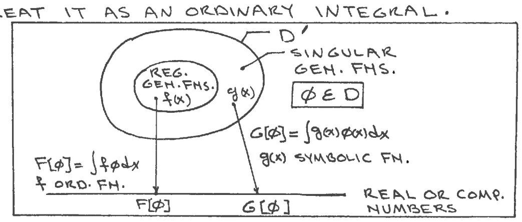
DEFINITION OF GENERALIZED FUNCTIONS (CONT'D)

– IT IS INCONVENIENT TO WORK WITH FUNCTIONAL NOTATION IN MATHEMATICAL MANIPULATIONS. FOR THIS REASON, WE INTRODUCE THE NOTATION OF SYMBOLIC FUNCTIONS FOR THOSE GENERALIZED FUNCTIONS WHICH ARE NOT ORDINARY FUNCTIONS. ORDINARY FUNCTIONS ARE CALLED REGULAR GENERALIZED FUNCTIONS. OTHER GENERALIZED FUNCTIONS ARE CALLED SINGULAR GENERALIZED FUNCTIONS. FOR SINGULAR GENERALIZED FUNCTION $F[\phi]$, WE DEFINE THE SYMBOLIC FUNCTION $f(x)$ SO THAT $\int f(x)\,\phi(x)\,dx \equiv F[\phi]$ FOR $\phi \in D$. IT IS IMPORTANT TO RECOGNIZE THAT THE INTEGRAL ON THE LEFT IS A SYMBOL STANDING FOR $F[\phi]$ AND ONE SHOULD NOT TREAT IT AS AN ORDINARY INTEGRAL.

– THIS IS THE PICTURE OF SPACE D' WE SHOULD HAVE IN MIND.



SOME OPERATIONS ON GENERALIZED FUNCTIONS

NOTE: ALL TEST FUNCTIONS ARE IN SPACE $\mathcal{D}$ (C^∞ FNS WITH COMPACT SUPP.)

i) EQUALITY OF TWO GEN. FNS. ON AN OPEN INTERVAL I : $F[\phi] = G[\phi]$ ON I IF FOR ALL ϕ IN $\mathcal{D}$ SUCH THAT SUPP $\phi \subset I$, WE HAVE $F[\phi] = G[\phi]$ (SYMBOLICALLY $f(x) = g(x)$)

EXAMPLE. $\delta(x) = 0$ ON $(0,\infty)$ SINCE $\delta[\phi] = \phi(0) = 0$ FOR ALL ϕ SUCH THAT SUPP $\phi \subset (0,\infty)$. THIS MEANS THAT A SINGULAR GENERALIZED FUNCTION CAN BE EQUAL TO AN ORDINARY FUNCTION (HERE $f = 0$) ON AN OPEN INTERVAL.

ii) MULTIPLICATION OF A GEN. FN. $F[\phi]$ WITH A C^∞ FUNCTION $a(x)$: $aF[\phi] = F[a\phi]$ (LEFT SIDE IS DEFINED BY RIGHT SIDE)

EXAMPLE. $a\delta[\phi] = \delta[a\phi] = a(0)\phi(0)$ OR SYMBOLICALLY

$a(x)\delta(x) = a(0)\delta(x)$, AN IMPORTANT RESULT!

NOTE. MULTIPLICATION OF TWO SINGULAR GEN. FNS. OR A REGULAR AND A SINGULAR GEN. FNS MAY NOT BE DEFINED.

SOME OPERATIONS ON GENERALIZED FUNCTIONS (CONT'D)

iii) ADDITION OF GEN. FNS : $(F+G)[\phi] = F[\phi] + G[\phi]$ OR SYMBOLICALLY $(f+g)(x) = f(x) + g(x)$

iv) SHIFT OPERATION : $E_h F[\phi] = F[E_{-h}\phi]$ WHERE $E_{-h}\phi = \phi(x-h)$

EXAMPLE. $E_h\delta[\phi] = \delta[E_{-h}\phi] = \phi(-h)$ OR SYMBOLICALLY

$$\int E_h\delta(x)\,\phi(x)\,dx = \int \delta(x+h)\,\phi(x)\,dx = \phi(-h)$$

NOTE. GENERALIZED FUNCTIONS ARE NOT DEFINED AT A POINT BUT ON OPEN INTERVALS. IN PRACTICE, THIS DOES NOT CAUSE PROBLEMS.

– WE CAN DEFINE OTHER OPERATIONS SUCH AS DILATION :

$$\int \delta(\alpha x)\phi(x)\,dx = \frac{1}{|\alpha|}\phi(0) \Rightarrow \delta(\alpha x) = \frac{1}{|\alpha|}\delta(x), \text{ AND}$$

FOURIER TRANSFORM $\hat{F}[\phi] = F[\hat{\phi}]$, $\hat{\phi} = \text{F.T.}(\phi)$ WHERE ϕ NOW BELONGS TO SPACE OF RAPIDLY DECREASING TEST FUNCTIONS $\mathcal{S}$. FOR OUR PURPOSE, THE MOST IMPORTANT OPERATION ON GENERALIZED FUNCTIONS IS DIFFERENTIATION.

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DIFFERENTIATION OF GENERALIZED FUNCTIONS

ALL TEST FUNCTIONS ARE IN D.

– $f(x)$ ORDINARY FUNCTION, DIFFERENTIABLE, $F[\phi] = \int f\phi\,dx$, WE MUST IDENTIFY $F'[\phi]$ WITH $\int f'\phi\,dx$. BUT

$$F'[\phi] \equiv \int f'\phi\,dx = -\int f\phi'dx = -F[\phi'] \quad \text{SINCE } \phi' \in D.$$

THEREFORE, WE USE THE RELATION:

$$\boxed{F'[\phi] = -F[\phi']}$$

AS THE DEFINITION OF <u>DERIVATIVE</u> OF ANY GEN. FN $F[\phi]$.

SIMILARLY $\boxed{F^{(n)}[\phi] = (-1)^n F[\phi^{(n)}]}$, *i.e.*, GENERALIZED FUNCTIONS HAVE DERIVATIVES OF <u>ALL</u> ORDERS.

<u>EXAMPLES</u>. *i*) $\delta'[\phi] = -\delta[\phi'] = -\phi'(0)$, OR $\int \delta'(x)\phi(x)dx = -\phi'(0)$

ii) $\delta''[\phi] = (-1)^2\delta[\phi''] = \phi''(0)$, OR $\int \delta''(x)\phi(x)\,dx = \phi''(0)$

<u>NOTE</u>: IF AN ORDINARY FUNCTION IS DIFFERENTIABLE ON REAL LINE, THEN $f'_{gen.} = f'$. HOWEVER, GENERALIZED DERIVATIVE OF AN ORDINARY FUNCTION <u>CAN</u> BE A SINGULAR GENERALIZED FUNCTION.

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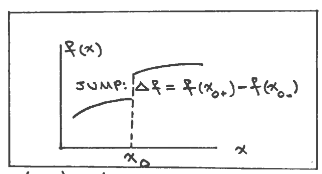
DIFFERENTIATION OF GENERALIZED FUNCTIONS (CONT'D)

<u>NOTATION</u>: FOR ORDINARY (REGULAR G.F.'S) FUNCTIONS, WE USE $\bar{f}'(x)$ OR $\frac{\bar{d}f}{dx}$ FOR $f'_{gen.}$ TO DISTINGUISH GENERALIZED FROM ORDINARY DERIVATIVE.

<u>EXAMPLE</u>. GENERALIZED DERIVATIVE OF AN ORD. FUNCTION WITH A JUMP.



$$F[\phi] = \int f\phi\,dx, \quad \phi \in D$$

$$F'[\phi] = -F[\phi'] = -\int f\phi'dx = -\left(\int_{-\infty}^{x_{0-}} + \int_{x_{0+}}^{\infty} f\phi'dx\right) = \int f'\phi\,dx + \Delta f\,\phi(x_0)$$

OR SYMBOLICALLY $\boxed{\bar{f}'(x) = f'(x) + \Delta f\,\delta(x - x_0)}$

<u>EXAMPLE</u>. GENERALIZED DERIVATIVE OF HEAVISIDE FUNCTION

$h(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$. $\bar{h}'(x) = \delta(x)$ SINCE $h'(x) = 0$ ON $(0,\infty) \cup (-\infty, 0)$.

<u>NOTE</u>: EVEN AT THIS LEVEL OF EXPOSITION, WE CAN DO A LOT WE COULD NOT DO BY USING ORDINARY FUNCTIONS. WE CAN DISCUSS GREEN'S FUNCTION OF AN O.D.E., FOR EXAMPLE.

SOME IMPORTANT RESULTS OF GENERALIZED FUNCTION THEORY

- STRUCTURE THEOREM OF D' : GENERALIZED FUNCTIONS IN D' ARE GENERALIZED DERIVATIVES OF FINITE ORDER OF CONTINUOUS FUNCTIONS.
- SEQUENCES OF GENERALIZED FUNCTIONS : A SEQUENCE $\{F_n[\phi]\}$ OF GENERALIZED FUNCTIONS IS CONVERGENT IF FOR ALL $\phi \in D$, THE SEQ. OF NUMBERS $\{F_n[\phi]\}$ IS CONVERGENT.

THEOREM : THE SPACE D' IS COMPLETE.

- EXCHANGE OF LIMIT PROCESSES : WE CAN EXCHANGE LIMIT PROCESSES WHEN WE ARE DEALING WITH GENERALIZED FUNCTIONS. THIS RESULT IS VERY IMPORTANT IN APPLICATIONS.

EXAMPLES : $\dfrac{\bar\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\bar\partial^2 f}{\partial x_j \partial x_i}$; $\sum_i \int_\Omega \cdots = \int_\Omega \sum_i \cdots$;
$\dfrac{\bar\partial}{\partial x_i}\int_\Omega \cdots = \int_\Omega \dfrac{\bar\partial}{\partial x_i}\cdots$; $\lim_{n\to\infty}\sum_m \cdots = \sum_m \lim_{n\to\infty}\cdots$; $\dfrac{\bar d}{dx}\lim_{m\to\infty}\cdots = \lim_{m\to\infty}\dfrac{\bar d}{dx}\cdots$

GREEN'S FUNCTION OF A 2ND ORDER LINEAR O.D.E.

GIVEN
$$\begin{cases} \ell u = f(x) & x \in [0,1] \quad \text{LINEAR 2ND ORDER O.D.E.} \\ BC_1[u] = a_1 u(0) + b_1 u'(0) + c_1 u(1) + d_1 u'(1) = 0 & \\ BC_2[u] = a_2 u(0) + b_2 u'(0) + c_2 u(1) + d_2 u'(1) = 0 & \end{cases} \Big\} \text{ LINEAR HOMOGENEOUS BC'S}$$

ASSUME THERE IS A FUNCTION $g(x,y)$ (GREEN'S FUNCTION) SUCH THAT

$$u(x) = \int_0^1 f(y)\, g(x,y)\, dy \qquad (1)$$

WE ARE INTERESTED IN SOLUTIONS WHERE $u \in C^1$ AND u IS TWICE DIFFERENTIABLE SO THAT $\bar u'' = u''$ AND $\bar u' = u$ AND, THEREFORE, $\bar\ell u = \ell u$. HERE $\bar\ell u$ STANDS FOR THE DIFFERENTIAL EQUATION WHERE ORDINARY DERIVATIVES ARE REPLACED BY GENERALIZED DERIVATIVES.

FROM EQ.(1), WE HAVE
$$\begin{aligned} \bar\ell u(x) &= \bar\ell_x \int_0^1 f(y)\, g(x,y)\, dy \\ &= \int_0^1 f(y)\, \bar\ell_x g(x,y)\, dy \quad \text{(EXCHANGE OF LIMIT PROCESS)} \\ &= f(x) \quad \text{(BY THE O.D.E.)} \end{aligned}$$

THEREFORE $\boxed{\bar\ell_x g(x,y) = \delta(x-y)}$. SIMILARLY, SINCE THE BOUNDARY CONDITIONS ARE LINEAR
$$\begin{aligned} BC_1[u] &= BC_{1,x}\int_0^1 f(y)\, g(x,y)\, dy \\ &= \int_0^1 f(y)\, BC_{1,x}[g(x,y)]\, dy = 0 \end{aligned}$$

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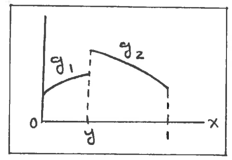
GREEN'S FUNCTION OF A 2ND ORDER LINEAR O.D.E. (CONT'D)

THE PREVIOUS RESULT MEANS THAT $\boxed{BC_{1,x}[g(x,y)]=0}$. ALSO $\boxed{BC_{2,x}[g(x,y)]=0}$, i.e. $g(x,y)$ IN VARIABLE x SATISFIES BOTH BC'S.

– WHAT IS THE INTERPRETATION OF $\bar{\ell}_x\, g(x,y)=\delta(x-y)$?

LET $\ell = A(x)\frac{d^2}{dx^2}+B(x)\frac{d}{dx}+C(x)$, THEN $g(x,y)$ AND $\frac{\partial g}{\partial x}(x,y)$ MUST HAVE SOME KIND OF DISCONTINUITY AT $x=y$.

LET $g(x,y)=\begin{cases} g_1(x,y) & x<y \\ g_2(x,y) & x>y \end{cases}$



THEN

$$\frac{\bar{\partial} g}{\partial x}=\frac{\partial g}{\partial x}+\Delta g\,\delta(x-y)$$

$$\frac{\bar{\partial}^2 g}{\partial x^2}=\frac{\partial^2 g}{\partial x^2}+\Delta\left(\frac{\partial g}{\partial x}\right)\delta(x-y)+\Delta g\,\delta'(x-y)$$

$$\bar{\ell}_x\, g(x,y)=\ell_x\, g(x,y)+\left[A(y)\Delta\left(\frac{\partial g}{\partial x}\right)+B(y)\Delta g\right]\delta(x-y)+A(x)\Delta g\,\delta'(x-y)$$

$$=\delta(x-y) \quad \text{(BY THE RESULT OF PREVIOUS PAGE)}$$

$\therefore\ \Delta g=0$ AT $x=y$ AND $\Delta\left(\frac{\partial g}{\partial x}\right)=\frac{1}{A(y)}$ AT $x=y$

THIS MEANS $\underline{\ell_x g_1(x,y)=\ell_x g_2(x,y)=0,\ g(x,y) \text{ IS CONTINUOUS AT}}$ $\underline{x=y \text{ AND } \frac{\partial g}{\partial x} \text{ HAS A JUMP EQUAL TO } 1/A(y) \text{ AT } x=y.}$

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GENERALIZED FUNCTIONS IN MULTIDENSIONS

– <u>SPACE D IN MULTIDENSIONS</u> : THIS SPACE IS FORMED BY C^∞ FUNCTIONS WITH BOUNDED SUPPORT. DEFINE

$$\phi(\vec{x};a)=\begin{cases}\exp\left[\frac{a^2}{a^2-|\vec{x}|^2}\right] & |\vec{x}|<a \\ 0 & |x|\geq a\end{cases},\quad |\vec{x}|=\left[\sum_{i=1}^{n}x_i^2\right]^{1/2}$$

THIS BELONGS TO D IN n DIMENSIONS. GIVEN ANY CONTINUOUS FUNCTION $g(\vec{x})$ AND ANY BOUNDED REGION Ω

$$\psi(\vec{x})=\int_\Omega g(\vec{y})\,\phi(\vec{x}-\vec{y};a)\,d\vec{y}\ \in D$$

– <u>GENERALIZED FUNCTIONS</u> IN n DIMENSIONS ARE CONTINUOUS LINEAR FUNCTIONALS ON n DIMENSIONAL TEST FUNCTION SPACE D.

<u>EXAMPLES</u>. i) $\int\delta(\vec{x})\,\phi(\vec{x})\,d\vec{x}=\phi(0)$; ii) $\int\frac{\partial}{\partial x_i}\delta(\vec{x})\,\phi(\vec{x})=-\frac{\partial\phi}{\partial x_i}(0)$

– FROM OUR POINT OF VIEW, THE MOST IMPORTANT GENERALIZED FUNCTIONS ARE DELTA FUNCTIONS WHOSE SUPPORTS ARE ON OPEN OR CLOSED SURFACES, e.g., $\delta(\xi)$. WE NEED TO INTERPRET INTEGRALS OF THE FORM $I_1=\int\delta(\xi)\,\phi(\vec{x})\,d\vec{x}$ AND $I_2=\int\delta'(\xi)\,\phi(\vec{x})\,d\vec{x}$.

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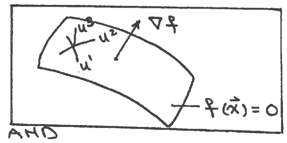
HOW DOES $\delta(f)$ APPEAR IN APPLICATIONS?

ASSUME $g(\vec{x})$ IS DISCONTINUOUS ACROSS THE SURFACE $f(\vec{x})=0$ WITH THE JUMP

$$\Delta g = g(f=0+) - g(f=0-)$$



SET UP COORDINATE SYSTEM (u^1,u^2) ON $f=0$ AND EXTEND THESE COORDINATES TO THE VICINITY OF $f=0$ ALONG LOCAL NORMALS. TAKE $u^3=f$ AS THIRD LOCAL VARIABLE. THEN

$$\frac{\bar{\partial} g}{\partial u^i} = \frac{\partial g}{\partial u^i} \quad i=1,2 \quad \text{AND} \quad \frac{\bar{\partial} g}{\partial u^3} = \frac{\partial g}{\partial u^3} + \Delta g\, \delta(u^3)$$

$$\therefore \quad \frac{\bar{\partial} g}{\partial x_j} = \frac{\bar{\partial} g}{\partial u^i}\frac{\partial u^i}{\partial x_j} = \frac{\partial g}{\partial u^i}\frac{\partial u^i}{\partial x_j} + \Delta g \frac{\partial u^3}{\partial x_j}\delta(u^3) = \frac{\partial g}{\partial x_j} + \Delta g \frac{\partial u^3}{\partial x_j}\delta(u^3)$$

SINCE $u^3 = f$, WE HAVE $\boxed{\bar{\nabla} g = \nabla g + \Delta g\, \nabla f\, \delta(f)}$.

SIMILARLY

$$\boxed{\begin{aligned} \bar{\nabla}\cdot\vec{g} &= \nabla\cdot\vec{g} + \Delta\vec{g}\cdot\nabla f\,\delta(f) \\ \bar{\nabla}\times\vec{g} &= \nabla\times\vec{g} + \Delta\vec{g}\times\nabla f\,\delta(f) \end{aligned}}$$

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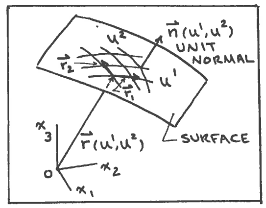
HOW DOES $\delta(f)$ APPEAR IN APPLICATIONS (CONT'D)?

- IN OUR WORK THE DISCONTINUITIES IN FUNCTIONS ARE EITHER REAL (e.g., SHOCK WAVES) OR ARTIFICIAL (e.g., ACROSS BLADE IN DERIVATION OF FW-H EQ.).
- EXAMPLE. SHOCK SURFACE SOURCES IN LIGHTHILL JET NOISE THEORY. LET THE SHOCK SURFACES BE DEFINED BY $f(\vec{x},t)=0$. WE WILL SHOW LATER THAT LIGHTHILL'S EQUATION IS VALID IN PRESENCE OF SHOCKS IF WE INTERPRET THE DERIVATIVES OF THE SOURCE TERM AS GENERALIZED DERIVATIVES:

$$\begin{aligned} \Box^2 p' &= \frac{\bar{\partial}^2 T_{ij}}{\partial x_i \partial x_j} \\ &= \frac{\bar{\partial}}{\partial x_i}\left[\frac{\partial T_{ij}}{\partial x_j} + \Delta T_{ij}\frac{\partial f}{\partial x_j}\delta(f)\right] \\ &= \underbrace{\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}}_{\text{TURBULENCE SOURCE}} + \underbrace{\Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right)\frac{\partial f}{\partial x_i}\delta(f) + \frac{\partial}{\partial x_i}\left[\Delta T_{ij}\frac{\partial f}{\partial x_j}\delta(f)\right]}_{\text{SHOCK SURFACE SOURCES}} \end{aligned}$$

SOME RESULTS FROM DIFFERENTIAL GEOMETRY

— INTRODUCE THE LOCAL SURFACE VARIABLES (u^1, u^2) ON A SURFACE. DEFINE LOCAL TANGENT VECTORS $\vec{r}_1 = \partial \vec{r}/\partial u^1$ AND $\vec{r}_2 = \partial \vec{r}/\partial u^2$. IN GENERAL, THESE ARE NOT OF UNIT LENGTH.



LET $g_{ij} = \vec{r}_i \cdot \vec{r}_j$, THE FIRST FUNDAMENTAL FORM IS $\boxed{dl^2 = g_{11}(du^1)^2 + 2g_{12}\, du^1 du^2 + g_{22}(du^2)^2}$, $\boxed{g_{12} = g_{21}}$.

THIS GIVES THE ELEMENT OF LENGTH OF A CURVE ON THE SURFACE.

IN THIS RELATION g_{ij}'S ARE KNOWN AS COEFFICIENTS OF THE FIRST FUNDAMENTAL FORM. WE DEFINE $g_{(2)}$ AS THE DETERMINANT OF COEFF. OF 1ST FUNDAMENTAL FORM $\boxed{g_{(2)} = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}g_{22} - g_{12}^2}$.

— WE CAN SHOW THAT THE ELEMENT OF SURFACE AREA dS IS $dS = |\vec{r}_1 \times \vec{r}_2|\, du^1 du^2$. SINCE $\boxed{g_{(2)} = |\vec{r}_1 \times r_2|^2}$, WE HAVE

$$\boxed{dS = \sqrt{g_{(2)}}\, du^1 du^2}$$

— NOTE : WE USE SUMMATION CONVENTION ON REPEATED INDEX BELOW.

SOME RESULTS FROM DIFFERENTIAL GEOMETRY (CONT'D)

— DEFINE g^{ij} AS ELEMENTS OF THE INVERSE OF THE MATRIX $G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$, i.e. $G^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} \Rightarrow g^{11} = \dfrac{g_{22}}{g_{(2)}}$, $g^{22} = \dfrac{g_{11}}{g_{(2)}}$, $g^{12} = g^{21} = \dfrac{-g_{12}}{g_{(2)}}$. WE HAVE $\boxed{g^{ij} g_{jk} = \delta^i_k}$ WHERE δ^i_k IS THE KRONECKER DELTA.

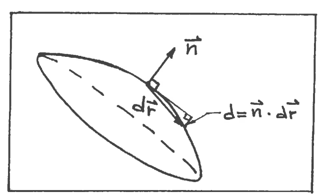
— DEFINE $\boxed{b_{ij} = \vec{r}_{ij} \cdot \vec{n}}$ WHERE $\vec{r}_{ij} = \partial^2 \vec{r}/\partial u^i \partial u^j$. THE SECOND FUNDAMENTAL FORM IS $\boxed{\Pi = b_{11}(du^1)^2 + 2b_{12}\, du^1 du^2 + b_{22}(du^2)^2}$.

NOTE THAT $b_{12} = b_{21}$ AND $\vec{n}$ IS THE LOCAL UNIT NORMAL. IN THIS RELATION b_{ij}'S ARE KNOWN AS COEFFICIENTS OF 2ND FUND. FORM.

— $\boxed{b = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}^2}$ IS THE DETERMINANT OF COEFF. OF 2ND FUND. FORM.

— WHAT IS THE GEOMETRICAL MEANING OF Π? $d = \vec{n} \cdot d\vec{r} = \vec{n} \cdot [\, \vec{r}_1\, du^1 + \vec{r}_2\, du^2 + \frac{1}{2}(\vec{r}_{11}(du^1)^2 + 2\vec{r}_{12}\, du^1 du^2 + \vec{r}_{22}(du^2)^2) + \ldots\,]$

$= \frac{1}{2}\Pi + O(du^i)^3 \quad \therefore \quad \boxed{\Pi \approx 2d}$



SOME RESULTS FROM DIFFERENTIAL GEOMETRY (CONT'D)

– ANOTHER RELATION FOR b_{ij} : $\boxed{b_{ij} = -\vec{r}_i \cdot \vec{n}_j}$, $\vec{n}_j = \partial \vec{n}/\partial u^j$

– WEINGARTEN FORMULA: $\boxed{\vec{n}_i = -b^j_i \, \vec{r}_j}$ WHERE $\boxed{b^j_i = g^{jk} b_{ki}}$

WE ARE USING SUMMATION CONVENTION HERE.

– GAUSS FORMULA : $\boxed{\vec{r}_{ij} = \Gamma^k_{ij} \vec{r}_k + b_{ij} \vec{n}}$ WHERE Γ^k_{ij} IS CHRISTOFFEL SYMBOL OF 2ND KIND.

– CHRISTOFFEL SYMBOLS : FIRST KIND Γ_{ijk} , SECOND KIND Γ^k_{ij}

$$\boxed{\Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right]} \quad \text{AND} \quad \boxed{\Gamma_{ijk} = \Gamma_{jik}}$$

$$\boxed{\Gamma^k_{ij} = g^{k\ell}\, \Gamma_{ij\ell}} \quad \text{AND} \quad \boxed{\Gamma^\ell_{ij} = \Gamma^\ell_{ji}}$$

NOTE : CHRISTOFFEL SYMBOLS ARE NOT TENSORS WHILE g_{ij} , g^{ij} , b_{ij} , b^i_j ARE.

– A USEFUL RESULT : $\boxed{\dfrac{\partial \sqrt{g_{(2)}}}{\partial u^i} = \Gamma^k_{ik} \sqrt{g_{(2)}}}$

– GAUSS FORMULA : A VERY SIGNIFICANT RESULT !

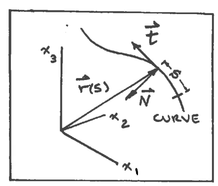
$$\boxed{\begin{aligned} b = \partial^2_{12}\, g_{12} - \tfrac{1}{2}(\partial^2_{22}\, g_{11} + \partial^2_{11}\, g_{22}) \\ -(\Gamma^i_{11}\Gamma^j_{12} - \Gamma^i_{12}\Gamma^j_{12})\, g_{ij} \end{aligned}}$$

$\partial_{ij} = \partial^2/\partial u^i \partial u^j$

SEE THEOREMA EGREGIUM OF GAUSS

SOME RESULTS FROM DIFFERENTIAL GEOMETRY (CONT'D)

– LET US PARAMETRIZE A CURVE IN SPACE BY LENGTH PARAMETER S. THE UNIT TANGENT $\vec{t}$ TO THE CURVE IS $\boxed{\vec{t} = \dfrac{d\vec{r}}{ds}}$ AND THE LOCAL CURVATURE k IS GIVEN BY



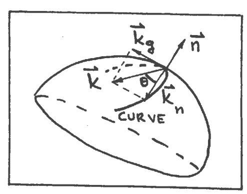
$$\boxed{k\vec{N} = \frac{d\vec{t}}{ds} = \vec{k}, k > 0} \quad \therefore \quad \boxed{k = \left|\frac{d\vec{t}}{ds}\right| = \left|\frac{d^2\vec{r}}{ds^2}\right|}$$

NOTE THAT $\vec{N}$ ALWAYS POINTS TO THE CENTER OF CURVATURE, i.e. $\vec{N}$ IS PARALLEL TO $d\vec{t}/ds$.

– FOR A CURVE ON A SURFACE $d^2\vec{r}/ds^2$ HAS COMPONENTS ALONG TANGENT AND NORMAL TO THE SURFACE.

$$\frac{d\vec{r}}{ds} = \vec{r}_i \frac{du^i}{ds}$$

$$\boxed{\begin{aligned} \vec{k} &= \frac{d^2\vec{r}}{ds^2} = \vec{r}_{ij}\frac{du^i}{ds}\frac{du}{ds} + \vec{r}_i \frac{d^2u^i}{ds^2} \\ &= \underbrace{\left(\frac{d^2u^k}{ds^2} + \Gamma^k_{ij}\frac{du^i}{ds}\frac{du^j}{ds}\right)\vec{r}_k}_{\vec{k}_g:\ \text{GEODESIC CURVATURE VECTOR}} + \underbrace{b_{ij}\frac{du^i}{ds}\frac{du^j}{ds}\vec{n}}_{\vec{k}_n:\ \text{NORMAL CURVATURE VECTOR}} \end{aligned}}$$



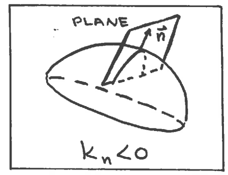
$$\boxed{k = \frac{k_n}{\sin\theta}}$$

MEUSNIER THEOREM

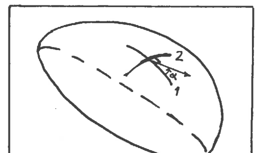
GEODESIC CURVATURE IS AN INTRINSIC WHILE NORMAL CURVATURE IS AN EXTRINSIC QUANTITY.

SOME RESULTS FROM DIFFERENTIAL GEOMETRY (CONT'D)

— THE NORMAL CURVATURE IS A SIGNED QUANTITY. IF $k_n > 0$, THEN THE CENTER OF CURVATURE OF THE CURVE OBTAINED BY INTERSECTION OF A PLANE CONTAINING $\vec{n}$ AND THE SURFACE, IS ON THE SIDE $\vec{n}$ POINTS TO. NOTE THAT $t^i = \frac{du^i}{ds}$ THE COMPONENTS OF UNIT TANGENT TO THIS CURVE AND $\boxed{k_n = b_{ij}\, t^i t^j}$. $g_{ij}\, t^i t^j = 1$.



— THERE ARE TWO DIRECTIONS AT A POINT ON A SURFACE ORTHOGONAL TO EACH OTHER WHERE k_n MAXIMUM AND MINIMUM. THESE ARE KNOWN AS THE PRINCIPAL DIRECTIONS WITH PRINCIPAL CURVATURES k_1 AND k_2 (NORMAL CURVATURES).



1 AND 2 PRINCIPAL DIRECTIONS

— MEAN CURVATURE $H = \frac{1}{2}(k_1 + k_2)$

GAUSSIAN CURVATURE $K = k_1 k_2$

EULER'S FORMULA

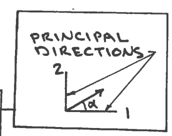
$$\boxed{k_n(\alpha) = k_1 \cos^2\alpha + k_2 \sin^2\alpha}$$

SOME RESULTS FROM DIFFERENTIAL GEOMETRY (CONT'D)

— THEOREMA EGREGIUM OF GAUSS: THE GAUSSIAN CURVATURE K DEPENDS ON g_{ij} AND THEIR FIRST AND SECOND DERIVATIVES.

— WE HAVE $\boxed{K = \frac{b}{g}}$, AND b WAS GIVEN IN TERMS OF g_{ij} AND THEIR FIRST AND 2ND DERIVATIVES. THUS THE GAUSSIAN CURVATURE IS AN INTRINSIC QUANTITY.

— $\boxed{\vec{n}_1 \times \vec{n}_2 = K\, \vec{r}_1 \times \vec{r}_2}$ $\quad \vec{r}_i = \frac{\partial \vec{r}}{\partial u^i}, \quad \vec{n}_i = \frac{\partial \vec{n}}{\partial u^i}$



— BY EULER'S FORMULA $\boxed{H = \frac{1}{2}\left[k_n(\alpha) + k_n(\alpha + \tfrac{\pi}{2})\right]}$

— WE CAN SHOW THAT $\boxed{H = \frac{1}{2} b^i_i = \frac{1}{2}(b^1_1 + b^2_2)}$ $\quad b^i_j = g^{ik} b_{kj}$

— THE PRINCIPAL DIRECTIONS CAN BE FOUND FROM SOLVING THE QUADRATIC EQUATION:

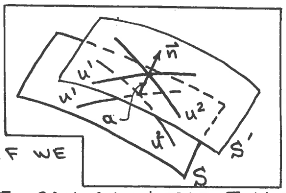
$$\begin{vmatrix} (du^2)^2 & -du^1 du^2 & (du^1)^2 \\ b_{11} & b_{12} & b_{22} \\ g_{11} & g_{12} & g_{22} \end{vmatrix} = 0$$

REMEMBER (du^1, du^2) DEFINES THE DIRECTION $\vec{r}_1 du^1 + \vec{r}_2 du^2$.

— $\boxed{H^2 \geq K}$, $H^2 = K$ IF AND ONLY IF THE TWO FUNDAMENTAL FORMS ARE PROPORTIONAL.

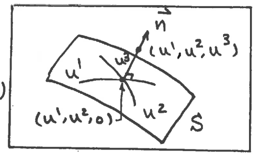
SOME RESULTS FROM DIFFERENTIAL GEOMETRY (CONT'D)

— LET US DISPLACE SURFACE S GIVEN BY $\vec{r}(u^1,u^2)$ BY DISTANCE α = CONST. ALONG LOCAL NORMAL TO GET S' GIVEN BY $\vec{R}(u^1,u^2) = \vec{r}(u^1,u^2) + \alpha\,\vec{n}(u^1,u^2)$. IF WE ASSOCIATE PRIME QUANTITIES TO S', WE CAN SHOW THE FOLLOWING



$$\boxed{\begin{aligned} g'_{(2)} &= (1-2H\alpha+K\alpha^2)^2 g_{(2)} \\ \sqrt{g'_{(2)}} &= (1-2H\alpha+K\alpha^2)\sqrt{g_{(2)}} \end{aligned}} \qquad \boxed{\begin{aligned} H' &= \frac{H-K\alpha}{1-2H\alpha+K\alpha^2} \\ K' &= \frac{K}{1-2H\alpha+K\alpha^2} \end{aligned}}$$

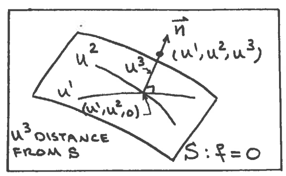
— IF WE NOW DEFINE $u^3=\alpha$ BE THE DISTANCE ALONG LOCAL NORMAL TO A SURFACE S, THEN THE THREE DIMENSIONAL SPACE NEAR S CAN BE PARAMETRIZED BY (u^1,u^2,u^3) AND $g_{(3)}$, THE DETERMINANT OF COEFF. OF 1ST FUND. FORM IN 3D IS GIVEN BY $\boxed{g_{(3)} = g'_{(2)}(u^1,u^2,u^3) = [1-2Hu^3+K(u^3)^2]^2 g_2(u^1,u^2)}$



FROM THIS WE FIND $\boxed{(\partial\sqrt{g'_{(2)}}/\partial u^3)_{u^3=0} = (\partial\sqrt{g'_{(2)}}/\partial n)_S = -2H\sqrt{g_{(2)}}}$

SOME RESULTS FROM DIFFERENTIAL GEOMETRY (CONT'D)

LET US NOW HAVE A VECTOR FIELD $\vec{Q}$ IN THE VICINITY OF SURFACE $S: f=0$. WE WANT TO WRITE $\nabla\cdot\vec{Q}$ IN A NEW WAY. FIRST PARAMETRIZE THE 3D SPACE



IN THE VICINITY OF S AS SHOWN. THEN, LET Q^i BE THE CONTRA-VARIANT COMPONENTS OF $\vec{Q}$. WE HAVE $\boxed{\nabla\cdot\vec{Q} = \frac{1}{\sqrt{g_{(3)}}}\frac{\partial}{\partial u^i}\left[\sqrt{g_{(3)}}\,Q^i\right]}$

NOW USING THE RESULT OF PREVIOUS PAGE THAT $g_{(3)} = g'_{(2)}(u^1,u^2,u^3)$, WE HAVE, USING $\alpha=1,2$

$$(\nabla\cdot\vec{Q})_S = \left[\frac{1}{\sqrt{g'_{(2)}}}\frac{\partial}{\partial u^\alpha}\left[\sqrt{g'_{(2)}}\,Q^\alpha\right] + \frac{\partial Q^3}{\partial u^3} + \frac{Q^3}{\sqrt{g'_{(2)}}}\frac{\partial\sqrt{g'_{(2)}}}{\partial u^3}\right]_S$$

SINCE $Q^3=Q_n$: $\boxed{(\nabla\cdot\vec{Q})_S = \nabla_2\cdot\vec{Q}_T + \frac{\partial Q_n}{\partial n} - 2HQ_n}$ $\quad \vec{Q}_T = \vec{Q} - Q_n\vec{n}$ SURFACE COMPONENT OF $\vec{Q}$ ON S

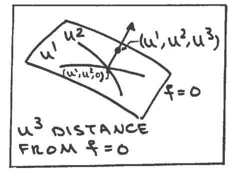
WHERE $\nabla_2\cdot\vec{Q}_T$ IS THE SURFACE DIVERGENCE OF $\vec{Q}_T = Q^1\vec{r}_1 + Q^2\vec{r}_2$:

$$\boxed{\nabla_2\cdot\vec{Q}_T = \frac{1}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^\alpha}\left[\sqrt{g_{(2)}}\,Q^\alpha\right]} \qquad \alpha=1,2$$

THE INTEGRATION OF $\delta(f)$ AND $\delta'(f)$

WE ASSUME $f(\vec{x})$ IS DEFIND SUCH THAT $|\nabla f| = 1$. ON THE SURFACE $f=0$. THIS CAN ALWAYS BE DONE. THIS MEANS $df = dn = du^3$

– PARAMETRIZE THE SPACE IN VICINITY OF SURFACE $f=0$ BY VARIABLES (u^1, u^2, u^3) AS SHOWN. THEN



$$I_1 = \int \phi(\vec{x})\,\delta(f)\,d\vec{x}$$

$$d\vec{x} = \sqrt{g_{(3)}}\,du^1 du^2 du^3$$

$$= \sqrt{g'_{(2)}(u^1,u^2,u^3)}\,du^1 du^2 du^3$$

$$I_1 = \int \phi(\vec{x})\,\delta(u^3)\sqrt{g'_{(2)}}\,du^1 du^2 du^3 = \int [\phi(\vec{x})]_{u^3=0}\sqrt{g_{(2)}}\,du^1 du^2$$

$$\boxed{I_1 = \int \phi(\vec{x})\,\delta(f)\,d\vec{x} = \int_{f=0} \phi(\vec{x})\,dS}$$

$$I_2 = \int \phi(\vec{x})\,\delta'(f)\,d\vec{x} = \int \phi(\vec{x})\,\delta'(u^3)\sqrt{g'_{(2)}}\,du^1 du^2 du^3$$

$$= -\int \frac{\partial}{\partial u^3}\left[\phi(\vec{x})\sqrt{g'_{(2)}}\right]_{u^3=0} du^1 du^2 = \int \left[-\frac{\partial\phi}{\partial u^3} + 2H_f\,\phi\right]\sqrt{g_{(2)}}\,du^1 du^2$$

WHERE WE USED $\left(\partial\sqrt{g'_{(2)}}/\partial u_3\right)_{u^3=0} = -2H_f\sqrt{g_{(2)}}$, H_f LOCAL MEAN CURVATURE ON $f=0$.

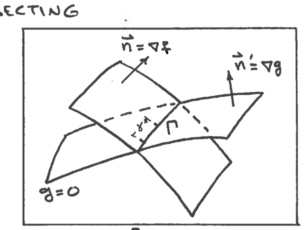
$$\boxed{I_2 = \int \phi(\vec{x})\,\delta(f')\,d\vec{x} = \int_{f=0}\left[-\frac{\partial\phi}{\partial n} + 2H_f\,\phi\right]dS}$$

INTEGRATION OF PRODUCT OF DELTA FUNCTIONS

LET $f(\vec{x}) = 0$ AND $g(\vec{x}) = 0$ BE TWO INTERSECTING SURFACES IN 3D. WE WANT TO INTEGRATE

$$I = \int \phi(\vec{x})\,\delta(f)\,\delta(g)\,d\vec{x}$$

LET THE TWO SURFACES INTERSECT ALONG THE CURVE Γ. ON LOCAL PLANE NORMAL



TO Γ, PARAMETRIZE SPACE BY $u^1 = f$, $u^2 = g$ AND $u^3 = \gamma$, WHERE γ IS THE DISTANCE ALONG Γ. EXTEND u^1 AND u^2 TO THE SPACE IN THE VICINITY OF THE PLANE ALONG LOCAL NORMAL TO THE PLANE. THEN

$$\boxed{d\vec{x} = \frac{du^1 du^2 du^3}{\sin\theta}} \qquad \boxed{\sin\theta = |\vec{n}\times\vec{n}'|}$$

$$I = \int \frac{\phi(\vec{x})}{\sin\theta}\,\delta(u^1)\,\delta(u^2)\,du^1 du^2 du^3 = \int_{f=g=0} \frac{\phi(\vec{x})}{\sin\theta}\,du^3$$

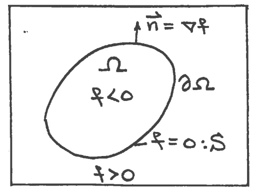
$$\boxed{I = \int_{\substack{f=0\\ g=0}} \frac{\phi(\vec{x})}{\sin\theta}\,d\gamma}$$

ALSO IF $|\nabla f| \neq 1$ OR $|\nabla g| \neq 1$

$$I = \int_{\substack{f=0\\ g=0}} \frac{\phi(\vec{x})}{|\nabla f||\nabla g|\sin\theta}\,d\gamma$$

ILLUSTRATION OF MANIPULATION OF GEN. FNS.

LET $\vec{Q}$ BE A VECTOR FIELD WHICH IS ZERO OUTSIDE Ω AND NONZERO INSIDE Ω.



$$\Delta \vec{Q} = \vec{Q}(f=0+) - \vec{Q}(f=0-) = -\vec{Q}|_S$$

$$\overline{\nabla} \cdot \vec{Q} = \nabla \cdot \vec{Q} + \Delta \vec{Q} \cdot \vec{n}\, \delta(f) = \nabla \cdot \vec{Q} - Q_n \delta(f)$$

NOW INTEGRATE $\overline{\nabla} \cdot \vec{Q}$ OVER THE ENTIRE 3D SPACE

$$\int \overline{\nabla} \cdot \vec{Q}\, d\vec{x} = 0 \quad \text{SINCE} \quad \int \frac{\overline{\partial} Q_1}{\partial x_1} dx_1 dx_2 dx_3 = \int \left(Q_1\Big|_{x_1=\infty} - Q_1\Big|_{x_1=-\infty}\right) dx_2 dx_3 = 0.$$

SIMILARLY FOR $\overline{\partial} Q_2/\partial x_2$ AND $\overline{\partial} Q_3/\partial x_3$. NOW, WE HAVE

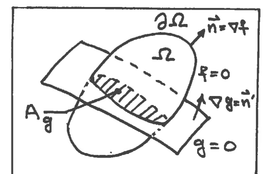
$$\int \overline{\nabla} \cdot \vec{Q}\, d\vec{x} = \int [\nabla \cdot \vec{Q} - Q_n \delta(f)]\, d\vec{x} = \boxed{\int_\Omega \nabla \cdot \vec{Q}\, d\vec{x} - \int_{\partial\Omega} Q_n\, dS = 0}$$

THIS IS <u>THE DIVERGENCE THM</u>. THIS RESULT IS VALID IF $\vec{Q}$ IS DISCONTINUOUS ACROSS A SURFACE $g=0$ INSIDE Ω.

$$\int_\Omega \overline{\nabla} \cdot \vec{Q}\, d\vec{x} = \int_\Omega [\nabla \cdot \vec{Q} + \Delta \vec{Q} \cdot \vec{n}'\, \delta(g)]\, d\vec{x}$$

$$= \int_\Omega \nabla \cdot \vec{Q}\, d\vec{x} + \int_{A_g} \Delta Q_{n'}\, dS = \int_{\partial\Omega} Q_n\, dS$$

$$\boxed{\int_\Omega \nabla \cdot \vec{Q}\, d\vec{x} = \int_{\partial\Omega} Q_n\, dS - \int_{A_g} \Delta Q_{n'}\, dS}$$



$$\Delta Q_{n'} = \vec{n}' \cdot [\vec{Q}(g=0+) - \vec{Q}(g=0-)]$$

ILLUSTRATION OF MANIPULATION OF GEN. FNS. (CONT'D)

IN DERIVING CONSERVATION LAWS IN DIFFERENTIAL FORM FROM FINITE VOLUMES INVOLVING DISCONTINUOUS FUNCTIONS, WHENEVER THE DIVERGENCE THEOREM IS USED TO CONVERT SURFACE INTEGRALS INTO VOLUME INTEGRALS, ONE SHOULD USE GENERALIZED DERIVATIVE. SUCH CONSERVATION LAWS HAVE THE JUMP CONDITIONS INCORPORATED IN THEM.

<u>EXAMPLE</u>. <u>SHOCK JUMP CONDITIONS</u>: LET THE SHOCK SURFACE BE GIVEN BY $f(\vec{x},t)=0$, $\nabla f = \vec{n}$, $\Rightarrow \frac{\partial f}{\partial t} = -v_n$ LOCAL SHOCK NORMAL SPEED

MASS CONTINUITY EQ.: $\frac{\overline{\partial} \rho}{\partial t} + \frac{\overline{\partial}}{\partial x_i}(\rho u_i) = 0$

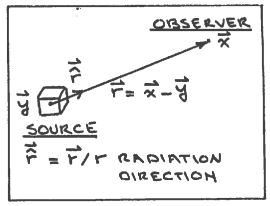
$$\frac{\overline{\partial}\rho}{\partial t} + \frac{\overline{\partial}}{\partial x_i}(\rho u_i) = \underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i)}_{=0} + \underbrace{\Delta\rho \frac{\partial f}{\partial t}\delta(f) + \Delta(\rho u_i)\frac{\partial f}{\partial x_i}\delta(f)}_{[-v_n \Delta\rho + \Delta(\rho u_n)]\,\delta(f)=0}$$

$$\therefore \quad \Delta[\rho(u_n - v_n)] = 0$$

SIMILARLY FOR MOMENTUM AND ENERGY EQUATIONS.

THINGS TO KNOW ABOUT GREEN'S FUNCTION OF WAVE EQUATION

– THE GREEN'S FUNCTION OF THE WAVE EQUATION IN THE UNBOUNDED SPACE IS



$$\boxed{G(\vec{y},\tau;\vec{x},t) = \frac{\delta(g)}{4\pi r}\ (\tau \le t);\ G=0\ (\tau > t)}$$

$$\boxed{g = \tau - t + \frac{r}{c}}\quad \text{OUTGOING WAVE}$$

$(\vec{y},\tau)$ SOURCE SPACE-TIME VARIABLES

$(\vec{x},t)$ OBSERVER SPACE-TIME VARIABLES

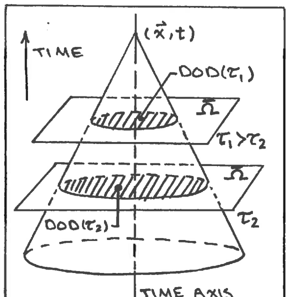
– THERE ARE MANY METHODS TO DERIVE $G(\vec{y},\tau;\vec{x},t)$ RIGOROUSLY. IT IS EASY TO SHOW THAT G DEPENDS ON $\vec{x}-\vec{y}$ AND $t-\tau$. USING $\vec{x}-\vec{y}=\vec{r}$, $\lambda = t-\lambda$, TAKE SPATIAL FOURIER TRANSFORM OF $\Box^2_{(\vec{r},\lambda)} G = \delta(\vec{r})\delta(\lambda)$ TO GET A SIMPLE PROBLEM INVOLVING FINDING THE GREEN'S FUNCTION OF AN O.D.E. IN λ. THE INVERSE SPATIAL FOURIER TRANSFORM OF THE GREEN'S FUNCTION OF THE O.D.E. GIVES GREEN'S FUNCTION OF THE WAVE EQUATION FOR BOTH THE OUTGOING AND INCOMING WAVES.

$$\boxed{\vec{r} = \vec{x}-\vec{y},\quad r = |\vec{x}-\vec{y}|,\quad \hat{\vec{r}} = \frac{\vec{r}}{r},\quad \frac{\partial r}{\partial x_i} = \hat{r}_i,\quad \frac{\partial r}{\partial y_i} = -\hat{r}_i}$$

USEFUL THINGS TO REMEMBER

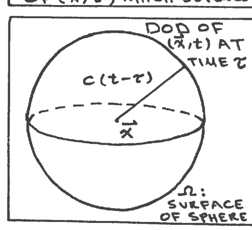
THINGS TO KNOW ABOUT GREEN'S FUNCTION OF WAVE EQUATION (CONT'D)

THE SUPPORT OF $\delta(g)$ IS ON THE SURFACE $g=0$. THE SURFACE $g=0$ IS $r = |\vec{x}-\vec{y}| = c(t-\tau)$. THIS THE CHARACTERISTIC CONE OF THE WAVE EQUATION WITH VERTEX AT $(\vec{x},t)$. SINCE $\Box^2$ IS A DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS, $g=0$ IS ALSO THE CHARACTERISTIC CONOID WITH VERTEX AT $(\vec{x},t)$. THIS GIVES US THE PICTURE ON THE RIGHT.



CHAR. CONE (CONOID) AT $(\vec{x},t)$. Ω IS THE UNBOUNDED 3-SPACE. DOD IS DOMAIN OF DEPENDENCE OF (x,t) WHICH DEPENDS ON τ

– VISUALIZATION OF DOMAIN OF DEPENDENCE OF $(\vec{x},t)$. FIX $(\vec{x},t)$ AND $\tau \Rightarrow r = c(t-\tau)$ IS A SPHERE WITH CENTER AT $\vec{x}$ AND RADIUS $c(t-\tau)$. ANY SOURCES ON THIS SPHERE AT TIME τ, CONTRIBUTES TO $\vec{x}$ AT TIME t.



AS τ INCREASES, THE RADIUS SHRINKS, HENCE WE HAVE A COLLAPSING SPHERE. RADIUS BECOMES ZERO AT $\tau = t$.

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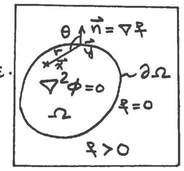
USE OF GREEN'S FUNCTIONS FOR DISCONTINUOUS SOLUTIONS

GREEN'S FUNCTION CAN BE USED TO FIND DISCONTINUOUS SOLUTIONS IF THE DERIVATIVES IN THE DIFFERENTIAL EQUATION ARE TREATED AS GENERALIZED DERIVATIVES. THIS ADDS TO USEFULNESS OF GREEN'S FUNCTION.



INTERIOR PROBLEM

EXAMPLE. GREEN'S IDENTITY FOR LAPLACE EQ.

LET $\tilde{\phi}(\vec{x}) = \begin{cases} \phi(\vec{x}) & \vec{x} \in \Omega \\ 0 & \vec{x} \notin \Omega \end{cases} \Rightarrow \nabla^2\tilde{\phi} = 0$ EVERYWHERE.

$$\bar{\nabla}\,\tilde{\phi} = \nabla\tilde{\phi} + \Delta\tilde{\phi}\,\vec{n}\,\delta(\varrho) = \nabla\tilde{\phi} - \phi\vec{n}\,\delta(\varrho)$$

$$\bar{\nabla}^2\tilde{\phi} = \nabla^2\tilde{\phi} - \nabla\phi\cdot\vec{n}\,\delta(\varrho) - \nabla\cdot[\phi\vec{n}\,\delta(\varrho)]$$

$$= -\frac{\partial\phi}{\partial n}\delta(\varrho) - \nabla\cdot[\phi\vec{n}\,\delta(\varrho)]$$

SINCE THIS EQUATION IS VALID IN THE UNBOUNDED SPACE, WE CAN USE THE GREEN'S FUNCTION $-\frac{1}{4\pi r}$ TO GET THE GREEN'S IDENTITY

$$4\pi\tilde{\phi}(\vec{x}) = \int \frac{1}{r}\frac{\partial\phi}{\partial n}\delta(\varrho)\,d\vec{y} + \nabla\cdot\int\frac{\phi\vec{n}}{r}\delta(\varrho)\,d\vec{y}$$

$$= \int_{\varrho=0}\frac{1}{r}\frac{\partial\phi}{\partial n}\,dS + \nabla\cdot\int_{\varrho=0}\frac{\phi\vec{n}}{r}\,dS = \int_{\varrho=0}\frac{\phi_n}{r}\,dS - \int_{\varrho=0}\frac{\phi\cos\theta}{r^2}\,dS$$

THIS METHOD TELLS US THAT WHEN $\vec{x} \notin \Omega$, $\tilde{\phi} = 0$ WHICH IS NOT OBVIOUS FROM THE CLASSICAL DERIVATION. THE EXTERIOR PROBLEM IS SIMILAR.

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THE TWO FORMS OF THE SOLUTION OF WAVE EQUATION (VOLUME SOURCES)

WE WANT TO FIND THE SOLUTION OF $\square^2\phi = Q(\vec{x},t)$

$$4\pi\,\phi(\vec{x},t) = \int\frac{1}{r}Q(\vec{y},\tau)\,\delta(g)\,d\vec{y}\,d\tau$$

ALL VOLUME INTEGRALS ARE OVER UNBOUNDED 3 SPACE AND ALL TIME INTEGRALS ARE OVER $(-\infty, t)$.

i) LET $\tau \to g \Rightarrow \frac{\partial g}{\partial\tau} = 1$ AND $4\pi\,\phi(\vec{x},t) = \int\frac{1}{r}Q(\vec{y}, g+t-\frac{r}{c})\delta(g)\,dg\,d\vec{y}$

INTEGRATE WRT g TO GET

$$\boxed{4\pi\,\phi(\vec{x},t) = \int\frac{1}{r}Q(\vec{y}, t-\frac{r}{c})\,d\vec{y}}$$ RETARDED TIME SOLUTION

ii) LET $y_3 \to g \Rightarrow \frac{\partial g}{\partial y_3} = -\frac{1}{c}\hat{r}_3$

$$4\pi\,\phi(\vec{x},t) = \int\frac{c\,Q(\vec{y},\tau)}{r}\delta(g)\,dg\,\frac{dy_1dy_2}{|\hat{r}_3|}\,d\tau$$

SINCE IN THE INNER INTEGRALS $(\vec{x},t)$ AND τ ARE FIXED, THEN

$\boxed{\frac{dy_1dy_2}{|\hat{r}_3|} = d\Omega}$ ELEMENT OF SURFACE AREA OF SPHERE $r = c(t-\tau)$.

INTEGRATE WRT g TO GET : $\boxed{4\pi\,\phi(\vec{x},t) = \int_{-\infty}^{t}\frac{d\tau}{t-\tau}\int_{r=c(t-\tau)}Q(\vec{y},\tau)\,d\Omega}$

THIS IS THE COLLAPSING SPHERE SOLUTION.

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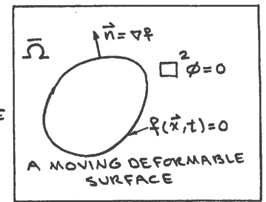
THE GOVERNING WAVE EQUATION FOR DERIVING KIRCHHOFF FORMULAS

WE CONSIDER THE EXTERIOR PROBLEM HERE.



$\bar\Omega$: THE EXTERIOR UNBOUNDED SPACE

LET $\tilde\phi(\vec x, t) = \begin{cases} \phi(\vec x, t) & \vec x \in \bar\Omega \\ 0 & \vec x \notin \bar\Omega \end{cases} \Rightarrow \Box^2\tilde\phi = 0$ EVERYWHERE

$$\frac{\bar\partial\tilde\phi}{\partial t} = \frac{\partial\tilde\phi}{\partial t} + \phi\frac{\partial f}{\partial t}\delta(f) = \frac{\partial\tilde\phi}{\partial t} - v_n\phi\,\delta(f)$$

WHERE $v_n = -\partial f/\partial t$ IS THE LOCAL NORMAL VELOCITY ON $f = 0$.

$$\frac{\bar\partial^2\tilde\phi}{\partial t^2} = \frac{\partial^2\tilde\phi}{\partial t^2} + \frac{\partial\phi}{\partial t}\frac{\partial f}{\partial t}\delta(f) - \frac{\partial}{\partial t}[v_n\phi\,\delta(f)]$$

$$= \frac{\partial^2\tilde\phi}{\partial t^2} - v_n\phi_t\,\delta(f) - \frac{\partial}{\partial t}[v_n\phi\,\delta(f)]$$

$$\bar\nabla\tilde\phi = \nabla\tilde\phi + \phi\vec n\,\delta(f), \quad \bar\nabla^2\tilde\phi = \nabla^2\tilde\phi + \phi_n\delta(f) + \nabla\cdot[\phi\vec n\,\delta(f)]$$

$$\bar\Box^2\tilde\phi = \frac{1}{c^2}\frac{\bar\partial^2\tilde\phi}{\partial t^2} - \bar\nabla^2\tilde\phi = \Box^2\tilde\phi - \left[\frac{v_n\phi_t}{c^2} + \phi_n\right]\delta(f) - \frac{1}{c^2}\frac{\partial}{\partial t}[v_n\phi\,\delta(f)] - \nabla\cdot[\phi\vec n\,\delta(f)]$$

SINCE $\Box^2\tilde\phi = 0$, WE GET USING $M_n = v_n/c$,

$$\boxed{\bar\Box^2\tilde\phi = -\left(\phi_n + \frac{1}{c}M_n\phi_t\right)\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}[M_n\phi\,\delta(f)] - \nabla\cdot[\phi\vec n\,\delta(f)]}$$

WE NOW SOLVE THIS WAVE EQUATION FOR STATIONARY, SUBSONIC AND SUPERSONIC SURFACES.

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DERIVATION OF THE CLASSICAL KIRCHHOFF FORMULA

THE KIRCHHOFF SURFACE $f(\vec x)$ IS NOW STATIONARY SO THAT $M_n = 0$.

THE GOVERNING WAVE EQUATION IS $\bar\Box^2\tilde\phi = -\phi_n\delta(f) - \nabla\cdot[\phi\vec n\,\delta(f)]$

$$4\pi\tilde\phi(\vec x, t) = -\int\frac{\phi_n}{r}\delta(f)\delta(g)\,d\vec y\,d\tau - \nabla_x\cdot\int\frac{\phi\vec n}{r}\delta(f)\delta(g)\,d\vec y\,d\tau$$

WHERE ϕ_n AND ϕ IN THE INTEGRANDS ARE FUNCTIONS OF $(\vec y, \tau)$. NOW LET $\tau \to g$, $\frac{\partial g}{\partial \tau} = 1$, AND INTEGRATE WRT g, TO GET

$$4\pi\tilde\phi(\vec x, t) = -\int\frac{\phi_n(\vec y, t - r/c)}{r}\delta(f)\,d\vec y - \nabla\cdot\int\frac{\phi(\vec y, t - r/c)\vec n}{r}\delta(f)\,d\vec y$$

WE HAVE DEALT WITH THESE INTEGRALS BEFORE. THE INTEGRATION OF $\delta(f)$ GIVES

$$4\pi\tilde\phi(\vec x, t) = -\int_{f=0}\frac{1}{r}\phi_n(\vec y, t - r/c)\,dS - \nabla\cdot\int_{f=0}\frac{\vec n}{r}\phi(\vec y, t - r/c)\,dS$$

TAKING THE DIVERGENCE OPERATOR IN AND USING SUBSCRIPT ret FOR RETARDED TIME, WE GET THE CLASSICAL KIRCHHOFF FORMULA

$$\boxed{4\pi\tilde\phi(\vec x, t) = \int_{f=0}\frac{[c^{-1}\dot\phi\cos\theta - \phi_n]_{ret}}{r}\,dS + \int_{f=0}\frac{\cos\theta}{r^2}[\phi]_{ret}\,dS} \quad \boxed{\cos\theta = \vec n\cdot\hat r}$$

AGAIN, OUR METHOD TELLS THAT $\tilde\phi(\vec x, t) = 0$ IN THE INTERIOR OF $f = 0$ WHICH IS NOT OBVIOUS FROM CLASSICAL DERIVATION.

DERIVATION OF THE SUBSONIC KIRCHHOFF FORMULA

WE NOW ASSUME A DEFORMABLE SURFACE MOVING AT SUBSONIC SPEED.

GOVERNING EQ.: $\bar{\Box}^2 \tilde{\phi} = -(\phi_n + c^{-1} M_n \phi_t)\,\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}[M_n \phi\, \delta(f)] - \nabla\cdot[\phi \vec{n}\, \delta(f)]$

$$4\pi \tilde{\phi}(\vec{x},t) = -\int \frac{1}{r}(\phi_n + c^{-1} M_n \phi_\tau)\,\delta(f)\delta(g)\, d\vec{y}\, d\tau \qquad \boxed{\phi_\tau = \frac{\partial \phi(\vec{y},\tau)}{\partial \tau}}$$
$$-\frac{1}{c}\frac{\partial}{\partial t}\int \frac{1}{r} M_n \phi\, \delta(f)\delta(g)\, d\vec{y}\, d\tau$$
$$-\nabla_{\vec{x}}\cdot \int \frac{1}{r}\phi \vec{n}\, \delta(f)\delta(g)\, d\vec{y}\, d\tau$$

IN THE LAST INTEGRAL, TAKE DIVERGENCE OPERATOR IN. IT ONLY MUST OPERATE ON $\frac{\delta(g)}{r}$ WHICH DEPEND ON $\vec{x}$. NOW USE THE FOLLOWING RESULT TO WRITE THE LAST INTEGRAL AS TWO INTEGRALS:

$$\boxed{\nabla\left[\frac{\delta(g)}{r}\right] = -\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{\hat{r}\,\delta(g)}{r}\right] - \frac{\hat{r}\,\delta(g)}{r^2}} \qquad \hat{r} = \frac{\vec{r}}{r}$$

$$\nabla\cdot\int \frac{1}{r}\phi\vec{n}\,\delta(f)\delta(g)\,d\vec{y}\,d\tau = \int \phi\,\delta(f)\,\vec{n}\cdot\nabla\left[\frac{\delta(g)}{r}\right] d\vec{y}\,d\tau$$
$$= -\frac{1}{c}\frac{\partial}{\partial t}\int \frac{1}{r}\phi\cos\theta\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$
$$-\int \frac{1}{r^2}\phi\cos\theta\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$

SUBSTITUTE IN EQ. FOR $\tilde{\phi}$ ABOVE.

DERIVATION OF THE SUBSONIC KIRCHHOFF FORMULA (CONT'D)

$$4\pi\tilde{\phi}(\vec{x},t) = -\int \frac{1}{r}(\phi_n + c^{-1}M_n\phi_\tau)\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$
$$+\int \frac{1}{r^2}\phi\cos\theta\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$
$$+\frac{1}{c}\frac{\partial}{\partial t}\int \frac{1}{r}(\cos\theta - M_n)\,\phi\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$

WE HAVE TWO KINDS OF INTEGRALS IN THE ABOVE EQUATION

$$I_1 = \int Q_1(\vec{y},\tau)\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$
$$I_2 = \frac{1}{c}\frac{\partial}{\partial t}\int Q_2(\vec{y},\tau)\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$

— LET US PARAMETRIZE $S: f(\vec{y},\tau)=0$ BY SURFACE COORDINATES (u^1,u^2) WITH DOMAIN $D(S)$. WE ASSUME $D(S)$ IS TIME INDEPENDENT. THIS IS ALWAYS POSSIBLE. BUT $g_{(2)}$, THE DET. OF COEF. OF 1ST FUND. FORM IS A FUNCTION OF TIME τ. PARAMETRIZE THE SPACE NEAR $f=0$ BY TAKING $u^3 = f$ AND EXTEND (u^1,u^2) ALONG LOCAL NORMAL TO $f=0$. NOW WE HAVE $d\vec{y} = \sqrt{g_{(2)}}\,du^1du^2du^3$ (STRICTLY SPEAKING, WE SHOULD USE $g'_{(2)}$ BUT IT MAKES NO DIFFERENCE HERE).

DERIVATION OF THE SUBSONIC KIRCHHOFF FORMULA (CONT'D)

WE USE $Q_1(u^1,u^2,u^3,\tau)$ FOR $Q_1[\vec{y}(u^1,u^2,u^3,\tau),\tau]$ IN I_1

$$I_1 = \int Q_1(u^1,u^2,u^3,\tau)\delta(u^3)\delta(g)\sqrt{g_{(2)}}\,du^1du^2du^3d\tau$$
$$= \int_{-\infty}^{t}\int_{D(S)} Q_1(u^1,u^2,0,\tau)\delta(g)\sqrt{g_{(2)}}\,du^1du^2d\tau$$

NOW LET $\tau \to g$, $\frac{\partial g}{\partial \tau} = 1 - M_r$ BECAUSE

$$g = \tau - t + |\vec{x} - \vec{y}(u^1,u^2,0,\tau)|/c \quad , \vec{M} = \frac{\partial \vec{y}(u^1,u^2,0,\tau)}{\partial \tau}$$

$$\boxed{I_1 = \int_{D(S)} \left[\frac{Q_1\sqrt{g_{(2)}}}{1-M_r}\right]_{\tau^*} du^1du^2}$$

τ^* EMISSION TIME OF POINT (u^1,u^2) ON $f=0$

WHERE τ^* IS THE SOLUTION OF $\boxed{\tau^* - t + |\vec{x} - \vec{y}(u^1,u^2,0,\tau^*)|/c = 0}$

SIMILAR PROCEDURE FOR I_2 GIVES

$$I_2 = \frac{1}{c}\int_{D(S)} \frac{\partial}{\partial t}\left[\frac{Q_2\sqrt{g_{(2)}}}{1-M_r}\right]_{\tau^*} dS$$

$$\boxed{I_2 = \frac{1}{c}\int_{D(S)} \left[\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\left\{\frac{Q_2\sqrt{g_{(2)}}}{1-M_r}\right\}\right]_{\tau^*} du^1du^2}$$

WE NOTE THAT $\tau^* = \tau^*(u^1,u^2;\vec{x},t)$ SO THAT $\left.\frac{\partial}{\partial t}\right|_{\vec{x}} = \frac{\partial \tau^*}{\partial t}\frac{\partial}{\partial \tau^*}$

FROM THE EQ. FOR EMISSION TIME $\boxed{\frac{\partial \tau^*}{\partial t} = \frac{1}{1-M_r}}$.

DERIVATION OF THE SUBSONIC KIRCHHOFF FORMULA (CONT'D)

FROM THE RESULTS FOR I_1 AND I_2 , WE GET

$$\boxed{\begin{aligned} 4\pi\tilde{\phi}(\vec{x},t) = &-\int_{D(S)} \left[\frac{(\phi_n + c^{-1}M_n\phi_\tau)\sqrt{g_{(2)}}}{r(1-M_r)}\right]_{\tau^*} du^1du^2 \\ &+ \int_{D(S)} \left[\frac{\phi\sqrt{g_{(2)}}\cos\theta}{r^2(1-M_r)}\right]_{\tau^*} du^1du^2 \\ &+ \frac{1}{c}\int_{D(S)} \left[\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\left\{\frac{(\cos\theta - M_n)\phi\sqrt{g_{(2)}}}{r(1-M_r)}\right\}\right]_{\tau^*} du^1du^2 \end{aligned}}$$

THIS RESULT WAS ORIGINALLY DERIVED BY W.R. MORGANS (PHIL. MAG., VOL. 9, 1930, 141-161). IT WAS REDERIVED BY FARASSAT AND MYERS USING THE ABOVE METHOD (JSV, VOL 123(3), 1988, 451-460). THESE AUTHORS HAVE GIVEN A USEFUL FORMULA FOR APPLICATIONS IN THE FOLLOWING FOR M $\boxed{4\pi\tilde{\phi}(\vec{x},t) = \int_{D(S)} \left[\frac{E_1\sqrt{g_{(2)}}}{r(1-M_r)}\right]_{\tau^*} du^1du^2 + \int_{D(S)} \left[\frac{\phi E_2\sqrt{g_{(2)}}}{r^2(1-M_r)}\right]_{\tau^*} du^1du^2}$

WHERE E_1 AND E_2 ARE LONG EXPRESSIONS GIVEN IN THE ABOVE REFERENCE. THIS FORMULA WAS VERIFIED BY USING ANALYTIC INPUT FOR RIGID SURFACES.

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A SIMPLE TRICK IN PREPARATION FOR SUPERSONIC K-FORMULA

TO REDUCE ALGEBRAIC MANIPULATIONS AND TO OBTAIN THE SIMPLEST FORM OF THE SUPERSONIC KIRCHHOFF FORMULA, WE INTRODUCE THE FOLLOWING TRICK. NOTE THAT IN THE GOVERNING WAVE EQUATION FOR DERIVING KIRCHHOFF FORMULA, WE HAVE TERMS INVOLVING TIME AND SPACE DERIVATIVES : $\frac{\partial}{\partial t}[M_n \phi \delta(f)]$ AND $\nabla \cdot [\phi \vec{n} \delta(f)]$. WE NEED TO TAKE THESE DERIVATIVES EXPLICITLY. WE PROPOSE THE FOLLOWING SIMPLIFICATION OF THIS PROCESS.

OBSERVATION : $\phi(x)\delta(x) = \phi(0)\delta(x)$ TAKE DERIVATIVES OF BOTH SIDES $\phi'(x)\delta(x) + \phi(x)\delta'(x) = \phi(0)\delta'(x)$. IT IS OBVIOUS THAT THE RIGHT SIDE IS SIMPLER THAN THE LEFT SIDE. WHAT IS SO SPECIAL ABOUT $\phi(0)\delta(x)$? HERE $\phi(x)$ IS RESTRICTED TO THE SUPPORT OF THE DELTA FUNCTION, i.e., $x=0$. CAN RESTRICTION OF $\phi(\vec{x})$ TO THE SUPPORT OF $\delta(f)$ IN $\phi(\vec{x})\delta(f)$ REDUCE MANIPULATIONS WHEN WE TAKE DERIVATIVES OF $\phi(\vec{x})\delta(f)$? THE ANSWER IS YES!

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A SIMPLE TRICK IN PREPARATION FOR SUPERSONIC K-FORMULA (CONT'D)

WE USE THE NOTATION $\underset{\sim}{\phi}(\vec{x})$ FOR RESTRICTION OF $\phi(\vec{x})$ TO THE SUPPORT OF $\delta(f)$. USING THE LOCAL PARAMETRIZATION OF SPACE NEAR $f=0$ ((u^1,u^2) ON $f=0$, u^3= DISTANCE FROM $f=0$), WE HAVE

$$\underset{\sim}{\phi}(\vec{x}) = \phi(u^1,u^2,0)$$

SIMILARLY $\underset{\sim}{\phi}(\vec{x},t) = \phi(u^1,u^2,0,t)$, NOTE $u^i = u^i(\vec{x},t)$

WE HAVE $\boxed{\phi(\vec{x},t)\delta(f) = \underset{\sim}{\phi}(\vec{x},t)\delta(f)}$

$$\nabla[\phi(\vec{x},t)\delta(f)] = \nabla\phi\,\delta(f) + \phi\,\nabla f\,\delta'(f) \qquad (A)$$

$$\nabla[\underset{\sim}{\phi}(\vec{x},t)\delta(f)] = \nabla_2\underset{\sim}{\phi}\,\delta(f) + \underset{\sim}{\phi}\,\nabla f\,\delta'(f) \qquad (B)$$

WHERE $\nabla_2\underset{\sim}{\phi}$ IS THE SURFACE GRADIENT OF $\underset{\sim}{\phi}$. AS EXPECTED, THE INTEGRATION OF THE RIGHT SIDE OF (A) IS ALGEBRAICALLY SOMEWHAT MORE COMPLICATED THAN INTEGRATION OF THE RIGHT SIDE OF (B). NOTE THAT $\boxed{\frac{\partial \underset{\sim}{\phi}}{\partial x_i} = \frac{\partial \phi}{\partial x_i} - n_i \frac{\partial \phi}{\partial n}}$, $\boxed{\frac{\partial \underset{\sim}{\phi}}{\partial n} = 0}$

$\boxed{\frac{\partial \underset{\sim}{\phi}}{\partial t} = \frac{\partial \phi}{\partial t} + v_n \frac{\partial \phi}{\partial n}}$ NOTE: WAVE PROPAGATION LITERATURE USE $\frac{\delta \phi}{\delta x_i}$ FOR $\frac{\partial \underset{\sim}{\phi}}{\partial x_i}$ AND $\frac{\delta \phi}{\delta t}$ FOR $\frac{\partial \underset{\sim}{\phi}}{\partial t}$.

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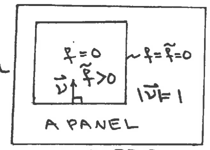
DERIVATION OF THE SUPERSONIC KIRCHHOFF FORMULA

WE ARE NOW INTERESTED TO DEVELOP THE SUPERSONIC KIRCHHOFF FORMULA FOR A PANEL ON THE KIRCHHOFF SURFACE. THIS IS ONLY BECAUSE OF PRACTICAL CONSIDERATION. ON THE SURFACE $f(\vec{x},t)=0$, WE DEFINE A PANEL BY ITS EDGE CURVE $\tilde{f}=0$ SUCH THAT $\tilde{f}>0$ ON THE PANEL AND $\nabla\tilde{f}=\vec{\nu}$ THE LOCAL UNIT GEODESIC NORMAL AT THE EDGE $f=\tilde{f}=0$. THE GEODESIC NORMAL IS TANGENT TO THE PANEL AND NORMAL TO THE EDGE.



DENOTING HEAVISIDE FUNCTION BY $H(\tilde{f})$, OUR GOVERNING DIFFERENTIAL EQUATION FOR FINDING THE KIRCHHOFF FORMULA FOR THE PANEL IS

$$\boxed{\begin{aligned}\bar{\Box}^2\tilde{\phi} = &-(\phi_n + c^{-1}M_n\phi_t)H(\tilde{f})\delta(f) - \frac{1}{c}\frac{\partial}{\partial t}\left[M_n\phi H(\tilde{f})\delta(f)\right] \\ &- \nabla\cdot\left[\phi\vec{n}\,H(\tilde{f})\,\delta(f)\right]\end{aligned}}$$

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DERIVATION OF THE SUPERSONIC KIRCHHOFF FORMULA (CONT'D)

$$\frac{1}{c}\frac{\partial}{\partial t}\left[M_n\phi H(\tilde{f})\delta(f)\right] = \frac{1}{c}\frac{\partial}{\partial t}(\underset{\sim}{M}_n\underset{\sim}{\phi})H(\tilde{f})\delta(f) - M_nM_\nu\phi\,\delta(f)\delta(\tilde{f}) - \underset{\sim}{M}_n^2\,\phi\,H(\tilde{f})\delta'(f)$$

WHERE $M_\nu=\vec{M}\cdot\vec{\nu}$ IS THE LOCAL MACH NUMBER OF THE EDGE IN THE DIRECTION OF $\vec{\nu}$. USING THE DIVERGENCE RESULT DERIVED EARLIER, WE HAVE

$$\nabla\cdot\left[\phi\vec{n}\,H(\tilde{f})\delta(f)\right] = -2H_f\,\phi H(\tilde{f})\delta(f) + \underset{\sim}{\phi}\,H(\tilde{f})\delta'(f)$$

THE GOVERNING EQUATION FOR DERIVING THE SUPERSONIC KIRCHHOFF FORMULA FOR A PANEL IS

$$\begin{aligned}\bar{\Box}^2\tilde{\phi} &= -\left[\phi_n + c^{-1}M_n\phi_t + c^{-1}(\underset{\sim}{M}_n\underset{\sim}{\phi})_t - 2H_f\phi\right]H(\tilde{f})\delta(f) \\ &\quad - (1-\underset{\sim}{M}_n^2)\,\underset{\sim}{\phi}\,H(\tilde{f})\delta'(f) + M_nM_\nu\,\phi\,\delta(f)\delta(\tilde{f}) \\ &\equiv q_1 H(\tilde{f})\delta(f) + \underset{\sim}{q}_2 H(\tilde{f})\delta'(f) + q_3\,\delta(f)\delta(\tilde{f})\end{aligned}$$

WE SEE THAT WE HAVE THREE KINDS OF SOURCES WHICH WE WILL TREAT BELOW.

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DERIVATION OF THE SUPERSONIC KIRCHHOFF FORMULA (CONT'D)

LET $\tilde{\phi} = \phi_1 + \phi_2 + \phi_3$ WHERE ϕ_i'S ARE SOLUTIONS OF WAVE EQUATION WITH SOURCES INVOLVING q_i ($i = 1-3$) ON P50.

$$\bar{\Box}^2 \phi_1 = q_1 H(\tilde{f}) \delta(f), \quad \bar{\Box}^2 \phi_2 = \underset{\sim}{q}_2 H(\tilde{f}) \delta'(f), \quad \bar{\Box}^2 \phi_3 = q_3 \delta(f)\delta(\tilde{f})$$

i) SOLUTION OF $\bar{\Box}^2 \phi_1 = q_1 H(\tilde{f}) \delta(f)$

$$4\pi \phi_1(\vec{x}, t) = \int \frac{q_1(\vec{y}, \tau)}{r} H(\tilde{f}) \delta(f) \delta(g) \, d\vec{y} \, d\tau$$

LET $\tau \to g$, $\frac{\partial g}{\partial \tau} = 1$, INTEGRATE WRT g

$$4\pi \phi_1(\vec{x}, t) = \int \frac{[q_1]_{ret}}{r} H(\tilde{F}) \delta(F) \, d\vec{y}$$

HERE $F(\vec{y}; \vec{x}, t) = [f(\vec{y}, \tau)]_{ret} = f(\vec{y}, t - r/c)$ AND

$$\tilde{F}(\vec{y}; \vec{x}, t) = [\tilde{f}(\vec{y}, \tau)]_{ret} = \tilde{f}(\vec{y}, t - r/c)$$

WE HAVE TREATED THIS INTEGRAL BEFORE. WE WRITE THE ELEMENT OF SURFACE AREA OF $F = 0$ BY $d\Sigma$

$$\boxed{4\pi \phi_1(\vec{x}, t) = \int_{\substack{F=0 \\ \tilde{F}>0}} \frac{[q_1]_{ret}}{r\Lambda} \, d\Sigma}$$

$$\boxed{\Lambda^2 = 1 + M_n^2 - 2M_n \cos\theta} \quad \boxed{\Lambda = |\nabla F|} \qquad \cos\theta = \vec{n} \cdot \vec{\hat{r}}$$

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DERIVATION OF THE SUPERSONIC KIRCHHOFF FORMULA (CONT'D)

ii) SOLUTION OF $\bar{\Box}^2 \phi_2 = \underset{\sim}{q}_2 H(\tilde{f}) \delta'(f)$

$$4\pi \phi_2(\vec{x}, t) = \int \frac{\underset{\sim}{q}_2}{r} H(\tilde{f}) \delta'(f) \delta(g) \, d\vec{y} \, d\tau$$

LET $\tau \to g$, $\frac{\partial g}{\partial \tau} = 1$ AND INTEGRATE WRT g

$$4\pi \phi_2(\vec{x}, t) = \int \frac{[\underset{\sim}{q}_2]_{ret}}{r} H(\tilde{F}) \delta'(F) \, d\vec{y}$$

$$= \int_{\substack{F=0 \\ \tilde{F}>0}} \left\{ -\frac{1}{\Lambda} \frac{\partial}{\partial N} \left[\frac{[\underset{\sim}{q}_2]_{ret}}{r\Lambda} \right] + \frac{2H_F [\underset{\sim}{q}_2]_{ret}}{r\Lambda^2} \right\} d\Sigma$$

$$- \int_{\substack{F=0 \\ \tilde{F}=0}} \frac{[\underset{\sim}{q}_2]_{ret} \cot\theta'}{r\Lambda^2} \, dL \qquad \text{(NASA TP 3428)}$$

$\vec{N} = \frac{\nabla F}{|\nabla F|} = \frac{\vec{n} - M_n \vec{\hat{r}}}{\Lambda}$, $\vec{\tilde{N}} = \frac{\nabla \tilde{F}}{|\nabla \tilde{F}|}$, $\cos\theta' = \vec{N} \cdot \vec{\tilde{N}}$, H_F MEAN CURVATURE OF Σ-SURFACE

iii) SOLUTION OF $\bar{\Box}^2 \phi_3 = q_3 \delta(f) \delta(\tilde{f})$

$$4\pi \phi_3(\vec{x}, t) = \int_{\substack{F=0 \\ \tilde{F}=0}} \frac{1}{r} \frac{[q_3]_{ret}}{\Lambda_0} \, dL \qquad \text{(NASA TP 3428}$$

WHERE $\Lambda_0 = |\nabla F \times \nabla \tilde{F}| = \Lambda \tilde{\Lambda} \sin\theta'$.

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DERIVATION OF THE SUPERSONIC KIRCHHOFF FORMULA (CONT'D)

NOW PUTTING THE SOLUTIONS FOR ϕ_1, ϕ_2 AND ϕ_3 TOGETHER IN $\tilde{\phi}$, WE GET THE SUPERSONIC KIRCHHOFF FORMULA

$$\boxed{\begin{aligned} 4\pi\tilde{\phi}(\vec{x},t) = & \int_{\substack{F=0\\ \tilde{F}>0}} \frac{1}{r\Lambda}\left[Q_1 + \frac{2H_f}{\Lambda}Q_2 + \frac{\vec{N}\cdot\nabla\Lambda}{\Lambda^2}Q_2 - \frac{\vec{N}\cdot\nabla Q_2}{\Lambda}\right]d\Sigma \\ & + \int_{\substack{F=0\\ \tilde{F}>0}} \frac{\vec{N}\cdot\nabla r}{r^2\Lambda^2}Q_2\, d\Sigma \\ & + \int_{\substack{F=0\\ \tilde{F}=0}} \frac{1}{r\Lambda_0}\left[Q_3 - \frac{\tilde{\Lambda}\cos\theta'}{\Lambda}Q_2\right]dL \end{aligned}}$$

WHERE $Q_i = [q_i]_{ret}$, $i = 1-3$

THIS EQUATION WAS DERIVED BY FARASSAT AND MYERS IN 1994. IT WAS PRESENTED AT ASME INT. MECH. ENG. CONGRESS AND EXPO., NOV. 6-11, 1994, CHICAGO, ILLINOIS.

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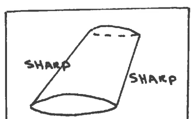
SELECTION OF THE KIRCHHOFF SURFACE FOR SUPERSONIC K-FORMULA

IT CAN BE SHOWN THAT WHEN THE COLLAPSING SPHERE LEAVES THE KIRCHHOFF SURFACE $f=0$ TANGENTIALLY AT A POINT WHERE $M_n = 1$, THE SUPERSONIC KIRCHHOFF FORMULA WILL DEVELOP A SINGULARITY. ONE CAN AVOID THIS SITUATION BY SELECTING A BICONVEX SHAPE FOR KIRCHHOFF SURFACE AVOIDING THE ABOVE SINGULARITY CONDITION.



SUPERSONIC KIRCH-HOFF SURFACE

IN ROTOR NOISE CALCULATIONS, IN-PLANE NOISE OF HIGH SPEED ROTORS IS THE MOST IMPORTANT. REASONABLE SHAPE OF KIRCHHOFF SURFACE IS POSSIBLE. FARASSAT AND MYERS HAVE SHOWN THAT THE SINGULARITY FROM LINE INTEGRAL IN KIRCHHOFF FORMULA IS INTEGRABLE

REFERENCES FOR DIFFERENTIAL GEOMETRY

MOST OF THE RESULTS ON DIFFERENTIAL GEOMETRY IN PREVIOUS PAGES WERE KNOWN BY THE END OF 19TH CENTURY. THE CONTRIBUTION OF 20TH CENTURY MATHEMATICIANS HAS BEEN DEVELOPMENT OF GENERAL AND POWERFUL TECHNIQUES TO SOLVE DIFFICULT PROBLEMSAS WELL AS A SUPPOSEDLY RIGOROUS STYLE OF WRITING MAKING DIFFERENTIAL GEOMETRY INACCESSABLE TO ENGINEERS. AVOID MODERN BOOKS ON DIFFERENTIAL GEOMETRY. REMEMBER GAUSS, ONE OF THE GREATEST MATHEMATICIAN OF ALL TIME, WROTE IN A STYLE THAT WE CAN ALL UNDERSTAND. HERE ARE A FEW USEFUL BOOKS.

1. DIRK J. STRUIK : LECTURES ON CLASSICAL DIFFERENTIAL GEOMETRY, 2ND ED., DOVER BOOKS, 1988
2. ERWIN KREYSZIG : DIFFERENTIAL GEOMETRY, DOVER BOOKS, 1991
3. A. J. McCONNELL : APPLICATIONS OF TENSOR ANALYSIS, DOVER, 1957
4. R. ARIS : VECTORS, TENSORS, AND THE BASIC EQS. OF FL. MECH., DOVER, 1989

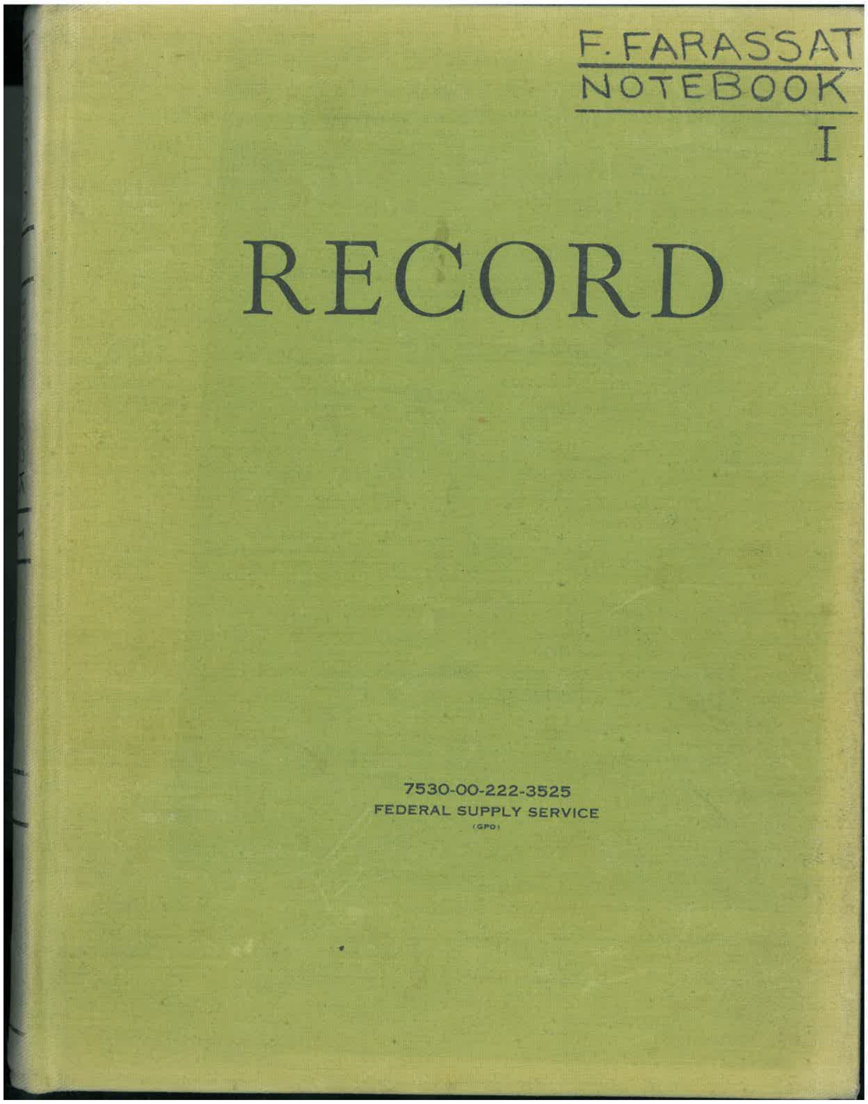
REFERENCES FOR GENERALIZED FUNCTIONS

1. M. J. LIGHTHILL : INTRODUCTION TO FOURIER ANALYSIS AND GENERALIZED FUNCTIONS, CAMB. UNIV. PRESS, 1964 (GEN. FNS. OF ONE VARIABLE, SEQUENTIAL APPROACH, EXCELLENT BOOK!)
2. D. S. JONES : THE THEORY OF GENERALIZED FUNCTIONS, 2ND ED., CAMB. UNIV. PRESS, 1982 (MULTIVARIABLE GEN. FNS., SEQ. APPROACH, HIGHLY TECHNICAL, FULL OF USEFUL RESULTS)
3. I. M. GEL'FAND AND G. E. SHILOV : GENERALIZED FUNCTIONS – VOL. 1, PROPERTIES AND OPERATIONS, ACADEMIC PRESS, 1964 (PROBABLY THE BEST BOOK EVER WRITTEN ON GEN. FNS., HIGHLY READABLE, FULL OF USEFUL RESULTS)
4. R. P. KANWAL : GENERALIZED FUNCTIONS – THEORY AND TECHNIQUE, ACADEMIC PRESS, 1983 (HIGHLY READABLE BUT ALSO ADVANCED)
5. R. S. STRICHARTZ : A GUIDE TO DISTRIBUTION THEORY AND FOURIER TRANSFORMS, CRC PRESS, 1994 (MASTERFUL EXPOSITORY BOOK)

ACKNOWLEDGEMENTS

I WOULD LIKE TO THANK MR. JACK PREISSER FOR HELPING TO ARRANGE THIS WORKSHOP AND HIS ENCOURAGEMENT. MY WORK ON KIRCHHOFF FORMULAS HAS BEEN IN COLLABORATION WITH DR. M. K. MYERS OF THE GEORGE WASHINGTON UNIVERSITY (JIAFS). OUR TECHNICAL DISCUSSIONS HAVE INFLUENCED AND SHAPED MY THINKING ON ACOUSTIC PROBLEMS.

10 Notebook One



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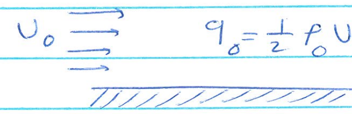
* BOUNDARY LAYER PRESSURE FLUCTUATIONS IN SUBSONIC RANGE



$q_0 = \frac{1}{2} \rho_0 U_0^2$

$1.5 \times 10^{-3} < p'_{rms}/q < 12 \times 10^{-3}$

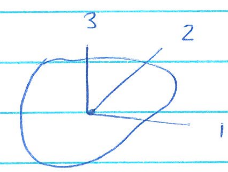
MOST GENERALLY ACCEPTED VALUE (BY WILLMARTH) $p_{rms}/q \approx 6 \times 10^{-3}$.

τ_0 : SHEAR STRESS AT THE WALL (MEAN STRESS)

* $p'_{rms}/\tau_0 \approx 3$ AT THE WALL
* p'/τ_0 VARIES LITTLE WITH MACH NO. IN SUBSONIC SPEEDS, AT SUPERSONIC SPEEDS, $p'/\tau_0 \uparrow$ AS $M \uparrow$.
* ABOVE $M = 1$, p'/τ_0 FAIRLY INSENSITIVE TO Re NO.

SHEAR STRESS FLUCTUATIONS

$$\tau'_0 = \mu \left[\left(\frac{\partial u'_2}{\partial x_3}\right)^2 + \left(\frac{\partial u'_1}{\partial x_3}\right)^2 \right]_{WALL}$$



τ'_0, u'_2, u'_1 : rms FLUCTUATIONS

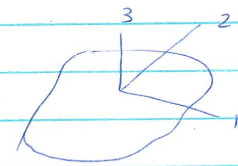
$\tau'_0 \approx 0.3 \tau_0$ [LAUFER, FLOW IN PIPES)

$\therefore \tau'_0 << p'_{rms}$

ORDER OF MAGNITUDE

$U = 100$ m/sec	$U = 300$ m/sec
$M = .29$	$M = .87$
$p'_{rms} \approx 36$ N/m^2	$p'_{rms} \approx 324$ N/m^2
$\tau_0 \approx 12$ N/m^2	$\tau_0 \approx 108$ N/m^2
$(\tau'_0)_{rms} \approx 3.6$ N/m^2	$(\tau'_0)_{rms} \approx 32.4$ N/m^2

SPACE-TIME CORRELATIONS AND SPECTRA OF B.L. PRESSURE



$$R_{pp}(\vec{\xi}_3,\tau) = \langle p(\vec{x}_3,t)\,p(\vec{x}_3+\vec{\xi}_3,t+\tau)\rangle$$

$$\vec{x}_3 = (x_1,x_2,0)\ ,\ \vec{\xi}_3 = (\xi_1,\xi_2,0)$$

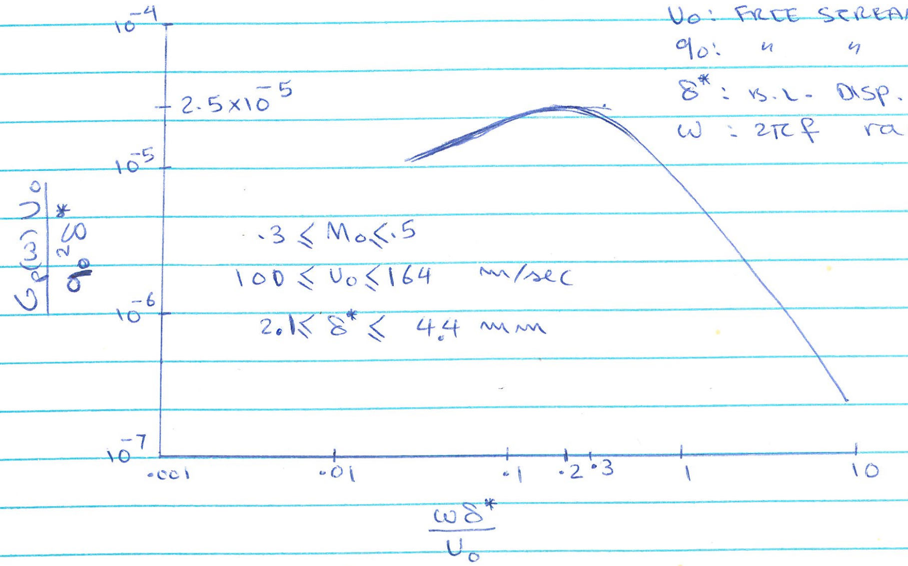
$$\rho_{pp} = R_{pp}/\langle p^2\rangle \qquad \text{CORRELATION COEFFICIENT}$$

$$(p'_{rms})^2 = R_{pp}(\vec{0},0)\quad,\quad \vec{0} = (0,0,0)$$

$$S_{pp}(\vec{\xi}_3,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{pp}(\vec{\xi}_3,\tau)\,e^{-i\omega\tau}\,d\tau$$

THE POWER SPECTRAL DENSITY $G_p(\omega)$

$$G_p(\omega) = S_{pp}(\vec{0},\omega) + S_{pp}(\vec{0},-\omega)$$
$$= \frac{2}{\pi}\int_0^{\infty} R_{pp}(\vec{0},\tau)\cos\omega\tau\,d\tau$$



U_0: FREE STREAM VELOCITY
q_0: " " DYN. PRES.
δ^*: B.L. DISP. THICKNESS
ω : $2\pi f$ rad/sec

$U_0 = 100$ m/sec
$\delta^* = 2.1$ mm
$f_{PEAK} = 2.27$ KHZ
$G_p(\omega_{PEAK}) = 3.15 \times 10^{-6}$

$U_0 = 164$ m/sec
$\delta^* = 4.4$ mm
$f_{PEAK} = 1.78$ KHZ
$G_p(\omega_{PEAK}) = 1.08 \times 10^{-5}$

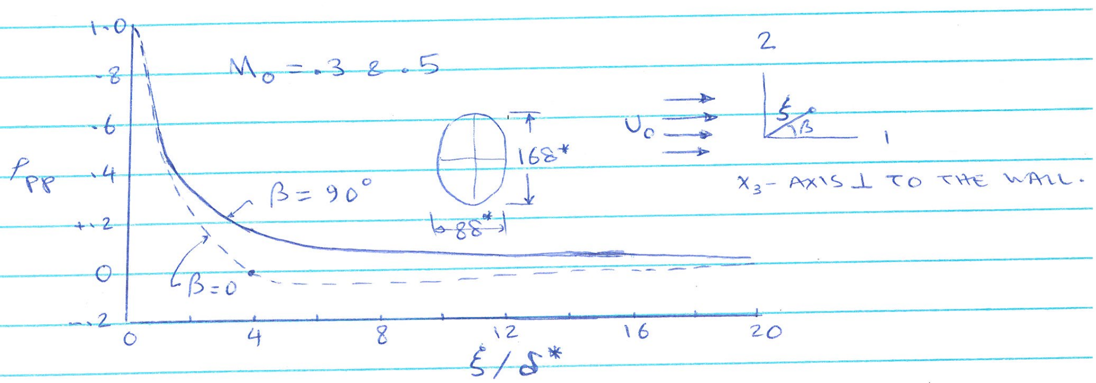
$$\frac{\omega_{PEAK}\,\delta^*}{U_0} \simeq .2 \text{ TO } .3$$

$$\frac{f_{PEAK}\,\delta^*}{U_0} \simeq .032 \text{ TO } .048$$

NOTE: THERE IS A LOT OF ENERGY IN THE HIGH FREQUENCY REGION OF THE SPECTRUM

SPACE CORRELATION

THE LATERAL INTEGRAL SCALE OF PRESSURE FIELD IS ABOUT TWICE THE LONGITUDINAL SCALE



$\rho_{PP} = \rho_{PP}(\xi\cos\beta, \xi\sin\beta, 0, 0)$

FOR OTHER VALUES OF β, THE CURVES FALL BETWEEN THOSE OF $\beta = 0$ AND $\beta = 90°$.

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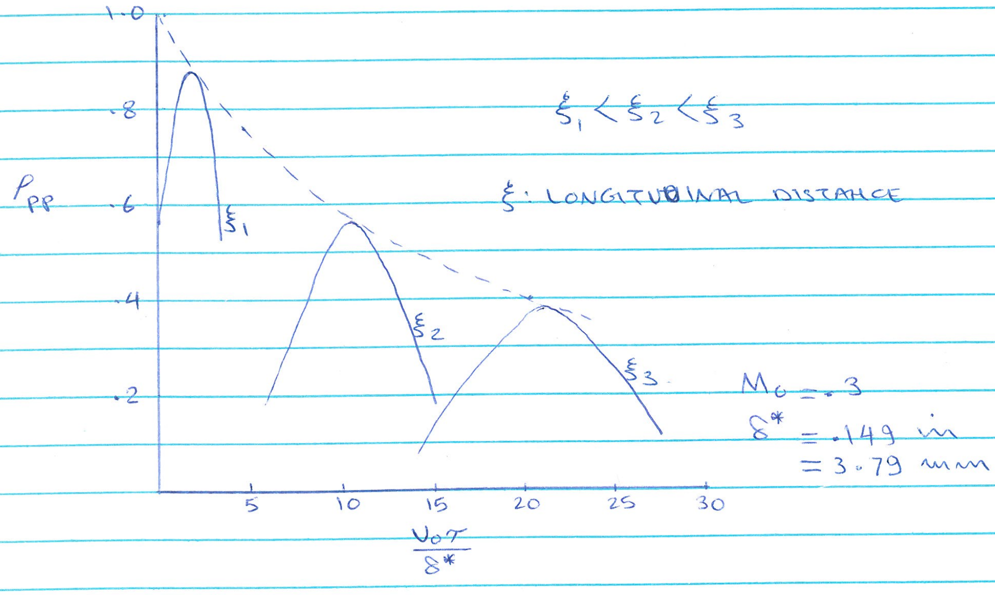
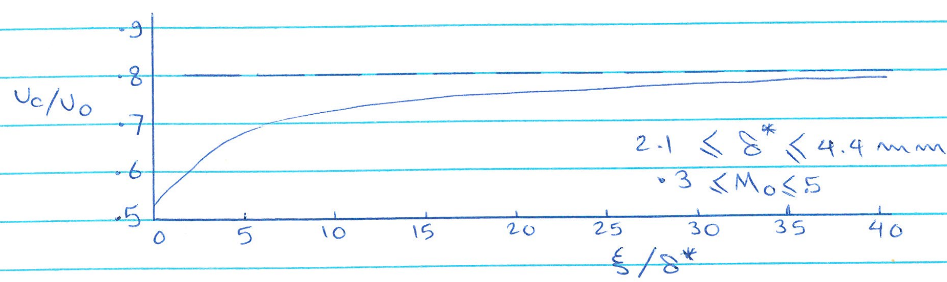
SPACE TIME CORRELATIONS & CONVECTION VELOCITIES





VARIATION OF CONVECTION SPEED WITH SPATIAL SEPARATION (LONGITUDINAL SEPARATION)

COMPONENTS OF PRESSURE FIELD WITH SMALL LONGITUDINAL SCALE TRAVEL AT SLOWER SPEED THAN LARGE SCALE COMPONENTS.

THE STATISTICAL DISTRIBUTION OF PRESSURE AMPLITUDES AT A POINT IS VERY NEARLY GAUSSIAN.

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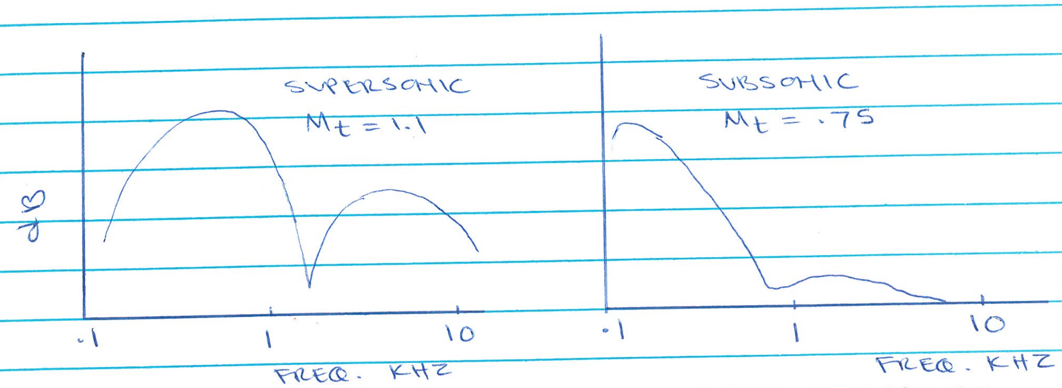
* PROPELLER AND HELICOPTER NOISE



ENVELOPE OF SPECTRUM OF ACOUSTIC PRESSURE FOR SUBSONIC AND SUPERSONIC TIP PROPELLERS

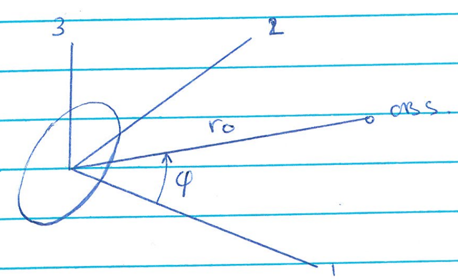
GUTIN'S RESULT (STATIC PROPELLER)

$$(p'_{rms})_m = \frac{mBW}{2\pi\sqrt{2}\,c\,r_0}\left|\left(-T\cos\varphi + \frac{1}{M_e}\frac{Q}{R_e}\right) J_{mB}(mBM_e\sin\varphi)\right|$$

T : THRUST / BLADE , $\omega = 2\pi f$, f : BPF

Q : TORQUE / BLADE , $M_e = R_e\omega/c$

R_e : EQUIVALENT RADIUS , φ : OBS. ANGLE WITH PROP. AXIS



BROADBAND NOISE (SHARLAND)

$$I = K\,(R_e)^{-.4}\,\frac{A f_0}{c^3 r_0^2}\,V_T^6\,\cos^2\varphi = \frac{K A f_0 R_e^{-.4}}{r_0^2}\,M_T^3 V_T^3 \cos^2\varphi$$

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$Re = \frac{\rho_o V_T b}{\mu}$, V_T : TIP SPEED

b : MEAN CHORD, A : TOTAL BLADE

K - CONST. : $O(10^{-4})$

TYPICAL VALUE FOR A 3-BLADED PROPELLER

$b = .16$ m, $R = 1.3$ m, $\rho_o = 1.2$ kg/m^3

$V_T = 290.$ m/sec, $c = 340$ m/sec

$r_o = 5$ m, $A = 3 \times .16 \times 1.3 = .62$ m^2

$Re = 1.2 \times 290 \times .16 / 1.75 \times 10^{-5} = 3.2 \times 10^6$

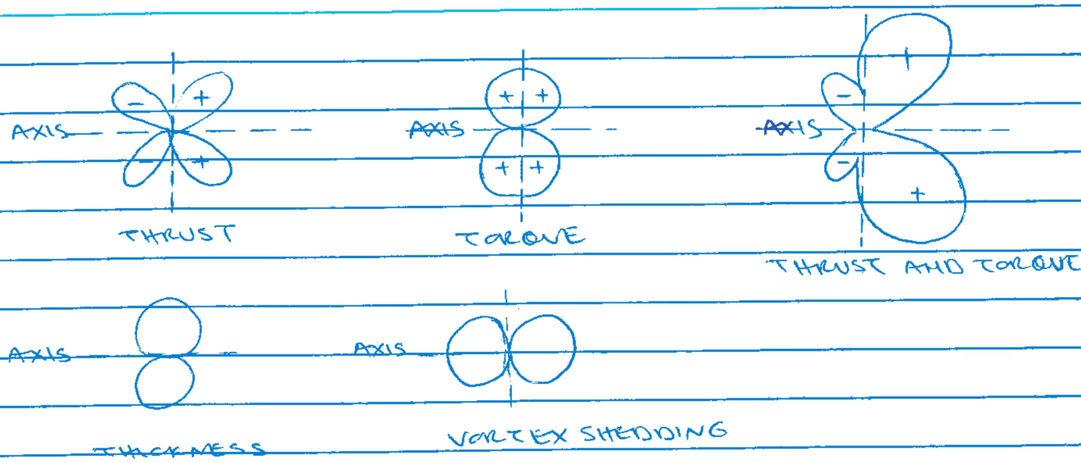
IN-PLANE INTENSITY TAKING $K = 10^{-4}$

$I = .1126$ W/m^2

$= 110.5$ dB re: 10^{-12} W/m^2

$p'_{rms} = 6.84$ N/m^2 $= 110.7$ dB re: 2×10^{-5} N/m^2 {TOO HIGH!

IDEALIZED DIRECTIVITY PATTERN



* NOTES ON TURBULENCE

IN AN INCOMPRESSIBLE FLUID, THE VELOCITY FIELD CAN BE DECOMPOSED INTO PLANE SHEAR LAYERS AS FOLLOWS. LET $\hat{\vec{v}}(\vec{k},t)$ BE THE F.T. OF $\vec{v}(\vec{x},t)$, THEN

$$\vec{v}(\vec{x},t) = \int_{-\infty}^{\infty} \hat{\vec{v}}(\vec{k},t)\, e^{-\imath \vec{x}\cdot\vec{k}}\, d\vec{k} \quad ; \imath = 2\pi i$$

$$d\vec{v} = \hat{\vec{v}}(\vec{k},t)\, e^{-\imath \vec{k}\cdot\vec{x}}\, d\vec{k}$$

WE WILL SHOW THAT $d\vec{v}$ IS PERPENDICULAR TO $\vec{k}$, THE WAVE NUMBER VECTOR

$$d\vec{v}\cdot\vec{k} = \vec{k}\cdot\hat{\vec{v}}(\vec{k},t)\, e^{-\imath \vec{k}\cdot\vec{x}}\, d\vec{k}$$

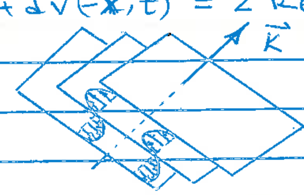
$$\vec{k}\cdot\hat{\vec{v}}(\vec{k},t) = \int_{-\infty}^{\infty} \vec{k}\cdot\vec{v}(\vec{x},t)\, e^{\imath \vec{k}\cdot\vec{x}}\, d\vec{x}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\imath}\vec{v}\cdot\nabla_{\vec{x}}\, e^{\imath \vec{k}\cdot\vec{x}}\, d\vec{x}$$

$$= \int_{-\infty}^{\infty} \nabla_{\vec{x}}\cdot\left(\vec{v}\, e^{\imath \vec{k}\cdot\vec{x}}\right) d\vec{x} \quad \text{SINCE } \nabla\cdot\vec{v} = 0$$

$$= 0 \quad \text{BY DIVERGENCE THM.}$$

$\therefore$ CORRESPONDING TO EVERY $\vec{k}$, THERE IS A SHEAR WAVE WITH TIME DEPENDENT AMPLITUDE $\hat{\vec{v}}(\vec{k},t)\,d\vec{k}$ WHICH IS COMPLEX. IF $\vec{k}$ IS TAKEN SUCH THAT IT ALWAYS POINTS INTO A HALF SPACE, THEN THE AMPLITUDES OF THESE SHEAR WAVES ARE $d\vec{v}_R(\vec{x},t) = d\vec{v}(\vec{x},t) + d\vec{v}(-\vec{x},t) = 2\,\mathrm{Re}\,\hat{\vec{v}}(\vec{k},t)\, e^{\imath \vec{x}\cdot\vec{k}}\, d\vec{k}$



LET $\hat{F}$ = F.T. (F), THEN IF $v = |\vec{V}|$, THE KINETIC ENERGY OF FLUID PER UNIT VOLUME IS

$$\frac{1}{2} \overline{v^2}(\vec{x},t) = \frac{1}{2}\int_{-\infty}^{\infty} \widehat{\overline{v^2}}(\vec{K},t)\, e^{-i\vec{K}\cdot\vec{x}}\, d\vec{K}$$

$$= \frac{1}{2}\int_0^{\infty} dK \int_{\substack{\text{SPHERE}\\ r=K}} \widehat{\overline{v^2}}(\vec{K},t)\, e^{-i\vec{K}\cdot\vec{x}}\, dS$$

$$= \int_0^{\infty} \mathcal{E}(K,\vec{x},t)\, dK$$

WHERE $\mathcal{E}(K,\vec{x},t) = \frac{1}{2}\int_{\substack{\text{SPHERE}\\ r=K}} \widehat{\overline{v^2}}(\vec{K},t)\, e^{i\vec{K}\cdot\vec{x}}\, dS$

NOTE THAT $\overline{v^2}(\vec{x},t)$ IS THE ENSEMBLE AVERAGE OF $v^2(\vec{x},t)$. IN A TIME DEPENDENT FLOW REPEATED WITH IDENTICAL INITIAL CONDITIONS, WE DEFINE

$$\overline{v^2}(\vec{x},t) = \frac{1}{M}\sum_{i=1}^{M} v^2(\vec{x},t,i)$$

WHERE i DENOTES THE EXPERIMENT AND M IS THE TOTAL NUMBER OF EXPERIMENTS. WE DEFINE

$$\widehat{\overline{v^2}}(\vec{K},t) = \int \overline{v^2}(\vec{x},t)\, e^{i\vec{K}\cdot\vec{x}}\, d\vec{x}$$

NOTE THAT IF WE USE $\langle\ \rangle$ FOR ENSEMBLE AVERAGE

$$\widehat{\overline{v^2}}(\vec{K},t) = \langle \widehat{v^2}(\vec{K},t)\rangle$$

$$= \frac{1}{M}\sum_{i=1}^{M} \widehat{v^2}(\vec{K},t,i)$$

$$= \text{F.T.}\ \frac{1}{M}\sum_{i=1}^{M} v^2(\vec{x},t,i)$$

$$= \widehat{\overline{v^2}}(\vec{K},t)$$

OR F.T. $\langle f\rangle = \langle$F.T. $f\rangle$.

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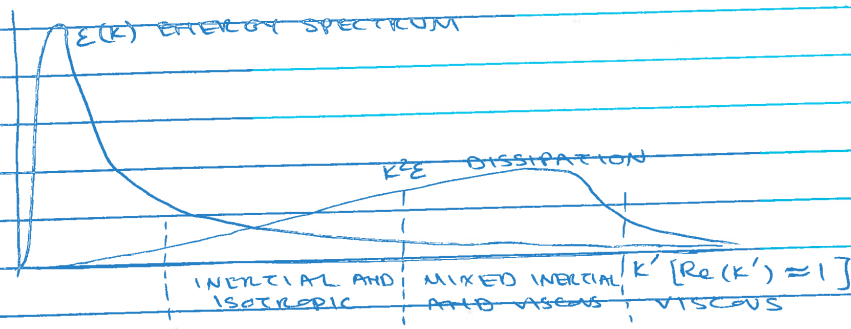
FOR HOMOGENEOUS TURBULENCE [AVERAGE PROPERTIES INDEPENDENT OF LOCATION]

$$\frac{\partial}{\partial t} E(k,t) = \underbrace{-2\nu k^2 E(k,t)}_{\text{[RATE OF VISCOUS DISSIPATION INTO INTERNAL ENERGY]}} + \underbrace{F(k,t)}_{\text{[RATE OF GAIN OF ENERGY AT } k \text{ DUE TO NONLINEAR INTERACTION WITH OTHER FOURIER COMPONENTS]}}$$

ν : KINEMATIC VISCOSITY



IN HOMOGENEOUS TURBULENCE WE MAY USE THE FOLLOWING DEFN. OF REYNOLDS NUMBER $Re(L) = \dfrac{u'L}{\nu}$

u' : RMS VELOCITY FLUCTUATIONS, L AVERAGE EDDY SIZE

WE MAY ALSO DEFINE SPECTRAL REYNOLDS NUMBER

$Re(k) = v_k \ell_k / \nu$: $v_k = [k E(k)]^{1/2}$, $\ell_k = 1/k$

$$Re(k) = \frac{1}{\nu}\left(\frac{E}{k}\right)^{1/2}$$

FOR "INERTIAL ISOTROPIC RANGE" ($k \gg 1/L$, $Re(k) \gg 1$) KOLMOGOROV HAS PREDICTED THE FORM OF $E(k)$. FOR EACH k IN THIS RANGE, THE RATE OF FLUX OF ENERGY IS THE SAME AS THE RATE OF DISSIPATION OF ENERGY ϕ, BY VISCOSITY : $\phi = 2\nu \int_0^\infty k^2 E(k)\, dk$. FROM E, k AND ϕ, THERE IS ONLY ONE NONDIMENSIONAL FORM POSSIBLE GIVING

$$E(k) \sim \phi^{2/3} \, k^{-5/3}$$

HEISENBERG'S THEORY PREDICTS THAT AS $k \to \infty$, $E \sim k^{-7}$.

* SOME USEFUL RESULTS FROM ELEMENTARY KINETIC THEORY OF GASES —

TYPICALLY FOR A GAS AT STP THE MEAN SEPARATION AND MEAN FREE PATH ARE 30 Å AND 1000 Å, RESPECTIVELY. THE DIAMETER OF A MOLECULE IS OF THE ORDER OF 3 Å.

$$\begin{cases} p = \frac{1}{3} m n \overline{c^2} \\ p = \frac{2}{3} (\text{KINETIC ENERGY OF MOLECULES IN UNIT VOLUME}) \end{cases}$$

MEAN VELOCITY (OF MOLECULES) SQUARED $\overline{c^2} = \frac{3\bar{R}T}{M} = 3RT$

$\bar{R}$ = UNIVERSAL GAS CONSTANT, M = MOLECULAR WEIGHT

R = GAS CONSTANT

MEAN FREE PATH $\lambda = \frac{1}{\pi \sigma^2 n}$, σ = DIAMETER OF MOLECULES, n = NO. DENSITY

VISCOSITY $\mu = \frac{1}{3} m n \bar{c} \lambda = \frac{1}{3} \rho \bar{c} \lambda$ WHERE m IS THE MASS OF EACH MOLECULE. ∴ VISCOSITY OF A GAS IS $\propto \bar{c}$ OR $\mu \propto \sqrt{T}$. WE ALSO CAN WRITE

$$\begin{cases} \mu = \frac{1}{3} \frac{m}{\pi \sigma^2} \bar{c} & \text{"INDEPENDENT OF DENSITY"} \\ \nu = \frac{1}{3} \lambda \bar{c} & \text{KINEMATIC VISCOSITY} \end{cases}$$

THERMAL CONDUCTIVITY $K = \frac{n \bar{c} \lambda}{3} c_v$, c_v = SPECIFIC HEAT AT CONSTANT VOLUME PER MOLECULE.

FROM ABOVE, WE HAVE $\boxed{\frac{K}{\mu C_v} = 1}$, C_v = SPECIFIC HEAT AT CONSTANT VOLUME PER kg OF GAS (~ 717 J/kg FOR AIR). FOR REAL GASES, THIS EQUALITY IS ONLY APPROXIMATELY SATISFIED (DUE TO MOLECULAR REPULSION, μ IS REDUCED AND THE ABOVE RATIO LIES BETWEEN 1.4 AND 2.5.)

SPEED OF SOUND $c_0 = \left[\frac{\gamma \overline{c^2}}{3}\right]^{1/2} = \sqrt{\gamma R T}$, $\gamma = \frac{C_p}{C_v}$

THE ABOVE RESULTS ARE ALL OBTAINED USING JOULE'S CLASSI-

FICATION (1/6 OF THE TOTAL NO. OF MOLECULES MOVE ALONG EACH DIRECTION OF XYZ - SPACE). FROM SOMEWHAT MORE ADVANCED THEORY, WE AGAIN OBTAIN $\mu = \frac{1}{3} m n \bar{c} \lambda$

COLLISION FREQUENCY $\nu_c = \frac{\bar{c}}{\lambda}$

IN JOULE'S CLASSIFICATION, THE AVERAGE NO. OF MOLECULES STRIKING EACH UNIT AREA PER SECOND IS $\frac{1}{6} n \bar{c}$ WHILE USING A MODEL WHERE MOLECULES MOVE IN ALL DIREC-TION, WE GET $\frac{1}{4} n \bar{c}$.

NOTE THAT FOR $K / \mu C_v$ CONSISTENT UNITS MUST BE USED FOR BOTH NUMERATOR AND DENOMINATOR.

* NONLINEAR ACOUSTICS

PLANE WAVE OF FINITE AMPLITUDE IN MEDIA WITHOUT DISPERSION — RIEMANN'S SOLUTION (1860)

THE GOVERNING EQS ARE

$$\begin{cases} \rho_t + u\rho_x + \rho u_x = 0 \\ \rho u_t + \rho u u_x + c^2 \rho_x = 0 \end{cases}$$

USING METHOD OF CHARACTERISTICS, WE HAVE

$$\begin{cases} \rho_t + u\rho_x + 0 + \rho u_x = 0 \\ 0 + c^2\rho_x + \rho u_t + \rho u u_x = 0 \\ \rho_t dt + \rho_x dx + 0 + 0 = d\rho \\ 0 + 0 + u_t dt + u_x dx = du \end{cases}$$

THE CHARACTERISTIC DIRECTIONS ARE GIVEN BY

$$\begin{vmatrix} 1 & u & 0 & \rho \\ 0 & c^2 & \rho & \rho u \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{vmatrix} = 0 \Rightarrow \frac{dx}{dt} = u \pm c$$

ON THESE CHARACTERISTICS, WE HAVE THE FOLLOWING CONDITION

$$\begin{vmatrix} 1 & 0 & 0 & \rho \\ 0 & 0 & \rho & \rho u \\ dt & d\rho & 0 & 0 \\ 0 & du & dt & dx \end{vmatrix} = 0 \Rightarrow du = \pm \frac{c(\rho)}{\rho} d\rho$$

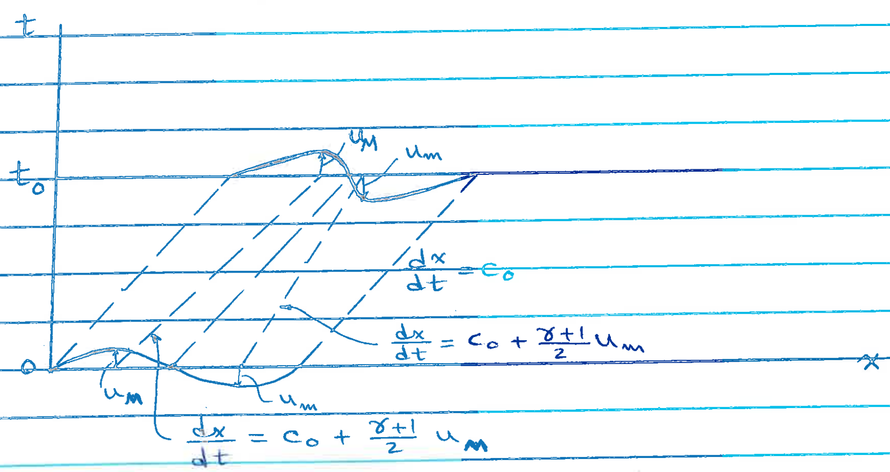
$$\therefore \quad u = \pm \int_{\rho_0}^{\rho} \frac{c(\rho)}{\rho} d\rho = \pm \frac{2}{\gamma - 1} (c - c_0)$$ USING REVERSIBLE ADIABATIC GAS EXPANSION

WHERE SUBSCRIPT 0 DENOTES UNDISTURBED CONDITION ($u = 0$). WE HAVE

$$c = c_0 \pm \frac{\gamma - 1}{2} u$$

FOR RIGHT MOVING WAVE, $c = c_0 + \frac{\gamma-1}{2} u$ AND

THE SPEED OF WAVE PROPAGATION $U = c_0 + \frac{\gamma+1}{2} u$.
IF $u = -\frac{2c_0}{\gamma+1}$, WE HAVE $U = 0$. THIS VELOCITY OF FLUID u IS CALLED CRITICAL VELOCITY: $u_{cr} = \frac{-2c_0}{\gamma+1}$.



SINCE $c = c(u) \Rightarrow$ ALONG CHARACTERISTICS WE HAVE u = CONST. AND c = CONST. AND THE CHARACTERISTICS THEMSELVES ARE STRAIGHT LINES(*). WE HAVE

$$\frac{c}{c_0} = \left(\frac{T}{T_0}\right)^{1/2} = \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{2}} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}}$$

$$\therefore \quad \rho = \left(1 \pm \frac{\gamma-1}{2} M\right)^{\frac{2}{\gamma-1}} \rho_0 \quad , \quad M = u/c_0$$

$$p = \left(1 \pm \frac{\gamma-1}{2} M\right)^{\frac{2\gamma}{\gamma-1}} p_0$$

* SEE P 15.

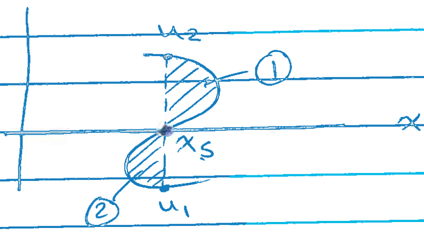
WHEN THE WAVE BECOMES MULTIVALUED, A GEOMETRICAL METHOD CAN GIVE THE POSITION OF THE SHOCK WAVE. IF WE USE CONTINUITY EQ. FOR THE SHOCKWAVE WHICH MOVES WITH SPEED U_s, WE GET

$$U_s = \frac{\rho_2 u_2 - \rho_1 u_1}{\rho_2 - \rho_1}$$

$$= \frac{\left(1+\frac{\gamma-1}{2}M_2\right)^{\frac{2}{\gamma-1}} M_2 - \left(1+\frac{\gamma-1}{2}M_1\right)^{\frac{2}{\gamma-1}} M_1}{\left(1+\frac{\gamma-1}{2}M_2\right)^{\frac{2}{\gamma-1}} - \left(1+\frac{\gamma-1}{2}M_1\right)^{\frac{2}{\gamma-1}}} c_0$$

$$\approx \left[1 + \frac{\gamma+1}{4}(M_1+M_2)\right] c_0$$

IN A MOVING FRAME WITH SPEED c_0, IF x_s IS THE LOCATION OF THE SHOCK, WE HAVE



$$A = \text{SHADED AREA} = \int_{u_1}^{u_2} (x - x_s)\, du$$

$$\frac{dA}{dt} = \int_{u_1}^{u_2} \left(\frac{dx}{dt} - \frac{dx_s}{dt}\right) du$$

$$= \int_{u_1}^{u_2} \left[\frac{\gamma+1}{2}u - \frac{\gamma+1}{4}(u_1+u_2)\right] du$$

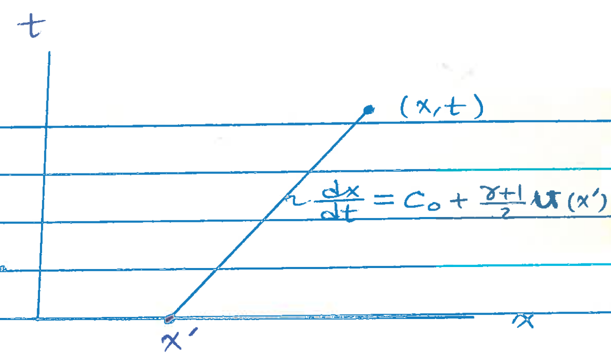
$$= 0 \quad \Rightarrow \quad A = \text{CONST.} = 0$$

BECAUSE BEFORE THE START OF THE SHOCK $A = 0$, SO WE HAVE THE RULE OF EQUALITY OF THE TWO AREAS 1 AND 2 ABOVE.

IF $u(x,0) = f(x)$ IS GIVEN, WE HAVE $u(x,t) = f(x')$. SINCE $x' = x - (c_0 + \frac{\gamma+1}{2}u)t$, WE GET THE FOLLOWING IMPLICIT RESULT FOR u : $u(x,t) = f[x - (c_0 + \frac{\gamma+1}{2}u)t]$

IF $f(x')$ IS SINUSOIDAL, THEN ONE CAN SEE THAT

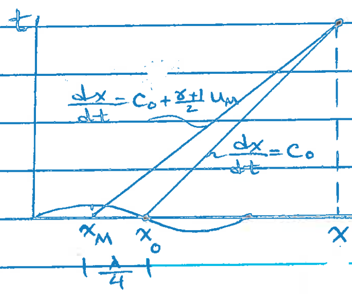
BESSEL FUNCTIONS WOULD APPEAR IN THE HARMONIC ANALYSIS OF THE DISTORTED WAVE.



HOW FAR DOES A SINUSOIDAL WAVE WHICH IS 120 dB (re: 2×10^{-5} Pa) AND HAS A FREQ. OF 1000 HZ HAVE TO GO TILL A SHOCK WAVE APPEARS IN EACH PERIOD?

$$u_M = \frac{p}{\rho_0 C_0}$$

$$= .048 \ \text{m/sec}$$



$$\frac{x - x_0}{t} = C_0$$

$$\frac{x - x_M}{t} = C_0 + \frac{\gamma+1}{2} u_M$$

$$\frac{x_0 - x_M}{t} = \frac{\gamma+1}{2} u_M$$

DISTANCE $x \approx x - x_0 = C_0 t = \frac{\lambda}{2(\gamma+1)} \frac{C_0}{u_M} = 500.\text{m}$

(TAKING $C_0 = 340.\text{m/sec}$)

* SINCE $du \pm \frac{c(\rho)}{\rho} d\rho$ IS AN EXACT DIFFERENTIAL, IT IS INTEGRATED AS ON P 12. IN PARTICULAR, IF $u =$ CONST, WE HAVE $\frac{dx}{dt} =$ CONST. AND THE CONDITION $du = \pm \frac{c(\rho)}{\rho} d\rho$ BECOMES $(1 \pm \frac{c(\rho)}{\rho} \frac{d\rho}{du}) du = 0$ OR $du = 0$, i.e. THE LINES $\frac{dx}{dt} =$ CONST ARE CHARACTERISTIC LINES ALONG WHICH $u =$ CONST.

* SOME INEQUALITIES AND OTHER ALGEBRAIC RESULTS

i) x_i REAL, $x_i > 0$, p, q REAL AND $pq > 0$

$$\Rightarrow \text{a) } x_i^{p+q} + x_j^{p+q} \geq x_i^p x_j^q + x_i^q x_j^p$$

$$\text{b) } m \sum_i^m x_i^{p+q} \geq \sum_i^m x_i^p \sum_i^m x_i^q$$

PROOF OF a) : $x_i^p - x_j^p$ HAS THE SIGN OF $(x_i - x_j)\,\mathrm{sig}\,p$

SINCE $pq > 0$, WE HAVE

$$(x_i^p - x_j^p)(x_i^q - x_j^q) \geq 0$$

FROM WHICH (a) FOLLOWS.

PROOF OF b) : x_i, $i = 1, \dots, m$. TAKE ALL THE POSSIBLE COMBINATIONS OF ANY TWO QUANTITIES AND USE (a), WE GET $(m-1)\sum_i^m x_i^{p+q} \geq \sum_i^m \sum_{j \neq i}^m x_i^p x_j^q$

ADD $\sum_i^m x_i^{p+q}$ TO BOTH SIDES TO GET

$$m \sum_i^m x_i^{p+q} \geq \sum_i^m \sum_j^m x_i^p x_j^q = \sum_i^m x_i^p \sum_i^m x_i^q$$

c) IF $pq < 0 \Rightarrow m \sum_i^m x_i^{p+q} \leq \sum_i^m x_i^p \sum_i^m x_i^q$

IN (c) IF $p = -q \Rightarrow \sum_i^m x_i^p \sum_i^m x_i^{-p} \geq m^2$

THM. 1: LET $b_i > 0$, $i = 1, \dots, m$, a_i/b_i GIVEN SET OF m FRACTIONS, $m = \operatorname{Min}_i (a_i/b_i)$, $M = \operatorname{Max}_i (a_i/b_i)$

$$\Rightarrow \quad m \leq \frac{\sum_i a_i}{\sum_i b_i} \leq M$$

PROOF, WE HAVE $m \leq a_i/b_i \ \forall i$

$$m b_i \leq a_i$$

$$m \sum b_i \leq \sum a_i$$

$$\Rightarrow m \leq \sum a_i / \sum b_i \quad \text{SINCE } b_i > 0$$

SIMILARLY FOR THE OTHER INEQUALITY.

COR. 1. a_i, b_i AS IN ABOVE THM., $f_i > 0, i = 1, 2, \ldots n$

$\Rightarrow \quad m \leq \sum_i f_i a_i / \sum_i f_i b_i \leq M$

THM 2: $a_i > 0, b_i > 0, f_i > 0, i = 1, \ldots, n$. $m = \underset{i}{Min}(a_i/b_i)$

$M = \underset{i}{Max}(a_i/b_i) \Rightarrow$

i) $\quad m \leq \left(\dfrac{\prod_i^n a_i}{\prod_i^n b_i}\right)^{1/n} \leq M$

ii) $\quad m \leq \left(\sum f_i a_i^m / \sum f_i b_i^m\right)^{1/m} \leq M$

THM 3: p, q INTEGERS, $x > 0, p > 0, q > 0 \Rightarrow$

$$\frac{x^p - 1}{p} > \frac{x^q - 1}{q} \quad \text{IF } p > q$$

PROOF: $\frac{x^y - 1}{y}$ IS AN INCREASING FN. OF y.

THM 4: $x > 0, x \neq 1 \Rightarrow m x^{m-1}(x-1) > x^m - 1 > m(x-1)$

IF $m \notin (0,1)$ AND $m x^{m-1}(x-1) < x^m - 1 < m(x-1)$ IF $m \in (0,1)$ (*)

COR. 2: IF $x > 0, y > 0, x \neq y \Rightarrow m x^{m-1}(x-y) > x^m - y^m > m y^{m-1}(x-y)$

IF $m \notin (0,1)$ AND $m x^{m-1}(x-y) < x^m - y^m < m y^{m-1}(x-y)$

IF $m \in (0,1)$. (*)

THM 5: THE ARITHMETIC MEAN OF n POSITIVE NUMBERS IS NOT LESS THAN THEIR GEOMETRIC MEAN: $\dfrac{\sum_1^n x_i}{n} \geq \left(\prod_1^n x_i\right)^{1/n}$

COR. 3: SEQ. $\{x_i\}$ GIVEN, $x_i > 0, i = 1, \ldots, n$, $\alpha_i > 0, i = 1, \ldots n$

$$\Rightarrow \frac{\sum \alpha_i x_i}{\sum \alpha_i} \geq \left(\prod x_i^{\alpha_i}\right)^{\frac{1}{\sum \alpha_i}}$$

PROOF: USE THE PRECEDING THM ASSUMING α_i RATIONAL.

(*) THIS THM AND ITS COR. ARE VERY USEFUL.

THM 6 $\alpha_i > 0,\ i=1,\dots m;\ x_i > 0,\ i=1,\dots m \Longrightarrow$

$$\frac{\sum \alpha_i x_i^m}{\sum \alpha_i} \geq \left(\frac{\sum \alpha_i x_i}{\sum \alpha_i}\right)^m \quad \text{IF } m \notin (0,1)$$

$$\frac{\sum \alpha_i x_i^m}{\sum \alpha_i} \leq \left(\frac{\sum \alpha_i x_i}{\sum \alpha_i}\right)^m \quad \text{IF } m \in (0,1)$$

PROOF: LET $\lambda_i = \dfrac{\alpha_i}{\sum \alpha_i}$ AND $y_i = \dfrac{x_i}{\sum \lambda_i x_i}$, THEN WE MUST SHOW THAT

$$\sum \lambda_i y_i^m \gtreqless 1 \quad \text{IF } m \notin (0,1) \text{ OR } \in (0,1)$$

BUT FROM THM. 4

$$y_i^m - 1 \gtreqless m(y_i - 1) \quad \text{IF } m \notin (0,1) \text{ OR } \in (0,1)$$

$$\sum \lambda_i (y_i^m - 1) \gtreqless m \sum (\lambda_i y_i - \lambda_i) \quad m \notin (0,1) \text{ OR } m \in (0,1)$$

BUT $\sum \lambda_i y_i = \sum \lambda_i = 1$

$$\sum \lambda_i (y_i^m - 1) \gtreqless 0 \quad m \notin (0,1) \text{ OR } m \in (0,1)$$

$$\therefore \sum \lambda_i y_i^m \gtreqless 1 \quad \text{" " "} \quad \text{Q.E.D.}$$

COR. 4: IN THE ABOVE THM, IF WE TAKE $\alpha_i = 1$, WE HAVE $\dfrac{\sum x_i^m}{m} \gtreqless \left(\dfrac{\sum x_i}{m}\right)^m$ IF $m \notin (0,1)$ OR $m \in (0,1)$

NOTE: IN THM 6 AND COR. 4, WE HAVE EQUALITY IF $x_1 = x_2 = \dots x_m$, OR IF $m = 0$ OR IF $m = 1$

THM 7: $x_i, \alpha_i, \beta_i, \gamma_i$ ALL POSITIVE, $i = 1, \dots, m$; $\sum_1^m \alpha_i x_i^{\beta_i}$ = CONSTANT $\Longrightarrow \prod x_i^{\gamma_i}$ IS MAXIMUM IF

$$\frac{\alpha_1 \beta_1 x_1^{\beta_1}}{\gamma_1} = \frac{\alpha_2 \beta_2 x_2^{\beta_2}}{\gamma_2} = \dots = \frac{\alpha_m \beta_m x_m^{\beta_m}}{\gamma_m}$$

PROOF $\prod x_i^{\gamma_i} = \prod \frac{(\alpha_i x_i^{\beta_i})^{\frac{\gamma_i}{\beta_i}}}{(\alpha_i)^{\gamma_i/\beta_i}} \Rightarrow \prod x_i^{\gamma_i}$ IS MAXIMUM IF $\prod (\alpha_i x_i^{\beta_i})^{\gamma_i/\beta_i}$ IS MAXIMUM. WE MUST THEREFORE DIVIDE $\alpha_i x_i^{\beta_i}$ INTO γ_i/β_i EQUAL PARTS, i.e.

$$\frac{\alpha_1 x_1^{\beta_1}}{(\gamma_1/\beta_1)} = \frac{\alpha_2 x_2^{\beta_2}}{(\gamma_2/\beta_2)} = \cdots = \frac{\alpha_m x_m^{\beta_m}}{(\gamma_m/\beta_m)}$$

THM. 8 : $x_i, \alpha_i, \beta_i, \gamma_i$ ALL POSITIVE, $i = 1, \ldots, m$; $\prod x_i^{\gamma_i} = \text{CONST.} \Rightarrow \sum \alpha_i x_i^{\beta_i}$ MINIMUM IF

$$\frac{\alpha_1 \beta_1 x_1^{\beta_1}}{\gamma_1} = \frac{\alpha_2 \beta_2 x_2^{\beta_2}}{\gamma_2} = \cdots\cdots = \frac{\alpha_m \beta_m x_m^{\beta_m}}{\gamma_m}.$$

COR. 5 i) IF $\sum \alpha_i x_i = \text{CONST.} \Rightarrow \prod x_i = \text{MAX.}$ IF $\alpha_1 x_1 = \alpha_2 x_2 = \cdots = \alpha_m x_m$

ii) IF $\prod x_i = \text{CONST.} \Rightarrow \sum \alpha_i x_i = \text{MIN.}$ IF $\alpha_1 x_1 = \alpha_2 x_2 = \cdots = \alpha_m x_m$

THM. 9 : α_i, β_i, x_i ALL POSITIVE, $i = 1, \ldots m$; $\sum \alpha_i x_i^{K} = \text{CONST.}$

$\Longrightarrow$ $\sum \beta_i x_i^{m}$ MINIMUM IF $\frac{m}{K} \notin (0,1)$

$\sum \beta_i x_i^{m}$ MAXIMUM IF $\frac{m}{K} \in (0,1)$

FOR $$\frac{\beta_1 x_1^{m}}{\alpha_1 x_1^{K}} = \frac{\beta_2 x_2^{m}}{\alpha_2 x_2^{K}} = \cdots\cdots = \frac{\beta_m x_m^{m}}{\alpha_m x_m^{K}}$$

THE PROOF OF THIS THM DEPENDS ON THE FOLLOWING

THM. 10 : α_i, x_i POSITIVE, $i = 1, \ldots, m$; $\sum \alpha_i x_i = \text{CONST.}$

$\Rightarrow$ $\sum \alpha_i x_i^{m}$ MINIMUM IF $m \notin (0,1)$,

$\sum \alpha_i x_i^{m}$ MAXIMUM IF $m \in (0,1)$

FOR $x_1 = x_2 = \cdots = x_m$

PROOF : THE PROOF CAN BE SEEN IMMEDIATELY USING THM. 6 AND THE FACT THAT EQUALITY HOLDS IN THAT THM FOR $x_1 = x_2 = \cdots = x_m$.

PROOF OF THM. 9: DEFINE γ AND $P_i \ni P_i y_i^{\gamma} = \beta_i x_i^m$
AND $P_i y_i = \alpha_i x_i^K \Rightarrow y_i^{\gamma_i - 1} = \frac{\beta_i}{\alpha_i} x_i^{m-K}$
TAKE $\gamma = \frac{m}{K}$, WE GET

$$y_i = (\beta_i / \alpha_i)^{m/(m-k)} x_i^m$$

BY THM. 10, IF $\sum P_i y_i = \text{CONST.}$, $\sum P_i y_i^{\gamma}$ IS MINIMUM (MAXIMUM) IF $\gamma \notin (0,1)$ ($\gamma \in (0,1)$) FOR $y_1 = y_2 = \dots = y_n$. THIS IS THE SAME AS THE CONDITION OF THE THM. 9.

A RECIPROCITY THM.: IF $\phi(\vec{x})$ HAS A MAXIMUM VALUE $\phi(\vec{x}_0) = B$ SUBJECT TO CONSTRAINT $f(\vec{x}) = A$, AND IF $B \uparrow$ WHEN $A \uparrow \Rightarrow f(\vec{x})$ HAS A MINIMUM $f(\vec{x}_0) = A$ SUBJECT TO THE CONSTRAINT $\phi(\vec{x}) = B$.

PROOF: IF $\min_{\phi = B} f(\vec{x}) = A' < A \Rightarrow \max_{f = A'} \phi = B' < B$ BECAUSE OF THE CONDITION $B \uparrow A \uparrow$. THIS IS A CONTRADICTION SINCE THE MIN. OF f WAS ACHIEVED FOR $\phi = B$. $\therefore f(\vec{x}) \geq A$. THIS THM HAS VERY USEFUL APPLICATIONS.

SOME RESULTS ON INDETERMINANT FORMS

THE FOLLOWING TWO THMS ARE BY CAUCHY (COURS D'ANALYSE DE L'ECOLE ROYALE POLYTECHNIQUE, PART I ANALYSE ALGÉBRIQUE, 1821). THESE ARE VERY INGENIOUS.

THM. 11: $\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} [f(x+1) - f(x)]$ PROVIDED THAT THE RIGHT SIDE IS NOT INDETERMINATE.

PROOF: LET $K = \lim_{x \to \infty} [f(x+1) - f(x)]$. THIS MEANS

THAT $\exists\, h(\epsilon) \ni \forall\, x \geq h$, $|f(x+1) - f(x) - k| < \epsilon$. NOW, WE HAVE

$$f(h+1) - f(h) < k + \epsilon$$
$$f(h+2) - f(h+1) < k + \epsilon$$
$$\vdots \qquad \vdots \qquad \vdots \quad \vdots$$
$$f(h+n) - f(h+n-1) < k + \epsilon$$

ADD BOTH SIDES $f(h+n) - f(h) < nk + n\epsilon$

OR $f(x) - f(h) < (x-h)k + (x-h)\epsilon$

$$\frac{f(x)}{x} - \frac{f(h)}{x} < \left(1 - \frac{h}{x}\right)k + \epsilon$$

NOW LET $x \to \infty$, WE GET

$$\frac{f(x)}{x} - k < \epsilon$$

SIMILARLY, WE CAN SHOW THAT $\frac{f(x)}{x} - k > -\epsilon$ AS $x \to \infty$,

SO THAT $k = \lim_{x \to \infty} \frac{f(x)}{x}$. THE VALUE OF k MAY BECOME ∞ AND THE SAME PROOF HOLDS.

THM. 12: $\lim_{x \to \infty} [f(x)]^{1/x} = \lim_{x \to \infty} \frac{f(x+1)}{f(x)}$ PROVIDED THAT THE RIGHT SIDE IS NOT INDETERMINATE.

PROOF: TAKE $\log$ OF BOTH SIDES AND USE THM. 11.

WE HAVE THE FOLLOWING LIMITS:

$$\lim_{x \to \infty} a^x / x = \infty \qquad a > 0$$

$$\lim_{x \to \infty} (\log_a x)/x = 0 \qquad a > 0$$

$$\lim_{x \to 0+} x \log_a x = 0 \qquad a > 0$$

$$\lim_{x \to 0+} x^x = 1$$

$$\lim_{x \to 0+} x^{x^n} = 1 \qquad n > 0$$

* IF $\lim_{x \to a} u = \lim_{x \to a} v = 0$ AND $\exists\, n > 0$, $n \neq \infty \ni$

$$\lim_{x\to a} \frac{v}{u^n} = \ell \Rightarrow \lim_{x\to a} u^v = \lim_{x\to a} (u^{u^n})^{v/u^n} = 1$$

* IF $\lim_{x\to a} u = 1$, $\lim_{x\to a} v = \infty$ AND $\lim_{x\to a} (u-1)v = \ell$

$$\Rightarrow \lim_{x\to a} u^v = \lim_{x\to a} \left\{ \left[1 + (u-1)\right]^{\frac{1}{u-1}} \right\}^{v(u-1)} = e^{\ell}.$$

WE NOTE THAT THE FUNDAMENTAL RESULTS FOR THE FORM 0^0 IS $\lim_{x\to 0} x^x = 1$ AND FOR THE FORM 1^∞ IS $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$.

FROM G. CHRYSTAL "ALGEBRA – AN ELEMENTARY APPROACH" PART II, 7TH EDITION 1964 (1ST EDITION 1889)

* SOME RESULTS AND METHODS FOR PDE'S.

1) UNIQUENESS OF THE SOLUTION - THE FOLLOWING TWO EXAMPLES ILLUSTRATE THE USE OF ENERGY IDENTITIES IN PROVING UNIQUENESS.

i)
$$\begin{cases} \frac{\partial}{\partial x}\left(e^x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(e^y \frac{\partial u}{\partial y}\right) = 0 & D: x^2+y^2<1 \\ u = x^2 & \text{ON } \partial D: x^2+y^2=1 \end{cases}$$

THIS PROBLEM HAS A UNIQUE SOLUTION SINCE IF U_1 AND U_2 ARE TWO SOLUTIONS, THEN $v = U_2 - U_1$ SATISFIES

$$\begin{cases} \frac{\partial}{\partial x}\left(e^x \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(e^y \frac{\partial v}{\partial y}\right) = 0 & \text{IN } D \\ v = 0 & \text{ON } \partial D \end{cases}$$

LET $\ell v = 0$ BE THE ABOVE DIFF. EQ. $\Longrightarrow$

$$v \ell v = \frac{\partial}{\partial x}\left[v e^x \frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[v e^y \frac{\partial v}{\partial y}\right] - \left[e^x \left(\frac{\partial v}{\partial x}\right)^2 + e^y \left(\frac{\partial v}{\partial y}\right)^2\right] = 0$$

$$0 = \int_D v \ell v \, dx\,dy = \int_{\partial D} v \left[e^x \frac{\partial v}{\partial x} n_1 + e^y \frac{\partial v}{\partial y} n_2\right] d\sigma - \int_D \left[e^x \left(\frac{\partial v}{\partial x}\right)^2 + e^y \left(\frac{\partial v}{\partial y}\right)^2\right] dx\,dy$$

WHERE (n_1, n_2) IS THE UNIT OUTWARD NORMAL TO ∂D AND $d\sigma$ IS ELEMENT OF THE LENGTH OF THE CURVE ∂D. WE HAVE

$$\int_D \left[e^x \left(\frac{\partial v}{\partial x}\right)^2 + e^y \left(\frac{\partial v}{\partial y}\right)^2\right] dx\,dy = 0$$

$\Longrightarrow$ $v = $ CONST. BECAUSE THE INTEGRAND IS ≥ 0 AND THE CASE >0 CANNOT OCCUR. THE CONSTANT IS 0 SINCE

$v = 0$ ON ∂D. BECAUSE WE ARE APPLYING THE DIVERGENCE THM, WE REQUIRE THAT $v \in C^1$ ON $D + \partial D$ AND THIS GUARANTEES $v \in C$ ON $D + \partial D$ SO THAT IF $v = \text{CONST}$ IN D AND $v = 0$ ON $\partial D \Rightarrow v = 0$ IN D.

ii) $\begin{cases} \ell u = \dfrac{\partial}{\partial x}\left[(1+x^2)\dfrac{\partial u}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[(1+x^2+y^2)\dfrac{\partial u}{\partial y}\right] - e^x u = 1 & \text{IN } D: x^2+y^2<2 \\ u = e^y & \text{IN } \partial D: x^2+y^2=2 \end{cases}$

HAS A UNIQUE SOLUTION. IF u_1 AND u_2 ARE TWO SOLUTIONS OF THIS PDE, THEN $v = u_2 - u_1$ SATISFIES THE EQ

$$\begin{cases} \ell v = 0 & \text{IN } D \\ v = 0 & \text{ON } \partial D \end{cases}$$

$$v \ell v = \frac{\partial}{\partial x}\left[(1+x^2) v \frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[(1+x^2+y^2) v \frac{\partial v}{\partial y}\right]$$
$$- e^x v^2 - (1+x^2)\left(\frac{\partial v}{\partial x}\right)^2 - (1+x^2+y^2)\left(\frac{\partial v}{\partial y}\right)^2 = 0$$

$$\int_D v \ell v \, dx\,dy = \int_{\partial D} v\left[(1+x^2)\frac{\partial v}{\partial x} n_1 + (1+x^2+y)\frac{\partial v}{\partial y} n_2\right] ds$$
$$- \int \left[e^x v^2 + (1+x^2)\left(\frac{\partial v}{\partial x}\right)^2 + (1+x^2+y^2)\left(\frac{\partial v}{\partial y}\right)^2\right] dx\,dy$$
$$= 0$$

SINCE $\int_{\partial D} = 0$ BY B.C. ON $v \Rightarrow v = 0$ IN D.

2. MAXIMUM PRINCIPLE FOR POISSON AND HEAT EQUATIONS.

(i) POISSON'S EQ. FOR A BOUNDED REGION: D

LET $f(\vec{x})$ $\forall \vec{x} \in D \Rightarrow M = \max_{\vec{x} \in D + \partial D} u$ CANNOT APPEAR IN D WHERE U IS THE SOLUTION OF

$\nabla^2 u = f(\vec{x})$ WITH SOME GIVEN B.C.'S

PROOF: IF $\vec{x}_0 \in D$ AND $u(\vec{x}_0) = M \Rightarrow$

$\frac{\partial^2 u}{\partial x^2} \le 0$, $\frac{\partial^2 u}{\partial y^2} \le 0$, $\frac{\partial^2 u}{\partial z^2} \le 0$ i.e. $\nabla^2 u \le 0$

CONTRADICTION SINCE $f > 0$ IN D. $\therefore$ M CAN ONLY APPEAR ON ∂D. THIS RESULT CAN BE EXTENDED TO THE CASE $f \ge 0$ $\forall \vec{x} \in D$ BY THE FOLLOWING SMART TECHNIQUE. LET

$$v = u + \varepsilon(x^2 + y^2 + z^2)$$

$$\nabla^2 v = \nabla^2 u + 6\varepsilon = f(\vec{x}) + 6\varepsilon > 0 \quad \forall \vec{x} \in D$$

$\therefore$ MAX v APPEARS ON ∂D. BUT

$$u \le v \le M + \varepsilon R^2$$

WHERE R IS THE RADIUS OF A SPHERE ENTIRELY ENCLOSING D. SINCE ε CAN BE ALLOWED TO APPROACH ZERO, THE MAXIMUM OF U ALSO IN ON ∂D. IF $f = 0$, THEN APPLYING THE ABOVE RESULT TO U & −U GIVES THAT THE MAX & MIN OF U $\in \partial D$.

(ii) THE HEAT EQUATION: $\frac{\partial u}{\partial t} - k\nabla^2 u = 0$

$\vec{x} \in D$, $0 \le t \le \bar{t}$. WE CAN SHOW IN TWO STEPS AS ABOVE THAT THE MAX AND MIN OF U ARE ON $D \times \{0\} \cup \partial D \times [0, \bar{t})$.

STEP 1: IF $\frac{\partial u}{\partial t} - k\nabla^2 u < 0$ $\forall (\vec{x}, t) \in D \times (0, \bar{t})$, THE MAX U CANNOT APPEAR IN THAT REGION SINCE $\frac{\partial u}{\partial t} = 0$ AND $-k\nabla^2 u \ge 0$, CONTRADICTION! MAX U CANNOT APPEAR IN $(\vec{x}, t) \in D \times \{\bar{t}\}$ EITHER SINCE

$\frac{\partial v}{\partial t} \geq 0$ AND $-k\nabla^2 u \geq 0$ $\therefore$ MAX U CAN APPEAR ONLY FOR $(\vec{x},t) \in D\times\{0\} \cup \partial D \times [0,T] = I$

STEP 2 : IF $\frac{\partial u}{\partial t} - k\nabla^2 u = 0$, LET $v = u + \varepsilon(x^2 + y^2 + z^2) \Rightarrow \frac{\partial v}{\partial t} - k\nabla^2 v = -6k\varepsilon < 0 \ \forall (\vec{x},t) \in D\times(0,T)$ $\therefore$ MAX v APPEARS IN REGION I. BUT

$$u \leq v \leq \text{Max } u + \varepsilon R^2$$

$\therefore$ Max U ALSO APPEARS FOR $(\vec{x},t) \in I$ SINCE WE CAN LET $\varepsilon \to 0$.

FROM H. F. WEINBERGER
"A FIRST COURSE IN PARTIAL DIFFERENTIAL EQS."

* SOME RESULTS FROM GEN. FN. THEORY

i) PROOF OF $\widehat{f(x-\alpha)} = e^{ix\xi}\hat{f}(\xi)$ FOR GEN. FNS $\in S'$. LET $\phi \in S$, THE SHIFTED G.F. IS DEFINED BY $F_{\alpha}[\phi] = F[\phi(x+\alpha)]$. WE HAVE

$$\hat{F}_{-\alpha}[\phi] = F_{-\alpha}[\hat{\phi}(\xi)]$$

$$= F[\hat{\phi}(\xi+\alpha)]$$

$$\hat{\phi}(\xi+\alpha) = \int \phi(x)\, e^{ix(\xi+\alpha)}\, dx$$

$$= \int \phi(x)\, e^{ix\alpha} \cdot e^{ix\xi}\, dx$$

$$= \widehat{\phi(x)\, e^{ix\alpha}}$$

$$\therefore \hat{F}_{-\alpha}[\phi] = F[\widehat{\phi(x)\, e^{ix\alpha}}]$$

$$= \hat{F}[\phi(x)\, e^{ix\alpha}]$$

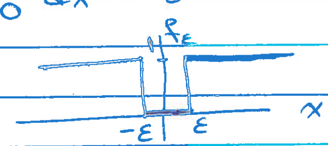
$$= e^{ix\alpha}\hat{F}[\phi(x)]$$

SYMBOLICALLY $\widehat{f(x-\alpha)} = e^{i\alpha\xi}\hat{f}(\xi)$

ii) GEN. FN. $\frac{1}{x^2}$: THIS FN. IS GEN. SECOND DERIVATIVE OF THE REGULAR GEN. FN. $\ln|x|$ i.e.

$$\text{G.F. } \frac{1}{x^2} = -\frac{d^2}{dx^2}[\ln|x|] = -\lim_{\varepsilon\to 0}\frac{d^2}{dx^2}[f_\varepsilon(x)\ln|x|]$$

WHERE $f_\varepsilon(x)$ IS SHOWN ON THE RIGHT.



$$\frac{d}{dx}[f_\varepsilon(x)\ln|x|] = [-\delta(x+\varepsilon)+\delta(x-\varepsilon)]\ln|x| + f_\varepsilon(x)\frac{1}{x}$$

$$= -\cancel{\ln|\varepsilon|} + \cancel{\ln|\varepsilon|} + f_\varepsilon(x)\frac{1}{x}.$$

$$\frac{d^2}{dx^2}\left[f_\varepsilon(x)\ln|x|\right] = \left[-\delta(x+\varepsilon)+\delta(x-\varepsilon)\right]\frac{1}{x} - f_\varepsilon(x)\frac{1}{x^2}$$

LET ϕ BE A TEST FN., THEN

$$I = \int_{-\infty}^{\infty}\left(G.F.\ \frac{1}{x^2}\right)\cdot\phi(x)\,dx = \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{\infty}\frac{f_\varepsilon(x)}{x^2}\phi(x) - \frac{2\phi(0)}{\varepsilon}\right\}$$

BUT

$$\frac{2\phi(0)}{\varepsilon} = \phi(0)\int_{-\infty}^{\infty}\frac{f_\varepsilon(x)}{x^2}\,dx$$

$$\therefore\quad I = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f_\varepsilon(x)\left[\phi(x)-\phi(0)\right]}{x^2}\,dx$$

$$= P.V\int_{-\infty}^{\infty}\frac{\phi(x)-\phi(0)}{x^2}\,dx$$

EVEN IF THE LAST INTEGRAL IS WELL-DEFINED, THE APPEARANCE OF PRINCIPAL VALUE IS SOMEWHAT UNSATISFACTORY. LET US USE ANOTHER APPROACH IN THE LAST STEP:

$$\frac{2\phi(0)}{\varepsilon} = \int_{-\infty}^{\infty}\frac{\phi(0)+x\phi'(0)}{x^2}f_\varepsilon(x)\,dx$$

WE NOTE THAT $\int_{-\infty}^{\infty}\frac{\phi'(0)}{x}f_\varepsilon(x)\,dx = 0$ SINCE THE INTEGRAND IS ODD. THEREFORE

$$I = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f_\varepsilon(x)\left[\phi(x)-\phi(0)-x\phi'(0)\right]}{x^2}\,dx$$

$$= \int_{-\infty}^{\infty}\frac{\phi(x)-\phi(0)-x\phi'(0)}{x^2}\,dx \quad \text{CONVERGENT INTEGRAL}$$

iii) GEN. FN. $\frac{1}{|x|}$: THIS GEN FN. IS THE SAME AS

$$\operatorname{sig}(x)\frac{d}{dx}\ln|x| = \lim_{\varepsilon\to 0}\operatorname{sig}(x)\frac{d}{dx}\left[f_\varepsilon(x)\ln|x|\right]$$

$$\phi(x)\,\mathrm{sig}(x)\,\frac{d}{dx}\left[f_\varepsilon(x)\ln|x|\right] = \mathrm{sig}\,x\left\{\left[-\delta(x+\varepsilon)+\delta(x-\varepsilon)\right]\ln|x| + f_\varepsilon(x)\frac{1}{x}\right\}\phi(x)$$

$$\int \phi(x)\,\mathrm{sig}(x)\,\frac{d}{dx}\left[f_\varepsilon(x)\ln|x|\right]dx = \left[+\phi(-\varepsilon)+\phi(\varepsilon)\right]\ln|\varepsilon| + \int_{-\infty}^{\infty} f_\varepsilon(x)\frac{\phi(x)}{|x|}\,dx$$

$$\left[\phi(-\varepsilon)+\phi(\varepsilon)\right]\ln|\varepsilon| = 2\phi(0)\ln|\varepsilon| + \varepsilon^2\phi''(0)\ln|\varepsilon| + \ldots$$

$$2\ln|\varepsilon| = -\int_{-\infty}^{\infty}\frac{\mathrm{sig}(x)\,f_\varepsilon(x)}{x}\,dx$$

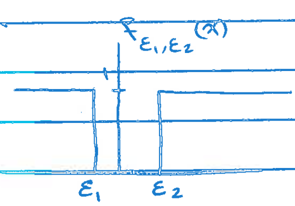
$$= -\int_{-\infty}^{\infty}\frac{f_\varepsilon(x)}{|x|}\,dx$$

$$\therefore\quad I = \int_{-\infty}^{\infty}\left(\mathrm{GF}\left(\frac{1}{|x|}\right)\right)\phi(x)\,dx$$

$$= \lim_{\varepsilon\to 0}\left\{\int_{-\infty}^{\infty}\frac{\left[\phi(x)-\phi(0)\right]}{|x|}f_\varepsilon(x)\,dx + O(\varepsilon)\right\}$$

$$= \int_{-\infty}^{\infty}\frac{\phi(x)-\phi(0)}{|x|}\,dx \qquad \text{CONVERGENT INTEGRAL}$$

A GENERAL APPROACH TO REGULARIZATION OF DIVERGENT INTEGRALS IS OUTLINED IN RESEARCH BOOK, P62. AS EXPLAINED ON PAGES 60-61 OF THAT BOOK, THERE IS NO NEED TO TAKE $f_\varepsilon(x)$ FOR THIS LIMIT PROCESS. THE FOLLOWING FUNCTION $f_{\varepsilon_1,\varepsilon_2}(x)$ IS JUST AS USEFUL.



SEVERAL INTERESTING RESULTS CAN BE OBTAINED USING $f_{\varepsilon_1,\varepsilon_2}(x)$ BY ASSUMING $\varepsilon_2/\varepsilon_1 = \alpha = $ CONST. AND LETTING $\varepsilon_1 \to 0$.

* DIFFERENCE EQUATIONS AND FINITE DIFFERENCES

* THE DIFFERENCE CALCULUS

THE DIFFERENCE OPERATOR $\Delta \equiv \underset{\rightarrow}{\delta}$

$$\Delta f = f(x+h) - f(x) \quad , \quad h \equiv \Delta x$$

THE SHIFT OPERATOR $E \equiv E_h$

$$E f(x) = f(x+h)$$

WE HAVE $\Delta = E - I$, $I \equiv$ IDENTITY OPERATOR

USUALLY 1 IS WRITTEN IN PLACE OF I.

THE DIFFERENTIAL OPERATOR d

$$d f(x) = f'(x)\,\Delta x$$
$$= D f(x)\, h$$

WHERE D IS THE DERIVATIVE OPERATOR. WE HAVE

$$D = \frac{d}{dx} = \lim_{\Delta x \to 0} \frac{\Delta}{\Delta x}$$

SOME RULES OF DIFF. CALCULUS

i) $\Delta(f+g) = \Delta f + \Delta g$; ii) $\Delta(\alpha f) = \alpha \Delta f$, α = CONST.

iii) $\Delta(fg) = f\Delta g + g\Delta f + \Delta f \cdot \Delta g$

iv) $$\Delta\left(\frac{f}{g}\right) = \frac{g\Delta f - f\Delta g}{g \cdot Eg}$$

WE DEFINE THE FACTORIAL FUNCTION $x^{(m)}$, $m = 1, 2, \ldots$

AS $x^{(m)} = x\cdot(x-h)\cdot(x-2h)\ldots\ldots[x-(m-1)h]$

FOR NEGATIVE INTEGERS, WE DEFINE (SEE P 33)

$$x^{(-m)} = \frac{1}{(x+h)(x+2h)\ldots(x+mh)} = \frac{1}{(x+mh)^{(m)}} \quad , m = 1, 2, 3, \ldots$$

GENERALIZED FACTORIAL FUNCTION FOR A FUNCTION $f(x)$ IS DEFINED AS FOLLOWS

$$[f(x)]^{(m)} = f(x)\, f(x-h)\, f(x-2h) \ldots\ldots f[x-(m-1)h] \; , \; m = 1, 2, \ldots$$

$$[f(x)]^{(-m)} = \frac{1}{f(x+h)\ldots\ldots f(x+mh)} = \frac{1}{[f(x+mh)]^{(m)}} \quad m = 1, 2, 3, \ldots$$

WE HAVE $x^{(0)} = 1$, $[f(x)]^{(0)} = 1$

FOR POSITIVE m, $x^{(m)}$ IS A POLYNOMIAL OF DEGREE m. CONVERSELY, EVERY POLYNOMIAL OF DEGREE m CAN BE WRITTEN AS THE SUM OF FACTORIAL POLYNOMIALS OF UPTO DEGREE m. WE HAVE

$$x^{(m)} = \sum_{k=1}^{m} s_k^m \, h^{m-k} \, x^k \quad ; m > 0$$

WHERE s_k^m ARE STIRLING NUMBERS OF THE FIRST KIND HAVING THE RECURSION FORMULA

$$s_k^{m+1} = s_{k-1}^m - m s_k^m$$

WE DEFINE $s_m^m = 1$; $s_k^m = 0$ FOR $k \leq 0$ OR $k \geq m+1$

SIMILARLY, WE HAVE

$$x^m = \sum_{k=1}^{m} S_k^m \, h^{m-1} \, x^{(k)} \quad ; m > 0$$

WHERE S_k^m ARE STIRLING NUMBERS OF THE SECOND KIND WITH THE RECURSION FORMULA

$$S_k^{m+1} = S_{k-1}^m + k S_k^m$$

WE DEFINE $S_m^m = 1$, $S_k^m = 0$ FOR $k \leq 0$ OR $k > m$

DIFFERENCES OF SOME FUNCTIONS

i) $\Delta c = 0$ $\quad c = \text{CONST.}$

ii) $\Delta [x^{(m)}] = m h \, x^{(m-1)}$ $\quad m > 0$ OR $m < 0$

iii) $\Delta (px+q)^{(m)} = mph \, (px+q)^{(m-1)}$

iv) $\Delta (b^x) = b^x (b^h - 1)$

v) $\Delta (e^{rx}) = e^{rx} (e^{rh} - 1)$

TAYLOR SERIES IN OPERATOR FORM

$$E f(x) = e^{hD} f(x)$$

$$\therefore \quad E = e^{hD} \; , \quad \Delta = E - 1 = e^{hD} - 1$$

LEIBNITZ RULE FOR DIFFERENCES

$$\Delta^n (fg) = f\,\Delta^n g + \binom{n}{1} \Delta f\,(\Delta^{n-1} E g) + \binom{n}{2} \Delta^2 f\,(\Delta^{n-2} E^2 g) + \cdots + \Delta^n f\, E^n g$$

OTHER DIFFERENCE OPERATORS

$\nabla = \delta f = f(x) - f(x-h)$ BACKWARD DIFF. OP.

$\delta f = f(x+\frac{h}{2}) - f(x-\frac{h}{2})$ CENTRAL DIFF. OP.

WE HAVE

$$\nabla = \Delta E^{-1} = 1 - E^{-1}$$
$$\delta = E^{1/2} - E^{-1/2}$$
$$= \Delta E^{-1/2}$$
$$= \nabla E^{1/2}$$

AVERAGING OPERATORS
$$M f(x) = \frac{1}{2}[f(x+h) + f(x)]$$
$$\mu f(x) = \frac{1}{2}[f(x+\frac{h}{2}) + f(x-\frac{h}{2})]$$

WE HAVE

$$M = \frac{1}{2}(E+1)$$
$$= 1 + \frac{1}{2}\Delta$$
$$\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$$

SOME USEFUL RESULTS

i) $Dx - xD = I$ IDENTITY OPERATOR

ii) $\Delta^n = (E-1)^n = E^n - \binom{n}{1} E^{n-1} + \cdots + (-1)^n$

iii) $\Delta E = E \Delta$

iv) $E^{-1} f(x) = f(x-h)$, $E^{-1}E = EE^{-1} = I$

v) $E^m E^n = E^{m+n}$ WHERE $E^m f = f(x+mh)$

THIS RESULT HOLDS FOR POSITIVE AND NEGATIVE m AND n. IT ALSO HOLDS FOR m AND n REAL.

vi) IF $f(x) = \sum_{k=0}^{m} a_k x^k \rightarrow \Delta^n f(x) = m! \, a_m h^m$,
AND
$$\Delta^m f(x) = 0 \qquad m > n$$

MOTIVATION BEHIND DEFN. OF $x^{(-m)}$

$$x^{(m+1)} = x^{(m)}(x - mh) \qquad (*)$$

NOW LET $m = -1$

$$x^{(0)} = 1 = x^{(-1)}(x+h)$$

OR $$x^{(-1)} = \frac{1}{x+h}$$

LET $m = -2$ IN THE ABOVE EQ. (*)

$$x^{(-1)} = x^{(-2)}(x+2h)$$

OR $$x^{(-2)} = \frac{1}{(x+h)(x+2h)}, \text{ ETC.}$$

THE SUBSCRIPT NOTATION

WE DEFINE $y_k = f(x+kh)$, THEN

$$E y_k = y_{k+1}, \quad \Delta y_k = y_{k+1} - y_k$$

$\therefore$ A UNIT CHANGE IN k CORRESPONDS TO A CHANGE OF x by h. k IS NOT NECESSARILY AN INTEGER. IT CAN BE A CONTINUOUS VARIABLE.

GREGORY-NEWTON FORMULA

$$f(x) = f(a) + \frac{\Delta f(a)}{\Delta x}\frac{(x-a)^{(1)}}{1!} + \frac{\Delta^2 f(a)}{\Delta x^2}\frac{(x-a)^{(2)}}{2!} + \cdots + \frac{\Delta^n f(a)}{\Delta x^n}\frac{(x-a)^{(n)}}{n!} + R_m$$

$$R_m = \frac{f^{(m+1)}(\eta)\,(x-a)^{(m+1)}}{(m+1)!}, \qquad a < \eta < x$$

IN SUBSCRIPT NOTATION

$$y_k = y_0 + \frac{\Delta y_0}{1!}k_1^{(1)} + \frac{\Delta^2 y_0}{2!}k_1^{(2)} + \cdots + \frac{\Delta^n y_0}{n!}k_1^{(n)} + R_m$$

WHERE $R_m = \dfrac{h^{m+1} f^{(m+1)}(\eta) K_1^{(m+1)}}{(m+1)!}$. WE HAVE USED THE FOLLOWING NOTATION IN THE ABOVE EQUATIONS

$$K_1^{(m)} = K(K-1)\ldots\ldots[K-(m-1)] \quad (*)$$

WE HAVE $(hK)^{(m)} = h^m K_1^{(m)}$

APPROXIMATE DIFFERENTIATION FORMULAS

WE HAVE $e^{hD} = 1+\Delta \Rightarrow D = \frac{1}{h}\ln(1+\Delta)$

$$D = \frac{1}{h}\left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right)$$

$$D^2 = \frac{1}{h^2}\left(\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \cdots\right)$$

$$D^3 = \frac{1}{h^3}\left(\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \cdots\cdots\right)$$

$$D = \frac{1}{h}\left(\delta - \frac{1}{24}\delta^3 + \frac{3}{640}\delta^5 - \cdots\right)$$

THE LAST EQUATION IS OBTAINED FROM

$$\delta = E^{1/2} - E^{-1/2}$$

$$= \frac{E-1}{E^{1/2}}$$

$$= \frac{e^{hD}-1}{e^{hD/2}}$$

$$= 2\sinh\left(\frac{hD}{2}\right)$$

$$D = \frac{2}{h}\sinh^{-1}(\delta/2)$$

(*) THE NOTATION OF THE SOURCE OF THESE NOTES IS CHANGED HERE. THE SUBSCRIPT 1 IS ADDED TO $K_1^{(m)}$ TO REDUCE CONFUSION WITH THE DERIV. OF $x^{(m)}$ BEFORE. THE SUBSCRIPT 1 STANDS FOR $h=1$ HERE.

* THE SUM CALCULUS

THE OPERATOR Δ^{-1} IS THE INVERSE OF Δ, i.e. $\Delta\Delta^{-1} = \Delta^{-1}\Delta = I$. THIS OPERATOR IS DENOTED BY Σ, THE SUM OPERATOR. WE NOTE THAT $\frac{\Delta}{\Delta x}$ AND $\Sigma(\ )\Delta x$ ARE INVERSE OF EACH OTHER. IF $\Sigma f(x) = F(x)$, THEN $f(x) = \Delta F(x)$ NOW LET $G(x) = F(x) + C(x)$ WHERE $C(x)$ IS A FUNCTION WHICH IS PERIODIC WITH PERIOD h, i.e. $\Delta C(x) = C(x+h) - C(x) = 0$ $\Rightarrow$ $\Delta G(x) = \Delta F(x) = f(x)$, i.e. $\Sigma f(x) = G(x)$ $\therefore$ TWO FUNCTIONS F AND G DIFFERING BY $C(x)$ ARE BOTH THE SUM OF f. ANY FUNCTION F SUCH THAT $\Sigma f = F$ IS CALLED THE INDEFINITE SUM OF f. THE FUNCTION $C(x)$ IS CALLED A PERIODIC CONSTANT. WE HAVE, UPTO A PERIODIC CONSTANT

i) $\Sigma\, \alpha = \dfrac{\alpha x}{h}$

ii) $\Sigma\, x^{(m)} = \dfrac{x^{(m+1)}}{(m+1)h}$ $\quad m \neq -1$

iii) $\Sigma\, (px - q)^{(m)} = \dfrac{(px+q)^{m+1}}{(m+1)\,ph}$ $\quad m \neq -1$

iv) $\Sigma\, b^{x} = \dfrac{b^{x}}{b^{h}-1}$

v) $\Sigma\, e^{rx} = \dfrac{e^{rx}}{e^{rh}-1}$

THE DEFINITE SUM: $\displaystyle\sum_{a}^{a+(m-1)h} f(x) = f(a) + f(a+h) + \cdots + f[a+(m-1)h]$

THE FUNDAMENTAL THM OF THE SUM CALCULUS IS

$$\sum_{a}^{a+(m-1)h} f(x) = F(x)\Big|_{a}^{a+mh}$$

WHERE F IS ANY INDEFINITE SUM OF f.

DIFFERENTIATION AND INTEGRATION OF SUMS

i) IF $\sum f(x) = F(x) \Rightarrow \sum \frac{df}{dx} = \frac{dF}{dx}$

WHERE AGAIN THE RIGHT SIDE IS DETERMINED UPTO A PERIODIC CONSTANT

ii) IF $\sum f(x,\alpha) = F(x,\alpha) \Rightarrow \int_{\alpha_1}^{\alpha_2} \sum f(x,\alpha)\, d\alpha =$

$\sum \int_{\alpha_1}^{\alpha_2} F(x,\alpha)\, d\alpha$

iii) IF $\sum f(x,\alpha) = F(x,\alpha) \Rightarrow \sum \frac{\partial f}{\partial \alpha}(x,\alpha) = \frac{\partial F}{\partial \alpha}(x,\alpha)$

SOME SUMMATION RULES

i) $\sum (f(x) + g(x)) = \sum f(x) + \sum g(x)$

ii) $\sum \alpha f = \alpha \sum f$

iii) $\sum f(x) \Delta g = f(x) g(x) - \sum g(x+h) \Delta f$

(SUMMATION BY PARTS)

IN SUBSCRIPT NOTATION

$$\sum y_k \Delta z_k = y_k z_k - \sum z_{k+1} \Delta y_k$$

THE PERIODIC CONSTANT BECOMES A CONSTANT HERE.

iv) $\sum_{k=1}^{m} y_k = \Delta^{-1} y_k \Big|_1^{m+1}$

v) ABEL'S TRANSFORMATION

$$\sum_{k=1}^{m} u_k v_k = u_{m+1} \sum_{k=1}^{m} v_k - \sum_{k=1}^{m} \left[\Delta u_k \sum_{p=1}^{k} v_p \right]$$

A USEFUL SUMMATION OPERATOR RESULT

IF $\beta \neq 1$, WE HAVE, SINCE $E - 1 = \Delta$

$$\sum \beta^k p(k) = \Delta^{-1}[\beta^k p(k)] = \frac{1}{E-1} \beta^k p(k)$$

$$= \beta^k \frac{1}{\beta E - 1} p(k)$$

$$= \beta^k \frac{1}{\beta(E-1) + \beta - 1} p(k)$$

$$= \frac{\beta^k}{\beta - 1} \frac{1}{1 + [\beta\Delta/(\beta-1)]} p(k)$$

$$\therefore \sum \beta^k P(k) = \frac{\beta^k}{\beta-1}\Big[1 - \frac{\beta\Delta}{\beta-1} + \frac{\beta^2\Delta^2}{(\beta-1)^2} - \frac{\beta^3\Delta^3}{(\beta-1)^3} + \cdots\Big] P(k) + C$$

WHERE C IS AN ARBITRARY CONSTANT. IN THE SECOND STEP, WE HAVE USED THE FOLLOWING RESULT

$$\phi(E)\beta^k P(k) = \beta^k \phi(\beta E) P(k)$$

EXTENSION OF THE DEFN. OF $x^{(m)}$ USING Γ-FUNCTION

IF m IS AN INTEGER, WE CAN SHOW THAT

$$x^{(m)} = \frac{h^m \Gamma(\frac{x}{h}+1)}{\Gamma(\frac{x}{h}-m+1)}$$

SINCE THE RIGHT SIDE HAS A MEANING WHEN m IS NOT AN INTEGER, WE TAKE IT AS THE DEFN. OF $x^{(m)}$ FOR ALL m. WE CAN SHOW THAT

$$\Delta x^{(m)} = mhx^{(m-1)},$$

AND $$\sum x^{(m)} = \frac{x^{(m+1)}}{h(m+1)} \quad \text{FOR } m \neq -1.$$

FOR $m = -1$, WE HAVE

$$\sum x^{(-1)} = \sum \frac{1}{x+h} = \frac{\Gamma'(\frac{x}{h}+1)}{h\Gamma(\frac{x}{h}+1)} = \psi(x)$$ THE DIGAMMA FUNCTION

ACTUALLY, WE SEE THAT

$$\psi(x) = \frac{d}{dx}\ln\Gamma(\frac{x}{h}+1) = D\ln\Gamma(\frac{x}{h}+1)$$

TO PROVE THE LAST RESULT, WE MUST SHOW THAT

$$\Delta\psi(x) = \frac{1}{x+h}$$

APPROXIMATE INTEGRATION

GREGORY-NEWTON FORMULA CAN BE INTEGRATED TERM BY TERM TO GIVE THE FOLLOWING APPROXIMATE INTEGRATION FORMULA :

$$\int_a^{a+mh} f(x)\,dx = mh\Big[f(a) + \frac{m}{2}\Delta f(a) + \frac{m(2m-3)}{12}\Delta^2 f(a) + \frac{m(m-2)^2}{24}\Delta^3 f(a) + \frac{m(6m^3-45m^2+110m-90)}{120}\Delta^4 f(a) + \dots\Big]$$

SPECIAL CASES OF THIS FORMULA CAN BE OBTAINED AS FOLLOWS :

i) TRAPEZOIDAL RULE : LET $m=1$ IN G-N FORMULA

$$\int_a^{a+h} f\,dx = \frac{h}{2}\left[f(a)+f(a+h)\right]$$

ERROR : $R = -\frac{h^3}{12} f''(\xi)$, $\xi \in [a, a+h]$

ii) SIMPSON'S RULE: LET $m=2$ IN G-N FORMULA

$$\int_a^{a+2h} f\,dx = \frac{h}{3}\left[f(a)+4f(a+h)+f(a+2h)\right]$$

ERROR : $R = -\frac{h^5}{90} f^{(iv)}(\xi)$, $\xi \in [a, a+2h]$

iii) SIMPSON'S $\frac{3}{8}$ RULE : TAKE $m=3$ IN G-N FORMULA

$$\int_a^{a+3h} f\,dx = \frac{3h}{8}\left[f_0 + 3f_1 + 3f_2 + f_3\right]$$

WHERE $f_0 = f(a)$, $f_k = f(a+kh)$

ERROR : $R = -\frac{3h^5}{80} f^{(iv)}(\xi)$, $\xi \in [a, a+3h]$

iv) WEDDLE'S RULE : TAKE $m = 6$ IN G-N FORMULA

$$\int_a^{a+6h} f\,dx = \frac{3h}{10}\left[(f_0+f_2+f_4+f_6)+5(f_1+f_5)+6f_3\right]$$

THIS IS SLIGHTLY DIFFERENT FROM THE RESULT FROM G-N FORMULA FOR $m=6$. THE COEFFICIENT OF $\Delta^6 f(a)$ WHICH IS $\frac{41}{140}$ IS REPLACED BY $\frac{42}{140} = \frac{3}{10}$

ERROR : $R = -\left[\frac{h\Delta^6 f(a)}{140} + \frac{9h^9}{1400} f^{(8)}(\xi)\right], \xi \in (a, a+6h)$

FROM THE RELATION $e^{hD} = E$, WE GET

$D = \frac{1}{h}\ln E$ AND

$$\int_a^{a+mh} f\,dx = D^{-1} f(x)\Big|_a^{a+mh}$$

$$= \frac{h}{\ln E} f(x)\Big|_a^{a+mh}$$

$$= \frac{h[f(a+mh) - f(a)]}{\ln E}$$

$$= h\left(\frac{E^m - 1}{\ln E}\right) f(a).$$

WE ALSO HAVE $E = 1+\Delta$, $E^{-1} = 1-\nabla$ (. . .)

$\ln E = \ln(1+\Delta) = -\ln(1-\nabla)$

THE PREVIOUS FORMULA FOR INTEGRATION BECOMES

$$\int_a^{a+mh} f\,dx = h\frac{1}{\ln E}(f_m - f_0)$$

$$= h\left[\frac{-1}{\ln(1-\nabla)} f_m - \frac{1}{\ln(1+\Delta)} f_0\right]$$

EXPANDING THE TWO LOGARITHMIC TERMS, WE GET GREGORY'S FORMULA FOR APPROXIMATE INTEGRATION :

$$\int_a^{a+mh} f\,dx = \frac{h}{2}[f_0 + 2f_1 + 2f_2 + \cdots + 2f_{m-1} + f_m]$$

$$-\frac{h}{12}[\nabla f_m - \Delta f_0] - \frac{h}{24}[\nabla^2 f_m + \Delta^2 f_0]$$

$$-\frac{19h}{720}[\nabla^3 f_m - \Delta^3 f_0] - \frac{3h}{160}[\nabla^4 f_m + \Delta^4 f_0]$$

$$- \cdots$$

ANOTHER IMPORTANT RESULT IS THE EULER-MAC-LAURAN FORMULA:

$$\int_a^{a+mh} f\,dx = \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + \cdots + 2f_{m-1} + f_m\right]$$
$$- \frac{h^2}{2}\left[f'_m - f'_0\right] + \frac{h^4}{720}\left[f'''_m - f'''_0\right]$$
$$- \frac{h^6}{30{,}240}\left[f^{(v)}_m - f^{(v)}_0\right] + \cdots$$

WHICH CAN ALSO BE WRITTEN AS

$$\sum_{k=0}^{m} f_k = \frac{1}{h}\int_a^{a+mh} f\,dx + \frac{1}{2}\left[f_m + f_0\right]$$
$$+ \frac{h}{12}\left[f'_m - f'_0\right] - \frac{h^3}{720}\left[f'''_m - f'''_0\right] + \cdots$$

THE ERROR CAN BE WRITTEN IN TERMS OF BERNOULLI NUMBERS.

* DIFFERENCE EQUATIONS

A DIFFERENCE EQ. IS A RELATION OF THE FORM

$$F\left(x, y, \frac{\Delta y}{\Delta x}, \frac{\Delta^2 y}{\Delta x^2}, \ldots, \frac{\Delta^n y}{\Delta x^n}\right) = 0$$

OR $$G\left(x, f(x), f(x+h), \ldots, f(x+nh)\right) = 0$$

WHERE $y = f(x)$ IS THE UNKNOWN FUNCTION. THE ORDER OF THE DIFFERENCE EQUATION IS $n - m$ WHERE n AND m ARE THE LARGEST AND SMALLEST COEFFICIENTS OF h IN THE SECOND FORM ABOVE. FOR EXAMPLE, IN THE ABOVE CASE $m = 0$ AND THE ORDER OF THE DIFFERENCE EQ. IS n.

THE SOLUTION OF A DIFFERENCE EQ. IS ANY FN. SATISFYING THE EQ. A GENERAL SOLUTION OF A DIFFERENCE EQ. OF ORDER n IS ANY SOLUTION INVOLVING n ARBITRARY PERIODIC CONSTANT. A PARTICULAR SOLUTION IS A SOLUTION OBTAINED BY ASSIGNING PARTICULAR PERIODIC CONSTANTS IN THE GENERAL SOLUTION. WE NEED n INDEPENDENT BOUNDARY CONDITIONS FOR THE UNKNOWN FN. y DESCRIBED BY A DIFFERENCE EQ. OF ORDER n.

USING SUBSCRIPT NOTATION WITH $x = a + kh$, $y_k = f(x + kh)$, THE DIFFERENCE EQ. OF ORDER n CAN BE WRITTEN AS $$H(k, y_k, y_{k+1}, \ldots y_{k+n}) = 0,$$

OR $$H(k, y_k, Ey_k, \ldots E^n y_k) = 0.$$

LINEAR DIFFERENCE EQS.

A LIN. DIFFERENCE EQ. OF ORDER n IS OF THE FORM

$$[a_0(k)E^n + a_1(k)E^{n-1} + \ldots + a_n(k)]\, y_k = R(k)$$

WHERE $a_0(k) \neq 0$. WE CAN WRITE THE LEFT SIDE AS $\phi(E)\, y_k$ WHERE $\phi(E) = \sum_{i=0}^{n} a_i(k) E^{n-i}$ IS A LINEAR OPERATOR. LINEAR DIFFERENCE EQS. WITH CONSTANT

COEFFICIENTS ARE RELATIVELY EASY TO SOLVE.

HOMOGENEOUS LIN. DIFF. EQS. WITH CONSTANT COEFFS.

ASSUME $y_k = r^k \Rightarrow$

$$\phi(E) r^k = (a_0 r^n + a_1 r^{n-1} + \dots + a_n) r^k = 0$$
$$= \phi(r) \cdot r^k$$

$$\therefore \quad \phi(E) = a_0 (E - r_1)(E - r_2) \dots (E - r_n)$$

WHERE $r_1, r_2, \dots r_n$ ARE THE ROOTS OF THE AUXILIARY EQ. $\phi(r) = 0$. WE HAVE THE FOLLOWING CASES

i) ROOTS ARE ALL REAL

$$y_k = C_1 r_1^k + C_2 r_2^k + \dots + C_n r_n^k$$

WHERE C_i'S ARE ARBITRARY PERIODIC CONSTANTS

ii) SOME ROOTS ARE COMPLEX

CORRESPONDING TO ROOTS $\alpha \pm \beta i$, THE FOLLOWING IS A SOLUTION

$$y_k = \rho^k (C_1 \cos k\theta + C_2 \sin k\theta)$$

WHERE $\rho = (\alpha^2 + \beta^2)^{1/2}$, $\theta = \tan^{-1}(\beta/\alpha)$

iii) REPEATED ROOTS.

IF THE ROOT $r = \alpha$ IS A MULTIPLE ROOT OF $\phi(r) = 0$ OF ORDER j, THEN CORRESPONDING TO THIS ROOT

$$y_k = (C_1 + C_2 k + \dots C_j k^{j-1}) \alpha^k$$

LINEARLY INDEPENDENT SOLUTIONS

THE SET OF n FNS $f_1(k), \dots f_n(k)$ IS LINEARLY DEPENDENT IF A SET OF n CONSTANTS $A_1, \dots A_n$ CAN BE FOUND, NOT ALL ZERO, SUCH THAT

$$A_1 f_1(k) + \dots + A_n f_n(k) = 0.$$

THM: A SET OF n FNS $f_1(k), \dots, f_n(k)$ ARE LINEARLY INDEPENDENT IF THE DETERMINANT

$$\begin{vmatrix} f_1(0) & f_2(0) & \cdots & f_m(0) \\ f_1(1) & f_2(1) & \cdots & f_m(1) \\ \vdots & \vdots & & \vdots \\ f_1(m-1) & f_2(m-1) & \cdots & f_m(m-1) \end{vmatrix} \neq 0$$

OTHERWISE THE FNS ARE LINEARLY DEPENDENT. THIS DETERMINANT IS CALLED THE CASORATI AND IS ANALOGOUS TO WRONSKIAN FOR DIFFERENTIAL EQS.

THM: FOR HOM. LIN. DIFFERENCE EQ. $\phi(E)y_k = 0$, IF $f_1(k), \ldots, f_m(k)$ ARE m LINEARLY INDEPENDENT SOLUTIONS, THEN THE GENERAL SOLUTION IS

$$y_k = c_1 f_1(k) + \cdots + c_m f_m(k)$$

(NOTE $\phi(E)$ DOES NOT NECESSARILY HAVE CONSTANT COEFFS.)

SOLUTION OF INHOMOGENEOUS EQ. $\phi(E)y_k = R(k)$

FOR THIS EQ., THE GENERAL SOLUTION OF $\phi(E)y_k = 0$ IS CALLED THE COMPLEMENTARY FUNCTION OR SOLUTION $Y_c(k)$. ANY SOLUTION WHICH SATISFIES THE COMPLETE EQ., IS CALLED A PARTICULAR SOLUTION $Y_p(k)$.

THM: THE GENERAL SOLUTION OF LIN. INHOM. EQ $\phi(E)y_k = R_k$ IS $y_k = Y_c(k) + Y_p(k)$.

METHODS FOR FINDING PARTICULAR SOLUTIONS

THERE ARE MANY METHODS TO FIND P.S. HERE ARE SOME IMPORTANT METHODS FOR EQS. WITH CONST. COEFFS.

1) METHOD OF UNDETERMINED COEFFS.

DEPENDING ON THE FORM OF $R(k)$, THE PARTICULAR SOLUTION CAN BE GUESSED EASILY EXCEPT FOR SOME

COEFFICIENTS WHICH ARE DETERMINED BY SUBSTITUTION IN THE EQ. HERE ARE SOME CASES

i) IF β^K IS A TERM IN $R(K)$, TRY $A\beta^K$

ii) IF $\sin \alpha K$ OR $\cos \alpha K$ ARE TERMS IN $R(K)$, TRY $A \cos \alpha K + B \sin \alpha K$

iii) IF $R(K)$ IS A POLYNOMIAL OF DEGREE m TRY $A_0 K^m + A_1 K^{m-1} + \ldots + A_m$. IF IT HAPPENS THAT $\sum_{i=0}^{m} a_i = 0$ IN $\phi(E) = \sum_{i=0}^{m} a_i E^i$, THEN $y_K = \text{CONST.}$ IS A SOLUTION OF THE HOM. EQ. $\phi(E) y_K = 0$ AND NOW TRY

$$A_0 K^{m+1} + A_1 K^m \ldots + A_m K$$

iv) IF $R(K) = \beta^K P(K)$, TRY

$$\beta^K [A_0 K^m + A_1 K^{m-1} + \ldots A_m]$$

v) IF $\beta^K \sin \alpha K$ OR $\beta^K \cos \alpha K$ ARE TERMS IN $R(K)$, TRY $\beta^K (A \cos \alpha K + B \sin \alpha K)$

NO TERM OF TRIAL SOLUTION CAN APPEAR IN COMPLEMENTARY FUNCTION. IF THIS HAPPENS, THEN THE ENTIRE TRIAL SOLUTION MUST BE MULTIPLIED BY AN INTEGER POWER OF K, LARGE ENOUGH SO THAT NO TERMS OF THE NEW TRIAL SOLUTION IS IN THE COMPLEMENTARY FN.

2) <u>OPERATOR TECHNIQUE</u>

FORMALLY, IF $\phi(E) y_K = R(K)$, WE HAVE $y_K = \frac{1}{\phi(E)} R(K)$. THE OPERATOR $\psi(E) = 1/\phi(E)$ IS THE INVERSE OF $\phi(E)$ AND IS A LINEAR OPERATOR. WE CAN SHOW THAT

i) $\frac{1}{\phi(E)} \beta^K = \frac{1}{\phi(\beta)} \beta^K$.

ii) $\frac{1}{\phi(E)} p(k) = \frac{1}{\phi(1+\Delta)} p(k)$

$= (b_0 + b_1\Delta + \dots + b_m\Delta^m + \dots)\, p(k)$

FOR $p(k)$ A POLYNOMIAL OF ORDER m, THE EXPANSION SHOULD BE CARRIED OUT UPTO Δ^m SINCE $\Delta^{m+1} p(k) = 0$.

iii) $\frac{1}{\phi(E)} \beta^k p(k) = \beta^k \frac{1}{\phi(\beta E)} p(k)$ THEN USE THE RESULT IN (ii)

iv) FOR $\frac{1}{\phi(E)} \sin\alpha k$ OR $\frac{1}{\phi(E)} \cos\alpha k$ USE COMPLEX NOTATION FOR $\sin\alpha k$ AND $\cos\alpha k$ THEN USE (i).

3) VARIATION OF PARAMETERS TECHNIQUE

FIND COMPLEMENTARY SOLUTION OF $\phi(E) y_k = R(k)$, SAY $y_k = C_1 u_1 + \dots C_m u_m$, THEN REPLACE CONSTANTS $C_1, C_2, \dots, C_m$ BY FUNCTIONS OF k, $K_1, K_2, \dots K_m$: $y_k = K_1 u_1 + \dots K_m u_m$. WE NEED m CONDITIONS TO DETERMINE $K_1, \dots K_m$. ONE OF THESE IS THAT y_k MUST SATISFY THE ORIGINAL DIFFERENCE EQ. FOR THE REMAINING $m-1$ CONDITIONS, THE FIRST $(m-1)$ EQS. ARE IMPOSED BELOW:

$$\begin{cases} u_1\Delta K_1 + u_2\Delta K_2 + \dots + u_m\Delta K_m = 0 \\ \Delta u_1\Delta K_1 + \Delta u_2\Delta K_2 + \dots + \Delta u_m \Delta K_m = 0 \\ \vdots \\ \Delta^{m-2}u_1\Delta K_1 + \Delta^{m-2}u_2\Delta K_2 + \dots + \Delta^{m-2}u_m\Delta K_m = 0 \\ \Delta^{m-1}u_1\Delta K_1 + \Delta^{m-1}u_2\Delta K_2 + \dots + \Delta^{m-1}u_m\Delta K_m = R(k-1)/a_0 \end{cases}$$

THE EQS ARE OBTAINED BY APPLYING OPERATOR Δ REPEATEDLY TO BOTH SIDE $y_k = K_1 u_1 + \dots K_m u_m$ AND SETTING THE ABOVE FIRST $(m-1)$ EQS. TO ZERO TO GET $\Delta^j y_k = \sum_i K_i \cdot \Delta^j u_i$, $j = 1, 2, \dots, m-2$

$$\Delta^{m-1} y_k = \sum_{i=1}^{m} K_i \Delta^{m-1} u_i + \underbrace{\sum_{i=1}^{m} \Delta K_i \Delta^{m-2} u_i}_{=0} + \underbrace{\sum_{i=1}^{m} \Delta K_i \Delta^{m-1} u_i}_{R(k-1)/a_0}$$

WHERE $\phi(E) = a_0 E^n + a_1 E^{n-1} + \cdots + a_n$. NOW

LET $\psi(\Delta) = \phi(\nabla+1) = \underset{\overset{\|}{a_0}}{b_0}\Delta^n + b_1\Delta^{n-1} + \cdots + b_n$.

THEN

$$\Delta^n y_k = \sum_{i=1}^{n} K_i \Delta^n u_i + \underbrace{\sum \Delta^{n-1} u_i \Delta K_i}_{R(K-1)/a_0}$$

$$+ \sum_{i=1}^{n} \Delta^n u_i \Delta K_i + \frac{\Delta R(K-1)}{a_0}$$

$$a_0 \Delta^n y_k = \sum_{i=1}^{n} a_0 K_i \Delta^n u_i + \underbrace{R(K-1) + \Delta R(K-1)}_{R(K)}$$

$$- \sum \left[b_1 \Delta^{n-1} u_i + b_2 \Delta^{n-2} u_i + \cdots + b_n \right] \Delta K_i$$

$$= \sum_{i=1}^{n} a_0 K_i \Delta^n u_i + R(K) - b_1 \frac{R(K-1)}{a_0}$$

WE HAVE USED THE FACT THAT $\psi(\Delta) u_i = 0$ AND THUS $a_0 \Delta^n u_i = -\left[b_1 \Delta^{n-1} u_i + \cdots + b_n \right]$. THEN THE EQUATIONS IN THE PREVIOUS PAGE WERE USED TO GET THE LAST TERM. NOW

$$\psi(\Delta) y_k = \sum_{j=0}^{n} \sum_{i=1}^{n} b_j K_i \Delta^{n-j} u_i + R(K) - b_1 \frac{R(K-1)}{a_0}$$

$$+ b_1 \frac{R(K-1)}{a_0} = \sum_{i=1}^{n} K_i \psi(\Delta) u_i + R(K)$$

THIS TERM COMES FROM $\Delta^{n-1} y_k$

$$= R(K) \quad \text{Q.E.D.}$$

4) <u>METHOD OF REDUCTION OF ORDER</u>

IF $\phi(E) y_k = a_0 (E - r_1)(E - r_2) \cdots (E - r_n) y_k = R(K)$, WE CAN WRITE $Z_k = a_0 (E - r_2)(E - r_3) \cdots (E - r_n)$ AND WE MUST SOLVE FOR Z_k FROM

$$(E - r_1) Z_k = R(K)$$

OR

$$Z_k = \frac{1}{E - r_1} R(K)$$

NOW WE USE OPERATOR METHOD AS FOLLOWS

$$Z_K = \frac{1}{E-r_1}\left[r_1^K\left(\frac{R(K)}{r_1^K}\right)\right]$$

$$= r_1^K \frac{1}{\underbrace{(E-1)}_{\Delta} r_1}\left[\frac{R(K)}{r_1^K}\right]$$

$$= r_1^{K-1}\Delta^{-1}\frac{R(K)}{r_1^K}$$

$$= r_1^{K-1}\sum_{P=1}^{K-1}\frac{R(P)}{r_1^P} + C_1 r_1^K$$

WHERE C_1 IS A PERIODIC CONSTANT. THIS PROCESS SHOULD BE CONTINUED TO GET A PARTICULAR SOLUTION OF $\phi(E)\,y_K = R(K)$.

5) METHOD OF GENERATING FUNCTIONS

THE GENERATING FN. FOR y_K IS DEFINED AS

$$G(t) = \sum_{K=0}^{\infty} y_K t^K$$

THIS FN. CAN BE USED TO SOLVE LINEAR DIFFERENCE EQS. AS FOLLOWS. LET THE DIFFERENCE EQ. BE

$$y_{K+2} - 3y_{K+1} + 2y_K = 0,\quad y_0 = 2,\quad y_1 = 3$$

TO FIND y_K, MULTIPLY THE EQ. BY t^K AND SUM FROM $K=0$ TO ∞:

$$(y_2 + y_3 t + y_4 t^2 \ldots) - 3(y_1 + y_2 t + y_3 t^2 + \ldots) + 2(y_0 + y_1 t + y_2 t^2 + \ldots) = 0$$

OR IN TERMS OF GENERATING FN. $G(t)$

$$\frac{G(t) - y_0 - y_1 t}{t^2} - 3\left(\frac{G(t) - y_0}{t}\right) + 2G(t) = 0$$

PUTTING $y_0 = 2$ AND $y_1 = 3$ IN THIS EQ., WE GET

$$G(t) = \frac{2-3t}{1-3t+2t^2} = \frac{1}{1-t} + \frac{1}{1-2t}$$

$$G(t) = \sum_{K=0}^{\infty} (1+2^K)\, t^K$$ {BY EXPANDING THE RIGHT SIDE OF PREVIOUS EQ.

$$= \sum_{K=0}^{\infty} y_K t^K \quad \text{BY DEFN.}$$

$$\therefore \quad y_K = 1 + 2^K$$

LINEAR DIFFERENCE EQS. WITH VARIABLE COEFFICIENTS

ANY FIRST ORDER LIN. EQ. WITH VARIABLE COEFFICIENTS CAN BE SOLVED AS FOLLOWS. WRITE THE EQUATION IN THE FORM

$$y_{K+1} - A_K y_K = R_K \qquad (*)$$

OR $$(E - A_K)\, y_K = R_K$$

LET $\rho_K = A_1 A_2 \ldots A_{K-1}$, THEN $\rho_{K+1} = \rho_K A_K$

MULTIPLY BOTH SIDES OF $(*)$ BY $1/\rho_{K+1}$ TO GET

$$\frac{y_{K+1}}{\rho_{K+1}} - \frac{y_K}{\rho_K} = \Delta\left(\frac{y_K}{\rho_K}\right) = \frac{R_K}{\rho_{K+1}}$$

$$\therefore \quad y_K = C\rho_K + \rho_K \Delta^{-1}\left(\frac{R_K}{\rho_{K+1}}\right)$$

$$= C\rho_K + \rho_K \sum_{P=1}^{K-1} \left(\frac{R_P}{\rho_{P+1}}\right)$$

FOR HIGHER ORDER LINEAR EQS. WITH VARIABLE COEFFICIENTS, THE FOLLOWING METHODS ARE AVAILABLE FOR FINDING A SOLUTION:

1) FACTORIZATION OF THE OPERATOR: $\phi(E) y_K = (E - A_K) \times (E - B_K) \ldots (E - U_K)\, y_K = R_K$. THEN USE THE METHOD OF REDUCTION OF ORDER APPLYING THE ABOVE RESULT IN EACH STEP.

2) VARIATION OF PARAMETERS. IF THE COMPLEMENTARY

SOLUTION IS KNOWN, THIS METHOD CAN BE TRIED

3) GENERATING FUNCTIONS. THE METHOD IS SIMILAR TO THE CASE OF EQS. WITH CONSTANT COEFFICIENTS. THIS TIME A DIFFERENTIAL EQ. FOR $G(t)$ IS OBTAINED WHICH MUST BE SOLVED. THIS CAN BE SEEN AS FOLLOWS: $G(t) = \sum_0^\infty y_K t^K$, $G'(t) = \sum_0^\infty K y_K t^{K-1}$

NOW A TERM SUCH AS $(K-1) y_{K+2}$ IN A DIFFERENCE EQ. CAN BE WRITTEN AS (AFTER MULTIPLYING BY t^K AND SUMMATION)

$$\sum_{K=0}^\infty (K-1) y_{K+2} t^K = \sum_{K=0}^\infty \left[(K+2) y_{K+2} \frac{t^{K+1}}{t} - 3 y_{K+2} \frac{t^{K+2}}{t^2} \right]$$

$$= \frac{G'(t) - y_1}{t} - 3 \frac{G(t) - y_0 - y_1 t}{t^2}$$

4) SERIES SOLUTION: WRITE $\phi(E) = \phi(\Delta + 1) = \psi(\Delta)$

THEN ASSUME A SOLUTION OF THE FORM

$$y_K = \sum_0^\infty C_p K_1^{(P)}$$

WHERE AS BEFORE

$$K_1^{P} = K(K-1) \cdots (K-P+1)$$

WE USE THE RESULT THAT $\Delta y_K = \sum_0^\infty p C_p K_1^{(P-1)}$. ANY TERM SUCH AS $K K_1^{(m)}$ CAN BE WRITTEN AS

$$K K_1^{(m)} = K_1^{(m+1)} + m K_1^{(m)}$$

THE REST OF THE METHOD WHICH INVOLVES FINDING C_p'S IS SIMILAR TO SERIES SOLUTION OF ORDINARY DIFFERENTIAL EQS.

NONLINEAR DIFFERENCE EQUATIONS

TRY SOME TRANSFORMATION TO CHANGE THE EQ. INTO A LINEAR DIFFERENCE EQ. SOME EXAMPLES FOLLOW.

FOR $y_{k+1} - y_k + k\, y_{k+1} y_k = 0$, $y_1 = 2$ DIVIDE THE EQ. BY $y_k y_{k+1}$ TO GET

$$\frac{1}{y_{k+1}} - \frac{1}{y_k} = k$$

NOW LET $v_k = 1/y_k$, WE GET

$$\Delta v_k = k \qquad v_1 = 1/2$$

$$v_k = \Delta^{-1} k$$
$$= \frac{k^{(2)}}{2} + c_1$$
$$= \frac{k(k-1)}{2} + c_1$$

WE GET $c_1 = \frac{1}{2}$ $\therefore$ $v_k = \dfrac{k^2+k+1}{2}$ AND THUS

$$y_k = \frac{2}{k^2-k+1}$$

FOR $y_{k+2}\, y_k^{\,2} = y_{k+1}^{\,3}$, $y_1 = 1$, $y_2 = 2$, WE TAKE LOGARITHM OF BOTH SIDES OF THE EQ. TO GET

$$\ln y_{k+2} - 3 \ln y_{k+1} + 2 \ln y_k = 0$$

NOW LET $v_k = \ln y_k \Rightarrow v_{k+2} - 3v_{k+1} + 2v_k = 0$, $v_1 = 0$, $v_2 = \ln 2$. THE GENERAL SOLUTION OF THIS DIFFERENCE EQ. IS $v_k = c_1 + c_2 2^k$; WE FIND $c_1 = -\ln 2$, $c_2 = \frac{1}{2}\ln 2$. WE FINALLY GET

$$v_k = \ln y$$
$$= \ln 2\,(2^{k-1} - 1)$$

OR $\qquad y_k = 2^{(2^{k-1}-1)}$

SIMULTANEOUS DIFFERENCE EQUATIONS

USE A PROCEDURE TO GET A SINGLE DIFFERENCE EQ. FOR ONE OF THE UNKNOWNS. THEN USE THE METHODS DISCUSSED BEFORE. OPERATOR TECHNIQUE IS AN IDEAL TOOL FOR EQUATIONS WITH CONSTANT COEFFICIENTS. FOR EXAMPLE TO SOLVE THE SYSTEM

$$\begin{cases} y_{k+1} + z_k - 3y_k = k \\ 3y_k + z_{k+1} - 5z_k = 4^k \end{cases} ; \quad \begin{matrix} y_1 = 2 \\ z_1 = 0 \end{matrix}$$

WRITE

$$\begin{cases} (E-3)\,y_k + z_k = k \\ 3\,y_k + (E-5)\,z_k = 4^k \end{cases}$$

THE EQ. FOR y_k BECOMES

$$(E^2 - 8E + 12)\,y_k = 1 - 4k - 4^k$$

WHICH CAN EASILY BE SOLVED. THEN WE USE THE EQ. $z_k = k + 3y_k - y_{k+1}$ TO FIND z_k.

MIXED DIFFERENCE EQUATIONS

IN SOME EQS., DERIVATIVES AND INTEGRALS IN ADDITION TO DIFFERENCES ALSO APPEAR. THERE IS NO GENERAL METHOD OF ATTACK. SPECIAL TECHNIQUES MUST BE USED TO SOLVE THESE PROBLEMS. FOR DIFFERENTIAL-DIFFERENCE EQUATIONS, THE METHOD OF GENERATING FUNCTIONS IS A USEFUL TECHNIQUE TO CONSIDER.

THESE NOTES WERE PREPARED FROM M.R. SPIEGEL: " CALCULUS OF FINITE DIFFERENCES AND DIFFERENCE EQUATIONS" SCHAUM'S OUTLINE SERIES, 1971

* MIKUSIŃSKI'S OPERATIONAL CALCULUS FOR NUMBER SEQUENCES

* PRELIMINARIES — A RING HAS TWO OPERATIONS: ADDITION AND MULTIPLICATION. IT HAS THE FOLLOWING PROPERTIES:

i) IT IS A COMMUTATIVE (ABELEAN) GROUP UNDER ADDITION

ii) MULTIPLICATION IS ASSOCIATIVE (A GROUPOID)

iii) DISTRIBUTIVE LAW $a(b+c) = ab + ac$ IS VALID

IF IN ADDITION MULTIPLICATION IS COMMUTATIVE, THE RING IS CALLED A COMMUTATIVE RING.

IF $ab = 0$ BUT $a \neq 0$ AND $b \neq 0$, THESE ELEMENTS ARE CALLED THE ZERO DIVISORS OF THE RING.

A COMMUTATIVE RING WITH NO ZERO DIVISORS IS CALLED AN INTEGRAL DOMAIN. CANCELATION LAW HOLDS FOR INTEGRAL DOMAINS: $ab = ac$, $a \neq 0 \Rightarrow a(b-c) = 0$. SINCE THE INTEGRAL DOMAIN HAS NO ZERO DIVISORS AND $a \neq 0 \Rightarrow b - c = 0$ OR $b = c$.

FOR EVERY INTEGRAL DOMAIN, THERE IS A FIELD CONTAINING IT. THIS FIELD IS CALLED THE QUOTIENT FIELD OF THE INT. DOMAIN.

PROOF: R INTEG. DOMAIN, CONSIDER THE SET

$$K = \{(a,b) \mid a \in R, b \in R, b \neq 0\}$$

WITH THE FOLLOWING RULES:

EQUALITY: $(a,b) = (c,d)$ IFF $ad = bc$

[THE EQUIVALENCE PROPERTIES OF SRT, SYMMETRY, REF-

LEXIVITY AND TRANSITIVITY HOLD]

ADDITION : $(a,b)+(c,d)=(ad+bc,bd)$
THIS RULE IS $\ni$ IF $(a,b)=(a',b')$; $(c,d)=(c',d')$
$\Rightarrow (ad+bc,bd)=(a'd'+b'c',b'd')$. WE ALSO
NOTE THAT SINCE $b\neq 0, d\neq 0 \Rightarrow bd\neq 0$ AND
$(ad+bc,bd)\in K$.

MULTIPLICATION : $(a,b)(c,d)=(ac,bd)\in K$.
AGAIN IF $(a,b)=(a',b')$; $(c,d)=(c',d') \Rightarrow (ac,bd)=$
$(a'c',b'd')$.

ZERO ELEMENT : $(0,a)$, $a\neq 0$ IS THE ZERO OF K

THE UNIT ELEMENT : (a,a), $a\neq 0$ IS THE UNIT ELEMENT OF K.

TO PROVE K IS A FIELD WE MUST SHOW THAT
$$(a,b)\,x=(c,d)\quad , a\neq 0$$
HAS A UNIQUE SOLUTION. THIS SOLUTION IS
$$x=(bc,ad)$$
TO PROVE THAT THE SOLUTION IS UNIQUE, LET
x_1 AND x_2 BE TWO SOLUTIONS $\Rightarrow (a,b)(x_1-x_2)=0$
LET $x_1-x_2=(\xi,\eta)$, WE HAVE
$$(a\xi,b\eta)=0 \quad \text{i.e. } a\xi=0$$
BUT SINCE $a\neq 0$ AND R IS AN INTEGRAL DOMAIN
$\Rightarrow \xi=0$ AND $x_1-x_2=0$ i.e. $x_1=x_2$.

WE HAVE SHOWN THAT K IS A FIELD. WE MUST
SHOW THAT $R\subset K$. THIS IS DONE BY IDENTIFYING

THE ELEMENT $(ac, a) \in K$, $a \neq 0$, WITH ELEMENT c OF R. WE CAN NOW DENOTE (a, b) AS a/b AND ALL KNOWN RULES OF CALCULATION FOR FRACTIONS BECOME SIMILAR TO THOSE IN K, e.g.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}, \text{ ETC.}$$

* THE OPERATIONAL CALCULUS OF NUMBER SEQUENCES

CONSIDER THE SET OF ALL SEQUENCES OF NUMBERS $\{a_m\}$. WE USE THE NOTATION $a(m)$, $b(m)$ FOR THE WHOLE SEQUENCES $\{a_n\}$ AND $\{b_n\}$ AS SINGLE MATHEMATICAL OBJECTS IN THIS SET. SOMETIMES WHEN THERE IS NO CONFUSION, WE USE a AND b AS ELEMENTS OF THIS SET. ALSO WE USE SEQUENCES SUCH AS $\{2^m\}$ AS AN ELEMENT OF THIS SET IN SOME OF THE EXPRESSIONS BELOW.

WE NOW FORM THE RING R FROM THE SET OF NUMBER SEQUENCES AS FOLLOWS:

i) ADDITION IN R: $a(m) + b(m) = c(m)$ WHERE

$$c(m) = \{a_m + b_m\} \quad \forall m = 0, 1, 2, \ldots$$

ii) MULTIPLICATION IN R: WE INTRODUCE CONVOLUTION OPERATION $a(m) * b(m)$:

$$a(m) * b(m) = \left\{ \sum_{j=0}^{m} a_{m-j}\, b_j \right\} = b(m) * a(m)$$

iii) $a(m) = b(m)$ IFF $a_m = b_m$

IF c IS A CONSTANT, THEN WE USE $c\,a(m)$ TO DENOTE THE SEQ. $\{c\,a_m\}$

IT IS EASY TO PROVE THAT R IS A COMMUTATIVE RING. NOW WE PROVE THAT R IS AN INTEGRAL DOMAIN. IF $a(n) * b(n) = 0$ AND a_p AND b_q ARE THE FIRST ELEMENTS OF a AND b WHICH ARE NOT ZERO, THEN THE $(p+q+1)$TH TERM OF $a*b$ IS $\sum_{k=0}^{p+q} a_{p+q-k}\, b_k = a_p b_q = 0$. SINCE THE SET OF NUMBERS IS AN INTEGRAL DOMAIN ITSELF $\Rightarrow$ EITHER a_p OR b_q IS ZERO, CONTRADICTION. THEREFORE EITHER $a(n) = 0$ OR $b(n) = 0$.

WE USE $a^2 = a^2(n)$ TO DENOTE $a*a$. BY INDUCTION, WE DEFINE $a^k = \underbrace{a*a*\cdots*a}_{k \text{ TIMES}}$.

SOME USEFUL RESULTS

i) THE INTEGRAL DOMAIN R HAS THE FOLLOWING SEQ. AS THE UNIT ELEMENT.

$$\delta = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}$$

WE HAVE $\forall a \in R$, $a * \delta = \delta * a = \{a_n\} = a$

ii) LET $p = \{0, 1, 0, 0, \ldots\}$ THIS SEQ. HAS THE FOLLOWING INTERESTING PROPERTY:

$$p^k = \begin{cases} 1 & n = k \\ 0 & n \neq 0 \end{cases}$$

i.e. p^k IS THE KRONECKER SEQ. $\{\delta_{kn}\}$

iii) LET $\ell = \{1, 1, 1, \ldots\}$, WE HAVE

$\ell^2 = \{n+1\} = \{1, 2, 3, \ldots\}$ AND IN GENERAL

$\ell^k = \left\{\binom{n+k-1}{k-1}\right\}$ WHERE $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

ALSO WE HAVE, IF $a(m) = u^m$

$$\ell * a = \{1 + u + \dots u^m\} = \left\{\frac{u^{m+1} - 1}{u - 1}\right\}$$

WE HAVE THE FOLLOWING INTERESTING RELATIONSHIP

$$\ell = \sum_{K=1}^{\infty} p^K + \delta$$

iv) THE SHIFT OPERATION: WE HAVE

$$p^K * a(m) = \begin{cases} 0 & m < K \\ a_{m-K} & m \geq K \end{cases} \quad \text{FORWARD SHIFT}$$

FOR EXAMPLE

$$p * a(m) = \{0, a_0, a_1, \dots\}$$
$$p^3 * a(m) = \{0, 0, 0, a_0, a_1, \dots\}$$

THE QUOTIENT FIELD OF R

SINCE R IS AN INTEGRAL DOMAIN, WE CAN DEFINE INVERSE OF ALL ELEMENTS OF R EXCEPT THE ZERO ELEMENT $\{0\}$ WHICH WE SIMPLY WRITE AS 0.
IF a AND $b \in R$, $b \neq 0$, WE HAVE

$$\frac{a}{b} = c \quad \text{IF} \quad a = b * c.$$

WE DESIGNATE THIS FIELD BY Q. WE HAVE $R \subset Q$.
THE UNIT ELEMENT OF THE Q. FIELD OF R IS AGAIN $\delta = a/a$, $a \neq 0$, $a \in R$. ELEMENTS OF Q WILL BE CALLED OPERATORS.
NOW WE CAN DEFINE $p^{-\nu} = \delta / p^{\nu}$ WITH $p^0 = \delta$.
THE RESULT $p^{\nu} * p^{\mu} = p^{\nu+\mu}$ HOLDS FOR ALL INTEGERS μ AND ν.

IF $a \in R$ AND $a_0 = a_1 = \dots a_{n-\nu} = 0$, THEN $p^{-\nu} * a(m) \in R$ AND $p^{\nu} * a(m) = a(m+\nu)$

HOWEVER, IN GENERAL $p^{-\nu} * a \in Q$ AND NOT R.
WE WRITE

$$\phi(n) = a(n) - a_0 \delta - a_1 p - a_2 p^2 \ldots - a_{\nu-1} p^{\nu-1}$$

$$p^{-\nu} * \phi = p^{-\nu} * a - a_0 p^{-\nu} - a_1 p_1^{-1} \ldots \ldots a_{\nu-1} p^{-1}$$

$$= a(n+\nu)$$

$$= \{a_\nu, a_{\nu+1}, a_{\nu+2}, \ldots \ldots\}$$

THIS IS THE BACKWARD SHIFT THM. WE NOTE THAT $\phi_0 = \phi_1 = \ldots \ldots = \phi_{\nu-1} = 0$.

IMPORTANT SPECIAL CASES

SOME INTERESTING QUOTIENTS IN Q ARE ACTUALLY ELEMENTS OF R. WE CONSIDER $\{e^{\alpha n}\}$. WE NOTE

$$\{e^{\alpha(n+1)}\} = e^\alpha \{e^{\alpha n}\}$$

$$= p^{-1}\{e^{\alpha n}\} - p^{-1}$$

OR

$$e^\alpha p * \{e^{\alpha n}\} = \{e^{\alpha n}\} - \delta$$

$$\Rightarrow \quad \{e^{\alpha n}\} = \frac{\delta}{\delta - e^\alpha p} \in R$$

NOW LET $\alpha = 0$

$$\ell = \frac{\delta}{\delta - p}$$

AND

$$\ell^k = \frac{\delta}{(\delta - p)^k} = \left\{\binom{n+k-1}{k-1}\right\} \in R$$

WE ALSO HAVE

$$\{e^{\alpha n}\}^k = \frac{\delta}{(\delta - e^\alpha p)^k} = \left\{\binom{n+k-1}{k-1} e^{\alpha n}\right\} \in R$$

USING $\sinh \alpha m = \frac{e^{\alpha m} - e^{-\alpha m}}{2}$, $\cosh \alpha m = \frac{e^{\alpha m} + e^{\alpha m}}{2}$

$\sin \alpha m = \frac{e^{i\alpha m} - e^{-i\alpha m}}{2}$, $\cos \alpha m = \frac{e^{i\alpha m} + e^{-i\alpha m}}{2}$

WE OBTAIN NUMBER SEQUENCES IN R AS FOLLOWS.

$$\{\cos \alpha m\} = (\delta - p\cos\alpha)/(\delta - 2p\cos\alpha + p^2)$$

$$\{\sin \alpha m\} = i p \sin\alpha / (\delta - 2p\cos\alpha + p^2)$$

$$\{\cosh \alpha m\} = (\delta - p\cosh\alpha)/(\delta - 2p\cosh\alpha + p^2)$$

$$\{\sinh \alpha m\} = p\sinh\alpha / (\delta - 2p\cosh\alpha + p^2)$$

WE CAN WRITE $\beta = e^{\alpha}$ AND USING THE RESULT FOR $\{e^{\alpha m}\}$ ON PREVIOUS PAGE, WE GET

$$\{\beta^m\} = \frac{\delta}{\delta - \beta p}$$

THE DIFFERENCE OPERATOR $\Delta a(m)$

FROM SEQ. $\{a_m\} = a(m)$, WE FORM A NEW SEQ. $\Delta a(m) \in R$ WHICH BY AN OPRN OF OPERATORS (ELEMENTS OF Q) WILL BE CALLED THE DIFFERENCE OPERATOR. THIS OPRN IS, OF COURSE, DIFFERENT FROM WHAT WE USED IN DIFFERENCE CALCULUS SEC. PP30-51. WE CAN WRITE Δa IN VARIOUS FORMS

$$\Delta a(m) = \{a_{m+1} - a_m\}$$

WE CLAIM THAT

$$\Delta a(m) = \frac{a(m+1)}{\ell(m)} - a_0 \delta$$

TO PROVE WE MUST SHOW

$$a(m+1) = \Delta a * \ell - a_0 \ell$$

$$= \left\{\sum_{k=0}^{m} (a_{k+1} - a_k)\right\} + a_0 \ell$$

$$= \{a_{m+1} - a_0\} + a_0 \ell = \{a_{m+1}\}$$

NOW WE WRITE

$$\Delta a(m) = \frac{p^{-1} * a - a_0 p^{-1}}{\ell(m)} - a_0 \delta$$

$$= \frac{a - a_0 \delta}{p * \ell} - a_0 \delta$$

NOW LET $\delta / (p * \ell) = q$, THEN

$$\Delta a = q * a - a_0 (q + \delta)$$

$$\Delta^2 a = q * \Delta a - \Delta a_0 (q+\delta)$$

$$= q^2 * a - a_0 (q^2 + q) - \Delta a_0 (q + \delta)$$

$$= q^2 * a - (a_0 q + \Delta a_0)(q+\delta)$$

IN GENERAL

$$\Delta^K a = q^K * a - (a_0 q^{K-1} + \Delta a_0 q^{K-2} \cdots + \Delta^{K-1} a_0) * (q+\delta)$$

NOW SOME USEFUL RESULTS INVOLVING q WILL BE DERIVED.
AGAIN WE WILL START WITH $a(m) = \{e^{\alpha m}\}$:

$$\{\Delta e^{\alpha m}\} = \{e^{\alpha(m+1)} - e^{\alpha m}\}$$

$$= (e^{\alpha} - 1)\{e^{\alpha m}\}$$

$$= q * \{e^{\alpha m}\} - (q + \delta)$$

$$\Rightarrow \{e^{\alpha m}\} = \frac{q + \delta}{q - (e^{\alpha} - 1)\delta} \quad ; \quad \{\beta^m\} = \frac{q+\delta}{q - (\beta - 1)\delta}$$

$$\{e^{\alpha m}\}^K = \frac{(q+\delta)^K}{[q - (e^{\alpha} - 1)\delta]^K} = \left\{\binom{m+K-1}{K-1} e^{\alpha m}\right\}$$

IF WE TAKE $\alpha = 0$ IN THIS LAST RESULT, WE GET

$$\ell^K = \frac{(q+\delta)^K}{q^K}$$

AND IF WE TAKE $\alpha = \ln 2$, WE GET

$$\{2^n\}^k = \left\{\binom{n+k-1}{k-1} 2^n\right\} = \frac{(q+\delta)^k}{(q-\delta)^k}$$

DECOMPOSITION THEOREMS

THE FOLLOWING TWO THMS. PLAY FUNDAMENTAL ROLES IN APPLICATIONS. WE LET $P(x) = C_0 + C_1 x + \cdots + C_k x^k, C_k \neq 0$

THM. 1: THE OPERATOR $\omega = \delta / P(p^{-1})$ IS A NUMBER SEQ. i.e. $\omega \in R$

PROOF: THE PROOF IS CONSTRUCTIVE AS FOLLOWS. WE HAVE

$$\frac{\delta}{P(p^{-1})} = \frac{p^k}{C_0 p^k + C_1 p^{k-1} + \cdots + C_k \delta}$$

SINCE $C_0, C_1, \ldots C_{\nu-1}, \nu \leq k$ MAY BE ZERO, THE DEGREE OF THE POLYNOMIAL IN DENOMINATOR IS SUPPOSED TO BE ν. LET

$$C_{k-\nu} z^\nu + C_{k-\nu+1} z^{\nu-1} \cdots + C_k = C_k \prod_{j=1}^{m} (1 - \xi_j z)^{\nu_j}$$

NOW USING DECOMPOSITION BY PARTIAL FRACTIONS, WE WRITE

$$\frac{z^{\nu-1}}{C_{k-\nu} z^\nu + \cdots + C_k} = \sum_{r=1}^{m} \sum_{s=1}^{\nu_r} \frac{A_{rs}}{(1-\xi_r z)^s}$$

$$\frac{\delta}{P(p^{-1})} = p^{k-\nu+1} * p^{\nu-1} / (C_{k-\nu} p^\nu + \cdots + C_k \delta)$$

$$= p^{k-\nu+1} * \sum_{r=1}^{m} \sum_{s=1}^{\nu_r} \frac{A_{rs}\,\delta}{(\delta - \xi_r p)^s}$$

$$= p^{k-\nu+1} * \sum_{r=1}^{m} \sum_{s=1}^{\nu_r} A_{rs} \{\xi_r^n\}^s \in R$$

EXAMPLE: $P(x) = x^2 - e^2(e+1)x + e^5$

WE HAVE $e^5x^2 - e^2(e+1)x + 1 = (1-e^2x)(1-e^3x)$

AND WE GET

$$\frac{1}{e^5x^2 - e^2(e+1)x+1} = \frac{e}{1-e}\left[\frac{1}{e}\,\frac{1}{1-e^2x} - \frac{1}{1-e^3x}\right]$$

CONSEQUENTLY

$$\frac{\delta}{P(p^{-1})} = \frac{e}{1-e}\left[\frac{1}{e}\,\frac{\delta}{\delta - e^2p} - \frac{\delta}{\delta - e^3p}\right] * p^2$$

$$= \frac{e}{1-e}\,p^2 * \left[\frac{1}{e}\{e^{2n}\} - \{e^{3n}\}\right]$$

$$= p^2 * \left\{e^{2n}\,\frac{1-e^{n+1}}{1-e}\right\}$$

$$= \begin{cases} 0 & n = 0,1,2 \\ e^{2(n-2)}(1+e+\cdots+e^{n-2}) & n = 3,4,\ldots \end{cases}$$

THM. 2: THE OPERATOR $\delta / P(q)$ IS A NUMBER SEQUENCE, I.E. $\delta / P(q) \in R$

PROOF: WE FIRST SHOW THAT IF $P(x) = x - \lambda$, THEN

$$\frac{\delta}{P(q)} = \frac{\delta}{q - \lambda\delta} = p * \{(1+\lambda)^n\}$$

TO SHOW THIS, WE NOTE THAT

$$\frac{\delta}{q-\lambda\delta} = \frac{p*\ell}{\delta - \lambda p * \ell}$$

WE MUST SHOW THAT

$$(\delta - \lambda p * \ell) * \{(1+\lambda)^n\} = \ell$$

BUT THE LEFT SIDE IS

$$\{(1+\lambda)^n\} - \lambda p * \left\{\frac{-1+(1+\lambda)^{n+1}}{x - x + \lambda}\right\} = \{(1+\lambda)^n\} + p * \{1-(1+\lambda)^{n+1}\}$$

IT IS SEEN THAT THE R.H.S. IS ACTUALLY ℓ. NOW LET

$$\frac{1}{P(x)} = \sum_{\ell=1}^{\nu} \sum_{j=1}^{\alpha_\ell} \frac{A_{j\ell}}{(x-\lambda_\ell)^j}$$

$$\Rightarrow \frac{\delta}{P(q)} = \sum_{\ell=1}^{\nu} \sum_{j=1}^{\alpha_\ell} A_{j\ell}\, p^j * \{(1+\lambda_\ell)\}^j \in R$$

SOLUTION OF LINEAR DIFFERENCE EQS. WITH CONSTANT COEFFICIENTS

A DIFFERENCE EQ. OF ORDER K MAY BE WRITTEN IN TWO FORMS

$$c_k f(n+k) + c_{k-1} f(n+k-1) + \dots + c_0 f(n) = g(n)$$

OR

$$d_k \Delta^k f(n) + d_{k-1} \Delta^{k-1} f(n) + \dots + d_0 f(n) = g(n)$$

WHERE $g(n)$ IS ANY ARBITRARY NUMBER SEQ. WE ASSUME $c_k \neq 0$, $d_k \neq 0$.

THE FIRST EQ. CAN BE WRITTEN AS

$$(c_k p^{-k} + c_{k-1} p^{-k+1} + \dots + c_0 \delta) * f(n) - c_k f_0 p^{-k}$$
$$- (c_{k-1} f_0 + c_k f_1) p^{-k+1} - \dots = g(n)$$

OR IF $P(\alpha) = c_k \alpha^k + \dots + c_0$

$Q(\alpha) = c_k f_0 \alpha^k + (c_{k-1} f_0 + c_k f_1) \alpha^{k-1} + \dots$

THEN

$$f(n) = \frac{g(n)}{P(p^{-1})} + \frac{Q(-p)}{P(-p)}$$

THE RHS IS IN FACT AN ELEMENT OF R. THE REASON IS THAT FROM THM. 1, ABOVE $\delta / P(p^{-1}) \in R$ AND IF WE MULTIPLY $Q(-p)/P(-p)$ BY p^k/p^k, THEN ESSENTIALLY THE SAME PROOF AS THAT OF THM 1

CAN BE USED TO SHOW THAT $Q(-P)/P(-P) \in R$.

EXAMPLE: FIBONACCI NUMBERS, $f_0 = 0$, $f_1 = 1$

$$f(n+2) = f(n+1) + f(n)$$

$$p^{-2} * f(n) - p^{-1} = p^{-1} * f(n) - f(n)$$

$$f(n) = \frac{p^{-1}}{p^{-2} - p^{-1} - \delta}$$

$$= -\frac{p}{p^2 + p - \delta}$$

$$= \frac{p}{(\delta - \xi_1 p) * (\delta - \xi_2 p)}$$

WHERE $\xi_1 = \frac{2}{1+\sqrt{5}}$, $\xi_2 = \frac{2}{1-\sqrt{5}}$

WE HAVE

$$f(n) = \frac{1}{\sqrt{5}} \left[\frac{\delta}{\delta - \xi_1 p} - \frac{\delta}{\delta - \xi_2 p} \right]$$

$$= \frac{1}{\sqrt{5}} \left\{ \xi_1^n - \xi_2^n \right\}$$

IN A SIMILAR MANNER THE DIFFERENCE EQ. IN TERMS OF $\Delta^k f(n)$ MAY BE WRITTEN AS

$$f(n) = \frac{g(n)}{P(q)} + \frac{Q(q)}{P(q)}$$

WHERE NOW DEGREE OF $Q <$ DEGREE OF P, SO THAT USING PARTIAL FRACTIONS AND THM. 2, WE SEE THAT $f(n) \in R$.

EXAMPLE: $\Delta_3 f(n) = 0$, $f_0 = \Delta f_0 = 0$, $\Delta^2 f_0 = 1$.

WE HAVE $q^3 * f - (q+\delta) = 0$

$$f(m) = \frac{q+\delta}{q^3}$$

$$= \frac{\delta}{q^2} + \frac{\delta}{q^3}$$

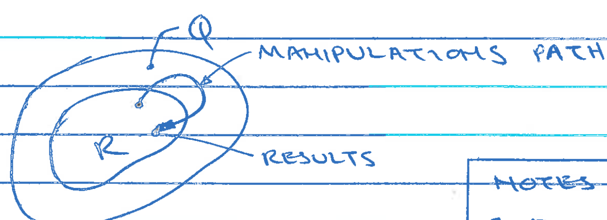
$$= p^2 * \ell^2 + p^3 * \ell^3$$

$$= p^2 * \ell^2 (\underbrace{\delta + p * \ell}_{\ell})$$

$$= p^2 * \ell^3$$

$$= \begin{cases} 0 & m = 0, 1 \\ \binom{m}{2} & m = 2, 3, \ldots \end{cases}$$

AS SEEN IN THE ABOVE SECTIONS, MIKUSIŃSKI OPERATIONAL CALCULUS IS A VERY POWERFUL TOOL BASED ON SIMPLE CONCEPTS FROM MODERN ALGEBRA, MAINLY THE EXTENSION OF AN INTEGRAL DOMAIN TO A QUOTIENT FIELD. NOTE THAT ALTHOUGH IN APPLICATIONS, THE FINAL RESULTS ARE IN R, IN THE PROCESS OF MANIPULATIONS, WE KEEP GETTING OUT OF R AS SHOWN BELOW. THIS IS JUST AS IN THE CASE OF



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S. FENYÖ ; T. FREY :
"MODERN MATHEMATICAL METHODS IN TECHNOLOGY"
CHAP. 2 . VOL. 1 .

GENERALIZED FUNCTIONS. I FIND MIKUSIŃSKI OPERATIONAL CALCULUS AS ONE OF THE MOST BEAUTIFUL PARTS OF MODERN MATHEMATICS.

* NOTES ON NUMBER THEORY

LET (a,b) BE THE GREATEST COMMON DIVISOR OF INTEGERS a AND b. THEN IF q AND r ARE INTEGERS SUCH THAT $a = bq + r \quad ; \quad 0 \le r < b$

WE HAVE $(a,b) = (b,r)$

PROOF: $a - bq$ IS DIVISIBLE BY (a,b) AND (a,b) IS THE GREATEST DIVISOR OF $a - bq \Rightarrow$ THE GREATEST DIVISOR OF r IS ALSO (a,b), i.e. $(a,b) = (b,r)$

EUCLEADIAN ALGORITHM FOR FINDING GCD OF a AND b USES THE ABOVE RESULT. LET $b = q_1 r + r_1$ WHERE $0 \le r_1 < r$, THEN

$$(a,b) = (b,r) = (r,r_1)$$

CONTINUE THIS PROCESS TILL $r_m = 0$, i.e.

$$(a,b) = (b,r) = (r,r_1) = (r_1,r_2) \dots = (r_{m-2}, r_{m-1}) = (r_m$$

THEN $r_{m-2} = q_m r_{m-1} + \cancel{r_m}$

$$\therefore (a,b) = (r_{m-1}, 0) = r_{m-1}$$

AN EXAMPLE: $(1804, 328) = ?$

$$1804 = 5 \times 328 + 164$$

$$328 = 2 \times 164 + 0$$

$$(1804, 328) = (328, 164) = (164, 0) = 164$$

THM 1.: a, b INTEGERS $\Rightarrow \exists\, x, y$ INTEGERS $\ni ax + by = (a,b)$

PROOF: REVERSE THE PROCESS OF EUCLIDEAN ALGORITHM

$$\begin{cases} r_{m-3} = q_{m-1} r_{m-2} + r_{m-1} & 0 < r_{m-1} < r_{m-2} \\ r_{m-4} = q_{m-2} r_{m-3} + r_{m-2} & 0 < r_{m-2} < r_{m-3} \\ \vdots \qquad \vdots \qquad \vdots & \vdots \\ a = q b + r & 0 < r < b \end{cases}$$

FROM THE FIRST EQ WE HAVE $r_{m-1} = r_{m-3} - q_{m-1} r_{m-2}$

FROM THE SECOND WE HAVE $r_{m-2} = r_{m-4} - q_{m-2} r_{m-3}$

$\therefore (a,b) = r_{n-1} = (1 + q_{n-1} q_{n-2}) r_{n-3} - q_{n-1} r_{n-4}$

CONTINUE THIS PROCESS TILL a AND b APPEAR ON THE RIGHT SIDE. IN THE ABOVE EXAMPLE

$$164 = 1804 - 5 \times 328$$

THE VALUES OF x AND y IN THE ABOVE THM ARE NOT UNIQUE. THERE ARE INFACT AN INFINITELY MANY PAIRS $(x,y) \ni (a,b) = xa + yb$. THIS CAN BE SEEN IF WE NOTE THAT REPLACING x BY $x + mb$ AND y BY $y - ma$ WILL RESULT IN

$$(x + mb)a + (y - ma)b = xa + yb = (a,b)$$

WHERE m IS AN INTEGER. INFACT, IF $a = (a,b)a_1$ AND $b = (a,b)b_2$, THEN $ab_1 = a_1b_1(a,b)$ AND $a_1b = a_1b_1(a,b)$. WE HAVE

$$(x + mb_1)a + (y - ma_1)b = xa + yb = (a,b).$$

WE SEE THAT IF $x_1a + y_1b = x_2a + y_2b = (a,b)$ THEN $x_1 \equiv x_2 \pmod{b_1}$ AND $y_1 \equiv y_2 \pmod{a_1}$.

WE WILL GIVE TWO APPLICATIONS OF THE ABOVE THM.

THM 2 : THE LINEAR DIOPHANTINE EQ. $ax + by = c$ HAS A SOLUTION IF AND ONLY IF (a,b) IS A DIVISOR OF c.

PROOF: SINCE (a,b) IS A DIVISOR OF a AND b $\Rightarrow$ (a,b) IS A DIVISOR OF $ax + by$, i.e. (a,b) IS A DIVISOR OF c. IF $c = k(a,b)$, FROM THM. 1 $\exists$ INTEGERS x_0 AND y_0 $\ni$ $ax_0 + by_0 = (a,b)$, MULTIPLYING BY k, WE GET $a(kx_0) + b(ky_0) = k(a,b) = c$, i.e. $x = kx_0$, $y = ky_0$.

THM 3 : IF m AND n ARE RELATIVELY PRIME AND

— REGULAR POLYGONS WITH m AND n SIDES CAN BE CONSTRUCTED $\Rightarrow$ A REGULAR POLYGON WITH mn SIDES CAN BE CONSTRUCTED.

PROOF: BY THM. 4 $\exists$ x AND y (INTEGERS) $\ni$
$1 = (m,n) = xm + yn$ OR $2\pi = x(2\pi m) + y(2\pi n)$
IT FOLLOWS $\frac{2\pi}{mn} = x\frac{2\pi}{n} + y\frac{2\pi}{m}$ $\therefore$ THE ANGLE
THE CONSTRUCTION OF A REGULAR POLYGON OF mn SIDES IS POSSIBLE. FOR $m=5$ AND $n=3$, WE HAVE $2\times5 - 3\times3 = 1 = (5,3)$, $\frac{2\pi}{15} = 2\cdot\frac{2\pi}{3} - 3\cdot\frac{2\pi}{5}$
$\therefore$ A REGULAR 15-SIDED POLYGON CAN BE CONSTRUCTED.

IN CONNECTION WITH CONSTRUCTION PROBLEMS, THE FOLLOWING THM. OF GAUSS IS VERY INTERESTING:

THM. 4: A REGULAR n-GON CAN BE CONSTRUCTED WITH RULER AND COMPASS IF AND ONLY IF EITHER:
1) n IS A PRIME NUMBER OF THE FORM $2^{2^k}+1$ OR A PRODUCT OF DISTICT PRIMES OF THIS FORM
2) n IS A POWER OF 2
3) n IS A PRODUCT OF NUMBERS SATISFYING (1) AND (2).

PRIME NUMBERS OF THE FORM $2^{2^k}+1$ ARE KNOWN AS FERMAT PRIMES. ONLY 5 HAVE BEEN DISCOVERRED: 3, 5, 17, 257 AND 65,537 CORRESPONDING TO $k=0,1,2,3,4$.

DEF: $a \equiv b \pmod m$ IF $a-b$ IS A MULTIPLE OF m.
THM. 5: $a \equiv b \pmod m \Leftrightarrow a$ AND b HAVE THE SAME REMAINDER WHEN DIVIDED BY m.

THM. 6: $a \equiv b \pmod{m}$, $c \equiv d \pmod{m} \Rightarrow a+c \equiv b+d \pmod{m}$, $ac \equiv bd \pmod{m}$, $a^n \equiv b^n \pmod{m}$, $(a,m) = (b,m)$

THE PROOF OF ALL THE ABOVE RESULTS ARE SIMPLE AND FOLLOW FROM THE DEFN OF THE CONGRUENCE RELATION $\equiv$. FOR $(a,m) = (b,m)$, WE NOTE THAT $a = b + qm$. BY THE RESULT USE IN EUCLIDEAN ALGORITHM $(a,m) = (m,b) = (b,m)$.

LET $N = a_n 10^n + a_{n-1} 10^{n-1} + \cdots \cdots + a_1 10 + a_0$
SINCE $10^k \equiv 1 \pmod{9}$, BY THM. 6 WE HAVE

$$N \equiv a_n + a_{n-1} + \cdots + a_1 + a_0 \pmod{9}$$

$\Rightarrow$ AN INTEGER IS DIVISIBLE BY 9 IF AND ONLY IF THE SUM OF ITS DIGITS IS DIVISIBLE BY 9.

MANY USEFUL RESULTS CAN BE OBTAINED BY USING THE CONGRUENCE RELATION DEFINED ABOVE. AN EXAMPLE FOLLOWS:

EXAMPLE:
FIND THE REMAINDER WHEN 15^{37} IS DIVIDED BY 13

$$15^{37} \equiv 2^{37} \equiv 2 \cdot (2^4)^9 \equiv 2 \cdot 3^9 \equiv 2 \cdot (27)^3$$
$$\equiv 2 \cdot 1 \equiv 2 \pmod{13}.$$

* NOTES ON PARTIAL COORDINATE CHANGE

CONSIDER A FUNCTION $\rho = f(\vec{x})$ WHERE $\vec{x} = (x_1, \dots, x_m)$. LET $\vec{x}_i$ AND $\vec{x}^i$ BE DEFINED AS FOLLOWS:

$$\vec{x}_i = (x_1, x_2, \dots, x_i) \qquad 1 \le i < m$$
$$\vec{x}^i = (x_{i+1}, \dots, x_m)$$

NOW LET US DEFINE NEW VARIABLES $\vec{y}_i$ AND $\vec{y}^i$ BY THE RELATIONS

$$\begin{cases} \vec{y}_i = (\phi_1(\vec{x}), \dots, \phi_i(\vec{x})) \\ \vec{y}^i = \vec{x}^i = (y_{i+1}, \dots\dots, y_m) \end{cases}$$

THIS SITUATION ARISES OFTEN IN APPLICATIONS. THE MEANING OF $\frac{\partial \rho}{\partial x_j}$ AND $\frac{\partial \rho}{\partial y_j}$ ARE CLEAR, i.e.

$$\frac{\partial \rho}{\partial x_j} = \left.\frac{\partial \rho}{\partial x_j}\right|_{(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m)}$$

$$\frac{\partial \rho}{\partial y_j} = \left.\frac{\partial \rho}{\partial y_j}\right|_{(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_m)}$$

NOW THE CHAIN RULE GIVES

$$\frac{\partial \rho}{\partial y_j} = \sum_{k=1}^{i} \frac{\partial \rho}{\partial x_k}\frac{\partial x_k}{\partial y_j} + \sum_{k=i+1}^{m} \frac{\partial \rho}{\partial x_k}\delta_{kj}$$

$$= \begin{cases} \sum_{k=1}^{i} \frac{\partial \rho}{\partial x_k}\frac{\partial x_k}{\partial y_j} & 1 \le j \le i \\ \sum_{k=1}^{i} \frac{\partial \rho}{\partial x_k}\frac{\partial x_k}{\partial y_j} + \frac{\partial \rho}{\partial x_j} & j \ge i+1 \end{cases}$$

WE NOTE THAT IF $\phi_j = \phi_j(\vec{x}_i)$, WE HAVE

$$\frac{\partial \rho}{\partial y_j} = \frac{\partial \rho}{\partial x_j} \qquad \text{FOR} \qquad j \ge i+1$$

SO THAT WE DO NOT HAVE TO RENAME $\vec{x}^i$ AS $\vec{y}^i$, i.e. WE CAN WRITE

$$p = f(\vec{x}_i, \vec{x}^i) = g(\vec{y}_i, \vec{x}^i)$$

THERE IS NO AMBIGUITY WHEN WE WRITE $\frac{\partial p}{\partial y_j}$ — IT MEANS $\left.\frac{\partial p}{\partial y_j}\right|_{\vec{x}^i,\, y_1, \dots y_{j-1}, y_{j+1}, \dots y_n}$.

HOWEVER, TO REDUCE CONFUSION, IN THE GENERAL CASE WHEN $\vec{y}_i = (\phi_1(\vec{x}), \dots, \phi_i(\vec{x}))$, IT IS BETTER TO RENAME THE VARIABLES THAT ARE KEPT FIXED. NOTE THAT IN THIS GENERAL CASE, WE HAVE

$$d\vec{x} = \frac{1}{\left|\frac{\partial \vec{y}}{\partial \vec{x}}\right|} d\vec{y}$$

BUT

$$\left|\frac{\partial \vec{y}}{\partial \vec{x}}\right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & & & \frac{\partial y_n}{\partial x_1} \\ \frac{\partial y_2}{\partial x_1} & \cdots & & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_i}{\partial x_1} & \cdots \frac{\partial y_i}{\partial x_i} & & \frac{\partial y_n}{\partial x_i} \\ & 0 & & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_i} \\ \\ \frac{\partial y_i}{\partial x_1} & \cdots & \frac{\partial y_i}{\partial x_i} \end{vmatrix}$$

$$\Rightarrow d\vec{x} = \frac{1}{\left|\frac{\partial \vec{y}_i}{\partial \vec{x}_i}\right|} d\vec{y}_i \, d\vec{y}^i = \frac{1}{\left|\frac{\partial \vec{y}_i}{\partial \vec{x}_i}\right|} d\vec{y}_i \, d\vec{x}^i$$

THIS MEANS THAT IN THIS CASE, NO RENAMING OF VARIABLES IS NEEDED.

EX. LET $g = g(\vec{x}) = \text{CONST.}$ BE A FAMILY OF SURFACES IN 3-D, THEN

$$\int f(\vec{x})\,\delta(g)\,d\vec{x} = \int f\,\delta(g)\,\frac{dg\,d\vec{x}\,'}{\left|\frac{\partial g}{\partial x_1}\right|}$$

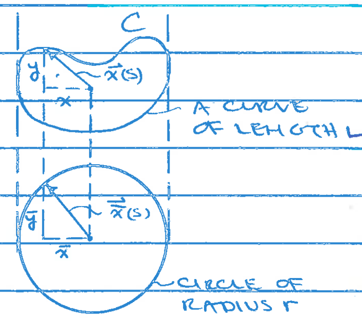
$$= \int_{g=0} \frac{f}{|\nabla g|}\,dS$$

WHERE $d\vec{x}\,' = dx_2\,dx_3$ AND dS IS THE ELEMENT OF THE SURFACE AREA OF $g=0$.

* NOTES ON ISOPERIMETRIC INEQUALITIES

THM. AMONG ALL CLOSED AND SIMPLE (i.e NO DOUBLE POINTS) CURVES OF LENGTH L, THE CIRCLE HAS THE LARGEST AREA.

PROOF (ERHARD SCHMIDT): DRAW TWO PARALLEL TANGENTS AS SHOWN TO THE GIVEN CURVE. CONSTRUCT A CIRCLE AS SHOWN. REPRESENT $\vec{x}$ AND $\vec{\bar{x}}$ AS A FUNCTION OF s, THE ARC LENGTH OF THE GIVEN CURVE C. IF THE CURVE C IS NONCONVEX, THEN PARTS OF THE CIRCLE WILL BE TRAVERSED MORE THAN ONCE AS $\vec{x}$ GOES ONCE AROUND C. NOW, WE HAVE FOR THE AREAS S AND S′ OF C AND THE CIRCLE, RESPECTIVELY



$$S = -\int_0^L x\,y'\,ds$$

$$S' = \int_0^L \bar{x}'\bar{y}\,ds = \pi r^2$$

WE HAVE $x(s) = \bar{x}(s)$, WE OBTAIN THE RELATION

$$S + \pi r^2 = +\int_0^L (x'\bar{y} - \bar{x}\,y')\,ds$$

WE NOW ESTABLISH THE UPPER AND LOWER BOUNDS FOR THIS RELATION. OBVIOUSLY

$$2r\sqrt{\pi S} \leq S + \pi r^2 \quad : \text{GEOM. MEAN} \leq \text{ARITH. MEAN}$$

CONSIDER THE VECTORS $\vec{a} = (x', y')$ AND $(\bar{y}, -\bar{x}) = \vec{b}$.

WE HAVE $x'\bar{y} - \bar{x}y' = \vec{a}\cdot\vec{b} \leq |\vec{a}|\,|\vec{b}|$. BUT $|\vec{a}| = 1$, $|\vec{b}| = r$. $\Rightarrow$

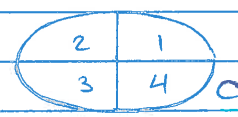
$$\left|\int_0^L (x'\bar{y} - \bar{x}y')\,ds\right| \leq rL$$

WE GET $2r\sqrt{\pi S} \leq rL$ OR

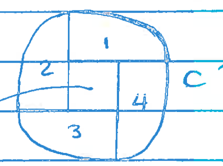
$$S \leq L^2/4\pi \quad \text{(ISOMPERIMETRIC INEQ.)}$$

TO GET THE CONDITION WHEN THE EQUALITY HOLDS, WE NOTE THAT $2r\sqrt{\pi S} = S + \pi r^2$ IF $r = \sqrt{S/\pi}$, i.e. C IS OF CONSTANT WIDTH, NOT NECESSARILY A CIRCLE. ALSO $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$ IF $\vec{a} \parallel \vec{b}$, i.e. THE TANGENT TO C IS PARALLEL TO THE TANGENT TO CIRCLE. THIS MEANS THAT C IS THE TRANSLATION OF THE CIRCLE, i.e. C IS A CIRCLE.

AN APPLICATION: IF a AND b ARE THE SEMIDIAMETERS OF AN ELLIPSE, ITS PERIMETER L SATISFIES THE INEQUALITY $L \geq \pi(a+b)$
[H. HADWIGER, ZUR SCHÄTZUNG DES ELLIPSENUMFANGS. ELEM. MATH. 4, PP 11-12, 1949]
PROOF [BY G.D. CHAKERIAN, ON ESTIMATING THE PERIMETER OF AN ELLIPSE, ELEM. MATH. 20, P 89 (1965)]
PUT THE ELLIPSE TOGETHER AS SHOWN. THE PERIMETER OF THE NEW FIGURE DOES NOT CHANGE: $L(C) = L(C')$



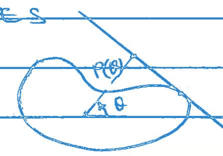
$$S(C') = \overbrace{\pi ab}^{\text{AREA OF THE ELLIPSE}} + \underbrace{(a-b)^2}_{\text{AREA OF THE CENTER SQUARE}}$$



$$L^2(C') \geq 4\pi S(C') = 4\pi^2 ab + 4\pi(a-b)^2$$
$$\geq 4\pi^2 ab + \pi^2(a-b)^2 = \pi^2(a+b)^2$$

ANOTHER INTERESTING METHOD OF DERIVING THE ISOPER. INEQ IS THE FOLLOWING. LET THE CURVE C BE DESCRIBED AS THE ENVELOPE OF ITS TANGENT LINES

$$x\cos\theta + y\sin\theta - p(\theta) = 0 \qquad (1)$$



TO GET THE COORDINATES OF POINTS ON C, WE MUST SOLVE FOR (x,y) BETWEEN EQ. (1) AND

$$-x\sin\theta + y\cos\theta - p'(\theta) = 0 \qquad (2)$$

THE RELATIONS ARE

$$\begin{cases} x = p\cos\theta - p'\sin\theta \\ y = p\sin\theta + p'\cos\theta \end{cases}$$

THE LENGTH OF THE CURVE C IS

$$L = \int_0^{2\pi} (x'^2 + y'^2)^{1/2}\, d\theta$$

$$= \int_0^{2\pi} (p + p'')\, d\theta$$

$$= \int_0^{2\pi} p\, d\theta \qquad (3)$$

AND THE AREA OF C IS

$$\vec{S} = \frac{1}{2}\oint \vec{x} \times (\vec{x} + d\vec{x}).$$

$$\vec{S} = \frac{1}{2}\oint \vec{x} \times d\vec{x}$$

$$S = \frac{1}{2}\int_0^{2\pi} (xy' - x'y)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} p(p + p'')\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)\, d\theta \quad , \ p \ \& \ p' \text{ PERIODIC}$$

NOW $p = p(\theta)$ IS A PERIODIC FUNCTION. LET

$$p = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos\nu\theta + b_\nu \sin\nu\theta)$$

THEN $$p' = \sum_{\nu=1}^{\infty} \nu(b_\nu \cos\nu\theta - a_\nu \sin\nu\theta)$$

$$\Longrightarrow \quad L = \pi a_0$$

AND $A = \frac{\pi}{2}\left(\frac{a_0^2}{2} - \sum_{\nu=2}^{\infty}(\nu^2-1)(a_\nu^2+b_\nu^2)\right)$

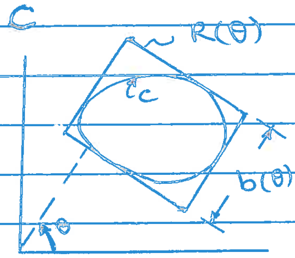
$$\therefore \quad A \leq \frac{\pi a_0^2}{4} = \frac{L^2}{4\pi}$$

WE NOTE THAT $A = \frac{L^2}{4\pi}$ IF $a_\nu = b_\nu = 0$ FOR $\nu \geq 2$. IN THIS CASE

$$p(\theta) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$$

WHICH DEFINES THE EQ. OF A CIRCLE BY ITS ENVELOPE. (COURANT, DIFF. & INT. CAL., VOL. II, P)

NOW CONSIDER A CONVEX CURVE C. LET C HAVE A WIDTH $b(\theta)$ AS SHOWN. LET $R(\theta)$ BE THE RECTANGLE CIRCUMSCRIBED ABOUT C AS SHOWN. THEN THE AREA OF $R(\theta)$ IS



$$S_R(\theta) = b(\theta)\, b(\theta + \tfrac{\pi}{2})$$

WE HAVE

$$\left[\operatorname*{Min}_{\theta} S_R(\theta)\right]^{1/2} \leq \frac{1}{2\pi}\int_0^{2\pi}\left[b(\theta)\, b(\theta+\tfrac{\pi}{2})\right]^{1/2} d\theta$$

THE EQUALITY HOLDS IF C IS OF CONSTANT WIDTH.

USING CAUCHY–SCHWARZ INEQUALITY

$$\left[\int_0^{2\pi}\left[b(\theta)\, b(\theta+\tfrac{\pi}{2})\right]^{1/2} d\theta\right]^2 \leq \int_0^{2\pi} b(\theta)\,d\theta \int_0^{2\pi} b(\theta+\tfrac{\pi}{2})\,d\theta$$

THE EQUALITY HOLDS IF $b(\theta)/b(\theta+\frac{\pi}{2}) = $ CONST. BOTH INTEGRALS ON THE RIGHT SIDE ARE EQUAL, SO THAT

$$\operatorname*{Min}_{\theta} S_R(\theta) \leq \frac{1}{4\pi^2}\left[\int_0^{2\pi} b(\theta)\,d\theta\right]^2$$

BUT USING EQ. (3), WE GET, BY LETTING THE ORIGIN INSIDE C:

$$\int_0^{2\pi} p \, d\theta = \int_0^{\pi} [p(\theta) + p(\theta + \tfrac{\pi}{2})] \, d\theta$$
$$= \int_0^{\pi} b(\theta) \, d\theta$$
$$= L$$

or $$\int_0^{2\pi} b(\theta)\, d\theta = 2L$$

$$\Rightarrow \quad \min_{\theta} S_R(\theta) \leq \frac{L^2}{\pi^2}$$

THE EQUALITY HOLDS IF AND ONLY IF C IS OF CONSTANT WIDTH. THIS PROOF IS BY E. LUTWAK, AM. MATH. MONTHLY, NO. 6, VOL. 86 (JUNE-JULY 79), PP 476-477.

THE PROBLEM OF L. MOSER [L.M. KELLY, THE GEOMETRY OF METRIC & LIN. SPACES, 1975; P 244] IS AS FOLLOWS: CAN EVERY CLOSED CURVE OF LENGTH 2π BE ACCOMODATED IN A RECTANGLE OF AREA 4? THE ABOVE RESULT CAN BE USED TO GIVE AN AFFIRMATIVE ANSWER TO THIS QUESTION. EVERY CURVE OF LENGTH 2π HAS A CONVEX HULL (SMALLEST CONVEX SET CONTAINING THE CURVE) WHOSE BOUNDARY IS LESS THAN OR EQUAL TO 2π. SINCE $L^2/\pi^2 = 4 \Rightarrow \exists$ A RECTANGLE $R \ni S_R \leq 4$. THIS PROOF IS AGAIN BY LUTWAK, DEPT. OF MATH., POLYTECHNIC INST. OF N.Y., GIVEN IN AM. MATH. MONTHLY.

NOTE THAT EQ (3) CAN BE OBTAINED BY CAUCHY-CROFTON THM. ALSO SEE M.P. DO CARMO, "DIFF. GEOMETRY OF CURVES AND SURFACES", PP 41-46.

* AN INTERESTING THEOREM AND AN APPLICATION

THE FOLLOWING PROBLEM APPEARED IN THE AMERICAN MATHEMATICAL MONTHLY [PROB. E 2721, 1978, P496]:
LET $a_0, a_1 > 0$ AND DEFINE a_n $(n \geq 2)$ RECURSIVELY BY $a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$. SHOW THAT $\{a_n\}$ IS CONVERGENT AND COMPUTE ITS LIMIT. (PROPOSED BY A. EMERSON)

IT IS EASY TO SEE THAT IF THE LIMIT EXISTS, IT MUST BE EITHER 0 OR 4 SINCE FOR LARGE n, $a_n \approx a_{n-1} \approx a_{n-2}$, OR $a_n = 2\sqrt{a_n} \Rightarrow a_n = 0$ OR 4. HOWEVER, THE PROOF FOR CONVERGENCE OF THE ABOVE SEQUENCE BY CONVENTIONAL METHODS APPEARS DIFFICULT. THE FOLLOWING THM. CAN BE USED TO PROVE THE CONVERGENCE OF THE SEQUENCE $\{a_n\}$ QUITE EASILY.

THM: LET $g : R_+^k \to R_+$ BE INCREASING WRT EACH OF ITS ARGUMENTS. LET $f : R_+ \to R_+$ BE DEFINED BY $f(x) = g(x, x, \ldots, x)$. SUPPOSE

$$f(x) > x \text{ IF } 0 < x < \alpha, \quad f(x) < x \text{ IF } x > \alpha$$

LET $a_0 > 0, a_1 > 0, \ldots a_{k-1} > 0$ AND DEFINE

$$a_n = g(a_{n-1}, a_{n-2}, \ldots a_{n-k}), n = k, k+1, \ldots$$

$$\Longrightarrow \lim_{n \to \infty} a_n = \alpha$$

PROOF: LET $m = \text{MIN}(a_0, \ldots, a_{k-1}, \alpha)$, $M = \text{Max}(a_0, \ldots, a_{k-1}, \alpha)$. BY INDUCTION WE SHOW THAT $m \leq a_n \leq M \ \forall n$. THE INEQUALITIES HOLD WHEN $n = 0, 1, \ldots, k-1$ (BY THE DEFN. OF m AND M!). SUPPOSE THEY HOLD FOR $n < N$
$\Rightarrow a_N = g(a_{N-1}, \ldots, a_{N-k}) \geq g(m, m, \ldots, m) = f(m) \geq m$
THE FIRST PART OF INEQ. FOLLOWS FROM THE FACT THAT

g IS AN INCREASING FN. W.R.T. INDIVIDUAL VARIABLES. THE SECOND PART OF INEQ FOLLOWS FROM THE FACT THAT $m \leq \alpha$ (BY DEFN). NOW, THE SAME ARGUMENT CAN BE USED TO SHOW $a_n \leq f(M) \leq M$.
$\therefore \{a_n\}$ IS BOUNDED. LET $a = \liminf a_n$, $A = \limsup a_n \Rightarrow$

$$a = \liminf g(a_{n-1}, \ldots a_{n-k}) \geq g(\liminf a_n, \ldots, \liminf a_{n-k}) = f(a)$$

SINCE $a \geq f(a) \Rightarrow a \geq \alpha$ (BY DEFN OF α) SIMILARLY $A \leq f(A) \Rightarrow A \leq \alpha$. SINCE $a \leq A \Rightarrow a = A = \alpha = \lim a_n$

(PROOF COMPLETED)

NOW LET $K=2$, $g(x_1, x_2) = x_1^{1/2} + x_2^{1/2} \Rightarrow f(x) = 2x^{1/2}$
$f(x) > x$ FOR $x > 4$ AND $f(x) < x$ FOR $x < 4$
$\therefore \alpha = 4$ AND $\lim a_n = 4$.

* NOTES ON INEQUALITIES

THE FOLLOWING INEQUALITY

$$2(\sqrt{m+1}-\sqrt{m}) < \frac{1}{\sqrt{m}} < 2(\sqrt{m}-\sqrt{m-1}) \qquad m \text{ INTEGER} > 0$$

CAN BE USED TO FIND BOUNDS ON THE NUMBER

$$S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{9999}} + \frac{1}{100}.$$

WE HAVE

$$2(\sqrt{2}-1) < 1 < 2$$

$$2(\sqrt{3}-\sqrt{2}) < \frac{1}{\sqrt{2}} < 2(\sqrt{2}-1)$$

$$\vdots$$

$$2(\sqrt{10001}-100) < \frac{1}{100} < 2(100-\sqrt{9999})$$

$$198 < 2\sqrt{10001} - 2 < S < 199$$

THE PROOF OF THE ABOVE INEQUALITY IS SIMPLE.

WE NOTE THAT

$$2(\sqrt{m+1}-\sqrt{m}) = \frac{2(m+1-m)}{\sqrt{m+1}+\sqrt{m}} < \frac{1}{\sqrt{m}}$$

$$2(\sqrt{n}-\sqrt{m-1}) = \frac{2(m-m+1)}{\sqrt{m}+\sqrt{m-1}} > \frac{1}{\sqrt{n}}$$

FOR THE FOLLOWING NUMBER

$$S_m = \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\quad\frac{2m-1}{2m}$$

ANOTHER KIND OF INEQUALITY MUST BE USED. IT IS

$$\sqrt{\frac{4m-3}{4m+1}} = \sqrt{\frac{4(m-1)+1}{4m+1}} < \frac{2m-1}{2m} < \sqrt{\frac{3(m-1)+1}{3m+1}} = \sqrt{\frac{3m-2}{3m+1}}$$

BOTH PARTS OF THIS INEQUALITY MAY BE PROVED BY SQUARING THE TERMS AND SIMPLIFYING THE RESULTING EXPRESSIONS. WE THEN FIND THAT

$$\frac{1}{\sqrt{4m+1}} < S_m < \frac{1}{\sqrt{3m+1}}$$

THE FOLLOWING WELL-KNOWN THM. IS VERY USEFUL ;

THM. LET G AND A STAND FOR THE GEOMETRIC AND ARITHMETIC MEANS OF m POSITIVE NUMBERS, RESPECTIVELY : $A = \frac{\sum a_i}{m}$, $G = (\prod a_i)^{1/m}$

$\Rightarrow$ $G \leq A$. THE EQUALITY HOLDS IF AND ONLY IF $a_1 = a_2 = \cdots = a_m$ (* SEE NOTE ON P83)

WE USE THIS THM. TO PROVE

i) $(ab^m)^{\frac{1}{m+1}} \leq \frac{a+mb}{m+1}$ $a>0, b>0$

PROOF. OBVIOUS ! TAKE $a_1 = a$, $a_2 = a_3 \ldots = a_{m+1} = b$

THE LEFT SIDE IS GEO. MEAN AND THE RIGHT IS THE ARITH. MEAN. THE EQUALITY HOLDS ONLY IF $a = b$.

ii) $\forall m \geq 2$, $m! < \left(\frac{m+1}{2}\right)^m$

TAKE $a_1 = 1, a_2 = 2, \ldots, a_m = m$

$\Rightarrow$ $G = (m!)^{1/m}$, $A = \frac{1}{m}(1+2+\cdots+m) = \frac{m(m+1)}{2m}$ $= \frac{m+1}{2}$. FROM THE ABOVE THM., $(m!)^{1/m} < \frac{m+1}{2}$

OR $m! < \left(\frac{m+1}{2}\right)^m$. NOTE THAT SINCE NONE OF THE NUMBERS $a_1, a_2, \ldots a_m$ ARE EQUAL, EQUALITY CANNOT HOLD.

iii) $9abc \leq (a+b+c)(ab+bc+ca)$; $a,b,c>0$

WE HAVE $(abc)^{1/3} \leq \frac{a+b+c}{3}$

$(abc)^{2/3} \leq (ab+bc+ca)/3$

MULTIPLY BOTH SIDES OF THE ABOVE TWO INEQUALITIES

iv) $a_i > 0$, $\left(\sum^m a_i\right)\left(\sum^m \frac{1}{a_i}\right) \geq m^2$

WE HAVE $\sum^m a_i \geq m\left(\prod^m a_i\right)^{1/m}$

$\sum^m \frac{1}{a_i} \geq m\left(\prod^m \frac{1}{a_i}\right)^{1/m}$

$\Rightarrow$ $\left(\sum^m a_i\right)\left(\sum^m \frac{1}{a_i}\right) \geq m^2$.

THE FOLLOWING ARE SOME ELEMENTARY PROPERTIES OF INEQUALITIES:

i) $a > b,\ c > d \Rightarrow a + c > b + d,\ a + e > b + e,\ e = \text{CONST.}$

ii) $a > b,\ c > d,\ e = \text{CONST.} \Rightarrow a - d > b - c$ (NOTE ORDER)

$$a - e > b - e$$

iii) $a > b \geq 0,\ c > d > 0 \Rightarrow ac > bd$

iv) $a > b > 0,\ c > d > 0 \Rightarrow \frac{a}{d} > \frac{b}{c}$ (NOTE ORDER)

V) $a > b$, m, n POSITIVE INTEGERS, $a^{1/n}$, $b^{1/n}$ POSITIVE nTH ROOTS $\Rightarrow a^{m/n} \geq b^{m/n}$, $b^{-m/n} \geq a^{-m/n}$, EQUALITY HOLDS IFF i) $a = b$ OR ii) $m = 0$.

LET $\mathfrak{M}_r = \mathfrak{M}_r(a) = \left(\frac{1}{n} \sum^{n} a_\nu^r\right)^{1/r}$, $a = (a_1, \ldots, a_n)$, $a_\nu \geq 0$, r REAL, $r \neq 0$. WE DEFINE

$$\mathfrak{M}_r(a) = 0 \text{ IF } r < 0 \text{ AND } a_i = 0 \text{ FOR SOME } i$$

IT IS SEEN THAT $A(a) = \mathfrak{M}_1(a)$. THE HARMONIC MEAN $h(a)$ IS DEFINED AS

$$h(a) = \mathfrak{M}_{-1}(a)$$
$$= \left(\frac{1}{n} \sum \frac{1}{a_\nu}\right)^{-1}$$
$$= n \frac{1}{\sum \frac{1}{a_\nu}}$$

WE NOTE THE FOLLOWING RELATIONS:

THESE RELATIONS ALSO HOLD FOR WEIGHTED MEANS DEFINED NEXT PAGE.

$$\left\{\begin{aligned} &\mathfrak{M}_r(a) = \{A(a^r)\}^{1/r} \\ &G(a) = \exp\{A(\log a)\} \qquad a > 0 \\ &\mathfrak{M}_{-r}(a) = 1 / \mathfrak{M}_r\left(\tfrac{1}{a}\right) \\ &\mathfrak{M}_{rs}(a) = \{\mathfrak{M}_s(a^r)\}^{1/r} \end{aligned}\right.$$

WHERE $a^r = \{a_\nu^r\}_{\nu=1}^n$, $\log a = \{\log a_\nu\}_{\nu=1}^n$ AND $\frac{1}{a} = \left\{\frac{1}{a_\nu}\right\}_{\nu=1}^n$, $a > 0$ IF $a_\nu > 0\ \forall\ \nu = 1, \ldots, n$. THE ASSUMPTION OF $a > 0$ MUST BE USED IN ALL THE ABOVE FORMULAS

IF A SUFFICE IS NEGATIVE. ALSO NOTE THAT

i) $A(a+b) = A(a) + A(b)$

ii) $G(ab) = G(a)G(b)$

iii) $M_r(b) = K M_r(a)$ IF $b = K(a)$ (i.e. $b_\nu = Ka_\nu$ FOR SOME K INDEPENDENT OF ν.)

iv) $G(b) = K G(a)$ IF $b = K(a)$

v) $M_r(a) \leq M_r(b)$ IF $a \leq b$ (i.e. $a_\nu \leq b_\nu \ \forall \nu$)

vi) $\text{Min } a < M_r(a) < \text{Max } a$ UNLESS ALL a_ν'S ARE EQUAL, OR ELSE $r < 0$ AND $a_\nu = 0$ FOR SOME ν

vii) $\text{Min } a < G(a) < \text{Max } a$ UNLESS ALL a_ν'S ARE EQUAL OR $a_\nu = 0$ FOR SOME ν.

viii) $\lim_{r \to 0} M_r(a) = G(a)$

WE MUST ONLY PROVE THIS FOR $a > 0$ SINCE FOR SOME $a_\nu = 0$, BOTH SIDES ARE ZERO. THE PROOF CAN BE GIVEN FOR THE WEIGHTED MEANS DEFINED BELOW

$$M_r(a) = \left(\sum q_\nu a_\nu^r\right)^{1/r} \qquad \sum q_\nu = 1,\ q_\nu > 0$$

$$G(a) = \prod a_\nu^{q_\nu} \qquad \sum q_\nu = 1,\ q_\nu > 0$$

THE ORDINARY MEANS DEFINED EARLIER ARE SPECIAL CASES OF THE WEIGHTED MEANS WHERE $q_\nu = 1/m$. WE HAVE

$$M_r(a) = \exp\left\{\frac{1}{r}\log\left(\sum q_\nu a_\nu^r\right)\right\}$$

$$= \exp\left\{\frac{1}{r}\log\left(1 + r\sum q_\nu \log a_\nu + O(r^2)\right)\right\}$$

$$= \exp\left\{\sum q_\nu \log a_\nu + O(r^2)\right\}$$

$$\to \prod a_\nu^{q_\nu} \quad \text{AS } r \to 0$$

PROPERTIES i) TO vii) ALSO HOLD FOR WEIGHTED MEANS. SO DOES THE FOLLOWING

ix) $\lim_{r \to \infty} M_r(a) = \text{Max } a, \quad \lim_{r \to -\infty} M_r(a) = \text{Min } a$

WE NOTE THAT $q_k^{1/r} a_k \leq m_r(a) < a_k$ WHERE $a_k = \text{Max}\, a$

NOW LET $r \to \infty$ TO GET THE FIRST INEQUALITY. THE SECOND INEQUALITY FOLLOWS FROM $m_{-r}(a) = \dfrac{1}{m_r(\frac{1}{a})}$ IF $a > 0$. IF $a_\nu = 0$ FOR SOME ν, THE RESULT IS TRIVIAL.

x) $m_r(a) < m_{2r}(a)$ $r > 0$, UNLESS ALL a_ν'S ARE EQUAL.

PROOF: $(\sum q_\nu a_\nu^r)^2 < (\sum q_\nu)(\sum q_\nu a_\nu^{2r}) = \sum q_\nu a_\nu^{2r}$

(CAUCHY INEQ.)

OR

$$\left(\sum q_\nu a_\nu^r\right)^{\frac{1}{r}} < \left(\sum q_\nu q_\nu^{2r}\right)^{\frac{1}{2r}}$$

* NOTE — THE RESULT $G \leq A$ IS ONE OF THE MOST WELL-KNOWN INEQUALITY. THERE ARE MANY PROOFS BY FAMOUS MATHEMATICIANS. THERE ARE TWELVE PROOFS IN BECKENBACH AND BELLMAN "INEQUALITIES". A PROOF BY INDUCTION DUE TO EHLERS [COLLOQUIUM ON LINEAR EQS., OFFICE OF NAVAL RESEARCH TECH. REP. ONRL 35-54, 1954, (ED.) W.D. HAYS] APPEARS TO BE ONE OF THE SIMPLEST. THIS PROOF IS REPRODUCED IN THE BOOK BY BECKENBACH AND BELLMAN.

* DERIVATION OF AN IDENTITY RELATED TO WAVE EQ. WITH SOURCES ON A SURFACE IN MOTION

CONSIDER THE MOVING SURFACE $f(\vec{y},\tau)=0$. DEFINE $\Sigma: F(\vec{y};\vec{x},t) = [f(\vec{y},\tau)]_{ret} = f(\vec{y},t-r/c)=0$ WHERE $r=|\vec{x}-\vec{y}|$. LET $\vec{x}$ BE OUTSIDE THE BODY. WE DEFINE $f>0$ OUTSIDE THE BODY. IF $\vec{N}$ IS THE UNIT VECTOR (POINTING OUTWARD) ON THE SURFACE $F=0$, WE NEED TO FIND AN EXPLICIT EXPRESSION FOR

$$\frac{\partial}{\partial x_i}\int_{F=0}\frac{N_i\,d\Sigma}{r} = I$$

WE NOTE THAT

$$I = \frac{\partial}{\partial x_i}\int_{F<0}\frac{\partial}{\partial y_i}\left(\frac{1}{r}\right)d\Sigma$$

$$= \frac{\partial}{\partial x_i}\int_{F<0}\frac{\hat{r}_i}{r^2}\,d\vec{y} \qquad ; \quad \hat{r}_i = \frac{x_i-y_i}{r}$$

$$= \frac{\partial}{\partial x_i}\int\frac{\hat{r}_i}{r^2}\left[1-H(F)\right]d\vec{y} \qquad ; H[\cdot]\ \text{HEAVISIDE FUNCTION}$$

$$= \int\frac{\partial}{\partial x_i}\left(\frac{\hat{r}_i}{r^2}\right)\left[1-H(F)\right]d\vec{y}$$

$$-\int\frac{\hat{r}_i}{r^2}\frac{\partial F}{\partial x_i}\,\delta(F)\,d\vec{y}$$

WE HAVE $\frac{\partial}{\partial x_i}\left(\frac{\hat{r}_i}{r^2}\right)=0$; $\frac{\partial F}{\partial x_i} = -\frac{\hat{r}_i}{c}\frac{\partial F}{\partial t} = \frac{\hat{r}_i}{c}V_N|\nabla F|$

WHERE

$$V_N = -(\partial F/\partial t)/|\nabla F|$$

ALSO

$$d\vec{y} = \frac{dF\,d\Sigma}{|\nabla F|}$$

SUBSTITUTING THESE RESULTS IN THE EXPRESSION FOR I,

WE GET

$$I = -\int \frac{1}{cr^2} V_N |\nabla F| \delta(F)\, d\vec{y}$$

$$= -\int_{F=0} \frac{M_N}{r^2}\, d\Sigma$$

WHERE $M_N = V_N/C$. IN THE ABOVE DERIVATION, C IS THE SPEED OF THE PROPAGATION OF DISTURBANCE WHICH IS ASSUMED CONSTANT.

I DERIVED THIS IDENTITY WHILE CHECKING THE DERIVATION OF ISOM'S THICKNESS NOISE FORMULA. I COULD NOT GET THE SAME NUMERICAL RESULT FROM ISOM'S FORMULA AND AN EXPRESSION WHICH IS

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int_{F=0} \frac{1}{r}\left[\frac{\rho_0 v_m}{\Delta}\right]_{ret} d\Sigma$$

$$= \frac{\partial}{\partial t}\int_{F=0}\frac{\rho_0 V_N}{r}\, d\Sigma \qquad (*)$$

WHERE $v_m = -(\partial f/\partial t)/|\nabla f|$; $\Delta^2 = 1 + M_m^2 - 2M_m\cos\theta$, $M_m = v_m/c$, $V_N = [v_m/\Delta]_{ret} = -(\partial F/\partial t)/|\nabla F|$ AND θ IS THE ANGLE BETWEEN ∇f AND $\vec{r}$. LET $\vec{m} = \nabla f/|\nabla f|$. WE HAVE

$$(**) \qquad N_i = \frac{m_i - M_m \hat{r}_i}{\Delta} \quad ; \quad |\nabla F| = [\Delta |\nabla f|]_{ret}$$

WE NEED TO PROVE THAT ISOM'S RESULT

$$4\pi p'_I(\vec{x},t) = -\frac{\partial}{\partial x_i}\int_{F=0}\frac{\rho_0 c^2}{r}\left[\frac{m_i}{\Delta}\right]_{ret} d\Sigma$$

IS EQUIVALENT TO EQ. (*). WE HAVE, USING (**)

$$-\frac{\partial}{\partial x_i}\int_{F=0}\frac{1}{r}\left[\frac{M_i}{\Lambda}\right]_{ret}d\Sigma = -\frac{\partial}{\partial x_i}\left[\int_{F=0}\frac{N_i\, d\Sigma}{r} + \int_{F=0}\frac{\hat{r}_i}{r}\left[\frac{M_n}{\Lambda}\right]_{ret}d\Sigma\right]$$

$$= \int_{F=0}\frac{M_N}{r^2}\,d\Sigma - \frac{\partial}{\partial x_i}\int\frac{\hat{r}_i}{r}\left[\frac{M_n}{\Lambda}\right]_{ret}d\Sigma$$

NOTE $M_N = V_N/c$, $M_n = v_n/c$. IN THE NEXT NOTE, P87, WE WILL SHOW THAT

$$-\frac{\partial}{\partial x_i}\int_{F=0}\frac{\hat{r}_i}{r}\left[\frac{M_n}{\Lambda}\right]_{ret}d\Sigma = \frac{\partial}{\partial t}\int_{F=0}\frac{1}{r}\left[\frac{M_n}{\Lambda}\right]_{ret}d\Sigma$$

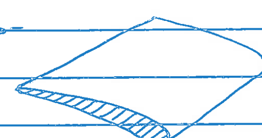
$$-\int_{F=0}\frac{M_N}{r^2}\,d\Sigma$$

IN THE ABOVE RESULTS WE SHOULD HAVE WRITTEN $[M_n/\Lambda]_{ret} = M_N$ SINCE $[v_n/\Lambda]_{ret} = V_N$. WE THUS HAVE

$$-\frac{\partial}{\partial x_i}\int\frac{1}{r}\left[\frac{M_i}{\Lambda}\right]_{ret}d\Sigma = \frac{\partial}{\partial t}\int_{F=0}\frac{V_N}{rc}\,d\Sigma$$

MULTIPLY BOTH SIDES BY $\rho_0 c^2$, THE LEFT SIDE IS $4\pi\, p'_I(\vec{x},t)$ AND THE RIGHT SIDE IS THE SAME AS THE INTEGRAL IN (*).

IT TURNED OUT THAT IN MY NUMERICAL WORK, I DID NOT INCLUDE THE SOURCE INTEGRAL ON THE AIRFOIL SECTION AT THE TIP. I FELT IT WAS NEGLIGIBLE BUT IT IS NOT. IT TOOK ME SIX WEEKS TO FIGURE THIS DURING WHICH I DOUBTED EVERYTHING I KNEW ABOUT WAVE EQ.!



CONVERSION OF SPACE DERIVATIVE INTO TIME DERIVATIVE IN THE SOLUTION OF WAVE EQ. WITH SOURCES ON MOVING SURFACES.

CONSIDER THE WAVE EQ

$$\Box^2 \phi = -\frac{\partial}{\partial x_i}\left[Q_i(\vec{x},t)\,|\nabla f|\,\delta(f)\right] \qquad (1)$$

THE SOLUTION OF THIS EQ. IS

$$4\pi\phi(\vec{x},t) = -\frac{\partial}{\partial x_i}\int_{-\infty}^{t}\int_{-\infty}^{\infty} \frac{Q_i}{r}\,|\nabla f|\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

WHERE $g = \tau - t + r/c$. WE CAN WRITE THIS EQ. AS

$$4\pi\phi(\vec{x},t) = -\frac{\partial}{\partial x_i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{Q_i\,H(t-\tau)}{r}\,|\nabla f|\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau \qquad (2)$$

WE NOTE THAT THE TERM INVOLVING $\vec{x}$ IS ONLY $\delta(g)/r$. WE HAVE

$$\frac{\partial}{\partial x_i}\left[\frac{\delta(g)}{r}\right] = -\frac{\hat{r}_i}{c}\,\frac{\delta'(g)}{r} - \frac{\hat{r}_i\,\delta(g)}{r^2}$$

$$= -\frac{\hat{r}_i}{cr}\,\frac{\partial}{\partial t}\delta(g) - \frac{\hat{r}_i\,\delta(g)}{r^2}, \qquad (3)$$

WE ALSO HAVE

$$\frac{\partial}{\partial t}\,\frac{H(t-\tau)\,\delta(g)}{r} = \frac{\delta(t-\tau)\,\delta(g)}{r} + \frac{H(t-\tau)}{r}\,\frac{\partial}{\partial t}\delta(g) \qquad (4)$$

USING EQ. (3), WE HAVE

$$-H(t-\tau)\frac{\partial}{\partial x_i}\left[\frac{\delta(g)}{r}\right] = \frac{\hat{r}_i\,H(t-\tau)\,\delta(g)}{r^2} + \frac{\partial}{\partial t}\left[\frac{\hat{r}_i\,H(t-\tau)}{cr}\,\delta(g)\right]$$

$$-\frac{\hat{r}_i\,\delta(t-\tau)\,\delta(g)}{cr} \qquad (5)$$

GOING BACK TO EQ. (2), WE HAVE

$$4\pi\,\phi(\vec{x},t) = -\int Q_i\, H(t-\tau)\, |\nabla f|\, \delta(f)\, \frac{\partial}{\partial x_i}\left(\frac{\delta(g)}{r}\right) d\vec{y}\, d\tau$$

$$= \int \hat{r}_i Q_i\, |\nabla f|\, \delta(f)\, \frac{\partial}{\partial t}\left[\frac{H(t-\tau)\,\delta(g)}{c\,r}\right] d\vec{y}\, d\tau$$

$$+ \int \frac{\hat{r}_i Q_i\, |\nabla f|\, H(t-\tau)}{r^2}\, \delta(f)\, \delta(g)\, d\vec{y}\, d\tau$$

$$- \int \frac{\hat{r}_i Q_i\, |\nabla f|}{r}\, \delta(t-\tau)\, \delta(f)\, \delta(g)\, d\vec{y}\, d\tau$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\int_{F=0} \frac{1}{r}\left[\frac{Q_r}{\Lambda}\right]_{ret} d\Sigma \quad ; Q_r = \hat{r}_i Q_i$$

$$+ \int_{F=0} \frac{1}{r^2}\left[\frac{Q_r}{\Lambda}\right]_{ret} d\Sigma$$

$$- \int \frac{Q_r\, |\nabla f|}{c\,r}\, \delta(t-\tau)\, \delta(f)\, \delta(g)\, d\vec{y}\, d\tau$$

LET THE LAST INTEGRAL BE CALLED I. THEN

LET $\tau \to g$

$$I = \int \frac{Q_r\, |\nabla f|}{c\,r}\, \delta\left(\frac{r}{c} - g\right) \delta(f)\, \delta(g)\, d\vec{y}\, dg$$

$$= \int \frac{1}{cr}\left[\frac{Q_r}{\Lambda}\right]_{ret} |\nabla F|\, \delta\left(\frac{r}{c}\right) \delta(F)\, d\vec{y}$$

$$= \int \frac{1}{r}\left[\frac{Q_r}{\Lambda}\right]_{ret} |\nabla F|\, \delta(r)\, \delta(F)\, d\vec{y}$$

IT IS OBVIOUS THAT IF THE OBSERVER IS NOT ON THE SURFACE $F=0$, THAT IS, IF THE OBSERVER IS

NOT ON $f=0$ AT THE TIME $t \implies I = 0$.
OTHERWISE, THE INTEGRAL IS HANDLED IN THE FOLLOWING WAY. LET

$$I_a = \int \frac{1}{r}\left[\frac{Q_r}{\Lambda}\right]_{ret} |\nabla F|\, \delta(r-a)\, \delta(F)\, d\vec{y}$$

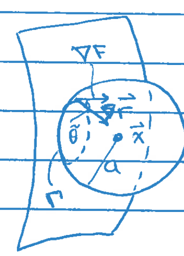
THEN $I = \lim_{a \to 0} I_a$. WE HAVE

$$d\vec{y} = \frac{dF\, dr\, dy_1}{\frac{\partial(F,r)}{\partial(y_2,y_3)}}$$

$$= \frac{dF\, dr}{|\nabla F \times \nabla r|} \quad \frac{dy_1}{\frac{\partial(F,r)}{\partial(y_2,y_3)} / |\nabla F \times \nabla r|}$$

$$= \frac{dF\, dr\, d\Gamma}{|\nabla F| \sin\tilde{\theta}} \quad ; \text{NOTE } |\nabla r| = 1$$

WHERE Γ IS THE CURVE OF INTERSECTION OF THE SPHERE OF RADIUS $r = a$ AND CENTER AT $\vec{x}$ WITH THE Σ-SURFACE $F = 0$. WE ALSO DEFINE $\tilde{\theta}$ AS THE ANGLE BETWEEN ∇F AND $\nabla r = \hat{r}_i$ AS SHOWN. WE HAVE



$$I_a = \int_{\substack{F=0 \\ r=a}} \frac{1}{a}\left[\frac{Q_r}{\Lambda}\right]_{ret_a} \frac{d\Gamma}{\sin\tilde{\theta}}$$

ASSUMING THAT F HAS A WELL-DEFINED TANGENT AT THE POINT WHERE $\vec{x}$ APPROACHES THE SURFACE*, THEN $\hat{\vec{r}} \to \vec{N} = \nabla F / |\nabla F|$ AND $\tilde{\theta} \to 0$ [$\therefore \theta \to 0$ IN Λ]. NOTE THAT THE SUBSCRIPT ret_a MEANS THAT τ IS REPLACED BY $t - \frac{a}{c}$.

(*) THIS IS NOT TRUE. THE STRUCTURE OF Σ SURFACE AS $\vec{x}$ APPROACHES $f=0$ AND THEN LIES ON IT IS CONICAL. THE RESULT APPEARS TO BE CORRECT F.F. 2/1/82

WE HAVE

$$I = \lim_{a\to 0} \left\{ \left[\frac{Q_r}{\Lambda}\right]_{ret_a} \int_{\substack{F=0\\ r=a}} \frac{1}{a \sin\tilde{\theta}}\, d\Gamma \right\}$$

$$= \left[\frac{Q_N}{|1-M_n|}\right]_{\tau=t} \frac{1}{a\sin\tilde{\theta}} \cdot 2\pi a \sin\tilde{\theta}$$

$$= \left[\frac{2\pi\, Q_N}{|1-M_n|}\right]_{\tau=t}$$ (*) (SEE NEXT PAGE)

WE DEFINE $Q_N = Q_i N_i$. NOTE $Q_N = Q_i M_i = Q_i N_i = Q_m$

WE HAVE SHOWN THAT IF THE OBSERVER IS NOT ON $F=0$ AT $\tau = t$, THEN

$$-\frac{\partial}{\partial x_i} \int_{F=0} \frac{1}{r}\left[\frac{Q_i}{\Lambda}\right]_{ret} d\Sigma = \frac{1}{c}\frac{\partial}{\partial t}\int_{F=0} \frac{1}{r}\left[\frac{Q_r}{\Lambda}\right]_{ret} d\Sigma$$
$$+ \int_{F=0} \frac{1}{r^2}\left[\frac{Q_r}{\Lambda}\right]_{ret} d\Sigma$$

SIMILARLY, WE CAN CONVERT THE SPACE DERIVATIVE INTO TIME DERIVATIVE IN AN INTEGRAL OF THE FORM

$$J = -\frac{\partial}{\partial x_i} \int_{F=0} \frac{\hat{r}_i}{r}\left[\frac{Q}{\Lambda}\right]_{ret} d\Sigma$$

WE ASSUME THAT THE OBSERVER IS NOT ON $F=0$ AT t. WE HAVE

$$J = \frac{\partial}{\partial x_i} \int Q\, |\nabla f|\, \delta(f)\, \frac{\hat{r}_i \delta(g)}{r}\, d\vec{y}\, d\tau$$

$$-\frac{\partial}{\partial x_i} \frac{\hat{r}_i \delta(g)}{r} = \frac{1}{c} \frac{\hat{r}_i \hat{r}_i}{r} \frac{\partial}{\partial t} \delta(g) - \frac{1}{r^2} \delta(g)$$

$$= \frac{1}{cr} \frac{\partial}{\partial t} \delta(g) - \frac{1}{r^2} \delta(g)$$

WE THEREFORE HAVE

NOTE SIGN!

$$J = \frac{1}{c} \frac{\partial}{\partial t} \int_{F=0} \frac{1}{r} \left[\frac{Q}{\Lambda}\right]_{ret} d\Sigma - \int_{F=0} \frac{1}{r^2} \left[\frac{Q}{\Lambda}\right]_{ret} d\Sigma$$

$$= -\frac{\partial}{\partial x_i} \int_{F=0} \frac{\hat{r}_i}{r} \left[\frac{Q}{\Lambda}\right]_{ret} d\Sigma$$

NOTE: WE CAN SHOW THAT IF $\cos\tilde{\theta} = N_i \hat{r}_i$, $M_N = \frac{-1}{c}(\partial F/\partial t)/|\nabla F| = [M_n/\Lambda]_{ret}$, WE HAVE

$$\Lambda = \frac{1}{[1 + M_N^2 + 2 M_N \cos\tilde{\theta}]^{1/2}}$$

I DERIVED THE ABOVE RESULTS FOR AERODYNAMIC LOAD CALCULATION OF PROPELLER BLADES.

(*) IF WE TAKE $\lim_{OBS. \to F=0} \int_{F=0} \frac{1}{r^2} \left[\frac{Q_r}{\Lambda}\right] d\Sigma$ WE GET EXACTLY I! THIS MEANS THAT ANOTHER METHOD OF FINDING THIS TERM, i.e. I, IS FINDING THE ABOVE LIMIT.

NOTE ADDED IN FEB. 13, 82 — I FINALLY SUCCEEDED TO USE ACOUSTIC EQ. (FW-H EQ.) OF MOVING BODIES FOR AERODYNAMIC CALCULATIONS. SEE P 161, THIS NOTEBOOK.

* AN INEQUALITY

THE FOLLOWING INEQUALITY APPEARED AS ONE OF THE PROBLEMS OF THE 9TH ANNUAL U.S.A MATHEMATICAL OLYMPIAD WHICH TOOK PLACE ON MAY 6, 1980 (MATHEMATICS MAGAZINE, VOL. 53, NO. 3, MAY 1980).

IF $0 \leq a, b, c \leq 1$ PROVE THAT

$$\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} + (1-a)(1-b)(1-c) \leq 1$$

PROOF: LET $a = \max(a,b,c)$, OTHERWISE RENAME THE VARIABLES a, b AND c. THIS IS POSSIBLE BECAUSE OF THE TOTAL SYMMETRY OF THE INEQUALITY. WE HAVE

$$\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} + (1-a)(1-b)(1-c) \leq$$

$$\frac{a}{1+b+c} + \frac{b}{1+c+b} + \frac{c}{1+c+b} + (1-a)(1-b)(1-c) =$$

$$\frac{a+b+c}{1+b+c} + (1-a)(1-b)(1-c) = \frac{a-1+1+b+c}{1+b+c} + (1-a)(1-b)(1-c)$$

$$= 1 + (1-a)\left[\underbrace{(1-b)(1-c)}_{1-b-c+bc} - \frac{1}{1+b+c}\right]$$

$$= 1 + (1-a)\,\frac{\cancel{1} - (b+c)^2 + bc(1+b+c) - \cancel{1}}{1+b+c}$$

$$= 1 + \frac{1-a}{1+b+c}\left[-(b+c)^2 + bc(1+b+c)\right]$$

WE WILL PROVE THAT THE SECOND TERM IS NEGATIVE. WE HAVE

$$b+c \geq 2\sqrt{bc} \Rightarrow (b+c)^2 \geq 4bc \geq (1+b+c)bc$$

THE LAST PART COMES FROM THE FACT THAT $1+b+c \leq 3 < 4$.

THE PROOF IS COMPLETE. SOLUTION BY F.F.

* TOTAL DIFFERENTIAL EQUATIONS

THE FOLLOWING NOTES ARE MADE FROM LOUIS BRAND'S ARTICLE "TOTAL DIFFERENTIAL EQS. IN THE LIGHT OF DIMENSIONAL ANALYSIS", THE AM. MATH. MONTHLY, VOL. 69, NO. 7 (AUG.-SEPT.) 1962, 618-623. IT IS INTERESTING BOTH FOR ITS USEFULNESS AND ALSO ITS CONNECTION WITH LIE GROUP THEORY (SEE NOTEBOOK ON LIE GROUPS).

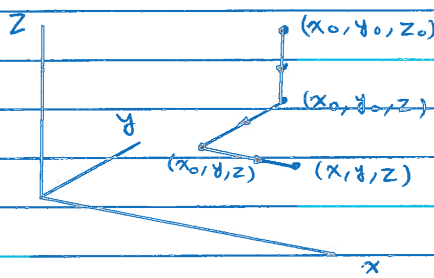
LET P, Q AND R BE FNS OF $\vec{r} = (x, y, z)$. THEN IF $\vec{F} = (P, Q, R)$, THE TOTAL DIFFERENTIAL EQ

$$P\,dx + Q\,dy + R\,dz = 0 \qquad (1)$$

CAN BE WRITTEN AS $\vec{F} \cdot d\vec{r} = 0$. THIS EQ IS EXACT IF AND ONLY IF $\exists\, \phi(x,y,z) \ni \vec{F} = \nabla\phi$. WE ALSO HAVE $\vec{F} = \nabla\phi$ IFF $\nabla \times \vec{F} = 0$. THE SCALAR ϕ IS GIVEN BY THE RELATION

$$\phi(x,y,z) = \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r}$$

WHERE THE PATH OF INTEGRATION IS ARBITRARY. TAKING THE PATH SHOWN ON THE RIGHT, WE HAVE



$$\phi(x,y,z) = \int_{x_0}^{x} P(t, y, z)\,dt + \int_{y_0}^{y} Q(x_0, t, z)\,dt + \int_{z_0}^{z} R(x_0, y_0, t)\,dt$$

IF $\nabla \times \vec{F} \neq 0$, EQ (1) IS INTEGRABLE IFF $\vec{F} \cdot \nabla \times \vec{F} = 0$

THE PROOF OF ($\Rightarrow$) IS SIMPLE. SUPPOSE $\exists\ \lambda(x,y,z) \ni$ $\lambda\vec{F}\cdot d\vec{r}=0$ IS EXACT $\Rightarrow \nabla\times(\lambda\vec{F}) = \nabla\lambda\times\vec{F} + \lambda\nabla\times\vec{F} = 0$. NOW TAKE $\vec{F}\cdot\nabla\times(\lambda\vec{F}) = \lambda\vec{F}\cdot(\nabla\times\vec{F}) = 0$. CONVERSELY, IF $\vec{F}\cdot\nabla\times\vec{F}=0$, WE HAVE $\forall\lambda$: $\lambda\vec{F}\cdot\nabla\times(\lambda\vec{F}) = 0$. WE SHOULD SHOW THAT $\vec{F}\cdot d\vec{r}=0$ IS INTEGRABLE. TAKE z TO BE CONSTANT $\Rightarrow$ $P\,dx + Q\,dy = 0$ POSSESSES A SOLUTION OF THE FORM $U(x,y,z) = C_1$ WHERE C_1 IS A FN OF Z. WE HAVE

$$\frac{\partial U}{\partial x} = \lambda P, \quad \frac{\partial U}{\partial y} = \lambda Q$$

$$\vec{F}\cdot d\vec{r} = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

$$+\left(\lambda R - \frac{\partial U}{\partial z}\right)dz$$

$$= dU + K\,dz \qquad (*)$$

WHERE $K = \lambda R - \dfrac{\partial U}{\partial z}$. NOW WE HAVE $\lambda\vec{F}\cdot\nabla\times(\lambda\vec{F}) = 0 \Rightarrow$

$$\lambda\vec{F}\cdot\nabla\times(\lambda\vec{F}) = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}+K\right)\cdot\left(\frac{\partial K}{\partial y}, \frac{\partial K}{\partial x}, 0\right)$$

$$= \frac{\partial U}{\partial x}\frac{\partial K}{\partial y} - \frac{\partial U}{\partial y}\frac{\partial K}{\partial x}$$

$$= \frac{\partial(U,K)}{\partial(x,y)} = 0$$

$\therefore\ U = \text{FN}(K)$, THIS FN IS INDEPENDENT OF x AND y BUT NOT OF Z, i.e. RELATION (*) ABOVE IS OF THE FORM

$$\frac{dU}{dz} + K(U,z) = 0.$$

FROM THE THEORY OF O.D.E., THIS EQ. HAS ALWAYS A SOLUTION OF THE FORM $\Phi(U,z) = C$. SUBSTITUTING FOR U IN TERMS OF x, y AND Z, WE GET

$F(x,y,z) = C$. THIS MEANS THAT $\vec{F} \cdot d\vec{r}$ IS INTEGRABLE. THIS PROOF IS FROM I.N. SNEDDON'S "ELEMENTS OF P.D.E."

IF μ AND λ ARE TWO INTEGRATING FACTORS OF $\vec{F} \cdot d\vec{r} = 0$, BY SUBTRACTING THE TWO EQS (AFTER MULTIPLYING BY μ AND λ, RESPECTIVELY)

$$\nabla \times (\lambda \vec{F}) = \nabla\lambda \times \vec{F} + \lambda \nabla \times \vec{F}$$

$$\nabla \times (\mu \vec{F}) = \nabla\mu \times \vec{F} + \mu \nabla \times \vec{F}$$

WE GET

$$(\mu \nabla\lambda - \lambda \nabla\mu) \times \vec{F} = 0$$

DIVIDE BY μ^2 TO GET

$$\nabla\left(\frac{\lambda}{\mu}\right) \times \vec{F} = 0$$

THIS MEANS THAT $\nabla\left(\frac{\lambda}{\mu}\right)$ IS PARALLEL TO $\vec{F}$ WHICH IS EQUIVALENT TO $\nabla\left(\frac{\lambda}{\mu}\right) \cdot d\vec{r} = d\left(\frac{\lambda}{\mu}\right) = 0$. $\therefore$ $\lambda/\mu =$ CONST. IS AN INTEGRAL OF $\vec{F} \cdot d\vec{r} = 0$.

IF THE VARIABLES x, y AND z HAVE DIMENSIONS U^a, U^b AND U^c RESPECTIVELY, THEN $\vec{F} \cdot d\vec{r} = P\,dx + Q\,dy + R\,dz$ IS SAID TO BE <u>ISOBARIC</u> IF $P\,dx$, $Q\,dy$ AND $R\,dz$ HAVE THE SAME DIMENSION. FOR EXAMPLE

$$2xz\,dx + 2yz^2\,dy + (x^2 + 2y^2z - 1)\,dz = 0$$

IS ISOBARIC WHEN $a = \text{wt } x = 0$, $b = \text{wt } y = -1$ AND $c = \text{wt } z = 2$, WHERE wt STANDS FOR WEIGHT. IF $\vec{F} \cdot d\vec{r} = 0$ IS ISOBARIC WHEN $a = b = c = 1$, THE EQ. IS SAID TO BE HOMOGENEOUS.

WE CAN PROVE THE FOLLOWING

THM. 1 — IF THE INTEGRABLE EQ. $\vec{f}\cdot d\vec{r}=0$ IS HOMOGENEOUS $\Rightarrow$ $\lambda = 1/(\vec{r}\cdot\vec{f})$ IS AN INTEGRATING FACTOR

PROOF: IF P, Q, R ARE HOMOGENEOUS FNS OF DEGREE m, THEN BY EULER'S THM

$$\vec{r}\cdot\nabla P = mP, \quad \vec{r}\cdot\nabla Q = mQ, \vec{r}\cdot\nabla R = mR$$

OR $\vec{r}\cdot\nabla\vec{f} = m\vec{f}$. WE HAVE $\vec{f}\cdot d\vec{r}$ INTEGRABLE, i.e. $\vec{f}\cdot(\nabla\times\vec{f}) = 0$. WE WILL SHOW THAT

$$\nabla\times\left(\frac{\vec{f}}{\vec{r}\cdot\vec{f}}\right) = 0 \qquad \vec{r}\cdot\vec{f}\neq 0.$$

WE HAVE THE FOLLOWING

$$\nabla\times\left(\frac{\vec{f}}{\vec{r}\cdot\vec{f}}\right) = \lambda\,\nabla\times\vec{f} - \lambda^2\,\nabla(\vec{r}\cdot\vec{f})\times\vec{f} \qquad (*)$$

$$\nabla(\vec{r}\cdot\vec{f}) = \vec{f}\times\underbrace{(\nabla\times\vec{r})}_{0} + \underbrace{\vec{f}\cdot\nabla r}_{\vec{f}} + \vec{r}\times(\nabla\times\vec{f}) + \underbrace{\vec{r}\cdot\nabla\vec{f}}_{m\vec{f}}$$

$$= (m+1)\vec{f} + \vec{r}\times(\nabla\times\vec{f})$$

$$\nabla(\vec{r}\cdot\vec{f})\times\vec{f} = \vec{r}\times(\nabla\times\vec{f})\times\vec{f}$$

$$= (\vec{r}\cdot\vec{f})\,\nabla\times\vec{f} - \underbrace{[\vec{f}\cdot(\nabla\times\vec{f})]}_{0}\,\vec{r}$$

$$= \lambda^{-1}\nabla\times\vec{f}$$

SUBSTITUTE THIS RESULT IN (*) TO GET $\boxed{\nabla\times(\lambda\vec{f}) = 0}$.

IN CONNECTION WITH ABOVE PROOF, WE NOTE THAT $P(x,y,z)$ IS HOMOGENEOUS OF DEGREE m IF

$$P(\alpha x, \alpha y, \alpha z) = \alpha^m P(x,y,z)$$

TAKE $\frac{d}{d\alpha}$ OF BOTH SIDES AND THEN LET $\alpha = 1$ TO

GET $\vec{r}\cdot\nabla P = mP$

WE NOTE THAT IF $\vec{r}\cdot\vec{P} = 0$, THE HOMOGENEOUS EQ. $\vec{P}\cdot d\vec{r} = 0$ IS ALWAYS INTEGRABLE. WE HAVE $\nabla(\vec{r}\cdot\vec{P}) = (m+1)\vec{P} + \vec{r}\times(\nabla\times\vec{P}) = 0$. IF $m+1 \neq 0$, TAKE DOT PRODUCT OF BOTH SIDES WITH $\nabla\times\vec{P}$ TO GET $\vec{P}\cdot(\nabla\times\vec{P}) = 0$. IF $m+1=0$, WE HAVE $\vec{r}\times(\nabla\times\vec{P}) = 0$ OR $\nabla\times\vec{P} = \lambda\vec{r}$ AND $\vec{P}\cdot(\nabla\times\vec{P}) = \lambda\vec{r}\cdot\vec{P} = 0$ $\therefore$ $\vec{P}\cdot d\vec{r} = 0$ IS INTEGRABLE.

THE RESULT OF THM 1 CAN BE EXTENDED TO ISOBARIC EQS. WE TAKE WT $x = a$, WT $y = b$, WT $z = c$. WE NOTE SEVERAL CASES:

CASE 1 - $a, b, c \neq 0$. INTRODUCE VARIABLES X, Y, Z AS FOLLOWS

$$x = X^a,\quad y = Y^b,\quad z = Z^c$$

THEN X, Y, AND Z WILL HAVE THE WEIGHT 1. THE D.E. BECOMES

$$aP X^{a-1}dX + bQ Y^{b-1}dY + cR Z^{c-1}dZ = 0$$

BY THM 1, THE INTEGRATING FACTOR λ IS

$$\lambda = (aX^aP + bY^bQ + cZ^cR)^{-1}$$

OR

$$\lambda = (axP + byQ + czR)^{-1}.$$

CASE 2 - $a, b \neq 0$, $c = 0$, DEFINE NEW VARIABLES

$$x = X^a,\quad y = Y^b,\quad z = Z/X$$

WE GET $\left(aPX^{a-1} - \frac{RZ}{X^2}\right)dX + bQY^{b-1}dY + \frac{R}{X}dZ = 0$

THIS EQ IS HOMOGENEOUS AND THE INTEGRATING FACTOR IS

$$\lambda = (axP + byQ)^{-1}$$

CASE 3: $a \neq 0,\ b = c = 0$, DEFINE

$$x = X^a,\quad y = Y/X,\quad z = Z/X$$

A METHOD SIMILAR TO CASE 2 GIVES

$$\lambda = (axP)^{-1}$$

WE HAVE PROVED THE FOLLOWING THM

THM II – IF THE INTEGRABLE EQ. $\vec{F} \cdot d\vec{r} = 0$ IS ISOBARIC WHEN x, y, z HAVE WEIGHTS a, b, c, RESPECTIVELY, NOT ALL ZERO $\Rightarrow \lambda = 1/(axP + byQ + czR)$ IS AN INTEGRATING FACTOR OF THE EQ.

EXAMPLES – 1 – $2xyz\,dx + z(1 - yz^2)\,dy + y(3 - 2yz^2)\,dz = 0$ IS ISOBARIC WHEN $a = 0,\ b = 2,\ c = -1$ AND AN INTEGRATING FACTOR IS

$$\lambda = 1/\left[2yz(1 - yz^2) - yz(3 - 2yz^2)\right] = \frac{-1}{yz}$$

MULTIPLYING BY λ THE ORIGINAL TOTAL D.E. RESULTS IN AN EXACT D.E.:

$$2x\,dx + (1 - yz^2)\frac{dy}{y} + (3 - 2yz^2)\frac{dz}{z} = 0$$

$$x^2 + \int_1^y \left(\frac{1}{y} - z^2\right) dy + \int_1^z \left(\frac{3}{z} - 2z\right) dz = \text{CONST.}$$

or

$$x^2 + \ln(yz^3) - yz^2 = C$$

2 – $(1 + yz)\,dx + x(z - x)\,dy - (1 + xy)\,dz = 0$ IS ISOBARIC WHEN $a = 1,\ b = -1,\ c = 1$ AND

$\lambda = 1/[(x-z)(1+xy)]$. THE INTEGRAL IS
$(1+xy)/(x-z) = C$.

3. $(2xz^2+y)\,dx + x\,dy + (2x^2z+3z^2)\,dz = 0$
IS ISOBARIC WHEN $a=1$, $b=5$, $c=2$ AND
$$\lambda = \frac{1}{6}\,\frac{1}{x^2z^2+xy+z^3}$$
ALSO $\mu = 1$ IS AN INTEGRATING FACTOR. THE
INTEGRAL IS THEREFORE $\mu/\lambda = 6(x^2z^2+xy+z^3) = C$.

* SECOND ORDER ORDINARY DIFFERENTIAL EQS WITH INHOMOGENEOUS B.C.'S.

CONSIDER THE 2ND ORDER DIFF. EQ.

$$l_u = Au'' + Bu' + C \qquad x \in [a,b]$$

A, B AND C FNS OF x, WITH THE B.C.'S

$$(*) \quad \begin{cases} \alpha_{11} u(a) + \alpha_{12} u'(a) + \beta_{11} u(b) + \beta_{12} u'(b) = 0 \\ \alpha_{21} u(a) + \alpha_{22} u'(a) + \beta_{21} u(b) + \beta_{22} u'(b) = 0 \end{cases}$$

WE CAN WRITE THE B.C.'S AS

$$\alpha \vec{u}_a + \beta \vec{u}_b = 0$$

WHERE $\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$, $\beta = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}$ AND

$$\vec{u}_a = \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix}, \quad \vec{u}_b = \begin{bmatrix} u(b) \\ u'(b) \end{bmatrix}.$$

WE LIKE TO FIND THE GENERAL FORM OF BC^*, THE BC FOR THE ADJOINT OPERATOR L^* OF

$$Lu : \begin{cases} l_u \\ BC[u] = 0 \end{cases}$$

WE DISTINGUISH TWO CASES:

i) $|\alpha| = \det(\alpha) = 0$, $|\beta| = 0$

ii) EITHER $|\alpha| \neq 0$ OR $|\beta| \neq 0$.

IN CASE i), WE NOTE THAT THERE WILL BE CONSTANTS p AND q SUCH THAT

$$\alpha_{21} u(a) + \alpha_{22} u'(a) = p[\alpha_{11} u(a) + \alpha_{12} u'(a)]$$

$$\beta_{21} u(b) + \beta_{22} u'(b) = q[\beta_{11} u(b) + \beta_{12} u'(b)]$$

THESE FOLLOW FROM THE FACT THAT SINCE $|\alpha| = 0 \Rightarrow$

THE 2ND ROW OF α IS A MULTIPLE OF ITS FIRST ROW AND SIMILARLY FOR β. NOW, WE AGAIN DISTINGUISH TWO CASES:

i-a) IF $p = q$, THEN THE TWO B.C.'S ARE IDENTICAL AND WE DO NOT HAVE SUFFICIENT B.C.'S TO FIND A UNIQUE SOLUTION

i-b) IF $p \neq q$, MULTIPLYING THE FIRST EQ. OF (*) BY q AND p, RESPECTIVELY AND SUBTRACTING FROM THE 2ND EQ., WE FIND

$$(p-q)\left[\alpha_{11} u(a) + \alpha_{12} u'(a)\right] = 0$$
$$(q-p)\left[\beta_{11} u(b) + \beta_{12} u'(b)\right] = 0$$

THAT IS, THE B.C.'S ARE SEPARABLE. IN THIS CASE, THE BC* CAN BE FOUND BY A METHOD SIMILAR TO THE ONE BELOW.

(*) SEE NOTE ON P107

CASE (ii): LET $|\beta| \neq 0 \Rightarrow \beta^{-1}$ EXISTS AND WE CAN WRITE

$$\vec{u}_b = -\beta^{-1}\alpha\,\vec{u}_a$$

NOW CONSIDER THE FOLLOWING INTEGRAL

$$\int_a^b v\,l u\, dx = \int_a^b v\left[A u'' + B u' + C\right] dx$$

WE HAVE

$$A v u'' = (A v u')' - (Av)' u'$$
$$= (A v u' - (Av)' u)' + (Av)'' u$$
$$B v u' = (B v u)' - (Bv)' u$$

$$\Rightarrow \quad l^* v = (A v)'' - (Bv)' + C v$$

THE B.C.* IS OBTAINED BY SETTING THE TERMS OBTAINED BY INTEGRATION EQUAL TO ZERO:

$$\left\{ A v u' + \left[B v - (A v)' \right] u \right\}_a^b = 0$$

$$\left\{ A v u' + \left[(B - A') v - A v' \right] u \right\}_a^b = 0$$

$$\left\{ \left[(B - A') v - A v' \quad A v \right] \begin{bmatrix} u \\ u' \end{bmatrix} \right\}_a^b = 0$$

$$\left\{ \left[v \quad v' \right] \underbrace{\begin{bmatrix} B - A' & A \\ -A & 0 \end{bmatrix}}_{\gamma_x} \begin{bmatrix} u \\ u' \end{bmatrix} \right\}_a^b = 0$$

LET $\vec{v}_a = \begin{bmatrix} v(a) \\ v'a \end{bmatrix}$, $\vec{v}_b = \begin{bmatrix} v(b) \\ v'(b) \end{bmatrix}$ AND LET

$$\gamma_a = \begin{bmatrix} B(a) - A'(a) & A(a) \\ -A(a) & 0 \end{bmatrix}, \text{ SIMILARLY FOR}$$

γ_b. THE ABOVE EQ MAY BE WRITTEN AS

$$\vec{v}_b^{\,T} \gamma_b \vec{u}_b - \vec{v}_a^{\,T} \gamma_a \vec{u}_a =$$

$$\left[\vec{v}_b^{\,T} \gamma_b \beta^{-1} \alpha + v_a^T \gamma_a \right] \vec{u}_a = 0$$

SINCE NO INFORMATION ON $\vec{u}_a$ IS AVAILABLE, WE REQUIRE

$$\vec{v}_a^{\,T} \gamma_a + \vec{v}_b^{\,T} \gamma_b \beta^{-1} \alpha = 0$$

or

$$\gamma_a^T \vec{v}_a + \alpha^T (\beta^{-1})^T \gamma_b^T \vec{v}_b = 0 \quad : \text{BC}^*$$

NOW WE CONSIDER THE INHOMO. B.C.'S

$$\alpha \vec{u}_a + \beta \vec{u}_b = \vec{d}$$

WHERE $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. AGAIN ASSUMING THAT $|\beta| \neq$, WE CAN WRITE

$$\vec{u}_b = \beta^{-1}\vec{d} - \beta^{-1}\alpha \vec{u}_a$$

NOW FOLLOWING THE ABOVE PROCEDURE, WE HAVE

$$\int_a^b v\, \ell u \, dx = \int_a^b u\, \ell^* v \, dx + \vec{v}_b^{\,T} \gamma_b \vec{u}_b - \vec{v}_a^{\,T} \gamma_a \vec{u}_a$$

$$= \int_a^b u\, \ell^* v\, dx + \vec{v}_b^{\,T} \gamma_b [\beta^{-1}\vec{d} - \beta^{-1}\alpha \vec{u}_a] - \vec{v}_a^{\,T}\gamma_a \vec{u}_a$$

$$= \int_a^b u\, \ell^* v \, dx + \vec{v}_b^{\,T} \gamma_b \beta^{-1} \vec{d} - (\vec{v}_b^{\,T} \gamma_b \beta^{-1}\alpha + \vec{v}_a^{\,T}\gamma_a)\vec{u}_a$$

IF WE NOW SELECT $v \ni BC^* v$ ARE SATISFIED, THEN

$$\int_a^b v\, \ell u \, dx = \int_a^b u\, \ell^* v\, dx + \vec{v}_b^{\,T}\gamma_b \beta^{-1}\vec{d}$$

WE NOTE THAT THE GREEN'S FN $G(x,\xi)$ SATISFIES

$$\begin{cases} \ell_\xi^* G(x,\xi) = \delta(x-\xi) \\ BC_\xi^*[G(x,\xi)] = 0 \end{cases}$$

$\therefore$ TO SOLVE $\ell u = f$, $B.C.[u] = 0$, WE

LET $v = G(x,\xi)$

$$\int_a^b \underbrace{u(\xi)\,\ell^* G(x,\xi)}_{\delta(x-\xi)}\, d\xi = u(x)$$

$$= \int_a^b \underbrace{\ell u(\xi)}_{f(\xi)}\, G(x,\xi)\, d\xi - \left[G(x,b)\ \frac{\partial G}{\partial \xi}(x,b)\right] \gamma_b \beta^{-1} \vec{d}$$

SINCE G SATISFIES BC^* IN ξ, THE LAST TERM MAY BE WRITTEN IN MANY DIFFERENT FORMS REPLACING EITHER $G(x,b)$ OR $\frac{\partial G}{\partial \xi}(x,b)$ (OR BOTH) IN TERMS OF $G(x,a)$ AND $\frac{\partial G}{\partial \xi}(x,a)$.

THE ABOVE PROCEDURE CAN BE MADE MORE SYSTEMATIC BY USING GENERALIZED FNS AS FOLLOWS. LET $[a,b] \subset [a',b']$ i.e. $a' < a$, $b' > b$. DEFINE

$$\tilde{u}(x) = \begin{cases} u(x) & x \in [a,b] \\ 0 & \text{OTHERWISE} \end{cases}$$

$$\langle \bar{\ell}\, \tilde{u}(\xi), G(x,\xi) \rangle \equiv \int_{a'}^{b'} \bar{\ell}\, \tilde{u}(\xi)\, G(x,\xi)\, d\xi$$

$$= \langle \tilde{u}(\xi), \bar{\ell}^* G(x,\xi) \rangle = \langle \tilde{u}(\xi), \delta(x-\xi) \rangle$$

$$= \tilde{u}(x)$$

HERE $\bar{\ell}$ AND $\bar{\ell}^*$ ARE SIMILAR TO ℓ AND ℓ^* EXCEPT GENERALIZED DIFFERENTIATION IS USED THROUGHOUT.

$$\bar{\ell}\, \tilde{u}(\xi) = \ell u(\xi) + [A(a) u'(a) + B(a) u(a)]\, \delta(\xi - a) -$$

$$- [A(b)u'(b) + B(b)u(b)]\,\delta(\xi - b)$$
$$+ A(\xi)u(a)\,\delta'(\xi - a) - A(\xi)u(b)\,\delta'(\xi - b)$$

NOW DIFFERENTIATE $A(\xi)\,\delta(\xi - a) = A(a)\,\delta(\xi - a)$:

$$A'(\xi)\,\delta(\xi - a) + A(\xi)\,\delta'(\xi - a) = A(a)\,\delta'(\xi - a)$$

or $A(\xi)\,\delta'(\xi - a) = A(a)\,\delta'(\xi - a) - A'(a)\,\delta(\xi - a)$

SIMILARLY $A(\xi)\,\delta'(\xi - b) = A(b)\,\delta'(\xi - b) - A'(b)\,\delta(\xi - b)$

WE THEREFORE HAVE

$$\bar{\ell}\tilde{u} = \ell\tilde{u} + \{A(a)u'(a) - [B(a) - A'(a)]u(a)\}\,\delta(\xi - a)$$
$$- \{A(b)u'(b) - [B(b) - A'(b)]u(b)\}\,\delta(\xi - b)$$
$$+ A(a)u(a)\,\delta'(\xi - a) - A(b)u(b)\,\delta'(\xi - b)$$

WE FIND

$$\langle \bar{\ell}\tilde{u}(\xi), G(x,\xi)\rangle = \int_{a'}^{b'} \ell\tilde{u}(\xi)\,G(x,\xi)\,d\xi \;+$$
$$+ \{A(a)u'(a) + [B(a) - A'(a)]u(a)\}\,G(x,a)$$
$$- \{A(b)u'(b) + [B(b) - A'(b)]u(b)\}\,G(x,b)$$
$$- A(a)u(a)\frac{\partial G}{\partial \xi}(x,a) + A(b)u(b)\frac{\partial G}{\partial \xi}(x,b)$$
$$= \int_a^b f(\xi)\,G(x,\xi)\,d\xi - \left[G(x,b) \quad \frac{\partial G}{\partial \xi}(x,b)\right] \times$$
$$\times \begin{bmatrix} B(b) - A'(b) & A(b) \\ -A(b) & 0 \end{bmatrix} \begin{bmatrix} u(b) \\ u'(b) \end{bmatrix}$$
$$+ \left[G(x,a) \quad \frac{\partial G}{\partial \xi}(x,b)\right] \begin{bmatrix} B(a) - A'(a) & A(a) \\ -A(a) & 0 \end{bmatrix} \begin{bmatrix} u(a) \\ u'(a) \end{bmatrix}$$

NOW WRITING $\vec{u}_b$ IN TERMS OF $\vec{u}_a$ AND $\vec{d}$, THE SAME RESULT AS IN THE PREVIOUS PAGE IS OBTAINED. IT IS INTERESTING TO NOTE THAT WE

HAVE FOUND THE FUNCTION $\tilde{U}(x)$ WHICH IS AN EXTENSION OF THE FN $u(x)$. WHEN $x = a$ OR b, STRANGE THINGS MAY HAPPEN.

EXAMPLE : CONSIDER THE 2ND ORDER D.E.

$$\begin{cases} y'' = f(x) & x \in [0,1] \\ y(0) - y'(1) = 1 \\ y'(0) - 2y(1) = -2 \end{cases}$$

THE GREEN'S FN. FOR THIS D.E. IS FOUND TO BE

$$G(x,\xi) = \begin{cases} \frac{1}{3}\left[(2\xi - 4)x + 2\xi - 1\right] & x < \xi \\ \frac{1}{3}\left[(2\xi - 1)x - \xi - 1\right] & x > \xi \end{cases}$$

THE BC* ARE FOUND AS FOLLOWS :

$$vy'' = (vy')' - v'y'$$
$$= (vy' - v'y)' + v''y$$

$$\left[vy' - v'y\right]_0^1 = v(1)y'(1) - v'(1)y(1) - v(0)y'(0) + v'(0)y(0)$$
$$= v(1)y'(1) - v'(1)y(1) - 2v(0)y(1) + v'(0)y'(1)$$
$$= \left[v'(0) + v(1)\right]y'(1) - \left[2v(0) + v'(1)\right]y(1) = 0$$

$$BC^* \begin{cases} v'(0) + v(1) = 0 \\ 2v(0) + v'(1) = 0 \end{cases}$$

WE NOTE THAT

$$\frac{\partial G}{\partial \xi}(x,0) + G(x,1) = \frac{2x-1}{3} + \frac{-2x+1}{3} = 0$$

$$2G(x,0) + \frac{\partial G}{\partial \xi}(x,1) = -\frac{2(x+1)}{3} + \frac{2(x+1)}{3} = 0$$

i.e. $G(x,\xi)$ SATISFIES THE BC*.

NOW LET $\tilde{y}(x) = \begin{cases} y(x) & x \in [0,1] \\ 0 & \text{OTHERWISE} \end{cases}$

$$\bar{\tilde{y}}' = \tilde{y}'(x) + y(0)\,\delta(x) - y(1)\,\delta(x-1)$$

$$\bar{\tilde{y}}'' = \tilde{y}''(x) + y'(0)\,\delta(x) - y'(1)\,\delta(x-1)$$
$$+ y(0)\,\delta'(x) - y(1)\,\delta'(x-1)$$

NOW $y(0) = 1 + y'(1)$
$y'(0) = -2 + 2y(1)$

$$\bar{\tilde{y}}'' = \tilde{y}''(x) + [-2+2y(1)]\,\delta(x) - y'(1)\,\delta(x-1)$$
$$+ [1+y'(1)]\,\delta'(x) - y(1)\,\delta'(x-1)$$

$$= \tilde{y}''(x) + [2\delta(x) - \delta'(x-1)]\,y(1)$$
$$+ [\delta'(x) - \delta(x-1)]\,y'(1) - 2\,\delta(x)$$
$$+ \delta'(x)$$

WE NOTE THAT $\langle 2\delta(\xi) - \delta'(\xi-1),\ G(x,\xi)\rangle$
$$= 2G(x,0) + \frac{\partial G}{\partial \xi}(x,1) = 0$$

SIMILARLY $\langle \delta'(\xi) - \delta(\xi - 1), G(x,\xi) \rangle =$

$$= -\frac{\partial G}{\partial \xi}(x,0) - G(x,1) = 0$$

$$\therefore \quad \tilde{y}(x) = \int_0^1 f(\xi)\, G(x,\xi)\, d\xi - 2\langle \delta(\xi), G(x,\xi) \rangle$$

$$+ \langle \delta'(\xi), G(x,\xi) \rangle$$

$$= \int_0^1 f(\xi)\, G(x,\xi)\, d\xi - 2\, G(x,0)$$

$$- \frac{\partial G}{\partial \xi}(x,0)$$

$$= \int_0^1 f(\xi)\, G(x,\xi)\, d\xi + 1$$

(*) NOTE: IT IS OBVIOUS FROM THE ABOVE EXAMPLE THAT THE TWO CASES COVERED FOR THE COEFFICIENTS OF THE BOUNDARY TERMS IN THE B.C.'S DO NOT INCLUDE ALL THE POSSIBILITIES. HOWEVER, THE ABOVE METHOD APPLIES IF WE CAN SOLVE FOR ANY TWO OF THE FOUR QUANTITIES $y(a)$, $y'(a)$, $y(b)$, $y'(b)$ IN TERMS OF THE OTHER TWO. THIS IS DONE IN THE ABOVE EXAMPLE.

* TWO INTEGRALS INVOLVING GENERALIZED FUNCTIONS

THE FOLLOWING TWO INTEGRALS APPEAR FREQUENTLY IN ACOUSTICS PROBLEMS. THE FIRST INTEGRAL WAS EVALUATED IN MY PH.D. THESIS BY A VERY LONG METHOD. THE METHOD USED HERE IS BASED ON THE TECHNIQUE USED FOR THE EVALUATION OF THE 2ND INTEGRAL. THIS LATTER INTEGRAL WAS EVALUATED IN A PH.D. THESIS BY HAL BLACKBURN. (*) I AM NOT SURE WHETHER THE TECHNIQUE AND THE RESULT ARE HIS. THE TWO INTEGRALS ARE

$$I_1 = \int F(\vec{y})\, \delta'(g)\, d\vec{y}$$

$$I_2 = \int F(\vec{y})\, \delta(f)\, \delta'(g)\, d\vec{y}$$

f, g HAVE FINITE SUPPORT.

WE ASSUME THAT THE INTEGRALS ARE n-DIMENSIONAL. IN THE FOLLOWING, WE ASSUME THAT $\vec{A}$ IS AN UNKNOWN VECTOR TO BE SPECIFIED LATER.

i) EVALUATION OF I_1 — APPLYING DIVERGENCE THM AND NOTING THAT f HAS FINITE SUPPORT, WE HAVE

$$\int \nabla\cdot[F\vec{A}\,\delta(g)]\, d\vec{y} = 0 \qquad (*)$$

$$\nabla\cdot[F\vec{A}\,\delta(g)] = \nabla\cdot(F\vec{A})\,\delta(g) + F\vec{A}\cdot\nabla g\,\delta'(g)$$

NOW LET $\vec{A}\cdot\nabla g = 1$. TAKE $\vec{A} \parallel \nabla g \Rightarrow \vec{A} = \frac{\nabla g}{|\nabla g|^2}$

SUBSTITUTE FOR $\nabla\cdot[F\vec{A}\,\delta(g)]$ IN (*) TO GET

$$I_1 = -\int \nabla\cdot\left[\frac{F(\vec{y})\,\nabla g}{|\nabla g|^2}\right]\delta(g)\, d\vec{y}$$

WE NOW USE $d\vec{y} = \frac{dg\, d\Sigma}{|\nabla g|}$ WHERE $d\Sigma$ IS THE ELEMENT OF THE SURFACE AREA OF $g=$ CONST TO GET

$$I_1 = -\int_{g=0} \nabla\cdot\left[\frac{F(\vec{y})\nabla g}{|\nabla g|^2}\right]\frac{d\Sigma}{|\nabla g|}$$

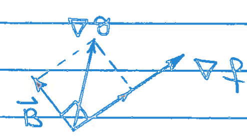
ii) EVALUATION OF I_2 – WITH THE SAME REASONING AS FOR I_1, WE HAVE

$$\int \nabla\cdot[F\vec{A}\,\delta(f)\delta(g)]\,d\vec{y} = 0 \qquad (**)$$

$$\nabla\cdot[F\vec{A}\,\delta(f)\delta(g)] = \nabla\cdot(F\vec{A})\,\delta(f)\delta(g) + F\vec{A}\cdot\nabla f\,\delta'(f)\delta(g) + F\vec{A}\cdot\nabla g\,\delta(f)\delta'(g)$$

NOW SELECT $\vec{A} \ni \vec{A}\cdot\nabla f = 0$, $\vec{A}\cdot\nabla g = 1$. WE CAN ASSUME THAT $\vec{A}$ IS IN THE PLANE CONTAINING (OR PARALLEL TO) ∇f AND ∇g

$$\vec{A} = \beta\vec{B}$$



$$\vec{B} = \nabla g - \frac{\nabla g\cdot\nabla f}{|\nabla f|}\frac{\nabla f}{|\nabla f|}$$

WE HAVE $\vec{A}\cdot\nabla f = \beta\vec{B}\cdot\nabla f = 0$

$$\vec{A}\cdot\nabla g = \beta\vec{B}\cdot\nabla g = 1$$

$$\beta = \frac{1}{\vec{B}\cdot\nabla g}$$

$$\vec{A} = \frac{|\nabla f|^2\,\nabla g - (\nabla f\cdot\nabla g)\,\nabla g}{|\nabla g|^2 - (\nabla f\cdot\nabla g)^2}$$

SUBSTITUTING THE RESULT FOR $\nabla \cdot [F \vec{A}\, \delta(f)\, \delta(g)]$ IN (**), WE GET

$$I_2 = - \int \nabla \cdot (F \vec{A})\, \delta(f)\, \delta(g)\, d\vec{y}$$

THE RESULT OF THIS INTEGRATION IS A SURFACE INTEGRAL OVER THE SUBSPACE $f = 0, g = 0$. IT WILL NOT BE REPEATED HERE.

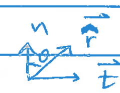
(*) HAL BLACKBURN IS PROF. FFOWCS WILLIAMS' STUDENT AT CAMBRIDGE. I READ A SECTION OF HIS PH.D. THESIS WHICH WAS GIVEN TO ME BY FW.

NOTE ADDED IN JUNE 9, 1991: FURTHER WORK ON I_2 HAS RESULTED IN OTHER APPROACHES FOR THE CASE $g = \tau - t + r/c$. THE ABOVE IDENTITY IS NO LONGER NEEDED. IN GENERAL, F IN I_2 MUST BE WRITTEN $\tilde{F} = F|_{f=0}$, i.e. RESTRICTION OF F TO $f = 0$. BELOW, WE WRITE Q FOR F IN I_2 SINCE WE WANT TO USE $F = [f(\vec{x}, t)]_{ret} = F(\vec{y}; \vec{x}, t)$. ALSO $g = \tau - t + r/c$ AND WE CONSIDER THE FOLLOWING INTEGRAL

$$I_3 = \int \tilde{Q}(\vec{y}, \tau)\, \delta'(f)\, \delta(g)\, d\vec{y}\, d\tau \quad \text{(NOTE PRIME ON } \delta'(f))$$

$$= \int \tilde{Q}(\vec{y}, t - r/c)\, \delta'(F)\, d\vec{y}$$

$$= - \int_{F=0} \frac{1}{|\nabla F|^2} \frac{\partial}{\partial N} \cdot \left[\tilde{Q}(\vec{y}, t - r/c)\right]_{F=0} d\Sigma$$



$$\frac{\partial}{\partial N} = \frac{\vec{n} - M_n \hat{\vec{r}}}{\Lambda} \cdot \nabla = \frac{(1 - M_n \cos\theta)\,\vec{n} - M_n \sin\theta\, \vec{t}}{\Lambda} \cdot \nabla \qquad \frac{\partial}{\partial n} \tilde{Q}(\vec{y}, t) = 0$$

$$\frac{\partial}{\partial N} \tilde{Q}(\vec{y}, t - r/c) = - \frac{M_n \sin\theta}{\Lambda} \frac{\partial \tilde{Q}}{\partial \sigma_t} - \frac{1}{c} \frac{\cos\theta - M_n}{\Lambda} \frac{\partial \tilde{Q}}{\partial \tau}$$

$$\frac{\partial}{\partial \sigma_t} = \vec{t} \cdot \nabla$$

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*IF THE NUMBER $2^n - 1$ IS A PRIME, THEN n IS PRIME.

PROOF: WE SHOW THAT IF n IS NOT A PRIME THEN $2^n - 1$ HAS A DIVISOR WHICH IS NOT 1 OR $2^n - 1$. LET $n = pq$ WHERE $p > 1$, $q > 1$ ARE POS. INTEGERS. THEN

$$2^n - 1 = \underbrace{1 + 2 + 2^2 + \cdots + 2^{pq-1}}_{pq \text{ TERMS}}$$

DIVIDE THE EXPRESSION ON THE RIGHT INTO q GROUPS AS FOLLOWS

$$2^n - 1 = \underbrace{1 + 2 + 2^2 + \cdots + 2^{p-1}}_{p \text{ TERMS}} + 2^p(1 + 2 + \cdots + 2^{p-1})$$

$$+ \cdots + 2^{pq-p}(1 + 2 + \cdots + 2^{p-1})$$

IT IS OBVIOUS THAT $2^n - 1$ IS DIVISIBLE BY

$$1 + 2 + \cdots + 2^{p-1} = 2^p - 1$$

SIMILARLY $2^n - 1$ IS DIVISIBLE BY $2^q - 1$

EXAMPLE $2^6 - 1$ IS DIVISIBLE BY $2^2 - 1 = 3$ AND $2^3 - 1 = 7$; $2^6 - 1 = 63 = 3 \times 21 = 7 \times 9$.

SOLUTION BY F.F.

PROBLEM TAKEN FROM "TEACHING PROBLEM-SOLVING SKILLS" BY ALAN H. SCHOENFELD, AM. MATH. MONTHLY, VOL. 87, NO. 10, DEC. 1980, 794-805

THIS PROBLEM IS TOO SIMPLE! WE HAVE $2^{pq} - 1 = (2^p)^q - 1 = (2^q)^p - 1$. SINCE $x^n - 1$ HAS $x - 1$ AS ITS DIVISOR $\Longrightarrow$ $(2^p)^q - 1$ HAS $2^p - 1$ AS ITS DIVISOR. SIMILARLY, $2^q - 1$ IS A DIVISOR OF $(2^q)^p - 1$.

(SOLUTION BY PAUL PAO. I AM GETTING OLD!)

* A NOTATION FOR THE DERIVATIVE OF A FUNCTION OF SEVERAL VARIABLES

C. H. EDWARDS IN "ADV. CALCULUS OF SEVERAL VARIABLES" INTRODUCES THE FOLLOWING MATRIX AS THE DEFN OF THE DERIVATIVE $\vec{F}'(\vec{x})$ WHERE $\vec{F} = (F_1, \ldots, F_m)$, $\vec{x} = (x_1, \ldots, x_n)$:

$$\vec{F}'(\vec{x}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & & \frac{\partial F_1}{\partial x_n} \\ & & & \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}, \quad m \times n \text{ MATRIX}$$

THIS DEFN HAS THE ADVANTAGE OF SIMPLIFYING THE KNOWN RESULTS AND ALSO DERIVING THEM. FOR EXAMPLE, WE HAVE

$$d\vec{F} = \vec{F}'(\vec{x})\, d\vec{x} \quad \text{(MATRIX MULTIPLICATION)}$$

ALSO LET $\vec{y} = (y_1, \ldots, y_m)$ BE NEW VARIABLES DEFINED AS FNS OF $\vec{x} = (x_1, \ldots, x_n)$ BY m RELATIONS

$$\vec{F}(\vec{x}, \vec{y}) = 0, \quad \vec{F} = (F_1, \ldots, F_m)$$

WE WANT TO FIND $\partial y_i / \partial x_j$. WE HAVE

$$d\vec{F} = \frac{\partial \vec{F}}{\partial \vec{x}} d\vec{x} + \frac{\partial \vec{F}}{\partial \vec{y}} d\vec{y} = 0$$

TO FIND $\partial y_i / \partial x_j$, WE NEED TO VARY ONLY x_j. SO WE HAVE

$$\frac{\partial \vec{F}}{\partial \vec{y}} d\vec{y} = -\frac{\partial \vec{F}}{\partial x_j} dx_j \quad \text{(NO SUM ON } j\text{)}$$

FROM THIS WE GET, BY CRAMER'S RULE

$$\frac{\partial y_i}{\partial x_j} = \frac{-\dfrac{\partial(F_1, F_2, \ldots F_j, \ldots F_m)}{\partial(y_1, \ldots, x_j, \ldots y_m)}}{\dfrac{\partial(F_1, F_2, \ldots, F_m)}{\partial(y_1, y_2, \ldots, y_m)}}$$

NOTE CANCELLATION OF SYMBOLS AND THE NEGATIVE SIGN OF THE FRACTION ON THE RIGHT.

THE CHAIN RULE OF DIFFERENTIATION CAN ALSO BE WRITTEN SIMPLY AS FOLLOWS. LET NEW COORDINATES $\vec{y}$ BE DEFINED BY $\vec{y} = \vec{\phi}(\vec{x})$, $\vec{y} = (y_1, y_2, \ldots, y_K)$

$\Longrightarrow$

$$\underset{(m \times K)}{\frac{d\vec{F}}{d\vec{y}}} = \underset{(m \times n)}{\frac{d\vec{F}}{d\vec{x}}} \; \underset{(n \times K)}{\frac{d\vec{x}}{d\vec{y}}} \qquad \text{(MATRIX MULTIPLICATION)}$$

$\vec{F} = (F_1, \ldots, F_m)$, $\vec{x} = (x_1, \ldots, x_n)$

* NOTES ON REGULARIZATION OF DIVERGENT INTEGRALS

IN REGULARIZATION OF DIVERGENT INTEGRAL A LIMITING PROCESS IS DEFINED USING HEAVISIDE FUNCTION AS FOLLOWS:

$$I(x) = \int_a^x \frac{f(y)}{y-x}\,dy = \lim_{\epsilon\to 0} \int_a^x f(y)\,\frac{\bar\partial}{\partial y}[H(\cdot)\ln|y-x|]\,dy$$

WHERE $H(\cdot) = H(y-x-\epsilon)$. ONE THEREFORE PROCEEDS FROM THIS STEP IN TWO WAYS TO GET $I(x)$

i) INTEGRATION BY PARTS GIVES

$$I(x) = \lim_{\epsilon\to 0}\left\{\left[f(y)\,H(\cdot)\ln|y-x|\right]_a^x - \int_a^{x-\epsilon} f(y)\ln|y-x|\,dy\right\}$$

$$= -f(a)\ln|x-a| - \int_a^x f(y)\ln|y-x|\,dy$$

ii) $\frac{\bar\partial}{\partial y}[H(\cdot)\ln|y-x|] = -\ln\epsilon\,\delta(y-x-\epsilon) + \frac{H(\cdot)}{y-x}$

$$-\ln\epsilon\int_a^x f(y)\,\delta(y-x-\epsilon)\,dy = -\ln\epsilon\, f(x-\epsilon)$$

$$= -\ln\epsilon\, f(x) + O(\epsilon)$$

$$= -f(x)\int_a^{x-\epsilon}\frac{dy}{y-x} + f(x)\ln|x-a| + O(\epsilon)$$

$$\therefore\ I(x) = \lim_{\epsilon\to 0}\left\{f(x)\ln|x-a| + \int_a^x H(\cdot)\,\frac{f(y)-f(x)}{y-x}\,dy + O(\epsilon)\right\}$$

$$= f(x)\ln|x-a| + \int_a^x \frac{f(y)-f(x)}{y-x}\,dy$$

WHERE NOW THE INTEGRAL IS CONVERGENT.

ONE NOW MUST JUSTIFY THE DEFINITION

$$\frac{1}{y-x} \equiv \lim_{\epsilon \to 0} \frac{\partial}{\partial y}\left[H(\cdot) \ln |y-x|\right] \qquad (*)$$

IN THE SOLUTION OF THE BOUNDARY VALUE PROBLEMS OF PHYSICS AND AERODYNAMICS, THE DIVERGENT INTEGRALS APPEAR AS A RESULT OF TAKING A DERIVATIVE INSIDE A CONVERGENT INTEGRAL:

$$-\frac{d}{dx}\int_a^x f(y) \ln |y-x|\, dy = J(x)$$

WE CAN WRITE

$$\int_a^x f(y) \ln |y-x|\, dy = \lim_{\epsilon \to 0} \int_a^\infty H(\cdot) f(y) \ln |y-x|\, dy$$

$$\therefore \quad J(x) = -\frac{d}{dx} \lim_{\epsilon \to 0} \int_a^\infty H(\cdot) f(y) \ln |y-x|\, dy$$

$$= + \lim_{\epsilon \to 0} \int_a^\infty f(y) \frac{\partial}{\partial y}\left[H(\cdot) \ln |y-x|\right] dy$$

THEREFORE, THE MOTIVATION FOR THE DEFINITION OF EQ. (*) ABOVE COMES FROM THE ABOVE PROCEDURE. WE NOTE THAT USING THIS DEFN, EVEN IF THE SINGULARITY IS AT THE POINT $y = x$, WE HAVE

$$\frac{d}{dx}\int_a^x \cdots\cdots = \int_a^x \frac{\partial}{\partial x} \cdots\cdots$$

FROM THE FOLLOWING EXAMPLE, IT IS SEEN THAT THE ORDER OF INTEGRATION CANNOT BE CHANGED ALWAYS FOR DOUBLE INTEGRALS(*). CONSIDER

$$I_1 = \int_0^x \int_0^\xi \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial \xi} \frac{1}{\sqrt{\xi - \eta}}\, d\eta\, d\xi$$

(*) NOT TRUE! SEE PAGE 119, F.F. 3/22/81

WE HAVE

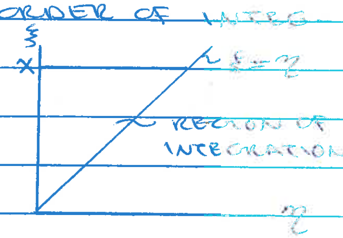
$$I = \int_0^{\xi} \frac{1}{\sqrt{\eta}} \frac{\partial}{\partial \xi} \frac{1}{\sqrt{\xi-\eta}} \, d\eta = \frac{\partial}{\partial \xi} \int_0^{\xi} \frac{d\eta}{\sqrt{\eta(\xi-\eta)}}$$

NOW LET $u = \frac{\xi}{\eta}$, SO THAT

$$I = \frac{\partial}{\partial \xi} \int_0^1 \frac{du}{\sqrt{u(1-u)}} = \frac{\partial}{\partial \xi} (\text{CONST.}) = 0 \quad (*)$$

$$\therefore \quad I_1 = 0.$$

HOWEVER, WHEN WE EXCHANGE THE ORDER OF INTEGRATION, WE OBTAIN



$$I_2 = \int_0^{x} \frac{d\eta}{\sqrt{\eta}} \int_{\eta}^{x} \frac{\partial}{\partial \xi} \frac{1}{\sqrt{\xi-\eta}} \, d\xi$$

$$= \lim_{\epsilon \to 0} \int_0^{x} \frac{d\eta}{\sqrt{\eta}} \left[\frac{H(\xi-\eta-\epsilon)}{\sqrt{\xi-\eta}} \right]_{\eta}^{x}$$

$$= \int_0^{x} \frac{d\eta}{\sqrt{\eta(x-\eta)}} = \int_0^1 \frac{du}{\sqrt{u(1-u)}} \quad \left(\text{BY TAKING } u = \frac{x}{\eta}\right)$$

NOW LET $u = \sin^2\theta$, $du = 2\sin\theta\cos\theta$

$$I_2 = 2\int_0^{\frac{\pi}{2}} d\theta = \pi \neq I_1 = 0$$

THIS RESULT MADE ME VERY UNEASY SINCE I ALWAYS THOUGHT THAT JUST AS DIFFERENTIATION AND INTEGRATION COMMUTE, TWO INTEGRALS ALSO COMMUTE. IN AERODYNAMIC PROBLEMS, THE ORDER OF INTEGRATION IS WELL-DEFINED SO THAT ONE IS ABLE TO SELECT THE CORRECT VALUE OF THE DOUBLE INTEGRAL. THIS IS BECAUSE, THE FIRST (INNER) INTEGRAL, FOR EXAMPLE, REPRESENTS THE PRESSURE AT A POINT ON AN AIRFOIL. THE OUTER INTEGRAL CAN THEN GIVE THE NET LOAD.

(*) THIS STEP IS WRONG. SEE P 119, F.F. 3/22/81

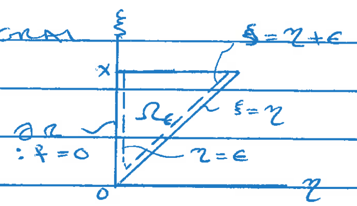
ON THE AVERAL. IT IS SEEN THAT THERE IS NO DOUBT ABOUT THE ORDER OF INTEGRATION IN THIS CASE. IT APPEARS THAT IN PRACTICAL PROBLEMS THIS IS ALWAYS THE CASE. ALSO, IT IS ENCOURAGING THAT THE DOUBLE INTEGRALS DISCUSSED ABOVE ARE ALWAYS DEFINED IN TERMS OF WELL-DEFINED LIMIT OF CONVERGENT INTEGRALS WHOSE BEHAVIOR CAN BE STUDIED BY CLASSICAL ANALYSIS. IT IS ALSO CLEAR THAT EVEN IF THE HEAVISIDE FUNCTION AND ITS DERIVATIVE ARE USED IN THE ABOVE STUDY, THE FUNCTIONS INVOLVED IN THE INTEGRAND DO <u>NOT</u> ALLOW THE USE OF GENERALIZED FUNCTION THEORY AS LINEAR FUNCTIONALS ON SPACE OF C^∞ FNS (OR EVEN C^n FNS) WITH BOUNDED SUPPORT. I SUSPECTED THAT THIS MAY BE THE REASON WHY EXCHANGE OF ORDER OF INTEGRATION DOES NOT RESULT IN A COMMON VALUE. BUT THIS IS NOT REALLY SO. THE BEST POLICY SEEMS TO BE TO STUDY EACH INTEGRAL INDIVIDUALLY. IT IS IMPORTANT TO GO TO THE ORIGIN OF THE INTEGRAL AND APPLY THE RESULTS OF CLASSICAL ANALYSIS, FOR EXAMPLE CONDITIONS FOR EXCHANGE OF ORDER OF INTEGRATION, BEFORE TAKING THE LIMIT PROCESS USED IN REGULARIZATION.

READ THE NEXT NOTE.
ALSO NOTE ON PAGE 128
F.F. 3/22/81
4/3/81

* NOTE ON CHANGE OF ORDER OF INTEGRATION DISCUSSED IN PRECEDING NOTE

WE WILL AGAIN CONSIDER THE INTEGRAL

$$\tilde{I} = \int_{\Omega} \frac{\bar{\partial}}{\partial \xi} \frac{1}{\sqrt{\eta(\xi-\eta)}}\, dS$$



LET $f=0$ DESCRIBE $\partial\Omega$ AND $g>0$ DESCRIBE THE REGION Ω_ϵ AS SHOWN. THEN WE CAN APPLY DIVERGENCE THM TO GET

$$\tilde{I} = \lim_{\epsilon\to 0}\int_{\partial\Omega} \frac{n_2 H(g)}{\sqrt{\eta(\xi-\eta)}}\, d\ell \quad , \{(n_1,n_2) \text{ UNIT OUTWARD NORMAL}, \; H(g)=0 \text{ ON } \xi=0 \text{ AND } \eta=\xi.\}$$

$$= \int_0^x \frac{d\eta}{\sqrt{\eta(x-\eta)}} = \pi = I_2! \quad (P\ 117)$$

$$g = (\eta-\epsilon)(\xi-\epsilon)(\xi-\eta+\epsilon)$$

WHERE DID WE MAKE A MISTAKE IN EVALUATION OF I_1?

WE NOTE THAT THE FIRST STEP IN EVALUATION I_1 MUST BE WRITTEN AS (IN THE NOTATION OF PRECEDING NOTES):

$$I = \frac{\bar{\partial}}{\partial\xi}\int_{\epsilon}^{\xi-\epsilon} H(\tilde{g}) \frac{d\eta}{\sqrt{\eta(\xi-\eta)}} \quad (*) \quad , \quad u = \frac{\eta}{\xi}$$

$$= \frac{\bar{\partial}}{\partial\xi}\int_{\frac{\epsilon}{\xi}}^{\frac{\xi-\epsilon}{\xi}} H(\tilde{g}) \frac{du}{\sqrt{u(1-u)}} \quad , \; H(\tilde{g}) \text{ IS NECESSARY HERE.}$$

$$\tilde{g} = \xi - \epsilon$$

$$= 2\frac{\bar{\partial}}{\partial\xi}\left\{\left[\sin^{-1}\sqrt{\frac{\xi-\epsilon}{\xi}} - \sin^{-1}\sqrt{\frac{\epsilon}{\xi}}\right] H(\tilde{g})\right\}$$

NOW $I_1 = \lim_{\epsilon\to 0}\int_0^x I\, d\xi = 2\lim_{\epsilon\to 0}\left[\sin^{-1}\sqrt{\frac{\xi-\epsilon}{\xi}}\right]_x = \pi = I_2.$

WE NOTE THAT WE CANNOT LET $\epsilon \to 0$ IN THE FIRST STEP, EQ.(*), SINCE DEPENDENCE ON ξ DISAPPEARS. HERE WE MUST CONVINCE OURSELVES THAT $\frac{\bar{\partial}}{\partial\xi}$ CAN BE

BROUGHT OUT OF THE INTEGRAL IN I. THE STEPS ARE AS FOLLOWS:

$$I = \int_0^{\xi} \frac{\partial}{\partial \xi} \frac{1}{\sqrt{\eta(\xi-\eta)}}\, d\eta \equiv \int_0^{\xi} \frac{\partial}{\partial \xi} \frac{H(g)}{\sqrt{\eta(\xi-\eta)}}\, d\eta,$$

WE HAVE

$$\frac{\partial}{\partial \xi} \int_0^{\xi} \frac{H(g)}{\sqrt{\eta(\xi-\eta)}}\, d\eta = \frac{\partial}{\partial \xi} \int_0^{\infty} \frac{H(g)}{\sqrt{\eta(\xi-\eta)}}\, d\eta$$

$$= \int_0^{\infty} \frac{\partial}{\partial \xi} \frac{H(g)}{\sqrt{\eta(\xi-\eta)}}\, d\eta$$

$$= \int_0^{\xi} \frac{\partial}{\partial \xi} \frac{H(g)}{\sqrt{\eta(\xi-\eta)}}\, d\eta$$

THE ABOVE RESULTS INDICATES THAT GENERALIZED DERIVATIVE OF $f(\vec{x})$ CAN BE DEFINED AS

$$\bar{D}^K f(\vec{x}) = \lim_{\epsilon \to 0} \bar{D}^K \left[H(g_\epsilon) f(\vec{x}) \right]$$

WHERE $g_\varepsilon = 0$ IS THE SURFACE SURROUNDING THE POINTS (OR SURFACES) OF DISCONTINUITY OF $f(\vec{x})$ AND ε IS A PARAMETER WHICH LETS g_ε COINCIDE WITH THE POINTS (OR SURFACES) OF DISCONTINUITY AS $\varepsilon \to 0$. g_ε IS DEFINED SO THAT $g_\varepsilon < 0$ IN THE REGION WHERE THE DISCONTINUITIES ARE. ALSO NOTE THAT

$$H(g_\varepsilon) f(\vec{x}) = \begin{cases} 0 & g_\varepsilon < 0 \\ f(\vec{x}) & g_\varepsilon > 0 \end{cases}$$

i.e. IF $f(\vec{x})$ BECOMES INFINITE AT SOME POINT IN THE REGION $g_\varepsilon < 0$, WE ARE TAKING $H(g_\varepsilon) f(\vec{x}) = 0$ THERE. THIS IS SIMPLY OUR DEFN.

* NOTE ON A NONLINEAR FIRST ORDER EQUATION

CONSIDER THE FIRST ORDER EQ.

$$p^2 + qy - z = 0 \quad , \quad p = \frac{\partial z}{\partial x} , \; q = \frac{\partial z}{\partial y}$$

THE CHARPIT'S RELATION OF THIS EQ. IS

$$\frac{dx}{2p} = \frac{dy}{y} = \frac{dz}{2p^2 + qy} = \frac{dp}{-p} = \frac{-dq}{0}$$

FROM $dq = 0$, WE GET $q = a$, (CONST.). FROM $\frac{dx}{2p} = \frac{dp}{p}$, WE GET $p = \frac{1}{2}(x+b)$ SO THAT

$$dz = \frac{1}{2}(x+b)\,dx + a\,dy$$

$$z = \frac{1}{4}(x+b)^2 + ay$$

NOW IF WE HAD USED THE RELATION $\frac{dp}{p} = \frac{dy}{y}$, WE WOULD GET $p = cy$ AND WE MAY BE TEMPTED TO SUBSTITUTE $q = a$ AND $p = cy$ IN THE D.E. TO GET $c^2y^2 + ay - z = 0$. HOWEVER, WHEN WE CHECK THIS SOLUTION, WE DO NOT SATISFY THE D.E.! WE HAVE DONE SOMETHING WRONG.

WHAT DOES CHARPIT'S RELATION GIVE US? TAKING x AS THE INDEPENDENT VARIABLE OF THE FOUR 1ST ORDER O.D.E.'S FOR y, z, p, q, WE GET

$$\begin{cases} y = \phi(x, c_1, c_2, c_3, c_4) \\ z = \psi(x, c_1, c_2, c_3, c_4) \\ p = \chi(x, c_1, c_2, c_3, c_4) \\ q = \eta(x, c_1, c_2, c_3, c_4) \end{cases}$$

WE HAVE A RELATION BETWEEN THE FOUR CONSTANTS THRU $F(x, y, z, p, q) = p^2 + qy - z = 0$. NOW IF WE SPECIFY A POINT (x_0, y_0, z_0) WHERE A SOLUTION

CURVE OF CHARPIT'S RELATION SHOULD PASS, THEN WE ARE LEFT WITH ONE PARAMETER TO DEAL WITH. THE VALUE OF p AT (x_0, y_0, z_0) IS ARBITRARY. CORRESPONDING TO EACH p, A DIFFERENT CURVE WILL PASS THRU (x_0, y_0, z_0). THESE CURVES A SURFACE WHICH IS TANGENT TO THE CHARACTERISTIC CONE AT THE POINT (x_0, y_0, z_0). ALONG EACH CURVE THE VALUES OF LOCAL p AND q ARE KNOWN SO THAT WE, IN FACT, HAVE A STRIP ASSOCIATED WITH EACH CURVE.

AN INTEGRAL SURFACE WHICH WE ARE SEARCHING FOR IS FORMED BY STRIPS (CALLED CHARACTERISTIC STRIP) DISCUSSED ABOVE. NOW IF WE TAKE

$$q = a, \quad y = cp$$

THE EQ. $z = c^2y^2 + ay$ GIVES THE RELATION BETWEEN y AND z ON THE CHARACTERISTIC STRIP AND NOT THE INTEGRAL SURFACE ITSELF.

IF WE NOW TRY TO PASS AN INTEGRAL SURFACE THRU $z = 0, 2y - x = 0$, WE PROCEED AS FOLLOWS. FROM THE COMPLETE INTEGRAL $(x+b)^2 + 4ay - 4z = 0$ WE GET

$$\begin{cases} f = (x+b)^2 + 2ax = 0 \\ x + b = 0 \end{cases} \quad \because \frac{\partial f}{\partial b} = 0$$

$\Rightarrow a = -2b$. WE MUST FIND THE ENVELOPE OF 1-PARAMETER FAMILY OF FNS $(x+b)^2 - 8by - z = 0$. IT IS $z + 4y^2 - 2xy = 0$.

CAN WE FIND THIS SURFACE IF WE HAD SOLVED

ALL THE O.D.E.'S OBTAINED FROM CHARPIT'S RELATION. WE HAVE

$$p = \frac{1}{2}(x+b)$$
$$q = a$$
$$y = cp = \frac{c}{2}(x+b) \equiv c'(x+b) \quad \text{(DROP PRIME!)}$$

TO FIND AN EQ. FOR Z, WE NOTE THAT

$$\frac{dz}{2p^2+qy} = \frac{dz}{p^2+\underbrace{(p^2+qy)}_{z \text{ FROM P.E.}}} = \frac{dz}{p^2+z} = \frac{dp}{p}$$

$$= \frac{dz-2p\,dp}{(p^2+z)-2p^2} = \frac{d(z-p^2)}{-p^2+z}$$

$$\therefore \quad z-p^2 = \frac{1}{2}e\,p, \quad z = p^2+\frac{1}{2}e\,p, \quad e = \text{CONST.}$$
$$z = \frac{1}{4}(x+b)^2 + \frac{e}{4}(x+b)$$

WE HAVE $p^2+qy-z=0$. THIS GIVES US $\underline{e = 4ac}$

LET t BE THE VARIABLE FOR x ON THE CURVE $2y-x=0$, $z=0$, WE MUST HAVE

$$\frac{1}{4}(t+b)^2 + ac(t+b) = 0 \qquad \text{i.e. } z=0$$

$$\Rightarrow \quad t+b+4ac = 0 \qquad (1)$$
$$y = \frac{t}{2} = c(t+b)$$
$$\Rightarrow \quad t-2c(t+b) = 0 \qquad (2)$$
$$\vec{T} = (1, \tfrac{1}{2}, 0) \qquad \text{TANGENT TO THE GIVEN CURVE}$$
$$\vec{T}\cdot(p,q,-1) = 0 \qquad \text{CONDITION OF THE CURVE TO LIE ON INTEGRAL SURFACE}$$
$$\Rightarrow \quad t+b+a = 0 \qquad (3)$$

FROM (1) AND (3) WE GET $c = \frac{1}{4}$. FROM (2) WE GET $b = t$ AND FROM (3) $a = -2t$. NOW

FROM THE EQS. FOR y AND z, WHICH ARE

$$\begin{cases} y = \frac{1}{4}(x+t) \\ z = \frac{1}{4}(x+t)^2 - \frac{t}{2}(x+t), \end{cases}$$

WE CAN ELIMINATE t TO GET

$$z = -4y^2 + 2xy .$$

THIS IS THE SAME AS THE SURFACE OBTAINED FROM THE COMPLETE INTEGRAL.

ALTHOUGH THE APPROACH IN SOLVING NONLINEAR FIRST ORDER PDE'S IS GEOMETRIC AND GEOMETRY IS ONE OF MY FAVORITE SUBJECTS, I HAVE HAD A LOT OF DIFFICULTIES LEARNING ABOUT THE 1ST ORDER PDE'S. THE WHOLE IDEA OF CONSTRUCTING THE CHARACTERISTIC STRIP IS VERY INGENIOUS. I FOUND COURANT AND HILBERT, VOL II VERY USEFUL. THIS BOOK HAS MUCH MATERIAL WHICH CANNOT BE FOUND IN MODERN AND ABSTRACT (i.e. USELESS!) BOOKS.

* NOTE ON GENERALIZED FOURIER TRANSFORM OF $\ln|x|$

IN THIS NOTE, THE F.T. OF $\ln|x|$ WILL BE DERIVED.

WE HAVE

$$\widehat{\ln|x|} = \int_{-\infty}^{\infty} \ln|x|\, e^{\kappa x \xi}\, dx \,, \quad \kappa = 2\pi i$$

$$= \int_{-\infty}^{\infty} \ln|x| \frac{d}{dx}\left(\frac{e^{\kappa x \xi}}{\kappa \xi}\right) dx$$

$$= \frac{-1}{\kappa \xi} \int_{-\infty}^{\infty} e^{\kappa x \xi} \frac{\bar{d}}{dx}(\ln|x|)\, dx$$

WHERE $\frac{\bar{d}}{dx}$ IS GENERERALIZED DERIVATIVE.

$H(g)$

$-\varepsilon \quad \varepsilon \quad x$

$g_\varepsilon(x) = (x-\varepsilon)(x+\varepsilon)$

$$\frac{\bar{d}}{dx} \ln|x| = \lim_{\varepsilon \to 0} \frac{d}{dx}\left[H(g_\varepsilon) \ln|x|\right]$$

$$= \lim_{\varepsilon \to 0}\left\{ \ln\varepsilon \left[-\delta(x+\varepsilon) + \delta(x-\varepsilon)\right] + \frac{H(g_\varepsilon)}{x}\right\}$$

$$\widehat{\ln|x|} = -\frac{1}{\kappa \xi} \lim_{\varepsilon \to 0}\left\{ \ln\varepsilon\left(e^{\kappa\xi\varepsilon} - e^{-\kappa\xi\varepsilon}\right)\right.$$

$$\left. + \int_{-\infty}^{\infty} \frac{H(g_\varepsilon)}{x} e^{\kappa x \xi}\, dx \right\}$$

WE HAVE

$$\lim_{\varepsilon \to 0} \ln\varepsilon\left(e^{\kappa\xi\varepsilon} - e^{-\kappa\xi\varepsilon}\right) = \lim_{\varepsilon \to 0} 2i \ln\varepsilon \sin 2\pi\xi\varepsilon$$

$$= 0$$

$$\therefore \quad \widehat{\ln|x|} = -\frac{1}{\kappa \xi} \widehat{PV\left(\frac{1}{x}\right)}$$

TO CALCULATE $\widehat{PV\left(\frac{1}{x}\right)}$ WE NOTE THAT

$$\widehat{\operatorname{sig} x} = \lim_{\eta \to 0}\left[\int_0^{\infty} e^{\kappa x \zeta}\, dx - \int_{-\infty}^{0} e^{\kappa x \bar{\zeta}}\, dx\right]$$

WHERE $\zeta = \xi + i\eta$ AND $\bar{\zeta} = \xi - i\eta$. HERE WE ARE USING A THM OF BREMERMANN WHICH SAYS

THAT A DISTRIBUTION IS THE BOUNDARY VALUE OF TWO ANALYTIC FUNCTIONS AS FOLLOWS:

$$f(x) = \lim_{y \to 0} [F(z) + G(\bar{z})] \quad , z = x + iy$$

WHERE F AND G ARE ANALYTIC ABOVE AND BELOW REAL AXIS, RESPECTIVELY. LET $f(x)$ BE AN ORDINARY FUNCTION WHICH BEHAVES AS $|x|^n$ AT INFINITY. THEN LET $f = f_+ + f_-$ WHERE

$$f_+(x) = \begin{cases} f & x > 0 \\ 0 & x < 0 \end{cases}, \quad f_-(x) = \begin{cases} 0 & x > 0 \\ f & x < 0 \end{cases}$$

THEN

$$\hat{f}(\xi) = \lim_{\eta \to 0} [\hat{f}_+(\xi) + \hat{f}_-(\bar{\xi})]$$

WHERE

$$\hat{f}_+(\xi) = \int_{-\infty}^{\infty} f_+(x)\, e^{ix\xi}\, dx$$

$$\hat{f}_-(\xi) = \int_{-\infty}^{\infty} f_-(x)\, e^{ix\xi}\, dx$$

WE THEREFORE HAVE

$$\widehat{\mathrm{sig}}(x) = \lim_{\eta \to 0} \frac{-1}{i} \left[\frac{1}{\xi} + \frac{1}{\bar{\xi}} \right]$$

IT CAN BE SHOWN EASILY

$$\lim_{\eta \to 0} \frac{1}{\xi + i\eta} = PV\left(\frac{1}{\xi}\right) - i\pi\delta(\eta)$$

$$\lim_{\eta \to 0} \frac{1}{\xi - i\eta} = PV\left(\frac{1}{\xi}\right) + i\pi\delta(\eta)$$

$$\therefore \widehat{\mathrm{sig}}(x) = -\frac{2}{i} PV\left(\frac{1}{\xi}\right) = -\frac{1}{\pi i} PV\left(\frac{1}{\xi}\right)$$

$$\therefore \widehat{PV\left(\frac{1}{x}\right)} = -\pi i\, \widehat{\widehat{\mathrm{sig}}}(\xi) = -\pi i\, \mathrm{sig}(-\xi)$$

$$= \pi i\, \mathrm{sig}(\xi)$$

$$\therefore \widehat{f_m(x)} = -\frac{\mathrm{sig}(\xi)}{2\xi} = -\frac{1}{2|\xi|}.$$

THE FUNCTION $\frac{1}{|\xi|}$ MUST BE TREATED AS $\mathrm{Sig}(\xi)\,PV(\frac{1}{\xi})$. THIS IS OBVIOUS FROM THE VERY FIRST STEP. WE SHOULD HAVE WRITTEN

$$e^{ix\xi} = \frac{d}{dx}\left[e^{ix\xi}\, PV\left(\frac{1}{i\xi}\right)\right]$$

* GENERALIZED FUNCTIONS ON THE SPACE OF PIECEWISE C^m FUNCTIONS

MOST OF THE PROBLEMS OF APPLIED MATHEMATICS INVOLVE FUNCTIONS WHICH WHEN THEY APPEAR UNDER AN INTEGRAL, DO NOT ALLOW INTERPRETATION OF THE RESULTING LINEAR FUNCTIONALS AS GENERALIZED FUNCTIONS ON SPACE D OR S. HERE D AND S ARE THE SPACES OF C^∞ FNS WITH BOUNDED SUPPORT AND RAPIDLY DECREASING C^∞ FNS, RESPECTIVELY. THIS IS VERY UNFORTUNATE SINCE THE OPERATIONAL PROPERTIES OF G.F.'S ARE VERY USEFUL TOOLS. STUDYING MANY POSSIBLE CASES, I FELT THAT ONE REALLY NEEDS TO DEFINE GENERALIZED FUNCTIONS ON THE SPACE OF PIECEWISE C^m FNS. ONE ENCOUNTERS SEVERAL DIFFICULTIES THAT CAN BE OVERCOME BUT THE RESULTS ARE MUCH MORE USEFUL THAN THOSE OF THE SPACES D' AND S'. OUR MAIN AIM IS TO INCLUDE δ-FN AND ITS DERIVATIVES INTO THE NEW SPACE OF FNS. WE NOTE THAT ONE WOULD NATURALLY WANT TO DEFINE $\int f(x)\delta(x)\,dx = f(0)$ IF $f \in C$ AT $x=0$. IF $f \notin C'$ AT $x=0$, THEN ONE SHOULD NOT EXPECT $\int f(x)\delta'(x)\,dx$ TO BE DEFINED WITHOUT SOME PROPER INTERPRETATION. THE REASON THAT L. SCHWARTZ USED THE SPACE D FOR TEST FUNCTIONS IS THAT ALL FUNCTIONS IN THE SPACE L (LOCALLY LEBESGUE INTEGRABLE FUNCTIONS) WILL THEN BELONG TO D' AND ALSO ALL SUCH FUNCTIONS WILL BE INFINITELY DIFFERENTIABLE IN THE SENSE OF DISTRIBUTIONS. WE KNOW THAT $D \subset S \subset S' \subset D'$. WE CAN ALSO SEE

THAT IF $A \subset B$, AND A' AND B' ARE THE SPACE OF CONTINUOUS LINEAR FUNCTIONALS ON A AND B, RESPECTIVELY, THEN $A \subset B \subset B' \subset A'$. THE SPACE D IS THE LARGEST SET OF C^∞ FUNCTIONS WHERE FNS IN L WILL BE INFINITELY DIFFERENTIABLE. IN MOST CASE IN APPLICATIONS, ONE DOES NOT NEED INFINITE DIFFERENTIABILITY AND THEREFORE, ONE DOES NOT NEED C^∞ FNS FOR TEST FUNCTIONS.

WE WANT TO USE FNS WHICH ARE PIECEWISE C^m FNS AS TEST FNS. WE MUST SOMEHOW DEFINE A VALUE FOR EACH FN AND ITS DERIVATIVES (UPTO ORDER m) AT POINTS OF DISCONTINUITY OF TEST FNS. WE START WITH C^m FNS (I.E. THE FNS AS WELL AS THEIR DERIVATIVES UPTO ORDER m ARE CONTINUOUS), WE DEFINE CONTINUITY OF LIN. FUNCTIONALS ON THIS SPACE AS FOLLOWS: $F[\phi]$ CONTINUOUS IF $F[\phi_\nu] \to 0$ IF $\phi_\nu \xrightarrow{C^m} 0$. WE DEFINE $\phi_\nu \xrightarrow{C^m} 0$ IF ϕ_ν AND ALL ALL ITS DERIVATIVES (UPTO ORDER m) GO TO ZERO UNIFORMLY IN THE SAME BOUNDED INTERVAL (RELEVANT TO A PROBLEM UNDER STUDY). IF THE PROBLEM HAS INFINITE DOMAIN, THEN $\phi_\nu^{(k)} \to 0$ UNIFORMLY $\forall k \leq m$ IN EVERY BOUNDED INTERVAL AND WE REQUIRE SOME BEHAVIOR AT $x = \pm\infty$ SO THAT THE FUNCTIONALS ARE DEFINED. WE DEFINE GENERALIZED FUNCTIONS ON SPACE C^m AS CONTINUOUS LINEAR FUNCTIONALS ON THIS SPACE.

LET $\phi \in C^m$ THEN $\Delta[\phi] = \phi(0)$ IS A CONT. LIN. FUN'L

THIS CAN BE IDENTIFIED WITH DELTA FUNCTION AND IF WE TAKE $\phi \in C^1(-a,a)$, $a>0$, WE CAN DEFINE $\Delta'[\phi] = -\Delta[\phi'] = -\phi'(0)$.

FOR GENERALIZED DIFFERENTIATION, WE FIRST CONSIDER FUNCTIONS SUCH THAT $|f\phi| \to 0$ AS $x \to \infty$, WE DEFINE

$$\int_{-\infty}^{\infty} \bar{f}' \phi \, dx = -\int_{-\infty}^{\infty} f \phi' \, dx, \quad \phi \in C^m$$

SIMILARLY

$$\int \bar{f}^{(k)} \phi \, dx = (-1)^k \int_{-\infty}^{\infty} f \phi^{(k)} \, dx, \quad \phi \in C^m, \ k \le m$$

FROM THIS DEFN, WE CAN SEE ALL LEBESGUE INTEGRABLE FNS FOR WHICH THE RIGHT SIDE OF THE ABOVE RELATION HAS A MEANING, HAVE GEN. DERIVATIVES OF UPTO ORDER m ON C^m FNS. IF f IS PIECEWISE CONTINUOUS WITH JUMPS OF Δf_i AT x_i, $i=1,2,\ldots$, THEN

$$\bar{f}' = f' + \sum_i \Delta f_i \, \delta(x - x_i)$$

IF f HAS BOUNDED SUPPORT, THEN $|f\phi| \to 0$ AS $x \to \infty$ AND WE DO NOT PUT ANY RESTRICTION ON ϕ. FOR INTEGRALS OVER BOUNDED DOMAIN, WE CAN ALWAYS USE A RESTRICTION $\tilde{f}$ OF f ON THE DOMAIN

$$F[\phi] = \int_a^b f \phi \, dx = \int_{-\infty}^{\infty} \tilde{f} \phi \, dx$$

$$F'[\phi] = \int \bar{\tilde{f}}' \phi \, dx$$

$$= -\int \tilde{f} \phi' \, dx$$

$$= -\int_a^b \tilde{f} \phi' \, dx$$

WE NOTE THAT

$$\bar{\tilde{f}}' = \tilde{f}' + f(a)\,\delta(x-a) - f(b)\,\delta(x-b)$$

NOW LET US ASSUME BOTH ϕ AND f ARE PIECE-WISE CONTINUOUS BUT THEIR POINTS OF DISCONTINUITY DO NOT COINCIDE. THEN ASSUMING THAT f & ϕ ALSO $\in$ PWC^1

$$\int_a^b f'\phi\,dx = \int_a^c + \int_c^d + \int_d^b f'\phi\,dx$$

WHERE $x=c$ IS THE SINGLE POINT OF DISCONTINUITY OF f. ASSUME $\Delta f = f(c+) - f(c-)$. SIMILARLY, $x=d$ IS THE ONLY POINT OF DISCONTINUITY OF ϕ AND $\Delta\phi = \phi(d+) - \phi(d-)$

$$\int_a^b f'\phi\,dx = f(b)\phi(b) - f(a)\phi(a) - \Delta f\,\phi(c) - \Delta\phi\,f(d) - \int_a^b \phi' f\,dx$$

LET $\tilde{f}$ BE THE RESTRICTION OF f TO $[a,b]$, THEN, AFTER REARRANGING TERMS ABOVE, WE HAVE

$$\int_a^b f'\phi\,dx - f(b)\phi(b) + f(a)\phi(a) + \Delta f\,\phi(c) = -\int_a^b \phi' f\,dx - \Delta\phi\,f(d)$$

$$\int_{-\infty}^{\infty}[\tilde{f}' + f(a)\delta(x-a) - f(b)\delta(x-b) + \Delta f\,\delta(x-c)]\phi(x)\,dx$$

$$= -\int [\phi' + \Delta\phi\,\delta(x-d)]\,f(x)\,dx$$

i.e. $\displaystyle\int_{-\infty}^{\infty} \bar{\tilde{f}}'\phi\,dx = -\int_{-\infty}^{\infty} \tilde{f}\,\bar{\phi}'\,dx$

NOTE THAT THE LEFT SIDE IS A GEN FN BASED ON ϕ WHILE THE RIGHT SIDE IS BASED ON $\tilde{f}$.

NOW WE ASSUME BOTH f AND $\phi \in$ PWC^1 AND THEY ONLY HAVE ONE JUMP AT $x=c$, $\Delta\phi = \phi(c+) - \phi(c-)$ AND $\Delta f = f(c+) - f(c-)$. THEN

$$\int_a^b f'\phi\,dx = \int_a^c + \int_c^b f'\phi\,dx$$

$$= f(b)\phi(a) - f(a)\phi(a) - \phi(c+)f(c+) + \phi(c-)f(c-) - \int_a^b f\phi'\,dx$$

WE CAN WRITE

$$f(c+)\phi(c+) - f(c-)\phi(c-) = \left[\frac{\Delta f}{2} + f_{av}\right]\phi(c+)$$
$$- \left[f_{av} - \frac{\Delta f}{2}\right]\phi(c-)$$
$$= f_{av}\left[\phi(c+) - \phi(c-)\right]$$
$$+ \Delta f \, \frac{\phi(c+) + \phi(c-)}{2}$$
$$= f_{av}\,\Delta\phi + \Delta f\,\phi_{av}$$

WHERE $f_{av} = \frac{1}{2}[f(c+) + f(c-)]$, $\phi_{av} = \frac{1}{2}[\phi(c+) + \phi(c-)]$.
LET US DEFINE $\phi(c) = \phi_{av}$ AND $f(c) = f_{av}$, THEN WE HAVE

$$f(c+)\phi(c+) - f(c-)\phi(c-) = f(c)\,\Delta\phi + \Delta f\,\phi(c)$$

SO THAT

$$\int_a^b f'\phi\,dx = f(b)\phi(b) + f(a)\phi(a) + \Delta f\,\phi(c)$$

$$= \int_{-\infty}^{\infty}[\tilde{f}' + f(a)\,\delta(x-a) - f(b)\,\delta(x-b) + \Delta f\,\delta(x-c)]\,\phi\,dx$$

$$= -\int_{-\infty}^{\infty}[\phi' + \Delta\phi\,\delta(x-c)]\,\tilde{f}\,dx$$

SO THAT AGAIN WE HAVE

$$\int \tilde{f}'\phi\,dx = -\int \tilde{f}\,\bar{\phi}'\,dx$$

WHERE AS BEFORE $\tilde{f}$ IS THE RESTRICTION OF f TO $[a,b]$. WE ARE NOW READY TO DEFINE G.F. ON THE SPACE OF PWC^m FUNCTIONS $\mathcal{C}_m$.

<u>DEFN.</u> THE FNS IN $\mathcal{C}_m$ ARE PIECEWISE CONTINUOUS

AND HAVE PIECEWISE CONTINUOUS DERIVATIVES OF UPTO ORDER m. WE DEFINE $\phi^{(k)}(x_0)$ AT THE POINT OF DISCONTINUITY x_0 OF $\phi^{(k)}(x)$ AS FOLLOWS

$$\phi^{(k)}(x_0) = \frac{1}{2}\left[\phi^{(k)}(x_0+) + \phi^{(k)}(x_0-)\right]$$

DEFN: SEQ $\{\phi_n\}$, $\phi_n \in \mathcal{C}_m$ GOES TO ZERO IN $\mathcal{C}_m$, i.e. $\phi_n \xrightarrow{\mathcal{C}_m} 0$, IF IN ANY BOUNDED INTERVAL

$$\phi_n^{(k)} \to 0 \quad \text{UNIFORMLY } \forall\ k = 0, 1, \ldots, m$$

DEFN: A FUNCTIONAL $F[\phi]$, $\phi \in \mathcal{C}_m$ IS CONTINUOUS IF $F[\phi_n] \to 0$ IF $\phi_n \xrightarrow{\mathcal{C}_m} 0$.

DEFN: A CONTINUOUS LINEAR FUNCTIONAL ON $\mathcal{C}_m$ IS CALLED A GENERALIZED FUNCTION. THIS SPACE WILL BE DENOTED BY $\mathcal{C}_m'$. WE INCLUDE IN THIS SPACE CONT. LIN. FUNCTIONALS DEFINED ON A SUBSET OF $\mathcal{C}_m$ ALSO.

THM: ANY PIECEWISE CONTINUOUS FUNCTION $f(x)$ WITH BOUNDED SUPPORT IS A GENERALIZED FUNCTION IN SPACE $\mathcal{C}_m$ BY THE RELATION $F[\phi] = \int_{-\infty}^{\infty} f\phi\,dx = \int_a^b f\phi\,dx$, $\phi \in \mathcal{C}_m$, WHERE $[a,b] = \operatorname{supp} f$.

WE WILL ALWAYS ASSUME THAT THE RANGE OF INTEGRATION IS OVER THE ENTIRE REAL LINE. FOR INTEGRALS OVER FINITE INTERVALS, WE DEFINE A RESTRICTION OF THE FUNCTION f ABOVE TO THE INTERVAL AS DONE ON PAGES 131 AND 132. THE CONTINUOUS LINEAR FUNCTIONALS GENERATED BY FUNCTIONS WITH BOUNDED SUPPORT ARE DEFINED ON THE ENTIRE SPACE $\mathcal{C}_m$.

FOR OTHER FUNCTIONALS, WE MUST RESTRICT THE TEST FUNCTIONS TO A SUBSET OF $\mathcal{C}^m$. FOR EXAMPLE, FOR $f(x) = x^n$, $F[\phi] = \int_{-\infty}^{\infty} x^n \phi(x)\,dx$ IS DEFINED IF $|x^{n+\alpha}\phi(x)| < C$ FOR LARGE x, $\alpha > 0$. WE DELIBERATELY DO NOT RESTRICT THE SPACE $\mathcal{C}_m$ SO THAT SO THAT WE HAVE THE LARGEST SET OF CONT. LINEAR FUNCTIONALS.

THM: EVERY PIECEWISE CONTINUOUS FUNCTION HAS GEN. DERIVATIVES OF UPTO ORDER m IN $\mathcal{C}_m'$. THIS DERIVATIVE IS DEFINED BY

$$F^{(k)}[\phi] = (-1)^k F[\phi^{(k)}(x)]. \quad \phi \in C^m \text{ [CONTINUOUS DERIVATIVES UPTO ORDER } m.]$$

IN FACT EVERY LEBESGUE INTEGRABLE FN HAS GEN. DERIVATIVES OF UPTO ORDER m IN $\mathcal{C}_m'$.

EXAMPLE: SINCE $\ln|x|$ IS INTEGRABLE, IT HAS A DERIVATIVE IN SPACE $\mathcal{C}_1'$. FOR EXAMPLE, LET

$$F[\phi] = \int_{-1}^{1} \ln|x|\,\phi(x)\,dx = \int_{-\infty}^{\infty} \widetilde{\ln}|x|\,\phi(x)\,dx$$

$$F'[\phi] = -\int_{-\infty}^{\infty} \widetilde{\ln}|x|\,\phi'(x)\,dx, \quad \phi \in C^1$$

$$= -\Big[\ln|x|\,\phi(x)\Big]_{-1}^{1} + PV\int_{-1}^{1} \frac{\phi(x)}{x}\,dx$$

HERE $\widetilde{\ln}|x|$ IS THE RESTRICTION OF $\ln|x|$ TO $[-1,1]$.

IN PROBLEMS INVOLVING FINITE INTERVALS, ONE CAN WRITE $\int_a^b f\phi\,dx = \int_{-\infty}^{\infty} \tilde{f}\phi\,dx = \int_{-\infty}^{\infty} f\tilde{\phi}\,dx = \int_{-\infty}^{\infty} \tilde{f}\tilde{\phi}\,dx$

THIS DOES NOT CAUSE ANY DIFFICULTIES AS SHOWN LATER.

WE NOTE THAT FOR FUNCTIONS WITH JUMP DISCONTINUI-

THUS WE ALWAYS HAVE

$$\int \bar{f}^{(m)} \phi \, dx = (-1)^m \int f \, \bar{\phi}^{(m)} \, dx$$

EVEN IF BOTH f AND ϕ DISCONTINUITIES AT THE SAME POINT. IN THIS CASE BOTH f AND ϕ SHOULD BELONG TO $\mathcal{E}_m$ AND THE TEST FUNCTION FOR THE LEFT SIDE IS ϕ WHILE FOR THE RIGHT SIDE IT IS f. IF $f \in C$ BUT $f^{(n)}$ HAS NONINTEGRABLE SINGULARITY AT $x = x_0$, THEN WE DEFINE $\bar{f}^{(n)}$ AS FOLLOWS:

$$\int \bar{f}^{(n)} \phi \, dx = (-1)^n \int f \, \bar{\phi}^{(n)} \, dx$$
$$= \lim_{\epsilon \to 0} \int \frac{\bar{d}^n}{dx^n} [f \, H(g_\epsilon)] \, \phi \, dx$$

WHERE IT IS ASSUMED THAT $\int f \, \bar{\phi}^{(n)} dx$ EXISTS. IT IS SUFFICIENT THAT $\phi^{(n)}$ HAS AN INTEGRABLE SINGULARITY AT $x = x_0$. HERE g_ϵ IS A FUNCTION WHICH IS LESS THAN ZERO ON AN INTERVAL OF LENGTH ϵ WHICH INCLUDES x_0 AND IT IS GREATER THAN ZERO OUTSIDE THIS INTERVAL. $H(\cdot)$ IS THE HEAVISIDE FUNCTION.

IN SUMMARY, IF $f^{(n)}$ AND $\phi^{(n)}$ BOTH ARE IN $\mathcal{E}_m$, THEN WE DEFINE $\int \bar{f}^{(n)} \phi \, dx = (-1)^n \int f \, \phi^{(n)} \, dx$
IF $f^{(n)}$ IS NOT INTEGRABLE AT SOME POINTS x_i, WE REQUIRE THAT $\int f \, \bar{\phi}^{(n)} dx$ BE DEFINED AND THIS REQUIRES OFTEN THAT $\phi^{(n-1)}$ BE CONTINUOUS AND $\phi^{(n)}$ HAS AT MOST A JUMP DISCONTINUITY AT POINTS x_i.
WE DEFINE ALL FUNCTIONALS ON THE ENTIRE REAL LINE USING RESTRICTION OF FUNCTIONS TO A FINITE INTERVAL IF NECESSARY. WE DO NOT WRITE THE INFINITE LIMITS ON INTEGRALS.

LET $F[\phi] = \int_a^b f \phi \, dx = \int \tilde{f} \phi \, dx = \int f \tilde{\phi} \, dx = \int \tilde{f} \tilde{\phi} \, dx$

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GPHM

LET $\phi, f \in C^1$, THEN $\tilde{f} = \begin{cases} f & x \in [a,b) \\ 0 & x \notin [a,b) \end{cases}$, $\tilde{f} \in \mathcal{C}_1$. WE HAVE

$$F'[\phi] = \int \bar{\tilde{f}}' \phi \, dx = + \int^b f' \phi \, dx + f(a)\phi(a) - f(b)\phi(b)$$

SINCE $\bar{\tilde{f}}' = \tilde{f}' + f(a)\delta(x-a) - {}^a f(b)\phi(b)$

WE HAVE

$$F'[\phi] = \int \bar{\tilde{f}}' \phi \, dx = -\int \tilde{f} \bar{\phi}' dx.$$

WE SEE IF $\phi \in C^1$, WE HAVE $\bar{\phi}' = \phi'$ AND WE CAN SHOW THE ABOVE RESULTS USING INTEGRATION BY PARTS.

IF WE NOW LOOK AT $\int \bar{f}' \tilde{\phi} \, dx$, WE HAVE

$$\int f' \tilde{\phi} dx = \int \bar{f}' \tilde{\phi} \, dx = -\int f \bar{\tilde{\phi}}' dx$$

$$= -\int f \tilde{\phi}' - f(a)\phi(a) + f(b)\phi(b)$$

WE SEE THAT $\int \bar{f}' \tilde{\phi} \, dx \neq \int \bar{f}' \phi = F'[\phi]$, ABOVE.
THEREFORE FOR OUR PURPOSE, THE FUNCTIONAL NOTATION IS NOT APPROPRIATE. AN INNER PRODUCT NOTATION IS MUCH BETTER. WE DEFINE

$$\int \bar{f}^{(n)} \phi \, dx = \langle \bar{f}^{(n)}, \phi \rangle = (-1)^n \langle f, \bar{\phi}^{(n)} \rangle$$

NOW WHICHEVER WAY WE DEFINE $\int_a^b f\phi \, dx$, WE HAVE

$$\langle \bar{\tilde{f}}', \phi \rangle = -\langle \tilde{f}, \bar{\phi}' \rangle$$

$$\langle \bar{\tilde{f}}', \tilde{\phi} \rangle = -\langle \tilde{f}, \bar{\tilde{\phi}}' \rangle$$

$$\langle \bar{f}', \tilde{\phi} \rangle = -\langle f, \bar{\tilde{\phi}}' \rangle$$

THE LEFT (AND RIGHT) SIDES OF THESE EQUALITIES ARE **NOT** THE SAME. IN THE ABOVE EXAMPLE, WITH $f, \phi \in C^1$, WE HAVE

$$\langle \bar{\tilde{f}}', \tilde{\phi} \rangle = \langle f' + f(a)\delta(x-a) - f(b)\delta(x-b), \tilde{\phi} \rangle$$

$$= \langle f', \tilde{\phi} \rangle + [f(a)\phi(a) - f(b)\phi(b)]/2$$

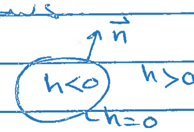
SINCE $\tilde{\phi}(a) = \frac{1}{2}[0 + \phi(a)] = \frac{1}{2}\phi(a)$, $\tilde{\phi}(b) = \frac{1}{2}\phi(b)$. HOWEVER

$$\langle \bar{f}', \tilde{\phi} \rangle = \langle f', \tilde{\phi} \rangle \quad \text{SINCE } f' \text{ HAS NO DISCONTINUITY.}$$

$$= \langle \bar{f}', \tilde{\phi} \rangle + \langle \tilde{\bar{f}}', \tilde{\phi} \rangle.$$

EXAMPLE LET $\vec{f}(\vec{x})$ BE A FUNCTION WHICH IS DISCONTINUOUS ACROSS THE SURFACE $g=0$ WITH THE JUMP $\Delta \vec{f} = \vec{f}(g=0+) - \vec{f}(g=0-)$. LET $h=0$ BE A SURFACE SUCH THAT IT ENCLOSES THE VOLUME V. ASSUME $g=0$ INTERSECTS THIS VOLUME. DEFINE $\tilde{f}$ AS FOLLOWS

$$\tilde{\vec{f}} = \begin{cases} \vec{f} & \vec{x} \in V \\ 0 & \vec{x} \notin V \end{cases}$$



$$\Rightarrow \int \bar{\nabla} \cdot \tilde{\vec{f}} \, d\vec{x} = - \int \tilde{\vec{f}} \cdot \bar{\nabla}(1) \, d\vec{x} = 0$$

$$\bar{\nabla} \cdot \tilde{\vec{f}} = \nabla \cdot \tilde{\vec{f}} - \vec{f} \cdot \nabla h \, \delta(h) + \Delta \tilde{\vec{f}} \cdot \nabla g \, \delta(g)$$

$$\underbrace{\int \nabla \cdot \tilde{\vec{f}} \, d\vec{x}}_{\int_V \nabla \cdot \vec{f} \, d\vec{x}} - \underbrace{\int \vec{f} \cdot \nabla h \, \delta(h) \, d\vec{x}}_{\int_{h=0} \vec{f} \cdot d\vec{S}} + \underbrace{\int \Delta \tilde{f} \cdot \nabla g \, \delta(g) \, d\vec{x}}_{\int_{h<0,\ g=0} \Delta \vec{f} \cdot d\vec{S}} = 0$$

OR

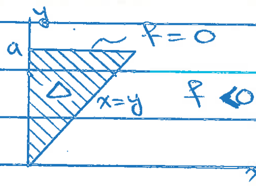
$$\int_V \nabla \cdot \vec{f} \, d\vec{x} = \int_{h=0} \vec{f} \cdot d\vec{S} - \int_{g=0,\ h<0} \Delta \vec{f} \cdot d\vec{S}$$

WITHOUT THE DISCONTINUITY IN $\vec{f}$, THIS RESULT IS, OF COURSE, THE DIVERGENCE THM! THE SECOND INTEGRAL IS A SURFACE INTEGRAL ON THE SET $\{g=0 \cap V\}$.

FOR FUNCTIONS WITH INFINITE DISCONTINUITY SUCH AS $\frac{1}{\sqrt{x(y-x)}}$, IT APPEARS THAT NO SATISFACTORY FUNCTIONAL VALUE CAN BE ASSIGNED TO GENERALIZED DERIVATIVES WHERE THE REGION OF INTEGRATION EXTENDS

TO THE CURVES OF DISCONTINUITY OF THE FUNCTION f. FOR EXAMPLE, IF Δ IS THE REGION SHOWN BELOW, THEN THE INTEGRAL

$$\int_{\Delta} \frac{\partial}{\partial y} \frac{1}{\sqrt{x(y-x)}} \, dx\,dy = I$$



CAN NOT BE SATISFACTORILY DEFINED.

WE HAVE TRIED TO GIVE A UNIQUE MEANING TO THIS INTEGRAL ON PAGES 116-120. WE MAY BE TEMPTED TO DEFINE I, AS IN PAGE 119, AS FOLLOWS

$$I = -\lim_{\epsilon \to 0} \int \frac{H(g_{\epsilon})}{\sqrt{x(y-x)}} \frac{\partial}{\partial y} H(f) \, dx\,dy \quad , \left\{ \text{SEE p 119 FOR } g_{\epsilon} \right.$$

HOWEVER,

$$\frac{\partial}{\partial y} H(f) = \frac{\partial f}{\partial y} \delta(f)$$

WE HAVE THE PRODUCT OF TWO GENERALIZED FNS $\frac{H(g_{\epsilon})}{\sqrt{x(y-x)}}$ AND $\frac{\partial f}{\partial y} \delta(f)$. WE SEE THAT $\sqrt{x(y-x)}$ IS ZERO ON $f=0$ CURVE. AS AN EXAMPLE OF DIFFICULTY OF DEFINING PRODUCTS OF GEN. FNS, WE LET $x_0 \in [x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2}]$. LET $g_{\epsilon}(x) = (x - x_0 + \frac{\epsilon}{2})(x - x_0 - \frac{\epsilon}{2})$, THEN $g_{\epsilon}(x) < 0$ WHEN $x \in [x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2}]$ AND $g_{\epsilon}(x) > 0$ OTHERWISE. THEN LET $f(x)$ BE A CONTINUOUS FN OF x, THEN IF $\phi \in C$, WE HAVE

$$\lim_{\epsilon \to 0} \langle f(x) H(g_{\epsilon}), \phi \rangle = \langle f, \phi \rangle$$

i.e. $$\lim_{\epsilon \to 0} f(x) H(g_{\epsilon}) = f(x)$$

BUT

$$\delta(x - x_0) f(x) H(g_{\epsilon}) = f(x_0) H[g_{\epsilon}(x_0)] = 0$$

$$\therefore \quad \lim_{\epsilon \to 0} \delta(x - x_0) f(x) H(g_{\epsilon}) = 0$$

HOWEVER,

$$f(x)\delta(x - x_0) = f(0)\delta(x - x_0) \neq \lim_{\epsilon \to 0} \delta(x - x_0) f(x) H(g_{\epsilon}) = 0$$

APR. 81
6PHM

I HAVE SPENT MANY DAYS TRYING TO COME UP WITH SOME SATISFACTORY INTERPRETATION OF INTEGRALS LIKE I. MOST OFTEN IN APPLICATIONS ONE HAS

$$I = \lim_{\varepsilon \to 0} \int \phi(\vec{x}) \frac{\partial}{\partial y} \frac{H(g_\varepsilon)}{\sqrt{x(y-x)}} \, dx\, dy$$

WHERE g_ε IS DEFINED ON P.119. $\phi(\vec{x})$ DOES NOT HAVE A DISCONTINUITY ON THE CURVE $f = 0$. WE THEN HAVE

$$I = \lim_{\varepsilon \to 0} - \int \frac{\partial \phi}{\partial y} \frac{H(g_\varepsilon)}{\sqrt{x(y-x)}} \, dx\, dy$$

$$= - \int_\Delta \frac{\partial \phi}{\partial y} \frac{1}{\sqrt{x(y-x)}} \, dx\, dy$$

THIS LAST RESULT IS NOW MEANINGFUL PROVIDED THAT THE ABOVE ASSUMPTION ON ϕ CONCERNING ITS SINGULARITY IS SATISFIED.

I WILL CONTINUE THIS LATER. I AM UNHAPPY ABOUT SOME OF THE METHODS I USED ABOVE.

JULY 81
GPHM

* A RESULT CONCERNING THE CURVATURE OF A CURVE

LET THE CURVE Γ BE DEFINED BY THE INTERSECTION OF TWO SURFACES $f=0$ AND $g=0$, PROVE THAT

$$K^2 \sin^2\theta = K_1^2 + K_2^2 - 2K_1K_2\cos\theta$$

WHERE K IS THE CURVATURE OF Γ, K_1 AND K_2 ARE THE NORMAL CURVATURES OF Γ ON $f=0$ AND $g=0$ AND θ IS THE ANGLE BETWEEN THE NORMALS TO THESE SURFACES. THIS PROBLEM IS FROM H. LASS "VECTOR AND TENSOR ANALYSIS" P 76, PROB. 4. K_1 AND K_2 ARE IN THE DIRECTION OF Γ.

PROOF 1: BY MEUSNIER THM

$$K = \frac{K_1}{\cos\theta_1} = \frac{K_2}{\cos\theta_2}$$

WHERE θ_1 AND θ_2 ARE THE ANGLES BETWEEN THE NORMAL $\vec{n}$ TO Γ AND THE NORMALS $\vec{n}_1$ AND $\vec{n}_2$ TO $f=0$ AND $g=0$, RESPECTIVELY. WE HAVE

$$K^2 = \frac{K_1^2}{\cos\theta_1^2} = \frac{K_2^2}{\cos^2\theta_2} = \frac{K_1K_2\cos\theta}{\cos\theta_1\cos\theta_2\cos\theta}$$

$$= \frac{K_1^2 + K_2^2 - 2K_1K_2\cos\theta}{\cos^2\theta_1 + \cos^2\theta_2 - 2\cos\theta_1\cos\theta_2\cos\theta}$$

OUR TASK IS TO PROVE

$$E = \cos^2\theta_1 + \cos^2\theta_2 - 2\cos\theta_1\cos\theta_2\cos\theta = \sin^2\theta$$

SINCE $\vec{n}$, $\vec{n}_1$ AND $\vec{n}_2$ ARE IN THE SAME PLANE, WE ASSUME θ_1 AND θ_2 ARE MEASURED IN TRIGONOMETRIC DIRECTION FROM $\vec{n}$ SO THAT $\theta = \theta_2 - \theta_1$.

WE HAVE

$E = \cos^2\theta_1(\cos^2\theta_2 + \sin^2\theta_2) + \cos^2\theta_2(\cos^2\theta_1 + \sin^2\theta_1)$

$\quad - 2\cos\theta_1\cos\theta_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$\quad = \sin^2(\theta_2 - \theta_1) = \sin^2\theta$. Q.E.D.

PROOF 2 – THE ABOVE METHOD, ALTHOUGH SATISFACTORY, DOES NOT LEAD ONE TO DISCOVER THE RESULT. PERHAPS THE FOLLOWING METHOD CAN HELP. WE CAN DECOMPOSE $\vec{n}$ AS: $\vec{n} = \alpha\vec{n}_1 + \beta\vec{n}_2$. BY TAKING DOT PRODUCT WITH $\vec{n}_1$ AND $\vec{n}_2$, RESPECTIVELY, WE OBTAIN

$$\alpha = \frac{\cos\theta_1 - \cos\theta_2\cos\theta}{\sin^2\theta}$$

$$\beta = \frac{\cos\theta_2 - \cos\theta_1\cos\theta}{\sin^2\theta}$$

NOW TAKING DOT PRODUCT OF $\vec{n} = \alpha\vec{n}_1 + \beta\vec{n}_2$ WITH $\vec{r}'' = \frac{d^2\vec{r}}{ds^2}$, S ALONG Γ, WE GET

$$K = \alpha K_1 + \beta K_2;$$

$$K^2 = \alpha^2 K_1^2 + \beta^2 K_2^2 + 2\alpha\beta K_1 K_2,$$

$$\alpha = \frac{\cos\theta_1(\cos^2\theta_2 + \sin^2\theta_2) - \cos\theta_1\cos(\theta_2-\theta_1)}{\sin^2\theta}$$

$$= \frac{\sin\theta_2}{\sin\theta},$$

$$\beta = \frac{-\sin\theta_1}{\sin\theta}$$

$$\Rightarrow K^2\sin^2\theta = K_1^2\underbrace{\sin^2\theta_2}_{1-\cos^2\theta_2} + K_2^2\underbrace{\sin^2\theta_1}_{1-\cos^2\theta_1} - 2K_1K_2\sin\theta_1\sin\theta_2$$

$$= K_1^2 + K_2^2 - 2K_1K_2[\overbrace{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}^{\cos\theta}]$$

$$- \underbrace{(K_1^2\cos^2\theta_2 + K_2^2\cos^2\theta_1 - 2K_1K_2\cos\theta_1\cos\theta_2)}_{(K_1\cos\theta_2 - K_2\cos\theta_1)^2 \,\equiv\, 0 \text{ BY MEUSNIER THM.}}$$

* NOTES ON TRANSFORMATION OF DEPENDENT AND INDEPENDENT VARIABLES

AFTER LEARNING ABOUT LIE GROUP THEORY FROM COHEN'S BOOK (1911), I AM CONVINCED THAT VARIABLE TRANSFORMATIONS CAN BE VERY USEFUL IN SOLVING NONLINEAR ORDINARY AND PARTIAL D.E.'S. ALTHOUGH ONE USES ELEMENTARY CALCULUS IN WORKING WITH THESE TRANSFORMATIONS, THE NECESSARY MANIPULATIONS ARE NOT SIMPLE AT ALL. THIS IS PARTLY BECAUSE VERY FEW BOOKS STRESS THE USEFULNESS OF THESE TRANSFORMATIONS. AN EXCEPTION IS COURANT'S 2 VOL. BOOK "DIFF. & INTEGRAL CALCULUS" OR ITS UPDATED EDITION WITH F. JOHN. HERE ARE SOME SIMPLE EXAMPLES:

i) CONSIDER $u_x - u u_y = 0$. THIS EQUATION HAS THE GENERAL SOLUTION $u = f(y + ux)$ WHERE f IS AN ARBITRARY FUNCTION. HERE IS A TECHNIQUE BASED ON TRANSFORMATION OF VARIABLES RESULTING IN THE ABOVE SOLUTION. LET

$$\begin{cases} x' = u \\ y' = y \\ u' = x \end{cases}$$

WE ASSUME u' TO BE THE NEW DEPENDENT VARIABLE. WE REWRITE dx' AND dy' IN TERMS OF dx AND dy:

$$\begin{cases} dx' = u_x\, dx + u_y\, dy \\ dy' = dy \\ du' = u'_{x'}\, dx' + u'_{y'}\, dy' = dx \end{cases}$$

$$du' = u'_{x'}(u_x dx + u_y dy) + u'_{y'} dy = dx$$

$\Rightarrow$

$$(u'_{x'}u_x - 1)\, dx + (u'_{x'}u_y + u'_{y'})\, dy = 0$$

SINCE dx AND dy ARE ARBITRARY, WE HAVE

$$u'_{x'} = \frac{1}{u_x} \; ; \; u'_{y'} = -\frac{u_y}{u_x} \Rightarrow \begin{cases} u_x = \dfrac{1}{u'_{x'}} \\ u_y = -\dfrac{u'_{y'}}{u'_{x'}} \end{cases}$$

THE D.E. IN THE NEW VARIABLES BECOME

$$\frac{1}{u'_{x'}} + \frac{x' u'_{y'}}{u'_{x'}} = 0$$

$$x' u'_{y'} = -1 \Rightarrow u'(x',y') = -\frac{y'}{x'} + \tilde{g}(x'), \; \tilde{g} \text{ ARB.}$$

OR

$$x'u' = -y' + x'\tilde{g}(x') \equiv y' + g(x'), \; g \text{ ARB.}$$

IN THE ORIGINAL VARIABLE

$$xu = -y + g(u)$$

OR

$$u = g^{-1}(y + xu) \equiv f(y + xu), \; f \text{ ARB.}$$

ii) CONSIDER THE NONLINEAR PDE

$$Lu = u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy} = 0$$

ALTHOUGH THERE IS SOME SYMMETRY IN THIS EQ., FINDING GENERAL SOLUTION SEEMS DIFFICULT. HOWEVER, THE VARIABLE TRANSFORMATION OF EX.(i) CAN BE USED AS FOLLOWS. WE HAVE SHOWN THAT

$$u_x = \frac{1}{u'_{x'}} \; , \; u_y = -\frac{u'_{y'}}{u'_{x'}}$$

WE MUST REMEMBER THAT $x' = x'[x, y, u(x,y)]$, $y' = y'[x, y, u(x,y)]$ WHILE u', $u'_{x'}$, $u'_{y'}$ ARE

FNS OF x', y', WE HAVE

$$u_{xx} = \left(\frac{\partial x'}{\partial x}\right)_t \frac{\partial}{\partial x'}\left(\frac{1}{u'_{x'}}\right) + \left(\frac{\partial y'}{\partial x}\right)_t \frac{\partial}{\partial y'}\left(\frac{1}{u'_{x'}}\right)$$

WHERE t STANDS FOR 'TOTAL' DEFINED AS

$$\left(\frac{\partial x'}{\partial x}\right)_t = \frac{\partial x'}{\partial x} + \frac{\partial x'}{\partial u} u_x$$

$$= u_x$$

$$\left(\frac{\partial y'}{\partial x}\right)_t = \frac{\partial y'}{\partial x} + \frac{\partial y'}{\partial u} u_x = 0$$

$$\Rightarrow \quad u_{xx} = -\frac{u'_{x'x'}}{u'^3_{x'}}$$

SIMILARLY

$$u_{xy} = \left(\frac{\partial x'}{\partial y}\right)_t \frac{\partial}{\partial x}\left(\frac{1}{u'_{x'}}\right) + \left(\frac{\partial y'}{\partial y}\right)_t \frac{\partial}{\partial y'}\left(\frac{1}{u'_{x'}}\right)$$

$$= \frac{u'_{y'} u'_{x'x'} - u'_{x'} u'_{x'y'}}{u'^3_{x'}},$$

$$u_{yy} = \frac{-u'^2_{y'} u'_{x'x'} - u'^2_{x'} u'_{y'y'} + 2u'_{x'} u'_{y'} u'_{x'y'}}{u'^3_{x'}}$$

$$\Rightarrow \quad \mathcal{L} u = \frac{1}{u'^5_{x'}} \Big[(\cancel{2u'^2_y} - \cancel{2u'^2_y}) u'_{x'x'} + (\cancel{2u'_x u'_y} \cancel{- 2u'_x u'_{y'}})$$

$$\times u'_{x'y'} - u'^2_{x'} u'_{y'y'} \Big] = 0$$

$$\Rightarrow \quad u'_{y'y'} = 0 \quad \therefore \quad u' = y' f(x') + g(x')$$

OR $\quad x = y f(u) + g(u)$, f & g ARBIT.

$\quad u = h[x - y f(u)]$, h ARB.

(iii) NOW WE CONSIDER $u(x,y)$, $v(x,y)$ AND WE CHANGE VARIABLES AS FOLLOWS

$$\begin{cases} x' = u \\ y' = v & u' = u'(x',y') \\ u' = x & v' = v'(x',y') \\ v' = y \end{cases}$$

WE HAVE

$$dx' = du = u_x\, dx + u_y\, dy$$
$$dy' = dv = v_x\, dx + v_y\, dy$$
$$du' = u'_{x'} dx' + u'_{y'} dy'$$
$$= u'_{x'}(u_x\, dx + u_y\, dy) + u'_{y'}(v_x\, dx + v_y\, dy)$$
$$= dx$$
$$dv' = v'_{x'} dx' + v'_{y'} dy'$$
$$= v'_{x'}(u_x\, dx + u_y\, dy) + v'_{y'}(v_x\, dx + v_y\, dy)$$
$$= dy$$

FROM $du' = dx$, WE GET, BY EQUATING COEF$(dx) = 0$

$$u'_{x'} u_x + u'_{y'} v_x - 1 = 0 \qquad (1)$$
$$u'_{x'} u_y + u'_{y'} v_y = 0 \qquad : \text{COEF.}(dy) = 0 \qquad (2)$$

SIMILARLY FROM $dv' = dy$, WE GET

$$v'_{x'} u_x + v'_{y'} v_x = 0 \qquad (3)$$
$$v'_{x'} u_y + v'_{y'} v_y - 1 = 0 \qquad (4)$$

FROM (1) AND (3), WE GET

$$u_x = \frac{v'_{y'}}{J'}$$

$$v_x = -\frac{v'_{x'}}{J'}$$

WHERE $J' = \dfrac{\partial(u',v')}{\partial(x',y')}$

FROM (2) AND (4), WE GET

$$u_y = -\frac{u'_{y'}}{J'}$$

$$v_y = \frac{u'_{x'}}{J'}$$

WE CAN NOW SOLVE THE FOLLOWING NONLINEAR SIMULTANEOUS EQ.

$$\begin{cases} v\,u_x - v_x = 0 \\ u_y - u\,v_y = 0 \end{cases}$$

THIS SEEMS TO BE DIFFICULT TO SOLVE ALTHOUGH I HAVEN'T SPENT MUCH TIME TO SOLVE IT.(*) THE ABOVE TRANSFORMATION GIVE THE FOLLOWING TWO EQS WHICH ARE NOT SIMULTANEOUS EQS:

$$\begin{cases} y'\,v'_{y'} + v'_{x'} = 0 \\ u'_{y'} + x'\,u'_{x'} = 0 \end{cases}$$

FOR THE FIRST EQ., THE CHARACTERISTIC DIRECTION IS GIVEN BY

$$\frac{dx'}{1} = \frac{dy'}{y'}, \quad x' = \ln c y'$$

$$\exp(x') = c y' \; ; \qquad v' = f\left(\frac{e^{x'}}{y'}\right), \; f \text{ (ARB.)}$$

SIMILARLY FOR THE 2ND EQ., THE CHAR. DIR. IS GIVEN BY

$$\frac{dx'}{x'} = \frac{dy'}{1} \Rightarrow u' = g\left(\frac{e^{y'}}{x'}\right), \; g \text{ ARB.}$$

IN THE OLD VARIABLE

$$y = f\left(\frac{e^u}{v}\right), \quad x = g\left(\frac{e^v}{u}\right)$$

(*) SEE NEXT PAGE. IT IS SIMPLE TO SOLVE THIS EQ. !

OR $\begin{cases} e^u = v\,\tilde{f}(y) & , \tilde{f} \text{ ARB.} \\ e^v = u\,\tilde{g}(x) & , \tilde{g} \text{ ARB.} \end{cases}$

$$e^u = \ln[u\,\tilde{g}(x)]\,\tilde{f}(y)$$

$$\underline{u = \ln\{\ln[u\,\tilde{g}(x)]\}\,\tilde{f}(y)}$$

$$e^v = \ln[v\,\tilde{f}(y)]\,\tilde{g}(x)$$

$$\underline{v = \ln\{\ln[v\,\tilde{f}(y)]\}\,\tilde{g}(x)}$$

IT IS SEEN THAT VARIABLE TRANSFORMATION IS A USEFUL TECHNIQUE IN SOLVING NONLINEAR PROBLEMS. NOW THAT THE SOLUTION OF THIS EXAMPLE IS KNOWN ABOVE, WE CAN USE A MUCH SIMPLER APPROACH:

$$u_x - \frac{v_x}{v} = 0$$

ALONG $y =$ CONST. LINE $\quad du - \frac{dv}{v} = 0$

OR $u - \ln cv = 0$. HERE $c = \tilde{f}(y)$ OR $e^u = v\,\tilde{f}(y)$. SIMILARLY FOR THE SECOND EQ. A NONTRIVIAL PROBLEM WOULD BE THE FOLLOWING:

$$\begin{cases} u_x + v\,u_y = 0 \\ v_x + u\,v_y = 0 \end{cases}$$

IN THE NEW VARIABLES, THE ABOVE EQ. BECOMES

$$\begin{cases} v_{y'}' - y'\,u_{y'}' = 0 \\ -v_{x'}' + x'\,u_{x'}' = 0 \end{cases}$$

TAKE $\partial/\partial x'$ OF THE FIRST EQ. AND $\partial/\partial y'$ OF THE 2ND,

THEN ADDING THE TWO EQS., WE GET

$$(x'-y')u'_{x'y'}=0 \Rightarrow u'_{x'y'}=0$$

OR $\quad u'(x',y') = f(x') + g(y')$, f, g ARB.

$$v'_{y'} = y'u'_{y'} \qquad \text{(FROM 1ST EQ.)}$$
$$= y'g'(y')$$
$$\therefore \quad v'(x',y') = \int y'g'(y')\,dy' + h(x')$$
$$= y'g(y') - \int g(y')\,dy' + h(x')$$

WHERE h IS ARBITRARY. IN TERMS OF ORIGINAL VARIABLES:

$$\begin{cases} x = f(u) + g(v) \\ y = v\,g(v) - \int g(v)\,dv + h(u) \end{cases} \qquad f, g, h \text{ ARB.}$$

AFTER TESTING THIS RESULT IT BECAME OBVIOUS THAT h IS NOT ARBITRARY AFTER ALL. GOING BACK TO $v'_{x'} = x'u'_{x'}$, WE SEE THAT

$$v'_{x'} = x'f'(x')$$
$$v'(x',y') = \int x'f'(x')\,dx' + \text{ARB. FN}(y')$$
$$= x'f(x') - \int f(x')\,dx' + \text{ARB. FN}(y')$$

SO THAT $h(u) = uf(u) - \int f(u)\,du \Rightarrow$

$$\begin{cases} x = f(u) + g(v) \\ y = uf(u) + v\,g(v) - \int f(u)\,du - \int g(v)\,dv \end{cases}$$

THIS PROBLEM CAN BE SOLVED NUMERICALLY BY THE METHOD OF CHARACTERISTICS. THE CHARACTERISTIC DIRECTIONS ARE GIVEN BY

$$\xi^+: \frac{dy}{dx} = u \quad , \quad \xi^-: \frac{dy}{dx} = v$$

AND THE COMPATIBILITY CONDITIONS ARE

$$v = \text{CONST ON } \xi^+$$
$$u = \text{CONST ON } \xi^-$$

ONE CAN FIND ANALYTIC SOLUTION BY THE METHOD DESCRIBED ABOVE. LET $u = x$ AND $v = -x$ ON THE LINE $y = 0$, i.e. THE x-AXIS. THEN

$$\begin{cases} x = f(x) + g(-x) \\ 0 = x f(x) - x\, g(-x) - \int f(x)\,dx - \int g(-x)\, d(-x) \end{cases}$$

$$g(-x) = x - f(x)$$

$$0 = x f(x) - x(x - f(x)) - \int f(x)\,dx + \int (x - f(x))\,dx$$

$$0 = 2x f(x) - \frac{x^2}{2} - 2\int f(x)\,dx$$

TAKE DERIVATIVE W.R.T. x

$$\cancel{2f(x)} + 2x f'(x) - x - \cancel{2f(x)} = 0$$

$$f'(x) = \frac{1}{2}$$

$$f(x) = \frac{1}{2}x + C$$

$$g(-x) = \frac{1}{2}x - C$$

$$g(x) = -\frac{1}{2}x - C$$

$$\therefore \begin{cases} x = \frac{1}{2}(u - v) \\ y = \frac{1}{4}(u^2 - v^2) \\ \quad = \frac{x}{2}(u + v) \end{cases}$$

$$\begin{cases} u = \frac{y}{x} + x & x \neq 0 \\ v = \frac{y}{x} - x & x \neq 0 \end{cases}$$

THE CHARACTERISTICS ARE

$$\xi^+ : v = C_1 = \frac{y}{x} - x = x_0$$

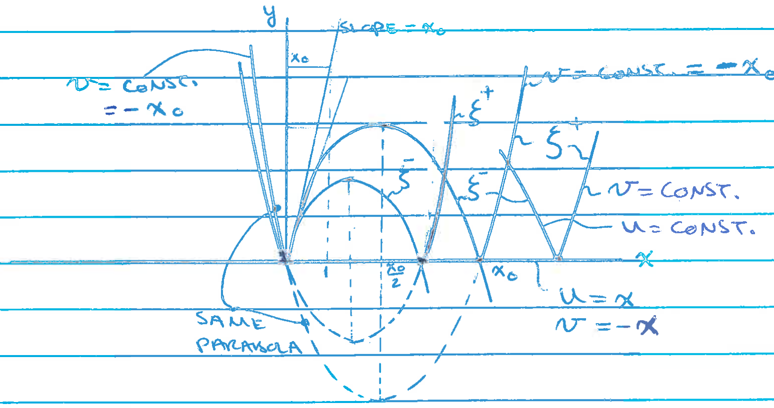
$$\therefore \quad y = x^2 + C_1 x = x^2 - x_0 x$$

$$\xi^- : u = C_2 = \frac{y}{x} + x = x_0$$

$$y = -x^2 + C_2 x = -x^2 + x_0 x$$

THE CHARACTERISTIC CURVES ARE PARABOLAS WHICH PASS THRU THE ORIGIN. THIS MEANS THAT THE

ORIGIN IS A POINT OF SINGULARITY OF THE DIFFERENTIAL EQUATION SINCE u OR v IS CONSTANT ON THE CHARACTERISTIC CURVES.



AS x_0 INCREASES, THE ξ^- PARABOLAS BECOME TANGENT TO Y-AXIS SINCE THE SLOPE OF THE ξ^- PARABOLAS AT $x=0$ IS x_0, AS SHOWN. ONE CAN SEE THAT IN THE VICINITY OF THE ORIGIN, NUMERICAL PROBLEMS WILL BE ENCOUNTERED.

WE CAN SEE THE BEHAVIOR OF u AND v BETTER IF WE USE THE EXACT SOLUTION. ALONG THE LINE $y = C_1 = $ CONST., WE HAVE

$$\begin{cases} u = \dfrac{C_1}{x} + x \\ v = \dfrac{C_1}{x} - x \end{cases} \qquad C_1 \neq 0$$

$\therefore$ BOTH u AND v INCREASE (OR DECREASE) BEYOND BOUND AS $x \to 0$. SIMILARLY, ALONG THE LINE $x = C_2 = $ CONST. $\neq 0$, WE HAVE

$$\begin{cases} u = \dfrac{y}{C_2} + C_2 \\ v = \dfrac{y}{C_2} - C_2 \end{cases}$$

$\therefore$ NEAR THE y-AXIS, u AND v INCREASE RAPIDLY AND LINEARLY W.R.T. y.

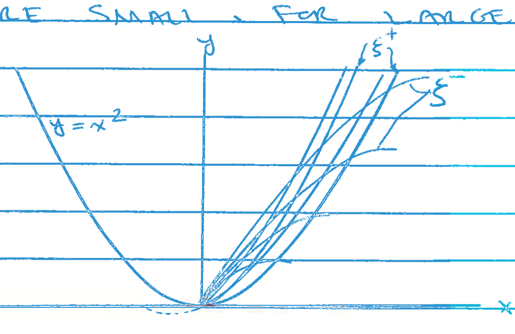
IT IS INTERESTING TO NOTE THAT WHEN A CHARACTERISTIC PASSES THRU THE ORIGIN, THE COMPATIBILITY CONDITION ON IT DOES NOT CHANGE. THIS CAN BE SEEN FROM THE EXACT SOLUTION

THE CHARACTERISTICS INSIDE THE PARABOLA $y = x^2$ ARE VERY INTERESTING. IN THIS REGIONS, BOTH CHARACTERISTICS PASS THRU THE ORIGIN. THE TWO TYPES OF CHARACTERISTICS HAVE DIFFERENT CONVEXITY. IF THE ORIGINAL DIFFERENTIAL EQ. ORIGINATES FROM A PHYSICAL PROBLEM, THEN IT WOULD BE DIFFICULT TO SEE HOW THE REGION INSIDE THE PARABOLA $y = x^2$ GETS THE DISTURBANCE FROM $y = 0$ (i.e THE B.C.). THE BEHAVIOR OF u AND v NEAR THE ORIGIN CAN BE ANALYSED IF WE USE POLAR COORDINATES (r, θ):

$$u = \tan\theta + r\cos\theta$$

$$v = \tan\theta - r\sin\theta$$

LET $r \ll 1$, THEN FOR SMALL θ, i.e $\theta \ll \frac{\pi}{4}$, BOTH u AND v ARE SMALL. FOR LARGE θ, $u \approx v \approx \tan\theta$.



ALL EXAMPLES BY F.F.

* GEOMETRIC ACOUSTIC RAY THEORY

CONSIDER THE WAVE EQ. $\Box^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi = 0$. ASSUME $c = c(\vec{x})$ AND $n = c_0/c$, THE INDEX OF REFRACTION. IF THE LENGTH SCALE ℓ OF VARIATION OF C IS MUCH GREATER THAN THE VARIATION LENGTH OF THE WAVE, i.e., THE WAVELENGTH, THEN WE CAN ASSUME ψ TO VARY AS

$$\psi(\vec{x},t) = A(\vec{x})\, \exp i\left[\omega t - k_0 S(\vec{x})\right]$$

WHERE $k_0 = \omega/c_0$. WE HAVE

$$\psi_{tt} = -\omega^2\psi = -k_0^2 n^2 \psi$$

$$\nabla\psi = \left[\frac{\nabla A}{A} - i k_0 \nabla S\right]\psi$$

$$\nabla^2\psi = \left(\frac{\nabla^2 A}{A} - \frac{|\nabla A|^2}{A^2} - i k_0 \nabla^2 S\right)\psi + \left|\frac{\nabla A}{A} - i k_0 \nabla S\right|^2 \psi$$

SUBSTITUTE THESE IN $\Box^2\psi = 0$ AND THEN SET THE REAL AND IM. PARTS TO ZERO. WE GET

$$\begin{cases} k_0^2\left[n^2 - |\nabla S|^2\right] + \nabla^2 A/A = 0 & (1)\\ A\nabla^2 S + 2\,\nabla A\cdot\nabla S = 0 & (2)\end{cases}$$

NOW $\nabla^2 A \sim A/\ell^2$ SO THAT $\nabla^2 A/A \sim 1/\ell^2$. THIS IS SMALL IN COMPARISON TO $k_0^2 n^2 \sim \frac{\omega^2}{c_0^2}$ BY ASSUMPTION SINCE $\frac{\omega}{c_0} = \frac{2\pi f}{c_0} = \frac{2\pi}{\lambda} \gg \frac{1}{\ell}$. $\therefore$ FROM EQ. 1

$$|\nabla S| = n \quad \text{OR} \quad \nabla S = n\,\vec{\hat{s}} \qquad (3)$$

WHERE $\vec{\hat{s}}$ IS THE UNIT VECTOR NORMAL TO THE SURFACE OF CONSTANT PHASE $S(\vec{x}) = $ CONST. THE VARIATION OF THE AMPLITUDE CAN BE FOUND FROM EQ. (2) AS FOLLOWS

$$2\nabla A\cdot\nabla S = \frac{2}{|\nabla S|}\frac{\partial A}{\partial s} = \frac{2}{n}\frac{\partial A}{\partial s} = -A\nabla^2 S$$

OR $$\frac{1}{A}\frac{\partial A}{\partial s} = -\frac{n}{2}\,\nabla\cdot(n\vec{\hat{s}}) \qquad (4)$$

HERE S IS THE ELEMENT OF LENGTH ALONG ALONG A RAY WHICH IS AN ORTHOGONAL TRAJECTORY OF THE SURFACES OF CONSTANT PHASE.

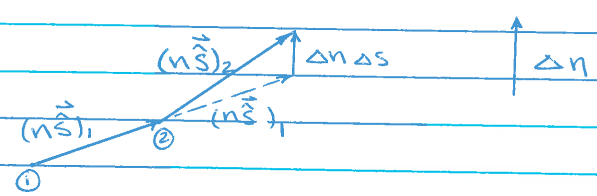
ANOTHER IMPORTANT EQ. IS OBTAINED AS FOLLOWS.

$$\frac{\partial}{\partial s}(n\vec{\hat{s}}) = \vec{\hat{s}}\cdot\nabla(\nabla S)$$
$$= \frac{\nabla S}{n}\cdot\nabla(\nabla S)$$
$$= \frac{1}{2n}\nabla(\nabla S)^2$$
$$= \frac{1}{2n}\nabla n^2$$
$$= \nabla n$$

OR $$\frac{\partial}{\partial s}(n\vec{\hat{s}}) = \nabla n \qquad (5)$$

WE CAN USE THIS RELATION TO DETERMINE THE CHANGE IN THE DIRECTION OF THE RAY AS FOLLOWS.

$$(n\vec{\hat{s}})_2 \approx (n\vec{\hat{s}})_1 + \nabla n\,\Delta s$$



DEC. 81
6PHM

* NOTES ON COMPLEX VARIABLE THEORY

i) CLASSIFICATION OF SINGLE-VALUED FUNCTIONS

FROM LIOUVILLE'S THM, WE KNOW THAT EVERY NONCONSTANT ENTIRE FN MUST HAVE A SINGULARITY AT INFINITY. THE TYPE OF THE SINGULARITY AT INFINITY IS RELATED TO THE TYPE OF THE FUNCTION INVOLVED. THEY CAN BE CLASSIFIED EASILY AS FOLLOWS. WE START WITH A THM.

THM: IF AN ENTIRE FUNCTION HAS A POLE OF ORDER n AT INFINITY $\Rightarrow$ THE FUNCTION IS A POLYNOMIAL OF ORDER n.

PROOF: LET $\xi = \frac{1}{z}$, BY ASSUMPTION

$$f\left(\frac{1}{\xi}\right) = \frac{b_1}{\xi} + \frac{b_2}{\xi^2} + \dots + \frac{b_n}{\xi^n} + \phi(\xi)$$

$$\therefore \quad f(z) = b_1 z + b_2 z + \dots + b z^n + \phi\left(\frac{1}{z}\right)$$

SINCE $\phi(\xi)$ IS ANALYTIC AT $\xi = 0$. NOW

$$\phi\left(\frac{1}{z}\right) = f(z) - (b_1 z + b_2 z + \dots + b z^n)$$

$\Rightarrow \phi\left(\frac{1}{z}\right)$ IS ALSO ENTIRE. SINCE IT IS ALSO ANALYTIC AT INFINITY $\Rightarrow \phi\left(\frac{1}{z}\right)$ IS CONSTANT BY LIOUVILLE'S THM

POLYNOMIALS ARE ALSO CALLED RATIONAL ENTIRE FUNCTIONS. RATIONAL FUNCTIONS ARE THE RATIO OF TWO POLYNOMIALS. ENTIRE FNS OTHER THAN POLYNOMIALS ARE KNOWN AS TRANSCENDENTAL ENTIRE FNS. A MEROMORPHIC FN IS A FN WITH ONLY POLES AS ITS SINGULARITIES IN A GIVEN REGION.

THM: A MEROMORPHIC FN FOR WHICH INFINITY IS AN ORDINARY POINT OR A POLE IS A RATIONAL FUNCTION (*)

PROOF: LET $f(z)$ HAVE m POLES $a_1, a_2, \ldots, a_m$ IN FINITE PLANE AND LET $\phi_r(z)$ BE THE PRINCIPAL PART OF $f(z)$ AT a_r, i.e.

$$\phi_r(z) = \frac{A_{r1}}{z-a_r} + \frac{A_{r2}}{(z-a_r)^2} + \ldots + \frac{A_{rp_r}}{(z-a_r)^{p_r}}$$

WHERE p_r IS THE ORDER OF THE POLE AT $z = a_r$.
$\Rightarrow$ $f(z) - \sum_{r=1}^{m} \phi_r(z)$ IS FINITE IN THE FINITE PLANE. SINCE $\phi_r(z) \to 0$ AS $z \to \infty$ $\Rightarrow$ $f(z) - \sum \phi_r(z)$ IS EITHER CONSTANT OR A POLYNOMIAL, SAY

$$\psi(z) = c_0 + c_1 z + \ldots + c_q z^q$$

$\therefore$ $f(z) = \sum_1^m \phi_r(z) + \psi(z)$, i.e. A RATIONAL FUNCTION

COROLLARY: ANY MEROMORPHIC FUNCTION WHICH IS NOT A RATIONAL FUNCTION HAS AN ESSENTIAL SINGULARITY AT INFINITY.

EXAMPLES: $\sin z$, e^z, $\operatorname{sech} z$ HAVE ESSENTIAL SINGULARITY AT INFINITY.

ii) MITTAG-LEFFLER'S THM

THIS THM STATES THAT GIVEN $a_1, a_2, \ldots$ $\ni$ $|a_1| < |a_2| < |a_3| \ldots$ AND $\ni$ $\sum_{r=1}^{\infty} \frac{1}{|a_r|^n}$ IS CONVERGENT FOR SOME n $\Rightarrow$ $\exists$ AN ANALYTIC FN WITH SIMPLE POLES AT $a_1, a_2, \ldots, a_n, \ldots$

(*) WE MUST FIRST PROVE THAT A MEROMORPHIC FUNCTION IN THE ENTIRE ~~COMPLEX PLANE HAS FINITE NO. OF SINGULARITIES IF INFINITY~~ IS AN ORDINARY POINT OR A POLE. SEE P 157.

PROOF: CONSIDER THE FUNCTION

$$w_r(z) = \frac{1}{z-a_r} + \frac{1}{a_r} + \frac{z}{a_r^2} + \dots + \frac{z^{n-2}}{a_r^{n-1}}$$

$$= \frac{1}{z-a_r} \frac{z^{n-1}}{a_r^{n-1}}$$

LET C BE THE CIRCLE $|z| = R$, $R < |a_{p+1}| \Rightarrow$

$\forall z \in$ THE REGION $|z| < R$

$$\left|\frac{z}{a_r} - 1\right| \geqslant 1 - \left|\frac{z}{a_r}\right| \geqslant 1 - \frac{R}{|a_r|} \geqslant 1 - \frac{R}{|a_{p+1}|} = \mu$$

$$\Rightarrow |w_r(z)| \leqslant \frac{R^{n-1}}{\mu} \frac{1}{|a_r|^n}, \quad r = p+1, p+2, \dots$$

BY WEIERSTRASS M-TEST THE SERIES $\sum_{p+1}^{\infty} w_r(z)$ CONVERGES ABSOLUTELY AND UNIFORMLY ON THE REGION $|z| < R$. $\Rightarrow \sum_{r=1}^{\infty} w_r(z)$ REPRESENTS A FN DESCRIBED IN THE THM SINCE R IS ARBITRARY.

COR. 1: LET THE TWO FNS $f(z)$ AND $\phi(z)$ HAVE SIMPLE POLES OF RESIDUE 1 AT $a_1, a_2, \dots a_n, \dots \Rightarrow$ $f(z) - \phi(z)$ IS AN ENTIRE FN. $\therefore$ THE GENERAL FORM OF FNS SUCH AS $f(z)$ AND $\phi(z)$ IS

$$f(z) = \sum_{r=1}^{\infty} w_r(z) + G(z)$$

WHERE $G(z)$ IS AN ENTIRE FN.

COR. 2: IF WE DIFFERENTIATE THE FN $\sum_{r=1}^{\infty} w_r(z)$ p TIMES, WE OBTAIN A FN WITH POLES OF ORDER p AT $a_1, a_2, \dots\dots$.

WE NOTE THAT THE FN CONSTRUCTED IN MITTAG-LEFFLER THM HAS AN ESSENTIAL SINGULARITY AT INFINITY.

THIS FOLLOWS FROM THE FIRST THM ON P155. BECAUSE OTHERWISE THE FUNCTION WILL BE A RATIONAL FUNCTION WITH FINITE NUMBER OF SINGULARITIES. IN THE PROOF OF THAT THM ON P155, WE NEED THE FOLLOWING THM:

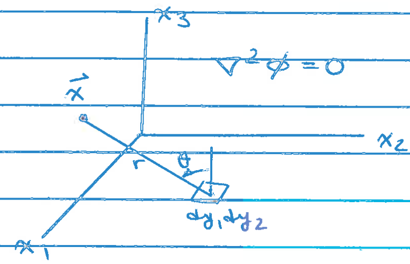
THM: IF INFINITY IS AN ORDINARY POINT OR A POLE FOR A FN WHICH IS MEROMORPHIC THROUGHOUT THE COMPLEX PLANE $\Rightarrow$ THE FN HAS ONLY A FINITE NUMBER OF SINGULARITIES.

PROOF: CONSIDER THE FN. $f(\frac{1}{\zeta})$. $\exists\, \varepsilon>0 \ni \exists$ AT MOST A POLE INSIDE $|\zeta|<\varepsilon$. i.e. ALL SINGULARITIES IN THE FINITE PLANE ARE INSIDE THE CIRCLE $|z|<\frac{1}{\varepsilon}$. THESE SINGULARITIES MUST BE FINITE IN NUMBER BECAUSE OTHERWISE THEY WILL HAVE A POINT OF ACCUMULATION WHICH IS IMPOSSIBLE. THIS IS BECAUSE THE FN $f(z)$ IS ASSUMED MEROMORPHIC IN THE ENTIRE Z PLANE AND POLES ARE ISOLATED SINGULARITIES.

BACK TO MITTAG-LEFFLER THM. HERE IS A SIMPLE EXAMPLE. WE KNOW THAT THE SERIES $\sum_{r=1}^{\infty}\frac{1}{r^2}$ CONVERGES $\Rightarrow$ $f(z)=\sum_{r=1}^{\infty}\left(\frac{1}{z-r}+\frac{1}{r}\right)$ IS ANALYTIC IN THE ENTIRE Z-PLANE WITH THE EXCEPTION OF SIMPLE POLES AT $z=1,2,\ldots$

NOTES PREPARED FROM
MACROBERT "FNS. OF A
COMPLEX VARIABLE"

* NORMAL DERIVATIVE OF A FUNCTION DESCRIBED BY A SINGULAR INTEGRAL



CONSIDER THE SOLUTION OF THE LAPLACE'S EQ IN THE HALF-SPACE $x_3 > 0$ WITH DIRICHLET B.C.:

$$\phi(x_1, x_2, 0) = Q(x_1, x_2)$$

THE SOLUTION IS

$$\phi(\vec{x}) = \frac{1}{2\pi} \int_{y_3=0} \frac{\hat{r}_3\, Q(y_1, y_2)}{r^2}\, dy_1\, dy_2$$

WHERE $\hat{r}_3 = x_3/r = \cos\theta$. THIS IS A CONVERGENT SINGULAR INTEGRAL. WE WANT TO FIND $\partial\phi/\partial x_3 \big|_{x_3=0}$.

WE KNOW THAT ACTUALLY $\phi(\vec{x})$ IS DEFINED ALSO FOR $x_3 < 0$ AND

$$\phi(x_1, x_2, -x_3) = -\phi(x_1, x_2, x_3) \quad , x_3 > 0$$

THIS MEANS THAT ϕ HAS A JUMP OF $2Q(x_1, x_2)$ ON THE PLANE $x_3 = 0$. LET $\tilde{\phi}(\vec{x})$ BE THE EXTENSION OF ϕ TO ENTIRE 3-D SPACE, i.e.

$$\tilde{\phi}(\vec{x}) = \frac{1}{2\pi} \int_{y_3=0} \frac{\hat{r}_3\, Q(y_1, y_2)}{r^2}\, dy_1\, dy_2$$

WE NOW CONSIDER GENERALIZED DERIVATIVE OF $\tilde{\phi}(\vec{x})$ WHICH CAN BE OBTAINED BY TAKING THE DERIVATIVE $\bar{\partial}/\partial x_3$ INSIDE THE INTEGRAL:

$$\frac{\bar{\partial}\tilde{\Phi}_3}{\partial x_3} = \frac{1}{2\pi}\int_{y_3=0} Q(y_1,y_2)\frac{\bar{\partial}}{\partial x_3}\left(\frac{\hat{r}_3}{r^2}\right)dy_1\,dy_2$$

SINCE WE KNOW THAT $\frac{\bar{\partial}\Phi_3}{\partial x_3}$ IS NOT A REGULAR G.F., THERE IS NO POSSIBILITY OF REGULARIZING THE INTEGRAL. WE NOTE THAT $\hat{r}_3/r^2 = \frac{\bar{\partial}}{\partial x_3}\left(\frac{-1}{r}\right)$

$$\frac{\bar{\partial}}{\partial x_3}\left(\frac{\hat{r}_3}{r^2}\right) = -\frac{\bar{\partial}^2}{\partial x_3^2}\left(\frac{1}{r}\right)$$

$$= -\bar{\nabla}^2\left(\frac{1}{r}\right) + \left(\frac{\bar{\partial}^2}{\partial x_1^2}+\frac{\bar{\partial}^2}{\partial x_2^2}\right)\left(\frac{1}{r}\right)$$

$$= 4\pi\delta(\vec{x}-\vec{y}) + \left(\frac{\bar{\partial}^2}{\partial y_1^2}+\frac{\bar{\partial}^2}{\partial y_2^2}\right)\left(\frac{1}{r}\right) \quad (*)$$

SUBSTITUTING THIS RESULT IN THE ABOVE INTEGRAL, WE GET, AFTER LETTING $y_3=0$ IN $(*)$:

$$\frac{\bar{\partial}\tilde{\phi}}{\partial x_3} = 2\,Q(x_1,x_2)\,\delta(x_3) + \frac{1}{2\pi}\int_{y_3=0}\frac{1}{r}\left(\frac{\bar{\partial}^2}{\partial y_1^2}+\frac{\bar{\partial}^2}{\partial y_2^2}\right)Q(y_1,y_2)\,dy_1\,dy_2$$

$$\equiv \underbrace{2\,Q(x_1,x_2)\,\delta(x_3)}_{\text{JUMP IN }\phi\text{ ON }x_3=0} + \frac{\partial\tilde{\phi}}{\partial x_3}$$

$$\therefore \frac{\partial\phi}{\partial x_3} = \frac{1}{2\pi}\int_{y_3=0}\frac{1}{r}\left(\frac{\bar{\partial}^2}{\partial y_1^2}+\frac{\bar{\partial}^2}{\partial y_2^2}\right)Q(y_1,y_2)\,dy_1\,dy_2 \qquad x_3>0$$

IT IS BETTER TO KEEP THE GENERALIZED DERIVATIVES WRT y_1 AND y_2 INSIDE THE INTEGRAL. IF $Q(y_1,y_2)\equiv 0$ OUTSIDE $g=0$ WHERE $g=0$ IS A CLOSED CURVE $\Longrightarrow$

$$\bar{\nabla}_2 Q \equiv \left(\frac{\bar{\partial}Q}{\partial y_1},\frac{\bar{\partial}Q}{\partial y_2}\right) = \nabla_2 Q + Q\,\nabla g\,\delta(g)$$

$\bar{\nabla}_2^2 = \frac{\bar{\partial}^2}{\partial y_1^2}+\frac{\bar{\partial}^2}{\partial y_2^2}$:

$$\bar{\nabla}_2^2 Q = \bar{\nabla}\cdot\bar{\nabla}_2 Q = \nabla_2^2 Q + \nabla_2 Q\cdot\nabla g\,\delta(g) + \bar{\nabla}\cdot[Q\,\nabla g\,\delta(g)]$$

NOW LET $g<0$ BE THE REGION INSIDE THE CURVE $g=0$ $\Longrightarrow$

$$\frac{\partial \phi}{\partial x_3} = \frac{1}{2\pi}\int_{g<0} \frac{1}{r}\nabla_2^2 Q \, dy_1 dy_2$$

$$+ \frac{1}{2\pi}\int_{g=0} \frac{1}{r}\frac{\partial Q}{\partial n}\, d\ell$$

$$- \frac{1}{2\pi}\int_{g=0} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) Q \, d\ell \qquad (**)$$

WHERE $\frac{\partial Q}{\partial n} = \frac{\nabla g}{|\nabla g|}\cdot \nabla Q$ AND

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \left(n_1\frac{\partial}{\partial y_1} + n_2\frac{\partial}{\partial y_2}\right)\left(\frac{1}{r}\right)$$

$$= \frac{n_1\hat{r}_1 + n_2\hat{r}_2}{r^2}$$

$\vec{n} = \frac{\nabla g}{|\nabla g|}$, WE NOTE THAT ONCE MORE BECAUSE OF THE ASSUMED DISCONTINUITY IN Q, $\frac{\partial \phi}{\partial x_3}$ IS NOT REGULAR SINCE IN (**) BOTH LINE INTEGRALS ARE NOT CONVERGENT AS $\vec{x}$ APPROACHES THE CURVE $g=0$. THIS NEEDS FURTHER STUDY

* REGULARIZATION OF A DIVERGENT SURFACE INTEGRAL OF AEROACOUSTICS FOR AERODYNAMIC APPLICATIONS

WE WOULD LIKE TO USE THE FOLLOWING ACOUSTIC EQUATION AS AN INTEGRAL EQUATION FOR FINDING THE AERODYNAMIC PRESSURE ON THE SURFACE OF A ROTATING BLADE

$$4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int_{f=0}\left[\frac{\rho_0 v_n c + p\cos\theta}{r(1-M_r)}\right]_{ret} dS$$

$$+ \int_{f=0}\left[\frac{p\cos\theta}{r^2(1-M_r)}\right]_{ret} dS \qquad (1)$$

TO AVOID NUMERICAL DIFFERENTIATION, WE CAN BRING THE TIME DERIVATIVE INSIDE. HOWEVER, THE RESULTING INTEGRALS ARE DIVERGENT. WE REGULARIZE THE RESULTING INTEGRALS AS FOLLOWS.

LET $\vec{x}$ BE ON THE SURFACE $f(\vec{y},t)=0$ WHICH DESCRIBES THE BODY (BLADE) SURFACE AT THE TIME t. LET $H(K_\varepsilon)$ BE THE HEAVISIDE FN WITH

$$K_\varepsilon = |\vec{x}-\vec{y}| - \varepsilon \qquad (2)$$

WHERE ε IS A SMALL POSITIVE NUMBER. NOTE THAT THE INTEGRATION IN EQ. (1) IS PERFORMED IN THE $\vec{\eta}$-FRAME FIXED WRT THE BODY. IN EQ. (2), THE $\vec{y}$ VECTOR IS THE SOURCE LOCATION IN THE FRAME FIXED WRT UNDISTURBED MEDIUM. WE KNOW THAT $\vec{y}=\vec{y}(\vec{\eta},\tau)$ SO THAT $K_\varepsilon = K_\varepsilon(\vec{x},\vec{\eta},\tau)$. WE

WRITE EQ (1) IN THE FOLLOWING FORM IN ORDER THAT THE TIME DERIVATIVE CAN BE BROUGHT INTO THE INTEGRAL:

$$4\pi p'(\vec{x},t) = \lim_{\varepsilon \to 0} \left\{ \frac{1}{c}\frac{\partial}{\partial t} \int_{f=0} \left[\frac{(\rho_0 c v_n + p\cos\theta) H(k_\varepsilon)}{r(1-M_r)} \right]_{ret} dS \right.$$

$$\left. + \int_{f=0} \left[\frac{p\cos\theta\, H(k_\varepsilon)}{r^2(1-M_r)} \right]_{ret} dS \right\} \qquad (3)$$

WE LET $\vec{x}$ TO LIE ON THE SURFACE SO THAT $p'(\vec{x},t) = p(\vec{x},t) = p(\vec{\eta})$. WE ALSO KNOW THAT

$$\left.\frac{\partial}{\partial t}\right|_{\vec{x}} = \left[\frac{1}{1-M_r} \left.\frac{\partial}{\partial \tau}\right|_{\vec{\eta}} \right]_{ret} \qquad (4)$$

WE TAKE THE DERIVATIVE INSIDE THE INTEGRAL AND WE GET

$$4\pi p(\vec{x},t) = \lim_{\varepsilon \to 0} \int_{f=0} \left[\frac{H(k_\varepsilon)}{c(1-M_r)} \frac{\partial}{\partial \tau} \left(\frac{\rho_0 c v_n + p\cos\theta}{r(1-M_r)} \right) + \frac{p\cos\theta\, H(k_\varepsilon)}{r^2(1-M_r)} \right]_{ret} dS$$

$$+ \lim_{\varepsilon \to 0} \int_{f=0} \left[\frac{(\rho_0 c v_n + p\cos\theta)}{c r (1-M_r)^2} \frac{\partial k_\varepsilon}{\partial \tau} \delta(k_\varepsilon) \right]_{ret} dS \qquad (5)$$

WE HAVE

$$\frac{\partial k_\varepsilon}{\partial \tau} = -\hat{r}_i v_i = -v_r$$

WE NOW PAY ATTENTION TO THE SECOND INTEGRAL. WE CONSIDER

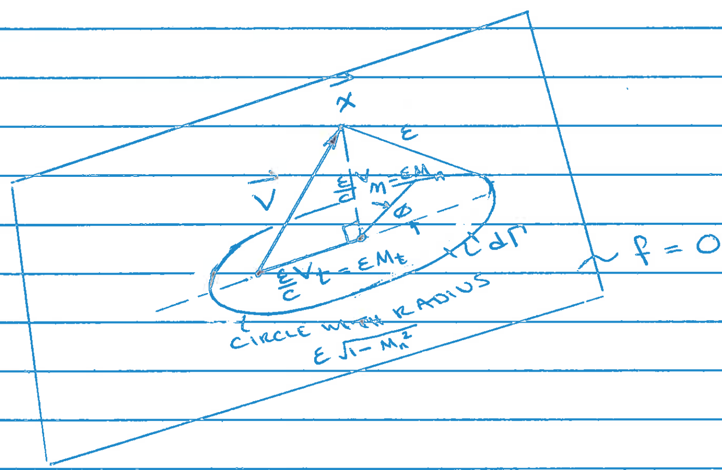
$$I_\varepsilon = \int_{f=0} \left[\frac{M_r (\rho_0 c v_n + p\cos\theta)\delta(k_\varepsilon)}{r(1-M_r)^2} \right]_{ret} dS \qquad (6)$$

WE WRITE THE INTEGRAL AS

$$I_\varepsilon = \int_{f=0} G\,[\delta(K_\varepsilon)]_{ret}\, ds$$

$$= \int_{f=0} G\,[\delta(K_\varepsilon)]_{ret}\, |\nabla f|\, \delta(f)\, d\vec{\eta} \qquad (7)$$

WE KNOW THAT WE HAVE A LINE INTEGRAL ALONG THE CURVE OF INTERSECTION OF SURFACES $K_\varepsilon = 0$ AND $f = 0$, BUT AT WHAT TIME? WE KNOW THAT THE OBSERVER IS ON THE SURFACE AT TIME t. SINCE $r = \varepsilon$ ON THE SPHERE $K_\varepsilon = 0$, WE HAVE $\tau = t - r/c = t - \varepsilon/c$. THE INTERSECTION OF $f=0$ AND $K_\varepsilon = 0$ WITH THE TANGENT PLANE TO $f=0$ (NOT EXACTLY AT $\vec{x}$!) IS SHOWN BELOW.



LET $\mathbb{K}_\varepsilon = [K_\varepsilon]_{ret}$, THEN

$$I_\varepsilon = \int_{f=0} G\,\delta(\mathbb{K}_\varepsilon)\, |\nabla f|\, \delta(f)\, d\vec{\eta} \qquad (8)$$

WE HAVE

$$d\vec{\eta} = \frac{df\, dK_\varepsilon\, d\eta_3}{\left|\frac{\partial(f, K_\varepsilon)}{\partial(\eta_1, \eta_2)}\right|}$$

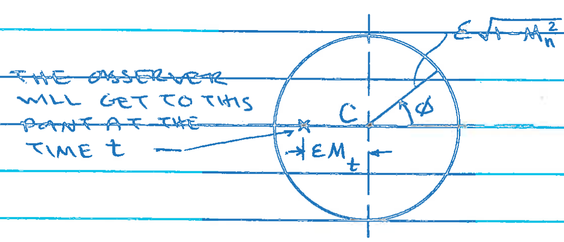
$$= \frac{df\, dK_\varepsilon}{|\nabla f \times \nabla K_\varepsilon|} \frac{d\eta_3}{\left|\frac{\partial(f, K_\varepsilon)}{\partial(\eta_1, \eta_2)}\right| / |\nabla f \times \nabla K_\varepsilon|}$$

$$= \frac{df\, dK_\varepsilon\, d\Gamma}{|\nabla f| \sin\theta} \qquad (9)$$

WHERE θ IS THE ANGLE BETWEEN $\vec{F}$ AND $\vec{n} = \nabla f / |\nabla f|$. SUBSTITUTING THIS EQUATION IN EQ. (8), WE GET

$$I_\varepsilon = \int_{\substack{f=0 \\ K_\varepsilon = 0}} G \frac{d\Gamma}{\sin\theta} \qquad (10)$$

THE CURVE OF INTERSECTION Γ IS GIVEN BY $f = 0$, $K_\varepsilon = 0$. THIS CURVE IS APPROXIMATED BY A CIRCLE OF RADIUS $\varepsilon\sqrt{1 - M_n^2}$ IN THE TANGENT PLANE. THE OBSERVER WILL BE LOCATED AT $\vec{x}$ AT TIME t. THE CENTER OF THE CIRCLE IS LOCATED AS SHOWN BELOW



WE WRITE

$$G = \left[(\rho_0 c v_n + p\cos\theta)_c + (\vec{\eta} - \vec{\eta}_c)\cdot\nabla(\rho_0 c v_n + p\cos\theta)_c + \cdots \right] \frac{M_r}{r(1-M_r)^2} \quad (11)$$

WHERE c STANDS FOR THE CENTER OF THE CIRCLE. WE ALSO HAVE

$$(\rho_0 c v_n + p\cos\theta)_c = (\rho_0 c v_n + p M_n)_c$$
$$= (\rho_0 c v_n + M_n p)_{\vec{x}} + O(\varepsilon)$$

$$d\Gamma = \varepsilon\sqrt{1-M_n^2}\, d\phi$$

$$\sin\theta = \frac{\varepsilon\sqrt{1-M_n^2}}{\varepsilon} = \sqrt{1-M_n^2}$$

$$M_r = -M_t \sin\theta\cos\phi + M_n\cos\theta$$
$$= -M_t\sqrt{1-M_n^2}\cos\phi + M_n^2$$

$$1 - M_r = 1 - M_n^2 + M_t\sqrt{1-M_n^2}\cos\phi$$
$$= (1-M_n^2)\left(1 + \frac{M_t}{\sqrt{1-M_n^2}}\cos\phi\right)$$
$$= \beta(1+\alpha\cos\phi)$$

WHERE

$$\beta = 1 - M_n^2$$
$$\alpha = \frac{M_t}{\sqrt{1-M_n^2}}$$

FROM THESE, WE GET

$$I_\varepsilon = (\rho_0 c v_n + p)_{\vec{x}} \int_0^{2\pi} \left[\frac{1}{r(1-M_r)^2} - \frac{1}{r(1-M_r)} \right] \frac{d\Gamma}{\sin\theta} + O(\varepsilon)$$

$$= (\rho_0 c v_n + M_n p)_{\vec{x}} \left\{ \int_0^{2\pi} \frac{d\phi}{\beta^2(1+\alpha\cos\phi)^2} - \int_0^{2\pi} \frac{d\phi}{\beta(1+\alpha\cos\phi)} \right\} + O(\varepsilon) \quad (12)$$

$$\int_0^{2\pi} \frac{d\phi}{1+\alpha\cos\phi} = \frac{2\pi}{\sqrt{1-\alpha^2}} \quad P_0\left(\frac{1}{\sqrt{1-\alpha^2}}\right) = \frac{2\pi}{\sqrt{1-\alpha^2}}$$

$$\int_0^{2\pi} \frac{d\phi}{(1+\alpha\cos\phi)^2} = \frac{2\pi}{1-\alpha^2} P_1\left(\frac{1}{\sqrt{1-\alpha^2}}\right)$$

$$= \frac{2\pi}{(1-\alpha^2)^{3/2}}$$

WHERE P_0 AND P_1 ARE LEGENDRE'S POLYNOMIALS.

$$I_\varepsilon = \frac{2\pi(\rho_0 c v_n + M_n p)_{\vec{x}}}{\beta(1-\alpha^2)^{1/2}}\left[\frac{1}{\beta(1-\alpha^2)} - 1\right] + O(\varepsilon) \quad (13)$$

WE HAVE

$$1 - \alpha^2 = \frac{1-M^2}{\beta}$$

$$\frac{1}{\beta(1-\alpha^2)} - 1 = \frac{1}{1-M^2} - 1$$

$$= \frac{M^2}{1-M^2}$$

$$I_\varepsilon = \frac{2\pi M^2(\rho_0 c v_n + M_n p)_{\vec{x}}}{(1-M_n^2)^{1/2}(1-M^2)^{3/2}} + O(\varepsilon)$$

$$= 4\pi\,\gamma(M, M_n)\,(\rho_0 c^2 + p)_{\vec{x}} + O(\varepsilon) \quad (14)$$

WHERE

$$\gamma(M, M_n) = \frac{M^2 M_n}{2(1-M_n^2)^{1/2}(1-M^2)^{3/2}} \quad (15)$$

SUBSTITUTING IN EQ. (5) AND REMEMBERING THAT THERE IS A NEGATIVE SIGN (FROM $\frac{\partial K_\varepsilon}{\partial \tau} = -v_r$) IN FRONT OF I_ε, WE HAVE, AFTER TAKING I_ε TO THE LEFT SIDE AND LETTING $\varepsilon \to 0$, THE FOLLOWING RESULT* :

(*) THERE ARE STILL SOME TERMS MISSING! WE HAVE REGULARIZED THE SURFACE INTEGRAL WITH $\vec{x}$ ON $f=0$ WHILE $\vec{x}$ APPROACHES $f=0$ FROM OUTSIDE. SEE NOTEBOOK II, P16

$$\boxed{\begin{aligned}4\pi[1+\gamma(M,M_n)]\,p(\vec{x},t) &= -4\pi\rho_0 c^2\gamma(M,M_n)\\ &+\lim_{\varepsilon\to 0}\iint_{f=0}\left[\frac{H(K_\varepsilon)}{c(1-M_r)}\frac{\partial}{\partial\tau}\left(\frac{\rho_0 c v_n+p\cos\theta}{r(1-M_r)}\right)+\frac{p\cos\theta\,H(K_\varepsilon)}{r^2(1-M_r)}\right]_{ret} dS\end{aligned}} \quad (16)$$

THE LAST INTEGRAL CAN BE WRITTEN AS FOLLOWS:

$$J_\varepsilon(\vec{x},t) = \frac{1}{c}\int_{f=0}\left[\frac{H(K_\varepsilon)\,p\,\Omega_r}{r(1-M_r)^2}\right]_{ret} dS$$

$$+\frac{1}{c}\int_{f=0}\left[\frac{H(K_\varepsilon)\dot{M}_r(\rho_0 c v_n+p\cos\theta)}{r(1-M_r)^3}\right]_{ret} dS$$

$$+\int_{f=0}\left[\frac{H(K_\varepsilon)\,p\,(\cos\theta-M_n)}{r^2(1-M_r)^2}\right]_{ret} dS$$

$$+\iint_{f=0}\left[\frac{(\rho_0 c v_n+p\cos\theta)(M_r-M^2)H(K_\varepsilon)}{r^2(1-M_r)^3}\right]_{ret} dS \quad (17)$$

WHERE $\Omega_r=\hat{r}\cdot(\vec{\omega}\times\vec{n})$, $\vec{\omega}$ ANGULAR VELOCITY ; $\dot{M}_r=\hat{r}\cdot\frac{\partial\vec{v}}{\partial\tau}$.
THE FIRST TWO INTEGRALS ARE CONVERGENT AS $\varepsilon\to 0$.
THE LAST TWO INTEGRALS MUST BE SUMMED IN A MANNER SIMILAR TO TAKING PRINCIPAL VALUE OF ONE DIMENSIONAL INTEGRALS. WE EXPAND p AND $\rho_0 c v_n + p\cos\theta$ AT $\vec{x}$ AND THEREFORE CONSIDER ONLY THE FOLLOWING TWO INTEGRALS:

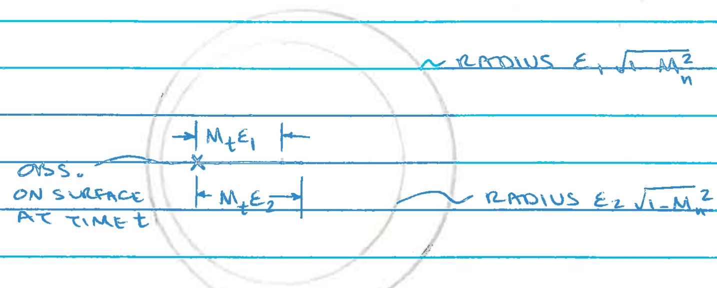
$$J_\varepsilon^1 = \int_{f=0}\left[\frac{H(K_\varepsilon)(\cos\theta-M_n)}{r^2(1-M_r)^2}\right]_{ret} dS \;, \quad (18)$$

$$J_\varepsilon^2 = \int_{f=0}\left[\frac{H(K_\varepsilon)(M_r-M^2)}{r^2(1-M_r)^3}\right]_{ret} dS \;, \quad (19)$$

WE CONSIDER TWO VALUES OF ε — ε_1 AND ε_2 SUCH THAT $\varepsilon_1 > \varepsilon_2$. WE NOTE THAT THE CIRCLE OF RADIUS $\varepsilon\sqrt{1-M_n^2}$ IS ACTUALLY THE INTERSECTION OF A SPHERE OF RADIUS ε WITH CENTER AT $\vec{x}$.
LET $t-\tau = \frac{\varepsilon}{c}$, THEN

$$\frac{dS}{1-M_r} = \frac{c\, d\tau\, d\Gamma}{\sin\theta}$$

THE TWO VALUES OF ε_1 AND ε_2 CORRESPOND TO THE TIMES (SOURCE TIMES) $\frac{\varepsilon_1}{c}$ AND $\frac{\varepsilon_2}{c}$.



2π

THEREFORE THE TWO INTEGRALS SHOULD BE ANALYZED AS FOLLOWS FOR THEIR CONVERGENCE PROPERTIES

$$J^1_{\varepsilon_1} - J^2_{\varepsilon_2} = \int_{\varepsilon_2/c}^{\varepsilon_1/c} \frac{d\tau}{c^2(t-\tau)^2} \int_\Gamma \frac{\cos\theta - M_n}{(1-M_r)\sin\theta}\, d\Gamma \qquad (20)$$

WE HAVE, ON Γ- CURVE

$$\cos\theta = M_n$$

SO THAT $J^1_{\varepsilon_1} - J^2_{\varepsilon_2} = 0$ ONLY IF WE SUM THE INTEGRAL IN THE ANNULAR REGION SHOWN ABOVE!

FOR J_ε^2 WE MUST STUDY

$$J_{\varepsilon_1}^2 - J_{\varepsilon_2}^2 = \int_{\varepsilon_2/c}^{\varepsilon_1/c} \frac{d\tau}{c^2(t-\tau)^2} \int_\Gamma \frac{M_r - M^2}{(1-M_r)^2 \sin\theta} d\Gamma \qquad (21)$$

REFERRING TO FIGURE IN PAGE 164, WE HAVE (AGAIN!)

$$d\Gamma = \varepsilon \sqrt{1-M_n^2}\, d\phi$$

$$= c(t-\tau)\sqrt{1-M_n^2}\, d\phi$$

$$\sin\theta = \sqrt{1-M_n^2}$$

$$M_r = M_n \cos\theta - M_t \sin\theta \cos\phi$$

$$= M_n^2 - M_t \sqrt{1-M_n^2} \cos\phi$$

$$1 - M_r = 1 - M_n^2 + M_t \sqrt{1-M_n^2} \cos\phi$$

$$= (1-M_n^2)\left[1 + \frac{M_t}{\sqrt{1-M_n^2}} \cos\phi\right]$$

$$\equiv \beta\,(1 + \alpha\cos\phi)$$

$$J_{\varepsilon_1}^2 - J_{\varepsilon_2}^2 = -\int_{\varepsilon_2/c}^{\varepsilon_1/c} \frac{d\tau}{c(t-\tau)} \int_0^{2\pi} \frac{d\phi}{\beta(1+\alpha\cos\phi)}$$

$$+ (1-M^2) \int_{\varepsilon_2/c}^{\varepsilon_1/c} \frac{d\tau}{c(t-\tau)} \int_0^{2\pi} \frac{d\phi}{\beta^2(1+\alpha\cos\phi)^2}$$

$$= \left\{ \frac{-2\pi}{\beta(1-\alpha^2)^{1/2}} + \frac{2\pi(1-M^2)}{\beta^2(1-\alpha^2)^{3/2}} \right\}$$

$$\times \int_{\varepsilon_2/c}^{\varepsilon_1/c} \frac{d\tau}{c(t-\tau)} \qquad (23)$$

THE QUANTITY IN THE BRACKET IS

$$\frac{2\pi}{\beta\sqrt{1-\alpha^2}}\left\{-1+\frac{1-M^2}{1-M^2}\right\}=0 \, !$$

AGAIN ONLY THIS PARTICULAR METHOD OF SUMMATION OF J_ε^2 GIVES CONVERGENT INTEGRAL.

SEE P 16 OF NOTEBOOK II FOR ADDITIONAL TERMS IN THE INTEGRAL EQ.

* THE THICKNESS PROBLEM FOR WINGS

THE SOLUTION OF THIS PROBLEM IS WELL-KNOWN. WE WILL DERIVE THE VELOCITY POTENTIAL AND THE SOLUTION USING ACOUSTIC (WAVE) EQUATION. CONSIDER THE WING SURFACE DESCRIBED BY EQUATION

$$\tilde{f}(\vec{x}) = \begin{cases} \tilde{x}_3 - h(\tilde{x}_1, \tilde{x}_2) = 0 \equiv \tilde{f}_+(\vec{\tilde{x}}) \\ -\tilde{x}_3 - h(\tilde{x}_1, \tilde{x}_2) = 0 = \tilde{f}_-(\vec{\tilde{x}}) \end{cases}$$

HERE WE ASSUME THAT THE CHORD AND SPAN OF THE WING ARE OF $O(1)$ WHILE $h(x_1, x_2)$ IS OF $O(\varepsilon)$, WHERE $\varepsilon \ll 1$. LET THIS WING MOVE WITH VELOCITY $-V$ ALONG NEGATIVE x_1-AXIS. THE VELOCITY POTENTIAL ϕ SATISFIES THE WAVE EQUATION $\Box^2\phi = 0$. THE BOUNDARY CONDITION IS v_n SPECIFIED OVER THE WING SURFACE. THE $\vec{\tilde{x}}$-FRAME IS FIXED TO THE WING AND $\vec{x}$-FRAME TO UNDISTURBED MEDIUM.

THE GOVERNING EQUATION

LET $\tilde{\phi}(\vec{x}, t)$ BE THE FUNCTION DEFINED AS FOLLOWS

$$\tilde{\phi}(\vec{x}, t) = \begin{cases} \phi(\vec{x}, t) & f > 0 \\ 0 & f < 0 \end{cases}$$

WHERE $f(\vec{x}, t) = \tilde{f}(x_1 + Vt, x_2, x_3)$

WE WRITE $\bar{\Box}^2\tilde{\phi}$:

$$\frac{\bar{\partial}\tilde{\phi}}{\partial t} = \frac{\partial\phi}{\partial t} + \phi\frac{\partial f}{\partial t}\delta(f)$$

$$= \frac{\partial\phi}{\partial t} - \phi v_n|\nabla f|\delta(f)$$

$$\frac{\bar{\partial}^2\tilde{\phi}}{\partial t^2} = \frac{\partial^2\tilde{\phi}}{\partial t^2} + \frac{\partial\phi}{\partial t}\frac{\partial f}{\partial t}\delta(f) - \frac{\bar{\partial}}{\partial t}[\phi v_n|\nabla f|\delta(f)]$$

$$= \frac{\partial^2\tilde{\phi}}{\partial t^2} - \frac{\partial\phi}{\partial t}v_n|\nabla f|\delta(f) - \frac{\bar{\partial}}{\partial t}[\phi v_n|\nabla f|\delta(f)]$$

$$\bar{\nabla}\tilde{\phi} = \nabla\tilde{\phi} + \phi\,\nabla f\,\delta(f)$$

$$\bar{\nabla}^2\tilde{\phi} = \nabla^2\tilde{\phi} + \nabla\phi\cdot\nabla f\,\delta(f) + \bar{\nabla}\cdot[\phi\nabla f\,\delta(f)]$$

$$\bar{\Box}^2\tilde{\phi} = \Box^2\tilde{\phi} - \left[\frac{\partial\phi}{\partial n} + \frac{v_n}{c^2}\frac{\partial\phi}{\partial t}\right]|\nabla f|\,\delta(f)$$

$$- \frac{1}{c^2}\frac{\bar{\partial}}{\partial t}\left[\phi\, v_n|\nabla f|\,\delta(f)\right] - \bar{\nabla}\cdot[\phi\,\nabla f\,\delta(f)]$$

WE HAVE $\Box^2\tilde{\phi} = 0$ AND ALSO THE FOLLOWING RELATION

$$\frac{\partial}{\partial t} = -V\frac{\partial}{\partial x_1}$$

$$\frac{\partial\phi}{\partial n} = v_n\,, \quad \frac{\partial\phi}{\partial x_1} = u\,.$$

$$\therefore\quad \bar{\Box}^2\tilde{\phi} = -\left[1 - \frac{uV}{c^2}\right]v_n|\nabla f|\,\delta(f)$$

$$+ \frac{V}{c^2}\frac{\bar{\partial}}{\partial x_1}\left[\phi\, v_n|\nabla f|\,\delta(f)\right] - \bar{\nabla}\cdot[\phi\,\nabla f\,\delta(f)] \qquad (*)$$

THIS IS THE EQUATION VALID IN THE ENTIRE UNBOUNDED SPACE. ITS FORMAL SOLUTION RESULTS IN AN INTEGRAL EQUATION FOR ϕ. HOWEVER, WE CAN SIMPLIFY THIS EQUATION KEEPING ONLY SOURCE TERMS OF ORDER ϵ AND THUS FIND THE SOLUTION FOR ϕ ITSELF. WE HAVE

$$\delta(f) = \delta(f_+) + \delta(f_-)$$

$$= \delta[x_3 - h(x_1+Vt, x_2)] + \delta[x_3 + h(x_1+Vt, x_2)]$$

$$= 2\delta(x_3) + h^2\,\delta''(x_3) + O(h^4)$$

$$|\nabla f| = [1 + h_1^2 + h_2^2]^{1/2} = 1 + \tfrac{1}{2}|\nabla h|^2 + O(|\nabla h|^4),$$

WHERE $|\nabla h| = [h_1^2 + h_2^2]^{1/2}$. ALSO

$$\nabla f\, \delta(f) = \nabla f_+ \delta(f_+) + \nabla f_- \delta(f_-)$$
$$= \nabla f_+ [\delta(x_3) - h\, \delta'(x_3) + \frac{h^2}{2} \delta''(x_3) + \ldots]$$
$$+ \nabla f_- [\delta(x_3) + h\, \delta'(x_3) + \frac{h^2}{2} \delta''(x_3) + \ldots]$$
$$= (\nabla f_+ + \nabla f_-)\, \delta(x_3) + (-\nabla f_+ + \nabla f_-) h \delta'(x_3)$$
$$+ \frac{1}{2} (\nabla f_+ + \nabla f_-) h^2 \delta''(x_3) + O(h^4)$$

$$\nabla f_+ = (-h_1, -h_2, 1)$$
$$\nabla f_- = (-h_1, -h_2, -1)$$

$$\nabla f\, \delta(f) = -2\nabla h\, \delta(x_3) - 2h\, \vec{e}_3\, \delta'(x_3)$$
$$- 2h^2 \nabla h\, \delta''(x_3) + O(h^4)$$
$$|\nabla f|\, \delta(f) = [1 + \tfrac{1}{2}|\nabla h|^2 + O(|\nabla h|^4)][2\delta(x_3) + h^2 \delta''(x_3) + O(h^4)]$$
$$= 2\delta(x_3) + |\nabla h|^2 \delta(x_3) + h^2 \delta''(x_3) + O(h^4)$$

$$v_n = -V\vec{e}_1 \cdot \nabla f / |\nabla f|$$
$$= V h_1 [1 + \tfrac{1}{2}|\nabla h|^2 + O(|\nabla h|^4)]$$
$$u = \frac{\partial \phi}{\partial n} = O(\varepsilon)$$

THE ONLY TERM OF ORDER ε ON THE RIGHT OF EQ. (*) IS $-2 v_n \delta(x_3)$. LET US WRITE

$$\tilde{\phi} = \tilde{\phi}_1 + \tilde{\phi}_2 + O(\varepsilon^3)$$

WHERE $\tilde{\phi}_1 = O(\varepsilon)$, $\tilde{\phi}_2 = O(\varepsilon^2)$, ETC.

THEN THE EQ. FOR $\tilde{\phi}_1$ IS

$$\Box^2 \tilde{\phi}_1 = -2 v_n \delta(x_3) = -2 V h_1 \delta(x_3)$$

THE EQUATION FOR $\tilde{\phi}_2$ IS

$$\bar{\Box}^2 \tilde{\phi}_2 = \left[2 \frac{M}{c} \frac{\partial \tilde{\phi}_1}{\partial x_1} \delta(x_3) - \cancelto{O(\varepsilon)^2}{|\nabla h|^2} \delta(x_3) - \cancelto{O(\varepsilon^2)}{h^2} \delta''(x_3) \right] v_n$$

$$+ \frac{M}{c} \frac{\bar{\partial}}{\partial x_1} \left[2 \tilde{\phi}_1 v_n \delta(x_3) \right]$$

$$+ 2 \bar{\nabla} \cdot \left\{ \tilde{\phi}_1 \left[\nabla h \, \delta(x_3) + h \vec{e}_3 \, \delta'(x_3) \right] \right\}$$

$$= \left[2 \frac{M}{c} v_n \frac{\partial \tilde{\phi}_1}{\partial x_1} + 2 \frac{M}{c} \frac{\bar{\partial}}{\partial x_1} (\tilde{\phi}_1 v_n) + 2 \bar{\nabla} \cdot (\tilde{\phi}_1 \nabla h) \right] \delta(x_3)$$

$$+ 2 h \frac{\partial \tilde{\phi}_1}{\partial x_3} \delta'(x_3) + 2 h \tilde{\phi}_1 \delta''(x_3)$$

WRITING $v_n = V h_1 + O(\varepsilon^3)$, WE GET

$$\bar{\Box}^2 \tilde{\phi}_2 = \Big(4 M^2 h_1 \frac{\partial \tilde{\phi}_1}{\partial x_1} + 2 M^2 \tilde{\phi}_1 h_{11} + 2 h_1 \frac{\partial \tilde{\phi}_1}{\partial x_1}$$

$$+ 2 h_2 \frac{\partial \tilde{\phi}_2}{\partial x_2} + 2 \tilde{\phi}_1 \nabla^2 h \Big) \delta(x_3)$$

$$+ 2 h \frac{\partial \tilde{\phi}_1}{\partial x_3} \delta'(x_3) + 2 h \tilde{\phi}_1 \delta''(x_3)$$

$$= 2 \Big\{ \left[(M^2 + 1) h_{11} + h_{22} \right] \tilde{\phi}_1 + (2 M^2 + 1) h_1 \frac{\partial \tilde{\phi}_1}{\partial x_1}$$

$$+ h_2 \frac{\partial \tilde{\phi}_2}{\partial x_2} \Big\} \delta(x_3) + 2 h \frac{\partial \tilde{\phi}_1}{\partial x_3} \delta'(x_3)$$

$$+ 2 h \tilde{\phi}_1 \delta''(x_3)$$

WHERE $M = V/c$.

IF THE SOLUTION FOR $\tilde{\phi}_1$ IS KNOWN, THEN $\tilde{\phi}_2$ CAN BE FOUND FROM THIS EQUATION. IF THE INTEGRAL EQUATION FROM EQ (*), P 172, IS SOLVED FOR ϕ, THEN ONE DOES NOT NEED TO SOLVE THE ABOVE

EQUATION FOR $\tilde{\phi}_2$. FOR BLUNT LEADING EDGE, $h_1 = \infty$ AT THE L.E. THIS CAUSES SOME DIFFICULTY IN NUMERICAL INTEGRATION WHICH CAN BE HANDLED. HOWEVER, THE NUMERICAL INVERSION OF EQ. (*) USING THE ACTUAL WING SURFACE $f(\vec{x},t)=0$ IS NOT A DIFFICULT PROBLEM.

THE SOLUTION OF THE D.E. FOR $\tilde{\phi}_1$ IS SIMPLE.

$$\Box^2 \tilde{\phi}_1 = -2Vh_1\,\delta(x_3)$$

WE NOTE THAT $h_1(\vec{x},t) = h_1(\tilde{x})$, THE SOLUTION IS

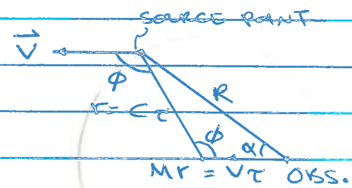
$$2\pi\,\tilde{\phi}_1(\vec{x},t) = \int_{S^*} \left[\frac{Vh_1}{r|1-M_r|}\right]_{ret} dS \quad ; dS = dx_1 dx_2$$

WHERE S^* IS THE REGION ON THE WING INFLUENCING THE POINT $\vec{x}$ ON THE BLADE. THE INTEGRATION IS ON THE ENTIRE WING IF $M<1$ AND ON PART OF THE WING INTERSECTED BY FORWARD LOOKING MACH CONE VIEWED FROM $\tilde{x}$-FRAME. THE RESULTS FOR SUBSONIC AND SUPERSONIC CASES IN $\tilde{x}$-FRAME WHICH WILL BE THE SAME AS THOSE IN AERODYNAMICS WILL BE DERIVED NEXT.

SUBSONIC CASE

WE WILL CALCULATE $r(1-M_r)$ IN THE FRAME FIXED TO THE WING. IN THIS FRAME THE DISTANCE R AND ANGLE α ARE THE MEANINGFUL PARAMETERS TO

USE TO FIND $r(1-M_r)$. REFERRING TO FIGURE BELOW, WE HAVE



$$r(1-M_r) = r - Mr\cos\phi$$

WE FIND r FROM

$$r^2 = R^2 + M^2r^2 - 2MRr\cos\alpha$$

$$(1-M^2)r^2 + 2MR\cos\alpha\, r - R^2 = 0 \quad (*)$$

$$r = \frac{1}{1-M^2}\left[-MR\cos\alpha + \sqrt{M^2R^2\cos^2\alpha + R^2(1-M^2)}\right]$$

$$= \frac{R}{1-M^2}\left[-M\cos\alpha + \sqrt{1-M^2\sin^2\alpha}\right]$$

ALSO WE HAVE

$$R^2 = r^2 + M^2r^2 - 2Mr^2\cos\phi$$

$$\therefore \quad Mr\cos\phi = \frac{(1+M^2)r^2 - R^2}{2r}$$

$$r - Mr\cos\phi = \frac{(1-M^2)r^2 + R^2}{2r}$$

$$= \frac{R^2 - MR\cos\alpha\, r}{r} \qquad \text{USING EQ (*), ABOVE}$$

$$\frac{1}{r - Mr\cos\phi} = \frac{r}{R(R - M\cos\alpha\, r)}$$

$$= \frac{1}{R}\,\frac{\frac{R}{1-M^2}\left[-M\cos\alpha + \sqrt{1-M^2\sin^2\alpha}\right]}{\frac{R}{1-M^2}\left[1-M^2+M^2\cos^2\alpha - M\cos\alpha\sqrt{1-M^2\sin^2\alpha}\right]}$$

$$= \frac{1}{R\sqrt{1-M^2\sin^2\alpha}}$$

$$R\sqrt{1-M^2\sin^2\alpha} = \sqrt{M^2R^2\cos^2\alpha + R^2(1-M^2)}$$

$$= \left\{M^2(\tilde{x}_1-\tilde{y}_1)^2 + (1-M^2)\left[(\tilde{x}_1-\tilde{y}_1)^2 + (\tilde{x}_2-\tilde{y}_2)^2\right]\right\}^{1/2}$$

$$\cancel{R\sqrt{1-M^2\sin^2\alpha}} = \sqrt{(\tilde{x}_1-\tilde{y}_1)^2+\beta^2(\tilde{x}_2-\tilde{y}_2)^2}, \quad \beta^2 = 1-M^2$$

$$\therefore \quad 2\pi\,\tilde{\phi}_1(\vec{\tilde{x}}) = -\int_{PF} \frac{V h_1(\tilde{y}_1,\tilde{y}_2)}{\sqrt{(\tilde{x}_1-\tilde{y}_1)^2+\beta^2(\tilde{x}_2-\tilde{y}_2)^2}}\, d\tilde{y}_1\, d\tilde{y}_2$$

THIS IS THE RESULT DERIVED IN AERODYNAMICS. TO FIND THE PRESSURE ON THE WING SURFACE, WE NEED $\frac{\partial \tilde{\phi}_1}{\partial \tilde{x}_1} = \frac{\bar{\partial}\tilde{\phi}_1}{\partial \tilde{x}_1}$ (SINCE $\tilde{\phi}_1$ HAS NO DISCONTINUITIES). LET THE PLANFORM BE DESCRIBED BY $K(x_1,x_2)=0$ SUCH THAT $K>0$ ON THE WING. WE HAVE

$$2\pi\frac{\partial\tilde{\phi}_1}{\partial x_1} = -\frac{\bar{\partial}}{\partial x_1}\int \frac{V h_1(y_1,y_2)H(K)}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}}\,dy_1 dy_2$$

$$= -\int V h_1 H(K)\frac{\bar{\partial}}{\partial x_1}\frac{1}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}}\,dy_1 dy_2$$

$$= \int V h_1 H(K)\frac{\bar{\partial}}{\partial y_1}\frac{1}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}}\,dy_1 dy_2$$

$$= -\int \frac{V}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}}\left[\frac{\partial h_1}{\partial y_1}H(K) + h_1\frac{\partial K}{\partial y_1}\delta(K)\right]dy_1 dy_2$$

WHERE WE HAVE DROPED THE TYLDE ON $\vec{\tilde{x}}$ AND $\vec{\tilde{y}}$. THE δ-FN WILL GIVE A LINE INTEGRAL AROUND THE EDGE OF PLANFORM. IF WE ASSUME $h_1=0$ ON THIS EDGE, THE LINE INTEGRAL IS ZERO. IN GENERAL,

$$2\pi\frac{\partial\tilde{\phi}_1}{\partial x_1} = -\int_{PF}\frac{V h_{11}\,dy_1 dy_2}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}} - \oint_{K=0}\frac{V h_1 n_1\,d\ell}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}}$$

WHERE $n_1 = (\partial K/\partial y_1)/|\nabla K|$ AND $d\ell$ IS ELEMENT OF LENGTH ON $K=0$. IF WE HAVE BLUNT LEADING EDGE, h_{11} WILL HAVE A NONINTEGRABLE SINGULARITY AT THE L.E. WHICH REQUIRES SPECIAL TREATMENT. (CONT'D IN NEW NOTEBOOK II).

5

$$\frac{\partial}{\partial x_j}(\sigma_{ij} u_i) = u_i \frac{\partial \sigma_{ij}}{\partial x_j} + \sigma_{ij} \frac{\partial u_i}{\partial x_j}$$

$$= u_i \left[\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} \right] + \sigma_{ij} \frac{\partial u_i}{\partial x_j}$$

$$= \underbrace{\rho \frac{D}{Dt}\left(\frac{1}{2} u^2\right)}_{\text{KINETIC ENERGY}} + \underbrace{\Phi}_{\text{DIS. FN.}}$$

- INVISCID ADIABATIC UNSTEADY FLOW

$$\boxed{\rho \frac{Dh_t}{Dt} = \frac{\partial p}{\partial t}}$$

$$h_t = h + \frac{1}{2}u^2$$

$$\boxed{\rho \frac{Dh}{Dt} = \frac{Dp}{Dt}}$$ (FROM BELOW $\Rightarrow \frac{Ds}{Dt} = 0$)

THE ABOVE RESULTS ARE IMPORTANT. THEY ARE USED IN UNSTEADY INVISCID AERODYNAMICS.

- SUMMARY

$$\rho T \frac{Ds}{Dt} = \rho \frac{De}{Dt} + p\,\nabla\cdot\bar{U}$$
$$= \rho \frac{Dh}{Dt} - \frac{Dp}{Dt}$$
$$= \Phi + \nabla\cdot(k\nabla T)$$

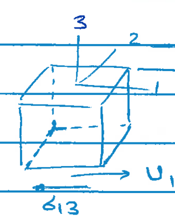
$\Phi = \sigma_{ij}\epsilon_{ij}$

$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

(O.K., OWCZAREK, P546)

- THE INTERPRETATION OF Φ

TAKE $\sigma_{13}\frac{\partial u_1}{\partial x_3}$. WORK DONE ON THE FLUID IN THE VOLUME SHOWN FOR ONE FORCE COMPONENT IS:



$$\left[\left(\sigma_{13} + \frac{\partial\sigma_{13}}{\partial x_3}dx_3\right)\left(U_1 + \frac{\partial U_1}{\partial x_3}dx_3\right) - \sigma_{13}U_1\right]dx_1dx_2 = \left[\sigma_{13}\frac{\partial U_1}{\partial x_3} + U_1\frac{\partial\sigma_{13}}{\partial x_3}\right]dx_1dx_2dx_3 + O(dx_1dx_2dx_3)$$

$$= \frac{\partial(\sigma_{13}U_1)}{\partial x_3}dx_1dx_2dx_3$$

$\therefore$ WORK DONE ON THE FLUID/UNIT VOLUME IS $\frac{\partial}{\partial x_j}(\sigma_{ij}u_i)$. HOWEVER, NOT ALL THIS WORK IS CONVERTED INTO INTERNAL ENERGY. SOME OF IT GOES INTO KINETIC ENERGY. WE NOTE THAT

THE ENERGY EQ.

IDEAL FLUID

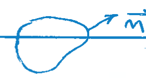
- $$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \rho e\right) = -\nabla\cdot\left[\rho\vec{u}\left(\tfrac{1}{2}u^2 + h\right)\right]$$

e : INTERNAL ENERGY ; h : ENTHALPY

VISCOUS HEAT CONDUCTING FLUID

- $$\boxed{\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \rho e\right) = -\nabla\cdot\left[\rho\vec{u}\left(\tfrac{1}{2}u^2 + h\right) - u_j\sigma_{ij} - K\nabla T\right]}$$

$\vec{F} = \sigma_{ij} m_j$ IS FORCE ON THE FLUID PARTICLE AT ITS SURFACE PER UNIT AREA.



THE EQUATION OF HEAT TRANSFER

- $$\rho T\left(\frac{\partial s}{\partial t} + \vec{v}\cdot\nabla s\right) = \overbrace{\sigma_{ij}\frac{\partial u_i}{\partial x_j}}^{\Phi} + \nabla\cdot(K\nabla T)$$

- $$\frac{d}{dt}\int_V \rho s\, d\vec{V} = \int_V \frac{K|\nabla T|^2}{T^2}\, d\vec{V} + \int_V \frac{\eta}{2T}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) d\vec{V} + \int_V \frac{\zeta}{T}(\nabla\cdot\vec{u})^2\, d\vec{V}$$

WE HAVE $$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left[\sigma_{ij}\frac{\partial u_i}{\partial x_j} + \sigma_{ji}\frac{\partial u_j}{\partial x_i}\right]$$
$$= \frac{1}{2}\sigma_{ij}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$
$$= \sigma_{ij}\epsilon_{ij}$$
$$= \Phi$$ THE DISSIPATION FN.

WHERE $\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$: RATE OF STRAIN TENSOR

- $$\rho\frac{De}{Dt} = \rho\left(\frac{\partial e}{\partial t} + \vec{v}\cdot\nabla e\right) = -p\,\nabla\cdot\vec{v} + \nabla\cdot(K\nabla T) + \Phi$$

- $$\rho\left(\frac{\partial h}{\partial t} + \vec{v}\cdot\nabla h\right) = \frac{\partial p}{\partial t} + \vec{v}\cdot\nabla p + \nabla\cdot(K\nabla T) + \Phi$$

$$\boxed{\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \nabla\cdot(K\nabla T) + \Phi}$$

REVERSIBLE ADIABATIC EXPANSION

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} \quad ; \quad \frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

$$C_p - C_v = R \quad ; \quad R = \frac{8314}{M} ,$$

M MOLECULAR WT

$M_{AIR} = 28.98 \approx 29$

(SEE THOMPSON, P86)

$$C_p = \frac{\gamma R}{\gamma - 1} \quad ; \quad C_v = \frac{R}{\gamma - 1}$$

CONT. EQ.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

MOM. EQ.

$$\boxed{\begin{aligned} &\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial P_{ij}}{\partial x_j} \\ &\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial P_{ij}}{\partial x_j} \end{aligned}}$$

$$P_{ij} = -p\,\delta_{ij} + \sigma_{ij}$$

CONVENTION:

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) + \xi \frac{\partial u_k}{\partial x_k}\delta_{ij}$$

$$\Pi_{ij} = p\,\delta_{ij} + \rho u_i u_j - \sigma_{ij}$$

MOMENTUM FLUX DENSITY TENSOR

$$\boxed{\frac{\partial (\rho u_i)}{\partial t} = -\frac{\partial \Pi_{ij}}{\partial x_j}}$$

1

THINGS TO REMEMBER

ALL UNITS MKS

GAS CONSTANT	8317 J/Kgmole-°C
MECH. EQUIV. OF HEAT	4186 J/Kcal
VOL. OF 1 Kgmole of GAS AT N.T.P.	22.42 m^3
AVOGADRO'S NUMBER	6.03×10^{26} $(\text{Kgmole})^{-1}$
PLANCK'S CONST.	6.63×10^{-34} J-sec
BOLTZMANN CONST.	1.38×10^{-23} J-°C^{-1}

VISCOSITY OF AIR AT 15°C AND 1.01×10^5 N/m^2 (0 m ALT.) $\}$ 1.79×10^{-5} Kg/m-sec

" " " AT −50°C AND 26,440 N/m^2 (10,000 m ALT.) $\}$ 1.46×10^{-5} Kg/m-sec

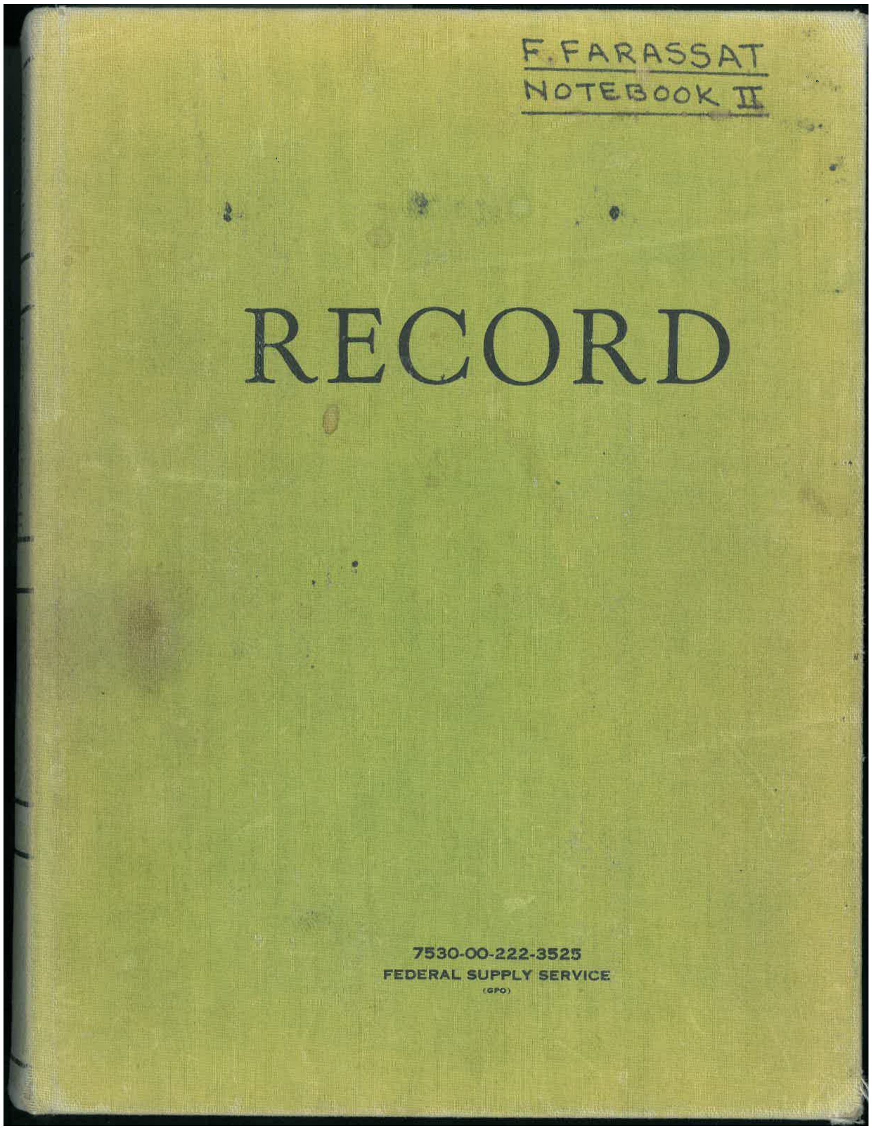
STANDARD ATM.

ALT. m	t °C	p N/m^2	ρ kg/m^3	SOUND SPEED c m/sec	K $\frac{\text{Kcal}}{\text{m-sec-°K}}$	g m/sec^2	NO. DENSITY m^{-3}	PARTICLE SPEED m/sec
0	15	101,330	1.225	340.	6.0×10^{-6}	9.81	2.55×10^{25}	459.
10,000	−50	26,440	.4127	299.	4.8×10^{-6}	9.78	$8.58 \times 10^{+24}$	404.

ALT. m	COLLISION FREQ. sec^{-1}	MEAN FREE PATH m	KIN. VISCOSITY m^2/sec
0	6.9×10^{9}	6.6×10^{-8}	1.46×10^{-5}
10,000	2.1×10^{9}	2.0×10^{-7}	3.53×10^{-5}

WATER	0°C	20°C	50°C
VISCOSITY	1.78×10^{-3}	1.00×10^{-3}	5.6×10^{-4}
KIN. VISCOSITY	1.78×10^{-6}	1.00×10^{-6}	5.6×10^{-7}

11 Notebook Two



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ANNIVERSARY

APOLLO 11

"THE EAGLE HAS LANDED"

F. FARASSAT
FEB. 8, 1982
HAMPTON

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1

THE THICKNESS PROBLEM FOR WINGS (CONT'D)

IT WAS SHOWN THAT THE VELOCITY IN x_1 DIRECTION FOR SUBSONIC WINGS IS DESCRIBED BY THE RELATION

$$2\pi \frac{\partial \tilde{\phi}_1}{\partial x_1} = -\int_{PF} \frac{v h_{11}\, dy_1 dy_2}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}} - \oint_{K=0} \frac{v h_1 n_1\, d\ell}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)}}$$

WHERE $K(y_1, y_2) = 0$ DESCRIBES THE WING PLANFORM,
(NOTE: ∇K POINTS INWARD)
$n_1 = (\partial K/\partial y_1)/|\nabla K|$ AND $d\ell$ IS THE ELEMENT OF LENGTH ON $K=0$. FOR BLUNT LEADING EDGES, BOTH h_1 AND h_{11} ARE INFINITE AT L.E. IT IS BETTER TO KEEP THE D.E. FOR THE THICKNESS PROBLEM AS

$$\square^2 \tilde{\phi}_1 = -2\, v_n\, \delta(x_3)$$

WITH THE FOLLOWING SOLUTION FOR $\frac{\partial \tilde{\phi}_1}{\partial x_1}$ IN THE FRAME FIXED TO THE WING

$$2\pi \frac{\partial \tilde{\phi}_1}{\partial x_1} = -\int_{PF} \frac{\partial v_n/\partial y_1\, dy_1 dy_2}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}} - \oint_{K=0} \frac{v_n n_1\, d\ell}{\sqrt{(x_1-y_1)^2+\beta^2(x_2-y_2)^2}}$$

SINCE $v_n = \vec{V}\cdot\vec{n} \Rightarrow$

$$\frac{\partial v_n}{\partial y_1} = \vec{V}\cdot\frac{d\vec{n}}{ds}\frac{ds}{dy_1}$$

$$= K\, v_t \frac{1}{\sin\theta}$$

WHERE K IS THE AIRFOIL SECTION CURVATURE (LOCAL CURVATURE), $v_t = \vec{V}\cdot\vec{t}$, $\vec{t}$ UNIT TANGENT TO THE AIRFOIL SURFACE AND θ IS THE ANGLE BETWEEN $\vec{t}$ AND x_1-AXIS. AT THE LEADING EDGE, $\theta = 90°$ AND $v_t = 0$

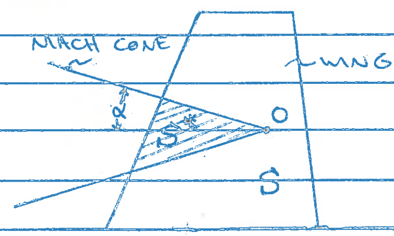
SO THAT THERE IS NO MORE A SINGULARITY THERE. THE CONVENTIONAL AIRFOILS HAVE FINITE CURVATURE AT THE L.E.

SUPERSONIC CASE

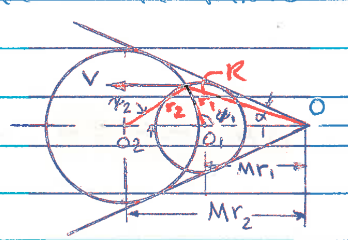
IN THIS CASE, THE COLLAPSING SPHERE $g=0$ WILL HAVE AN ENVELOPE IN THE FRAME FIXED TO THE WING WHICH IS A FORWARD FACING CIRCULAR CONE. (THE MACH CONE). THE SEMI-VERTEX ANGLE OF THIS CONE, α, IS FOUND FROM THE RELATION $\sin\alpha = \frac{1}{M}$. THE AREA S^* IN

$$2\pi\tilde{\phi}_1(\vec{x},t) = -\int_{S^*}\left[\frac{v_n}{r|1-M_r|}\right]_{ret} dS'$$

IS THE REGION ON THE WING SURFACE INSIDE THE MACH CONE AS SHOWN BELOW.



FROM THE FOLLOWING FIGURE, WE CAN SEE THAT EACH POINT ON S^* CONTRIBUTES TWICE TO THE OBSERVER AT O, WHEN IT IS AT POINTS O_1 AND O_2.



WE HAVE

$$r(1-M_r) = r - Mr\cos\psi$$

TO FIND r_1 AND r_2, WE MUST SOLVE

$$r^2 = R^2 + M^2 r^2 - 2MRr\cos\alpha$$

$$(M^2-1)r^2 - 2MRr\cos\alpha + R^2 = 0 \qquad (*)$$

$$\beta^2 = M^2 - 1$$

$$r = \frac{1}{\beta^2}\left[MR\cos\alpha \pm \sqrt{M^2R^2\cos^2\alpha - \beta^2R^2}\right]$$

$$= \frac{R}{\beta^2}\left[M\cos\alpha \pm \sqrt{M^2\cos^2\alpha - \beta^2}\right]$$

BOTH SOLUTIONS ARE ADMISSIBLE. TO FIND $\cos\psi$, WE US

$$R^2 = r^2 + M^2r^2 - 2Mr^2\cos\psi$$

$$Mr\cos\psi = \frac{(1+M^2)r^2 - R^2}{2r}$$

$$r - Mr\cos\psi = \frac{R^2 - \beta^2 r^2}{2r}$$

$$= \frac{R^2 - MRr\cos\alpha}{r} \qquad \text{USING } (*)$$

WE NOTE THAT

$$r_1 = \frac{R}{\beta^2}\left[M\cos\alpha - \sqrt{M^2\cos^2\alpha - \beta^2}\right]$$

$$r_2 = \frac{R}{\beta^2}\left[M\cos\alpha + \sqrt{M^2\cos^2\alpha - \beta^2}\right]$$

$$r_2 > r_1$$

$$|r_1 - Mr_1\cos\psi_1| = \frac{R}{r_1}(R - Mr_1\cos\alpha)$$

$$|r_2 - Mr_2\cos\psi_2| = \frac{R}{r_2}(Mr_2\cos\alpha - R)$$

THEREFORE

$$E = \left[\frac{1}{r\,||\!-\!M\cdot r|}\right]_1 + \left[\frac{1}{r\,||\!-\!M\cdot r|}\right]_2$$

$$= \frac{1}{R}\left(\frac{r_1}{R - Mr_1\cos\alpha} + \frac{r_2}{Mr_2\cos\alpha - R}\right)$$

$$= \frac{1}{R} \quad \frac{R(r_2 - r_1)}{-R^2 + MR\cos\alpha(r_2+r_1) - M^2\cos^2\alpha\, r_1 r_2}$$

WE HAVE $\quad r_1 + r_2 = \dfrac{2MR\cos\alpha}{\beta^2}$

$$r_2 - r_1 = \frac{2R\sqrt{M^2\cos^2\alpha - \beta^2}}{\beta^2}$$

$$r_1 r_2 = \frac{R^2}{\beta^2}$$

$$\Rightarrow \quad E = \frac{1}{R} \quad \frac{2R^2\sqrt{M^2\cos^2\alpha - \beta^2}}{R^2(M^2\cos^2\alpha - \beta^2)}$$

$$= \frac{2}{R\sqrt{M^2\cos^2\alpha - \beta^2}}$$

$$= \frac{2}{\sqrt{(\tilde{x}_1 - \tilde{y}_1)^2 - \beta^2(\tilde{x}_2 - \tilde{y}_2)^2}}$$

WE DROP TILDE AGAIN AND WRITE $\tilde{\phi}_1(\vec{x}, t)\{= \tilde{\phi}_1(\vec{\tilde{x}})\}$
$\equiv \tilde{\phi}_1(\vec{x})$ AS FOLLOWS

$$\pi\,\tilde{\phi}_1(\vec{x}) = -\int_{S^*} \frac{v_n(y_1, y_2)\,dy_1\,dy_2}{\sqrt{(x_1-y_1)^2 - \beta^2(x_2-y_2)^2}}$$

$$= -\int \frac{H(K)\,v_n(y_1, y_2)\,dy_1\,dy_2}{\sqrt{(x_1-y_1)^2 - \beta^2(x_2-y_2)^2}}$$

WHERE $K(\vec{y})=0$ IS THE CLOSED CURVE ON THE BOUNDARY OF S AND IS DEFINED SUCH THAT $K>0$ INSIDE THE CURVE. WE HAVE

$$\pi \frac{\partial \tilde{\phi}_1}{\partial x_1} = -\int H(K)\, v_n \frac{\partial}{\partial x_1}\left[\frac{H(K_1)H(x_1-y_1)}{\sqrt{(x_1-y_1)^2-\beta^2(x_2-y_2)^2}}\right] dy_1\, dy_2$$

$$= \int H(K)\, v_n \frac{\partial}{\partial y_1}\left[\frac{H(K_1)H(x_1-y_1)}{\sqrt{(x_1-y_1)^2-\beta^2(x_2-y_2)^2}}\right] dy_1\, dy_2$$

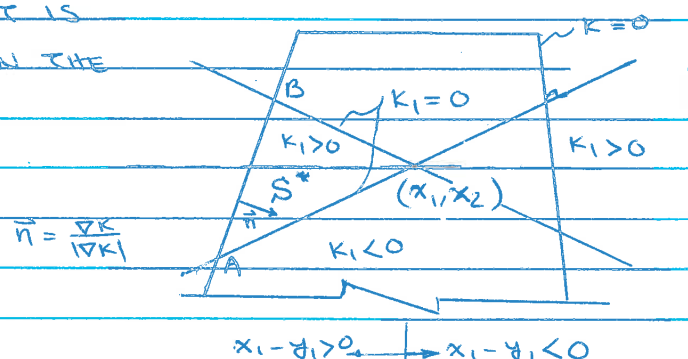
$$= \int H(K_1)H(x_1-y_1) \frac{(\partial v_n/\partial y_1)H(K) + v_n(\partial K/\partial y_1)\delta(K)}{\sqrt{(x_1-y_1)^2-\beta^2(x_2-y_2)^2}}\, dy_1\, dy_2$$

$$= -\int_{S^*} \frac{\partial v_n/\partial y_1}{\sqrt{(x_1-y_1)^2-\beta^2(x_2-y_2)^2}}\, dy_1\, dy_2$$

$$-\int_{\substack{K=0\\K_1>0}} \frac{v_n n_1 H(x_1-y_1)}{\sqrt{(x_1-y_1)^2-\beta^2(x_2-y_2)^2}}\, d\ell$$

WHERE $n_1 = (\partial K/\partial y_1)/|\nabla K|$, $K_1(\vec{x},\vec{y}) = (x_1-y_1)^2-\beta^2(x_2-y_2)^2$

THE LAST INTEGRAL IS OVER THE PORTION OF L.E. WHICH IS INSIDE THE MACH CONE AS SHOWN BELOW. THIS SEGMENT IS CALLED AB IN THE FIGURE.



THE ABOVE RESULT IS EQUIVALENT (EXCEPT FOR OBVIOUS FACTOR

FEB. 82

OF $= 2/\pi$ COMING FROM DEFN OF C_p) TO EQ. (8-7) OF ASHLEY & LANDAHL "AERODYNAMICS OF WINGS AND BODIES" WE NOTE THAT THEIR RESULT IS WRITTEN FOR WING WITH L.E. ALMOST NORMAL TO INFLOW VELOCITY, i.e. $n_1 \cong 1$. IT IS INTERESTING THAT OUR RESULT IS OBTAINED ALMOST EFFORTLESSLY.

FEB 82
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* A NEW RESULT FOR CALCULATION OF PROPELLER NOISE

THE RESULTS AND THE METHOD OF THE PREVIOUS NOTES ON AERODYNAMICS HAS LED TO TO A NEW FORMULATION OF PROPELLER NOISE. THIS RESULT HAS ELUDED ME FOR ALMOST EIGHT YEARS. IN MY EARLIER FORMULATION, THE ACOUSTIC PRESSURE $p'(\vec{x},t)$ IS WRITTEN AS

$$4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int\left[\frac{\rho_0 c v_n + p\cos\theta}{r|1-M_r|}\right]_{ret} dS + \int\left[\frac{p\cos\theta}{r^2|1-M_r|}\right]_{ret} dS$$

HERE THE TIME DERIVATIVE IS TAKEN NUMERICALLY APPART FROM VERY SIMPLE FORM, MY INTENTION IN WRITING THE SOLUTION OF FW-H EQ. WITH TIME DERIVATIVE OUTSIDE THE INTEGRAL WAS TO REDUCE THE ORDER OF SINGULARITY FOR SUPERSONIC CASE. FOR SUPERSONIC BLADES, THE COLLAPSING SPHERE TECHNIQUE WAS USED. AGAIN WE STUDY THE SUBSONIC AND SUPERSONIC CASES SEPARATELY.

STATIC SUBSONIC PROPELLERS

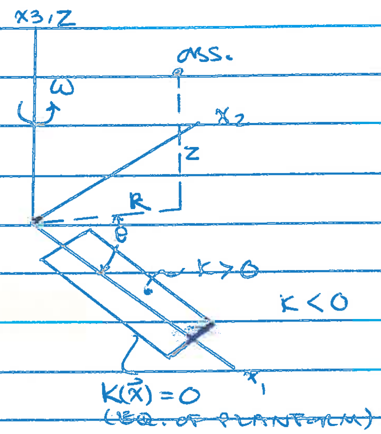
CONSIDER THE THICKNESS PROBLEM FIRST. THE ACOUSTIC PRESSURE IS

$$4\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int\left[\frac{\rho_0 v_n}{r|1-M_r|}\right]_{ret} dS$$

WE ASSUME THAT THE BLADE IS THIN AND THAT IT LIES IN x_1x_2-PLANE. TAKING THE SURFACE S AS THE MEAN SURFACE OF THE BLADE WITH v_n ON ONE SIDE USED IN THE CALCULATION, WE HAVE

$$2\pi p'(\vec{x},t) = \frac{\partial}{\partial t}\int_{K=0}\left[\frac{\rho_0 v_n}{r(1-M_r)}\right]_{ret} dS \qquad (1)$$

NOW LET (R, θ, Z) BE THE POLAR COORDINATES OF THE OBS. IN CYLINDRICAL COORDINATES AS SHOWN BELOW. WE KNOW THAT THE SOUND FIELD IS ROTATING WITH ANGULAR VELOCITY ω SO THAT θ AND t ARE GROUPED AS ONE VARIABLE θ − ωt. IT FOLLOWS THAT, WE CAN WRITE EQ. (1) AS



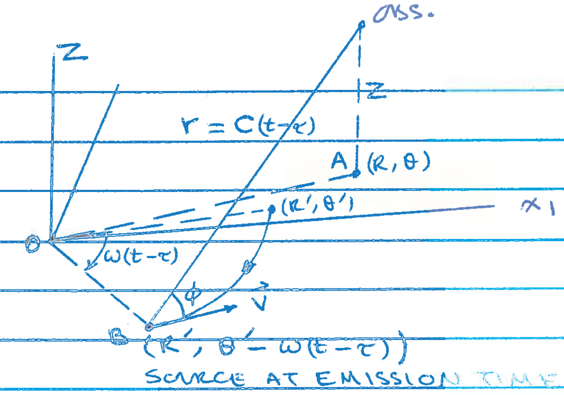
$$2\pi p'(\vec{x},t) = -\omega\frac{\partial}{\partial\theta}\int_{K=0}\left[\frac{\rho_0 v_n}{r(1-M_r)}\right]_{ret} dS$$

$$= -\omega\int_{K=0}\rho_0 v_n \frac{\partial}{\partial\theta}\left[\frac{1}{r(1-M_r)}\right]_{ret} dS \qquad (2)$$

WE NOW PROVE AN IMPORTANT AND CRUCIAL RESULT. WE SHOW THAT IF (R′, θ′) IS THE COORDINATES OF THE SOURCE, THEN $\left[\frac{1}{r(1-M_r)}\right]_{ret}$ IS A FN. OF θ − θ′. WE SEND OUT AN EXPANDING WAVE (SPHERICAL) AT TIME t. FROM THE OBSERVER AND LET THE SOURCE MOVE BACK IN TIME ALONG THE ARC OF THE CIRCLE OF RADIUS R′. WHEN IT INTERSECTS THE EXPANDING SPHERE, THE RETARDED TIME CAN BE CALCULATED. THE SOURCE IS ON THE CIRCLE OF RADIUS $\tilde{R} = [c^2(t-\tau)^2 - z^2]^{1/2}$ WITH CENTER AT A, THE PROJECTION OF THE OBS. ON x_1x_2 PLANE. THE EQ. OF THIS CIRCLE IN POLAR COORDINATES $(\bar{R}, \psi)$ IS

$$R^2 + \bar{R}^2 - 2R\bar{R}\cos(\psi-\theta) = \tilde{R}^2 = c^2(t-\tau)^2 - z^2$$

NOW THE POINT WITH COORDINATES $(R', \theta' - \omega(t-\tau))$ MUST BE ON THIS CIRCLE, i.e.



$$R^2 - R'^2 - 2RR'\cos[\theta - \theta' + \omega(t-\tau)] - c^2(t-\tau)^2 + z^2 = 0$$

THIS MUST BE SOLVED FOR $t-\tau$ TO GET THE RETARDED TIME. OF COURSE, $r = c(t-\tau)$. WE NOTE THAT IN THE ABOVE EQ., THE COMBINATION OF $\theta - \theta'$ APPEARS SO THAT

$$r = \text{FN}(\theta - \theta')$$

WE HAVE $r(1 - M_r) = r - \vec{r}\cdot\vec{M} = r - rM\cos\phi$

WE NOW SHOW THAT $\cos\phi = \text{FN}(\theta - \theta')$ ALSO. SINCE $\vec{V}$ HAS NO COMPONENT IN Z (OR x_3) DIRECTION, WE WRITE

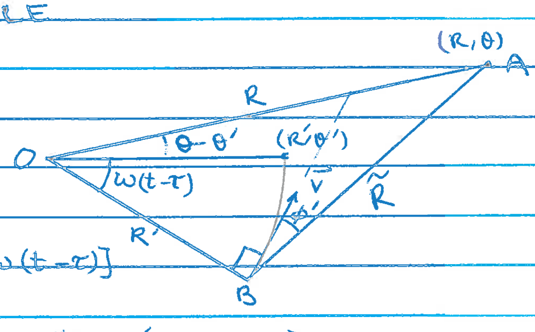
$$rM\cos\phi = rM\frac{|\vec{AB}|}{r}\cos\phi'$$

$$= M|\vec{AB}|\cos\phi', \quad |\vec{AB}| = \tilde{R}$$

WHERE ϕ' IS THE ANGLE BETWEEN $\vec{V}$ AND $\vec{AB}$

WE USE THE SINE RULE

$$\frac{R}{\sin(\frac{\pi}{2}+\phi')} = \frac{|\vec{AB}|}{\sin[\theta - \theta' + \omega(t-\tau)]}$$



$\therefore$

$$\cos\phi' = \frac{R}{|\vec{AB}|}\sin[\theta - \theta' + \omega(t-\tau)]$$

$$= \frac{R}{[c^2(t-\tau)^2 - z^2]^{1/2}}\sin[\theta - \theta' + \omega(t-\tau)]$$

$$= \text{FN}(\theta - \theta') \quad \text{SINCE } t - \tau = \text{FN}(\theta - \theta')$$

$$\therefore \quad r(1-M_r) = \text{FN}(\theta - \theta')\,!$$

NOW, WE USE THIS FACT AS FOLLOWS

$$2\pi p'(\vec{x},t) = -\int \rho_0 v_n H(K) \frac{\bar{\partial}}{\partial\theta'}\left[\frac{1}{r(1-M_r)}\right]_{ret} dS$$

$$= \int \rho_0 \frac{\bar{\partial}}{\partial\theta'}\left[v_n H(K)\right] \cdot \left[\frac{1}{r(1-M_r)}\right]_{ret} dS$$

$$= \int_{K>0} \rho_0 H(K) \frac{\bar{\partial} v_n}{\partial\theta'} \left[\frac{1}{r(1-M_r)}\right]_{ret} dS$$

$$+ \int \rho_0 v_n \frac{\partial K}{\partial \theta'} \delta(K) \left[\frac{1}{r(1-M_r)}\right]_{ret} dS$$

$$= \int_{K>0} \rho_0 \frac{\bar{\partial} v_n}{\partial\theta'} \left[\frac{1}{r(1-M_r)}\right]_{ret} dS$$

$$+ \oint_{K=0} \rho_0 R' n_\theta v_n \left[\frac{1}{r(1-M_r)}\right]_{ret} d\ell \quad (*)$$

WHERE n_θ IS THE COMPONENT OF INWARD UNIT NORMAL TO K=0 ALONG θ DIRECTION. THIS INTERESTING RESULT IS EQUIVALENT TO THE RESULT IN P1 OF THIS NOTEBOOK FOR WINGS. THIS RESULT CLARIFIES A LOT OF MYSTERIES IN MY MIND. IT IS SIMPLE AND COMPACT FORMULATION.

FOR LOADING NOISE, WE CONSIDER THE MEAN SURFACE OF AIRFOIL AGAIN. THE DEFN OF Δp IS $p_L - p_u$ AS BEFORE. OUR EQ. IS

$$4\pi p'(\vec{x},t) = -\frac{1}{c}\frac{\partial}{\partial t}\int \Delta p\, \vec{n} \cdot \left[\frac{\vec{r}}{r(1-M_r)}\right]_{ret} dS - \int \Delta p\, \vec{n} \cdot \left[\frac{\vec{r}}{r^2(1-M_r)}\right]_{ret} dS$$

HERE $\vec{n}$ IS THE NORMAL TO MEAN SURFACE POINTING TO THE SUCTION SIDE OF THE AIRFOIL. WE NOTE THAT

$$\hat{\vec{r}} = \frac{-\vec{AB} + z\vec{K}}{r}$$

$$= \frac{\tilde{R}(\cos\phi'\vec{e}_\theta + \sin\phi'\vec{e}_R) + z\vec{K}}{r}$$

SINCE BOTH ϕ' AND r ARE FNS OF $\theta - \theta' \Rightarrow \hat{\vec{r}}$ IS A FN OF $\theta-\theta'$. WE MUST REMEMBER THAT $\vec{n}$ MUST ALSO BE SPECIFIED AS A FN OF ($\vec{e}_\theta, \vec{e}_r, \vec{K}$) WHERE $\vec{K}$ IS THE UNIT VECTOR IN Z-DIRECTION.

WE, THEREFORE, HAVE

$$4\pi p'(\vec{x},t) = +\frac{\omega}{c}\int \Delta p\,\vec{n}\,H(K)\cdot\frac{\partial}{\partial\theta}\left[\frac{\hat{\vec{r}}}{r(1-M_r)}\right]_{ret} dS$$

$$-\int \Delta p\,\vec{n}\cdot\left[\frac{\hat{\vec{r}}}{r^2(1-M_r)}\right]_{ret} dS$$

$$= \frac{\omega}{c}\int \frac{\partial}{\partial\theta'}\left[\Delta p\,\vec{n}\,H(K)\right]\cdot\left[\frac{\hat{\vec{r}}}{r(1-M_r)}\right]_{ret} dS$$

$$-\int \Delta p\,\vec{n}\cdot\left[\frac{\hat{\vec{r}}}{r^2(1-M_r)}\right]_{ret} dS$$

AGAIN WE WRITE

$$\frac{\partial}{\partial\theta'}\left[\Delta p\,\vec{n}\,H(K)\right] = H(K)\frac{\partial}{\partial\theta'}(\Delta p\,\vec{n}) + \Delta p\,\vec{n}\frac{\partial K}{\partial\theta'}\delta(K)$$

$$\frac{\partial K}{\partial\theta'} = R' n_\theta \quad , \quad K = K(R', \theta)$$

WITH THE SAME DEFN OF n_θ AS ON P10. WE THUS HAVE

$$4\pi p'(\vec{x},t) = \int_{K>0}\left\{\frac{\omega}{c}\frac{\partial}{\partial\theta'}(\Delta p\,\vec{n})\cdot\left[\frac{\hat{\vec{r}}}{r(1-M_r)}\right]_{ret} - \Delta p\,\vec{n}\cdot\left[\frac{\hat{\vec{r}}}{r^2(1-M_r)}\right]_{ret}\right\} dS$$

$$+\frac{\omega}{c}\oint_{K=0} R' n_\theta\,\Delta p\,\vec{n}\cdot\left[\frac{\hat{\vec{r}}}{r(1-M_r)}\right]_{ret} d\ell \qquad (*)$$

(*) THESE RESULTS WERE EXTENDED TO PROPELLERS WITH TWIST AND PUBLISHED AS VKI LECTURE NOTE. FORWARD FLIGHT WAS ALSO INCLUDED. FF 7/8/82.

* INTERPRETATION OF PRODUCTS OF DELTA FUNCTIONS REPRESENTING SOURCES ON CURVES IN SPACE IN MOTION

WE WOULD LIKE TO INTERPRET GEOMETRICALLY THE PRODUCTS OF DELTA FUNCTIONS WHICH APPEAR AS A RESULT OF EDGE SOURCES IN THE SOLUTION OF THE WAVE EQUATION. WE DRAW FROM OUR KNOWLEDGE OF SURFACE SOURCES.

i) 3-DIMENSIONAL CASE

LET $f(\vec{y})=0$ AND $K(\vec{y})=0$ BE INTERSECTING SURFACES WITH CURVE OF INTERSECTION Γ. WE CONSIDER THE INTEGRAL I:

$$I = \int Q\,\delta(f)\,\delta(K)\,d\vec{y}$$

WE LET $(y_1, y_2) \to (f, K)$

$$d\vec{y} = \frac{df\,dK\,dy_3}{\left|\frac{\partial(f,K)}{\partial(y_1,y_2)}\right|} = \frac{df\,dK\,d\Gamma}{|\nabla f \times \nabla K|}$$

$$\Rightarrow \quad I = \int_{\substack{f=0\\K=0}} \frac{Q}{|\nabla f \times \nabla K|}\,d\Gamma$$

WE HAVE WRITTEN THE INTEGRAL INDEPENDENT OF THE COORDINATE SYSTEM. THIS IS DESIRABLE.

ii) 4-DIMENSIONAL CASE

LET THE SURFACES $f(\vec{y},\tau)=0$, $K(\vec{y},\tau)=0$ INTERSECT EACH OTHER. LET $g=\tau-t+r/c=0$.

WE FIRST ASSUME THAT $f=0$ AND $K=0$ ARE NOT RIGID SURFACES, i.e. $\nexists$ A MOVING $\vec{\eta}$-FRAME $\ni$ $f(\vec{y}(\vec{\eta},\tau),\tau) = \tilde{f}(\vec{\eta})$ AND $K(\vec{y}(\vec{\eta},\tau),\tau) = \tilde{K}(\vec{\eta}) = 0$.
WE WOULD LIKE TO INTERPRETE THE INTEGRAL

$$I = \int Q\,\delta(f)\,\delta(K)\,\delta(g)\,d\vec{y}\,d\tau$$

WE GIVE THREE INTERPRETATIONS AS FOLLOWS

A) LET $F = [f]_{ret} = f(\vec{y}, t-r/c) = 0$ AND $K = [K]_{ret} = K(\vec{y}, t-r/c) = 0$. NOW LET $\tau \to g \Rightarrow \partial g/\partial\tau|_{\vec{y}} = 1$ AND

$$I = \int [Q(\vec{y},\tau)]_{ret}\,\delta(F)\,\delta(K)\,d\vec{y}$$

$$= \int_{\substack{F=0\\K=0}} \frac{[Q(\vec{y},\tau)]_{ret}}{|\nabla F \times \nabla K|}\,d\Gamma \quad \left(\text{BY THE RESULT OF (i)}\right)$$

WE CAN FURTHER CALCULATE $\nabla F \times \nabla K$ AND RELATE THESE TO ∇f; ∇K AND THE NORMAL VELOCITIES OF $f=0$ AND $K=0$.

B) NOW LET $\vec{y} \to (f, K, g) \Rightarrow$

$$d\vec{y} = \frac{df\,dK\,dg}{\left|\dfrac{\partial(f,K,g)}{\partial(y_1,y_2,y_3)}\right|}$$

$$= \frac{df\,dK\,dg}{|\nabla f|\cdot|\nabla K||\nabla g|\,\sin\phi\,|\cos\psi|}$$

$$= \frac{c\,df\,dK\,dg}{|\nabla f||\nabla K||\cos\psi|\sin\phi}$$

WHERE ϕ IS THE ANGLE BETWEEN ∇f AND ∇K AND ψ IS THE ANGLE BETWEEN THE TANGENT TO Γ, i.e. $\nabla f \times \nabla K$ AND $\vec{r} = \vec{x} - \vec{y}$ $\Rightarrow$

$$I = c \int_\tau \left[\frac{Q(\vec{y},\tau)}{|\nabla f||\nabla K| \sin\phi |\cos\psi|} \right]_{\substack{f=0\\g=0\\K=0}} d\tau \qquad (*)$$

i.e. INTEGRATE AS LONG AS $f=0$, $g=0$, $K=0$ INTERSECT EACH OTHER.

C) NOW ASSUME $\exists$ FRAME $\vec{\eta} \ni f=0$ AND $K=0$ ARE INDEPENDENT OF TIME. LET $\vec{y}$ AND $\vec{\eta}$ FRAMES BE RELATED TO EACH OTHER BY THE RELATION

$$\vec{y} = \vec{y}_0 + A(\tau)\vec{\eta}$$

WHERE $A(\tau)$ IS A MATRIX WHOSE COEFFICIENTS ARE FNS OF τ AND $\ni \frac{\partial \vec{y}}{\partial \vec{\eta}} = 1$. THIS IS THE SITUATION OFTEN APPEARING FOR RIGID BODIES IN MOTION. WE WILL MISUSE THE NOTATION AND WRITE $\tilde{f}(\vec{\eta}) = f(\vec{y}(\vec{\eta},\tau),\tau) \equiv f(\vec{\eta})$ AND ALSO FOR $\tilde{K}(\vec{\eta})$. THIS IS NOT GOOD PRACTICE SINCE IS LEADS TO CONFUSION. WE LET $\vec{y} \to \vec{\eta}$ SO THAT

$$I = \int Q(\vec{y}(\vec{\eta},\tau),\tau)\,\delta(f)\,\delta(K)\,\delta(g)\,d\vec{\eta}\,d\tau$$

NOW $g = g(\vec{y}(\vec{\eta},\tau),\tau;\vec{x},t) \Longrightarrow$

$$\left.\frac{\partial g}{\partial \tau}\right|_{\vec{\eta}} = 1 - M_r, \quad M_r = \frac{v_i \hat{r}_i}{c}, \quad v_i = \left.\frac{\partial y_i}{\partial \tau}\right|_{\vec{\eta}}$$

$$\therefore \quad I = \int \left[\frac{Q}{|1-M_r|}\right]_{ret} \delta(f)\,\delta(K)\,d\vec{\eta}$$

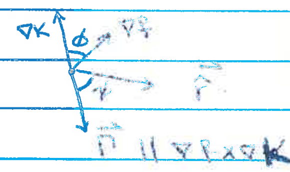
FROM PART (i)

$$I = \int_{\substack{f=0\\K=0}} \left[\frac{Q}{|1-M_r|}\right]_{ret} \frac{d\Gamma}{|\nabla f \times \nabla K|}$$

$$= \int_{\substack{f=0\\K=0}} \left[\frac{Q}{|1-M_r|}\right]_{ret} \frac{d\Gamma}{|\nabla f||\nabla K|\sin\phi}$$

COMPARING THE ABOVE RESULT AND EQ (*) ON PREVIOUS PAGE, WE NOTE THAT ONE IS OBTAINED FROM ANOTHER BY THE RELATION

$$\frac{d\Gamma}{\sin\phi|1-M_r|} = \frac{c\, d\tau}{|\cos\psi|\sin\phi}$$



AS IN THE CASE OF SURFACE SOURCES IN MOTION, ONE OF THESE RELATIONS IS PREFERABLE TO OTHER IN A SPECIFIC PROBLEM.

WE ALSO NOTE THAT IN 3-D, A LINE INTEGRAL ON THE CURVE OF INTERSECTION OF $f=0$, $K=0$ CAN BE WRITTEN IN TERMS OF δ-FNS AS FOLLOWS

$$\int_{\substack{f=0\\=0}} Q\, d\Gamma = \int |\nabla f \times \nabla g|\, Q\, \delta(f)\,\delta(K)\, d\vec{y}$$

IN THE 4-D CASE, WE HAVE

$$\int_{\substack{f=0\\K=0}} \left[\frac{Q}{|1-M_r|}\right]_{ret} d\Gamma = \int |\nabla f \times \nabla K|\, Q\, \delta(f)\,\delta(K)\,\delta(g)\, d\vec{y}\, d\tau$$

$$= c \int_\tau \left[\frac{Q}{|\cos\psi|}\right]_{f=0,\, g=0,\, K=0} d\tau$$

APR. 82
GPHM

* THE MISSING TERMS IN THE INTEGRAL EQUATION DERIVED FOR AERODYNAMICS OF MOVING BODIES

IN NOTEBOOK I, P 161–170, WE DERIVED AN INTEGRAL EQ. FOR AERODYNAMICS OF MOVING BODIES. WE NOTE THAT WHEN $v_n = 0$ AND $M = 0$, THE WELL-KNOWN KIRCHOFF'S RESULT FOR THE WAVE EQ. IS NOT OBTAINED.(*) THIS TELLS US THAT SOME TERMS ARE MISSING FROM OUR RESULT. HERE WE FIND THESE TERMS FOR AN ARBITRARY BODY IN SUBSONIC MOTION. FIRST WE TAKE A PLANE SHAPE MOVING NORMAL TO ITSELF. THE MISSING TERMS COME FROM CONTRIBUTION OF THE SURFACE OF THE REMOVED HOLE WHEN THE OBSERVER GETS VERY NEAR THE SURFACE. WE WILL CALL THIS SURFACE Δ AND CONSIDER THE FOLLOWING INTEGRALS WHEN THE OBSERVER COMES CLOSE TO BUT <u>NOT</u> ON THE SURFACE ITSELF:

$$I = \int_{\Delta}\left[\frac{p(\cos\theta - M_n)}{r^2(1-M_r)^2}\right]_{ret} dS + \int_{\Delta}\left[\frac{(\rho_0 c v_n + p\cos\theta)(M_r - M^2)}{r^2(1-M_r)^3}\right]_{ret} dS$$

$$= \int_{\Delta}\left[\frac{p(1-M^2)(\cos\theta - M_n)}{r^2(1-M_r)^3}\right]_{ret} dS + \int_{\Delta}\left[\frac{(\rho_0 c^2 + p)M_n}{r^2(1-M_r)^2}\left\{-1 + \frac{1-M^2}{1-M_r}\right\}\right]_{ret} dS$$

AS WE COME TOWARD THE SURFACE, WE MUST CONSIDER THE LIMITING FORMS OF THE INTEGRALS

$$I_1 = p(1-M^2)\int_{\Delta}\left[\frac{\cos\theta - M_n}{r^2(1-M_r)^3}\right] dS ,$$

* THIS WAS FOUND BY LYLE N. LONG, GWU STUDENT.

$$I_2 = (\rho_0 c^2 + p) M_n \int_{\Delta} \left[\frac{1}{r^2 (1-M_r)^2} \right]_{ret} dS$$

$$I_3 = M_n (1-M^2)(\rho_0 c^2 + p) \int_{\Delta} \left[\frac{1}{r^2 (1-M_r)^3} \right]_{ret} dS$$

WE NOTE THAT

$$I = I_1 - I_2 + I_3$$

WE CALL THE SURFACE INTEGRALS IN I_1, I_2 AND I_3 AS I_1', I_2' AND I_3', RESPECTIVELY. FOR CHECKING THE FINAL RESULT IN (ii), WE FIRST ASSUME THAT THE SURFACE Δ MOVES NORMAL TO ITSELF.

i) SURFACE Δ MOVING NORMAL TO ITSELF

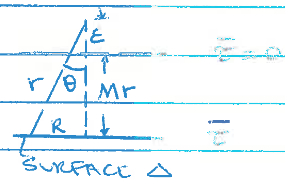
WE USE THE RELATION

$$\frac{dS}{1-M_r} = \frac{c\, d\Gamma\, d\bar{\tau}}{\sin\theta}$$

LET $\bar{\tau} = t - \tau$ AS $\bar{\tau}$ INCREASES FROM 0 TO SOME POSITIVE VALUE, WE LET Δ MOVE BACK IN TIME. WE ASSUME AT $\tau = t$ OR $\bar{\tau} = 0$, THE OBSERVER IS THE DISTANCE ε ABOVE THE SURFACE Δ. WE HAVE

$$I_1' = \int_{\bar{\tau}_1}^{\bar{\tau}_2} \frac{\cos\theta - M}{r^2 (1 - M\cos\theta)^2} \frac{2\pi R c\, d\bar{\tau}}{\frac{R}{r}}$$

$$= 2\pi \int_{r_1}^{r_2} \frac{\cos\theta - M}{r(1 - M\cos\theta)^2} dr$$



$$\cos\theta = \frac{\varepsilon + Mr}{r} = \frac{\varepsilon}{r} + M$$

$$1 - M\cos\theta = 1 - M^2 - \frac{M\varepsilon}{r}$$

$$I_1' = 2\pi \int_{r_1}^{r_2} \frac{\varepsilon \, dr}{[(1-M^2)r - M\varepsilon]^2}$$

$$= \frac{-2\pi \quad \varepsilon}{1 - M2} \quad \frac{1}{(1-M^2)r - M\varepsilon} \Bigg|_{r_1}^{r_2}$$

WE HAVE $r_1 = \varepsilon + Mr_1$ OR $r_1 = \dfrac{\varepsilon}{1-M}$

$$I_1' = \frac{-2\pi \quad \varepsilon}{1 - M2} \left[\frac{1}{(1-M^2)r_2 - M\varepsilon} - \frac{1}{\varepsilon} \right]$$

NOW TAKE THE LIMIT AS $\varepsilon \to 0$ TO GET

$$I_1' = \frac{2\pi}{1-M^2}$$

FOR I_2' WE HAVE

$$I_2' = \int_{\tau_1}^{\tau_2} \frac{1}{r^2(1 - M\cos\theta)} \; \frac{2\pi R\, c\, d\tau}{\frac{R}{r}}$$

$$= 2\pi \int_{r_1}^{r_2} \frac{dr}{(1-M^2)r - M\varepsilon}$$

WE LEAVE THIS AS IT IS NOW. WE CONSIDER I_3'.

$$I_3' = \int_{\tau_1}^{\tau_2} \frac{1}{r^2(1 - M\cos\theta)} \; \frac{2\pi R\, c\, d\tau}{\frac{R}{r}}$$

$$= 2\pi \int_{r_1}^{r_2} \frac{r\, dr}{[(1-M^2)r - M\varepsilon]^2}$$

$$= \frac{2\pi}{1-M^2} \int_{r_1}^{r_2} \frac{(1-M^2)r - M\varepsilon + M\varepsilon}{[(1-M^2)r - M\varepsilon]^2} dr$$

$$= \frac{2\pi}{1-M2} \left\{ \int_{r_1}^{r_2} \frac{dr}{(1-M^2)r - M\varepsilon} + M\varepsilon \int_{r_1}^{r_2} \frac{dr}{[(1-M^2)r - M\varepsilon]^2} \right\}$$

WE LEAVE THE FIRST INTEGRAL AS IT IS SINCE IT WILL CANCEL I_2'. THE LAST INTEGRAL HAS BEEN CALCULATED IN I_1'. WE HAVE, AS $\varepsilon \to 0$

$$I_3' = \frac{I_2'}{1-M^2} + \frac{2\pi M}{(1-M^2)^2}$$

$$\therefore \quad I = 2\pi p + \frac{2\pi M^2}{1-M^2}(\rho_0 c^2 + p)$$

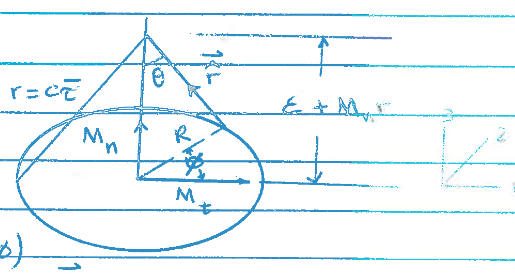
$$= \frac{2\pi M^2}{1-M^2}\rho_0 c^2 + \frac{2\pi}{1-M^2} p$$

AS $M \to 0$, WE GET KIRCHOFF'S EXPECTED RESULT.

ii) IF THE SURFACE DOES NOT MOVE NORMAL TO ITSELF, THE MANIPULATIONS ARE MESSY BUT MANAGEABLE AS SHOWN BELOW. AGAIN WE REPLACE $dS/(1-M_r)$ BY $c\,d\Gamma\,d\tau/\sin\theta$, $\sin\theta = \frac{R}{r}$. FROM THE FIGURE BELOW, WE HAVE

$$M_r = \vec{M}\cdot\hat{\vec{r}}$$

$$= -M_t \sin\theta\cos\phi + M_n\cos\theta$$



$$1 - M_r = (1 - M_n\cos\theta)(1 + \alpha\cos\phi)$$

$$\hat{\vec{r}} = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta)$$

$$\alpha = \frac{M_t\sin\theta}{1 - M_n\cos\theta}$$

WE HAVE $|\alpha| < 1$. WE NEED THIS IN THE INTEGRATION TO BE PERFORMED SOON. FOR I_1 WE HAVE

$$I_1' = \int \frac{\cos\theta - M_n}{r^2(1-M_r)^2} \frac{cR\,d\phi\,d\tau}{\frac{R}{r}}$$

$$\cos\theta = \frac{\varepsilon + M_n r}{r} = M_n + \frac{\varepsilon}{r}$$

$$I_1' = \int_{r_1}^{r_2} \frac{\varepsilon\,dr}{r^2(1-M_n\cos\theta)^2} \int_0^{2\pi} \frac{d\phi}{(1+\alpha\cos\phi)^2}$$

$$= 2\pi\varepsilon \int_{r_1}^{r_2} \frac{dr}{r^2(1-M_n\cos\theta)^2(1-\alpha^2)^{3/2}}$$

SEE NOTEBOOK I, P 166

$$1-\alpha^2 = 1 - \frac{M_t^2\sin^2\theta}{(1-M_n\cos\theta)^2}$$

$$= \frac{1+M_n^2\cos^2\theta - 2M_n\cos\theta - M_t^2 + M_t^2\cos^2\theta}{(1-M_n\cos\theta)^2}$$

$$= \frac{1-M_t^2+M^2\cos^2\theta - 2M_n\cos\theta}{(1-M_n\cos\theta)^2}$$

$$M^2\cos^2\theta - 2M_n\cos\theta = M^2\left(M_n^2 + \frac{\varepsilon^2}{r^2} + 2M_n\frac{\varepsilon}{r}\right)$$

$$- 2M_n^2 - 2M_n\frac{\varepsilon}{r}$$

$$= M_n^2(M^2-2) + 2M_n(M^2-1)\frac{\varepsilon}{r} + \frac{M^2\varepsilon^2}{r^2}$$

$$1-M_t^2+M^2\cos^2\theta - 2M_n\cos\theta = 1 - M^2 + M_n^2(M^2-1) + 2M_n(M^2-1)\frac{\varepsilon}{r}$$

$$+ \frac{M^2\varepsilon^2}{r^2}$$

$$= (1-M^2)\left[1 - M_n^2 - 2M_n\frac{\varepsilon}{r} + \frac{M^2\varepsilon^2}{(1-M^2)r^2}\right]$$

$$1 - M_n\cos\theta = 1 - M_n^2 - \frac{M_n\varepsilon}{r}$$

$$I_1' = 2\pi\varepsilon \int_{r_1}^{r_2} \frac{(1-M_n\cos\theta)\,dr}{r^2[1-M_t^2+M^2\cos^2\theta - 2M_n\cos\theta]^{3/2}}$$

$$= 2\pi\varepsilon \int_{r_1}^{r_2} \frac{(1-M_n^2)r - M_n\varepsilon}{(1-M^2)^{3/2}[(1-M_n^2)r^2 - 2M_n\varepsilon r + \beta\varepsilon^2]^{3/2}}\,dr$$

$$\beta = \frac{M^2}{1-M^2}$$

$$I_1' = -\frac{2\pi\varepsilon}{(1-M^2)^{3/2}} \frac{1}{[(1-M_n^2)r^2 - 2M_n\varepsilon r + \beta\varepsilon^2]^{1/2}}\Bigg|_{r_1}^{r_2}$$

$$r_1 = \frac{\varepsilon}{1-M_n}$$

AS $\varepsilon \to 0$, WE GET

$$I_1' = \frac{2\pi}{(1-M^2)^{3/2}} \frac{1}{\left[\frac{1+M_n}{1-M_n} - \frac{2M_n}{1-M_n} + \beta\right]^{1/2}}$$

$$= \frac{2\pi}{(1-M^2)^{3/2}} \frac{1}{\left(1+\frac{M^2}{1-M^2}\right)^{1/2}}$$

$$= \frac{2\pi}{1-M^2}$$

FOR I_2' WE HAVE

$$I_2' = \int \frac{1}{r^2(1-M_r)} \frac{c R\, d\phi\, d\bar{\tau}}{\frac{R}{r}}$$

$$= \int_{r_1}^{r_2} \frac{dr}{r(1-M_n\cos\theta)} \int_0^{2\pi} \frac{d\phi}{1+\alpha\cos\phi}$$

$$= 2\pi \int_{r_1}^{r_2} \frac{dr}{r(1-M_n\cos\theta)(1-\alpha^2)^{1/2}}$$

$$= 2\pi \int_{r_1}^{r_2} \frac{dr}{(1-M^2)^{1/2}[(1-M_n)^2 r^2 - 2M_n\varepsilon r + \beta\varepsilon^2]^{1/2}}$$

AGAIN WE LEAVE THIS AND GO TO I_3'

$$I_3 = \int \frac{1}{r^2(1-M_r)^2} \frac{c R\, d\phi\, d\bar{\tau}}{\frac{R}{r}}$$

$$I_3' = \int_{r_1}^{r_2} \frac{dr}{r(1-M_n\cos\theta)^2} \int_0^{2\pi} \frac{d\phi}{(1+\alpha\cos\phi)^2}$$

$$= 2\pi \int_{r_1}^{r_2} \frac{dr}{r(1-M_n\cos\theta)^2(1-\alpha^2)^{3/2}}$$

$$= 2\pi \int_{r_1}^{r_2} \frac{(1-M_n\cos\theta)\,dr}{r\left[1-M_t^2+M^2\cos^2\theta-2M_n\cos\theta\right]^{3/2}}$$

$$= 2\pi \int_{r_1}^{r_2} \frac{r\left[(1-M_n^2)r-M_n\varepsilon\right]dr}{(1-M^2)^{3/2}\left[(1-M_n^2)r^2-2M_n\varepsilon r+\beta\varepsilon^2\right]^{3/2}}$$

$$= \frac{2\pi}{(1-M^2)^{3/2}} \int_{r_1}^{r_2} \frac{dr}{\left[(1-M_n^2)r^2-2M_n\varepsilon r+\beta\varepsilon^2\right]^{3/2}}$$

$$+ \frac{2\pi}{(1-M^2)^{3/2}} \int_{r_1}^{r_2} \frac{M_n\varepsilon r-\beta\varepsilon^2}{\left[(1-M_n^2)r^2-2M_n\varepsilon r+\beta\varepsilon^2\right]^{3/2}}\,dr$$

$$= \frac{I_2'}{1-M^2} + \frac{2\pi}{(1-M^2)^{3/2}} I_3''$$

WHERE

$$I_3'' = \int_{r_1}^{r_2} \frac{M_n\varepsilon r-\beta\varepsilon^2}{\left[(1-M_n^2)r^2-2M_n\varepsilon r+\beta\varepsilon^2\right]^{3/2}}\,dr$$

$$= \frac{M_n\varepsilon}{1-M_n^2} \int_{r_1}^{r_2} \frac{(1-M_n^2)r-M_n\varepsilon}{\left[(1-M_n^2)r^2-2M_n\varepsilon r+\beta\varepsilon^2\right]^{3/2}}$$

$$+ \varepsilon^2 \int_{r_1}^{r_2} \frac{(\beta_n-\beta)\,dr}{\left[(1-M_n^2)r^2-2M_n\varepsilon r+\beta\varepsilon^2\right]^{3/2}}$$

$$\beta_n = \frac{M_n^2}{1-M_n^2}$$

$$I_3'' = \frac{M_n \varepsilon}{1-M_n^2} \frac{1}{[(1-M_n^2) r^2 - 2 M_n \varepsilon r + \beta\varepsilon^2]^{1/2}} \Bigg|_{r_1}^{r_2}, \quad r_2 > r_1$$

$$+ \varepsilon^2 (\beta_n - \beta) \frac{2[2(1-M_n^2) r - 2 M_n \varepsilon]}{A [(1-M_n^2) r^2 - 2M_n \varepsilon r + \beta\varepsilon^2]^{1/2}} \Bigg|_{r_1}^{r_2}$$

$$= J_1 + J_2$$

$$A = -4\beta\varepsilon^2(1-M_n^2) - 4M_n^2\varepsilon^2 \qquad \text{[P 83, GRADSHTEYN \& RYZHIK]}$$

$$= -4\varepsilon^2[(1-M_n^2)\beta - M_n^2]$$

AS $\varepsilon \to 0$, WE HAVE

$$J_1 = \frac{M_n}{1-M_n^2} (1-M^2)^{1/2}$$

$$J_2 = \frac{-\beta_n + \beta}{(1-M_n^2)\beta - M_n^2} \lim_{\varepsilon \to 0} \frac{(1-M_n^2) r - M_n \varepsilon}{[(1-M_n^2) r^2 - 2M_n\varepsilon r + \beta\varepsilon^2]^{1/2}} \Bigg|_{r_1}^{r_2}$$

$$= \frac{\beta - \beta_n}{(1-M_n^2)\beta - M_n^2} \left[(1-M_n^2)^{1/2} - (1-M^2)^{1/2} \right]$$

$$= \frac{1}{1-M_n^2} \left[(1-M_n^2)^{1/2} - (1-M^2)^{1/2} \right]$$

$$I_3'' = J_1 + J_2$$

$$= \frac{1}{(1-M_n^2)^{1/2}} - \frac{(1-M^2)^{1/2}}{1-M_n^2} [1 - M_n]$$

$$= \frac{1}{(1-M_n^2)^{1/2}} - \frac{(1-M^2)^{1/2}}{1+M_n}$$

$$I_3' = \frac{I_2'}{1-M^2} + \frac{2\pi}{(1-M_n^2)^{1/2}(1-M^2)^{3/2}} - \frac{2\pi}{(1+M_n)(1-M^2)}$$

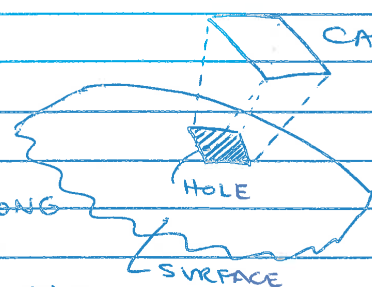
$$I = I_1 + I_2 + I_3$$

$$= 2\pi p + \frac{2\pi M_n(\rho_0 c^2 + p)}{(1-M_n^2)^{1/2}(1-M^2)^{1/2}} - \frac{2\pi M_n(\rho_0 c^2 + p)}{1 + M_n}$$

CHECK: IF $M_n = M$, WE DO GET PART (i).

NOTE ADDED IN SEP. 82.

IN THE PERIOD BETWEEN APRIL AND SEPT., I WAS BUSY WITH A LOT OF THINGS. THE VKI LECTURE NOTES WAS COMPLETED IN LATE MAY. I PUT A CHAP. ON THE RELATION BETWEEN AERODYNAMICS AND ACOUSTICS. IN THE MEAN TIME, MY THOUGHTS BECAME ORGANIZED ON THE ABOVE PROBLEM. I HAD REASONED TOO MUCH WITH ANALOGY TO LAPLACE'S EQ. WHEN WE COME TO THE SURFACE, WE DO NOT HAVE A PROPERLY CONVERGENT INTEGRAL BUT ONE THAT IS SEMI-CONVERGENT, i.e. CAN BE MADE CONVERGENT IF A HOLE OF PROPER SHAPE IS REMOVED FROM THE SURFACE. THIS IS ANALOGOUS TO CAUCHY'S PRINCIPAL VALUE. LYLE LONG AND I CAME TO THIS CONCLUSION AFTER HE THOUGHT



THAT IF A RECTANGULAR HOLE WAS ASSUMED, THERE WAS NO SINGULARITY. SINCE I WAS CONVINCED THAT THE SURFACE INTEGRAL WAS DIVERGENT, I STUDIED HIS INTEGRAND AND I SAW SEMI-CONVERGENCE PROPERTY. SO BOTH MY FINAL RESULT, TO BE GIVEN

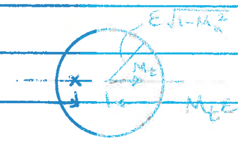
AND LYLE'S WERE CORRECT. THE INTEGRAL EQUATION I DERIVED USING OFF-CENTER CIRCULAR HOLE IS

$$2\pi\,(1+\Gamma(M,M_n)]\,p = -2\pi\,\Gamma(M,M_n)\rho_\infty c^2 + I_\epsilon$$

WHERE

$$\Gamma(M,M_n) = \frac{M_n}{\beta_n\beta} - \frac{M_n}{1+M_n}$$

$$\beta_n = (1-M_n^2)^{1/2}, \quad \beta = (1-M^2)^{1/2}$$



THE PROCEDURE IS AS FOLLOWS. FIRST FIND THE SHAPE OF THE HOLE THAT MAKES THE SURFACE INTEGRAL CONVERGENT. THEN LET THE OBSERVER MOVE TO THE CAP COVERING THE HOLE.

NOTE ADDED IN JULY 1983 (7/11/83)

ALTHOUGH LONG'S INTEGRAL EQ. IS MORE SUITABLE THAN MINE FOR SUBSONIC SURFACES, MY METHOD OF REGULARIZATION IS IDEAL FOR SUPERSONIC/TRANSONIC SURFACES USING NEW RESULT ON PAGE 90. THE REASON IS THAT COLLAPSING SPHERE METHOD IS SUITABLE FOR THIS SPEED RANGE. I INTEND TO COPY MY NOTES IN THIS BOOK ON DERIVATION OF THIS INTEGRAL EQUATION.

* SOME USEFUL RESULTS IN GENERALIZED FUNCTION THEORY

I HAVE DERIVED THE FOLLOWING NEW RESULTS IN THE LAST FEW MONTHS.

i) WE HAVE SHOWN BEFORE (NOTEBOOK I) THAT

$$\int K(\vec{y})\,|\nabla f|\,\delta'(f)\,d\vec{y} = -\int \bar{\nabla}\cdot(K\,\vec{n})\,\delta(f)\,d\vec{y}$$

WE ARE CONSIDERING THE 3-D PROBLEM HERE.

NOW $\bar{\nabla}\cdot(K\cdot\vec{n}) = \frac{\partial K}{\partial n} + K\,\nabla\cdot\vec{n}$

I HAD SHOWN IN MY PH.D. THESIS THAT

$$\nabla\cdot\vec{n} = 2K_{av} \quad \text{(SEE BELOW FOR SIGN OF THIS)}$$

WHERE K_{av} IS THE LOCAL MEAN CURVATURE OF SURFACE $f=0$. $\Longrightarrow$

THIS SIGN MUST CHANGE

$$K(\vec{y})|\nabla(f)|\,\delta'(f) = -\left(\frac{\partial K}{\partial n} + 2K_{av}\,K(\vec{y})\right)\delta(f)$$

THIS IS THE GENERALIZATION OF

$$K(x)\,\delta'(x) = -K'(0)\,\delta(x)$$

IF $f=0$ IS THE SPHERE $r-a=0$, THEN $|\nabla f| = 1$, $K_{av} = \frac{1}{a}$ AND

$$K(\vec{y})|\nabla f|\,\delta'(f) = K(\vec{y})\,\delta'(f) = -\left(\frac{\partial K}{\partial r} + \frac{2}{a}K\right)\delta(f)$$

ii) SUPPOSE WE HAVE A SURFACE WHICH IS NOT CLOSED. LET THIS SURFACE BE DEFINED BY THE EQUATIONS

$$f(\vec{y}) = 0, \quad g(\vec{y}) > 0$$

CONSIDER THE INTEGRAL

(*) IF WE ASSUME THAT K_1 & K_2 (PRINCIPAL CURVATURES) ARE POSITIVE IF THE CENTERS OF CURVATURE ARE ON THE SAME SIDE OF THE SURFACE AS $\vec{n}$ IS $\Rightarrow \nabla\cdot\vec{n} = -2K_{av}$. I LEARNED THIS AFTER I WROTE THE ABOVE FROM A PAPER IN AM. MATH. MONTHLY.

$$I = \int \phi(\vec{y}) \vec{\nabla} \cdot [\vec{K}(\vec{y}) |\nabla f| H(g) \delta(f)] \, d\vec{y} \quad (1)$$

IT IS INTUITIVELY OBVIOUS THAT ONLY THE KNOWLEDGE OF $\vec{K}$ ON THE SURFACE IS SUFFICIENT TO GET THE VALUE OF THE INTEGRAL. ONE ALSO SUSPECTS THAT THE RESULT IS INDEPENDENT OF g SINCE THERE ARE MANY g'S WHICH CAN BE USED TO SPECIFY THE SAME OPEN PIECE OF A SURFACE. WE SHOW BELOW THAT BOTH OF OUR GUESSES ARE RIGHT. WE HAVE

$$\vec{\nabla} \cdot [\vec{K} |\nabla f| H(g) \delta(f)] = H(g) \, \vec{\nabla} \cdot [\vec{K} |\nabla f|] \, \delta(f) + |\nabla f| \vec{K} \cdot \nabla g \, \delta(f) \delta(g)$$
$$|\nabla f| H(g) \, \vec{K} \cdot \nabla f \, \delta'(f) \quad (2)$$

NOW FROM (i), ABOVE

$$\phi |\nabla f|^2 K_n H(g) \delta'(f) = - \vec{\nabla} \cdot [\phi |\nabla f| K_n H(g) \vec{n}] \, \delta(f) \quad (3)$$

WHERE $K_n = \vec{K} \cdot \vec{n}$, $\vec{n} = \nabla f / |\nabla f|$.

THE RIGHT SIDE OF THE ABOVE EQ. IS

$$\vec{\nabla} \cdot [\phi |\nabla f| K_n H(g) \vec{n}] = |\nabla f| K_n H(g) \frac{\partial \phi}{\partial n} + \phi H(g) \vec{\nabla} \cdot (K_n |\nabla f| \vec{n}) + \phi |\nabla f| K_n \, \vec{n} \cdot \nabla g \, \delta(g) \quad (4)$$

MULTIPLY (2) BY ϕ AND REPLACE THE LAST EXPRESSION INVOLVING $\delta'(f)$ BY USING (3) AND (4)

$$\phi \, \vec{\nabla} \cdot [\vec{K} |\nabla f| H(g) \delta(f)] = - |\nabla f| K_n H(g) \frac{\partial \phi}{\partial n} \delta(f) + \phi H(g) \, \vec{\nabla} \cdot [|\nabla f| (\vec{K} - K_n \vec{n})] \, \delta(f) + \phi |\nabla f| |\nabla g| (\vec{K} - K_n \vec{n}) \cdot \vec{n}' \, \delta(f) \delta(g)$$

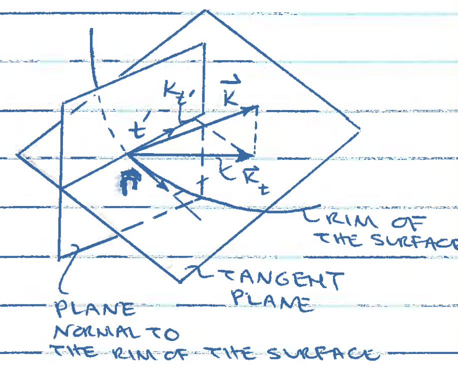
WHERE $\vec{n}' = \nabla g / |\nabla g|$. WE HAVE $\vec{K} - K_n \vec{n} = \vec{K}_t$ WHERE $\vec{K}_t$ IS THE PROJECTION OF $\vec{K}$ TO THE LOCAL TANGENT PLANE TO $f = 0$. ALSO LET g BE SO DEFINED THAT $\nabla g \times \nabla f \neq 0$ i.e. ∇g IS NOT PARALLEL TO ∇f. THEN DEFINE $\vec{\Gamma} = \nabla f \times \nabla g / |\nabla g \times \nabla f|$, i.e. UNIT TANGENT VECTOR TO THE RIM OF THE SURFACE. THEN LET US DECOMPOSE $\vec{K}_t$ IN THE TANGENT PLANE AS FOLLOWS

$$\vec{K}_t = K_r \vec{\Gamma} + K_{t'} \vec{t}'$$

WHERE $\vec{t}'$ IS UNIT VECTOR NORMAL TO $\vec{\Gamma}$. WE HAVE

$$\vec{K}_t \cdot \vec{n}' = K_{t'} \vec{t}' \cdot \vec{n}' = K_{t'} \sin\theta$$

WHERE θ IS THE ANGLE BETWEEN ∇f AND ∇g. THE COMPONENT $K_{t'}$ IS EASILY VISUALIZED AS SHOWN ON THE RIGHT. WE DEFINE $\vec{t}'$ IN SUCH A WAY THAT $\vec{t}' \cdot \vec{n}' > 0$. OUR FINAL RESULT IS OBTAINED IF WE GIVE A MEANING TO



$\vec{\nabla} \cdot [\vec{K}_t |\nabla f|]$. WE USE THE FOLLOWING DEFN OF DIVERGENCE

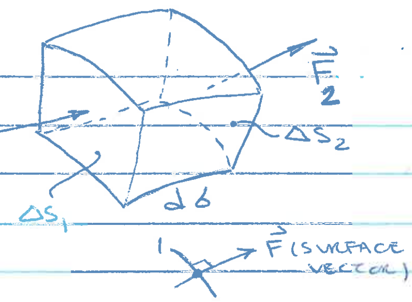
$$\nabla \cdot \vec{V} = \lim_{\Delta V \to 0} \frac{\int \vec{V} \cdot d\vec{s}}{\Delta V}$$

WE LET $\vec{K}_t |\nabla f| = \vec{F}$. $\vec{F}$ IS TANGENT TO THE SURFACE $f = 0$. WE PUT A BOX OVER $f = 0$ WITH TWO OPPOSING SURFACES $\perp$ TO $\vec{K}_t$. WE LET $dn = \frac{df}{|\nabla f|}$. THE ONLY NONVANISHING $\vec{V} \cdot d\vec{s}$ TERMS COME FROM THE TWO FACES $\perp$ TO $\vec{K}_t$. WE HAVE

$$\int \vec{F}\cdot d\vec{s} = F_2 \Delta s_2 - F_1 \Delta s_1$$

WHERE

$|\vec{F}_2| = F_2$, $|\vec{F}_1| = F_1$ $\vec{F} = \vec{F}_1$



$$F_2 = F + \frac{\partial F}{\partial \sigma} d\sigma$$

$\Delta s_1 = R_1\, d\theta\, dn$, R_1 RADIUS OF CURVATURE ALONG DIRECTION $\perp \vec{F}$ $\left(\equiv \frac{1}{|\kappa_1|}\right)$

$$\Delta s_2 = \left(R_1 + \frac{\partial R_1}{\partial \sigma} d\sigma\right) d\theta\, dn$$

$$\Delta V = R_1\, d\theta\, d\sigma\, dn$$

$$\int \vec{F}\cdot d\vec{s} = \left(F + \frac{\partial F}{\partial \sigma} d\sigma\right)\left(R_1 + \frac{\partial R_1}{\partial \sigma} d\sigma\right) d\theta\, dn - F R_1\, d\theta\, dn$$

(*) IN ERROR, SEE P 31

$$= \left(\frac{1}{R_1} F \frac{\partial R_1}{\partial \sigma} + \frac{\partial F}{\partial \sigma}\right) R_1\, d\theta\, d\sigma\, dn$$

$$\therefore \quad \nabla\cdot\vec{F} = \kappa_1 F \frac{\partial R_1}{\partial \sigma} + \frac{\partial F}{\partial \sigma}$$

HERE κ_1 IS THE SIGNED NORMAL CURVATURE, ASSUMING THAT THE SURFACE IS ONE-SIDED AND $\vec{n}$ ALWAYS POINTS TO THE DIRECTION OF ∇f. WE HAVE

$$\frac{\partial}{\partial \sigma} R_1 = \frac{\partial}{\partial \sigma} \frac{1}{|\kappa_1|} = - \frac{\mathrm{sig}\,\kappa_1}{\kappa_1^2} \frac{\partial \kappa_1}{\partial \sigma}$$

$$\therefore \quad \kappa_1 \frac{\partial R_1}{\partial \sigma} = -\frac{1}{|\kappa_1|} \frac{\partial \kappa_1}{\partial \sigma} = - R_1 \frac{\partial \kappa_1}{\partial \sigma}$$

WE HAVE FOUND THAT

$$\phi\, \bar{\nabla}\cdot\left[\vec{K}|\nabla f| H(g)\, \delta(f)\right] = -|\nabla f| K_n H(g) \frac{\partial \phi}{\partial n} \delta(f)$$

$$+ \phi H(g) \left[-R_1 K_t |\nabla f| \frac{\partial \kappa_1}{\partial \sigma} + \frac{\partial}{\partial \sigma}(K_t |\nabla f|)\right] \delta(f)$$

(SEE P 32)

$$+ \phi |\nabla f| |\nabla g| K_{t'} \sin\theta\, \delta(f)\, \delta(g)$$

WHERE $K_t = |\vec{K}_t|$. ALSO NOTE THAT IF WE DEFINE

$\vec{t}' = \vec{\Gamma} \times \vec{n}$; $\vec{\Gamma} = \vec{n}\times\vec{n}' / |\vec{n}\times\vec{n}'| \Rightarrow \vec{t}'\cdot\vec{n}' \geq 0$.

($\Rightarrow (\vec{n}, \vec{t}', \vec{\Gamma})$ MAKE A RIGHT-HANDED COORDINATE SYSTEM.)

(SEE NEXT PAGE)

WE HAVE ACHIEVED OUR GOAL SINCE ONLY THE INFORMATION AVAILABLE ON THE SURFACE $f=0, g>0$ IS USED. IT REMAINS TO SHOW THE INVARIANCE W.R.T. DEFN OF g. THIS CAN BE SEEN EASILY BY EVALUATING I:

$$I = -\int_{\substack{f=0\\g>0}} K_n \frac{\partial \phi}{\partial n}\, dS$$

$$+ \int_{\substack{f=0\\g>0}} \phi \left\{-RK_t \frac{\partial \kappa_1}{\partial \sigma} + \frac{1}{|\nabla f|}\frac{\partial}{\partial \sigma}[K_t |\nabla f|]\right\} dS \qquad (*)$$

(IN ERROR)
SEE NEXT PAGE

$$+ \int_{\substack{f=0\\g=0}} \phi K_{t'}\, d\Gamma$$

WHERE $d\Gamma$ IS THE ELEMENT OF ARC LENGTH OF THE RIM OF THE SURFACE.

IF $f = r-a = 0$, THEN THE OPEN SURFACE IS PART OF A SPHERE. $K_1 = \frac{1}{a}$ AND $\frac{\partial \kappa_1}{\partial \sigma} = 0$. THEN

$$I = -\int_{\substack{f=0\\g>0}} K_n \frac{\partial \phi}{\partial r}\, dS + \int_{\substack{f=0\\g>0}} \phi \frac{\partial K_t}{\partial \sigma}\, dS$$

$$+ \int_{\substack{f=0\\g=0}} \phi K_{t'}\, d\Gamma \qquad (**)$$

$\frac{\partial}{\partial \sigma}$ IS DIRECTIONAL DERIVATIVE ALONG THE GREAT CIRCLE TANGENT TO THE DIRECTION OF $\vec{K}_t$. THIS FORMULA LOOKS (IS!) VERY MUCH LIKE A DIVERGENCE THM ON A SURFACE. THEREFORE $\vec{t}'$ MUST HAVE A DEFINITE DIRECTION ON THE RIM OF $f=0, g>0$. INFACT,

$\vec{n}'$ MUST HAVE A PROJECTION ON THIS OPEN SURFACE BECAUSE $g > 0$ ON THIS SURFACE, NO MATTER HOW WE DEFINE $\vec{n}$ $\Rightarrow$ $\vec{t}'$ POINTS INTO THE SURFACE $f=0$, $g>0$. INFACT, IN THE EXAMPLE OF THE PREVIOUS PAGE FOR A SPHERE, IF WE LET $\phi = 1$, THEN WE HAVE

$$I = 0 = \int_{\substack{f=0 \\ g>0}} \frac{\partial K_t}{\partial \delta}\, ds + \int_{\substack{f=0 \\ g=0}} K_{t'}\, d\Gamma$$

OR

$$\int_{\substack{f=0 \\ g>0}} \frac{\partial K_t}{\partial \delta}\, dS = - \int_{\substack{f=0 \\ g=0}} K_{t'}\, d\Gamma$$

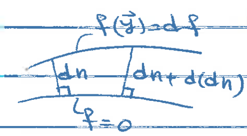
WHICH IS DIVERGENCE THM ON THE SURFACE AND $\vec{t}'$ IS THE INWARD NORMAL TO THE CURVE OF THE RIM WHICH IS TANGENT TO THE SURFACE $f=0$.

* A CORRECTION : THE CALCULATION OF $\nabla \cdot \vec{F} = \nabla \cdot [\vec{K}_t |\nabla f|]$ IS WRONG. THIS WAS FOUND BY NOTING THAT THE FINAL RESULT, EQ. (*) P 30, IS NOT INDEPENDENT OF THE WAY THE SURFACE IS DEFINED. THIS IS BECAUSE

$$\frac{1}{|\nabla f|} \frac{\partial}{\partial \delta} |\nabla f|$$

DEPENDS ON THE REPRESENTATION OF THE SURFACE AND $\therefore$ EQ. (*), P 30 CANNOT BE CORRECT. I FIRST BELIEVED THAT $\frac{1}{|\nabla f|} \frac{\partial}{\partial \delta} |\nabla f|$ IS A FUNCTION OF SURFACE PROPERTIES. AFTER ALMOST TEN DAYS OF FUTILE WORK, I DID WHAT I SHOULD HAVE DONE THE FIRST DAY — REPRESENT A SPHERE BY TWO FUNCTIONS $f_1 = r - a = 0$ AND $f_2 = z \pm \sqrt{a^2 - x^2 - y^2} = 0$ AND FOUND THAT $|\nabla f_1|^{-1} \frac{\partial}{\partial \delta} |\nabla f_1| = 0$ BUT $|\nabla f_2|^{-1} \frac{\partial}{\partial \delta} |\nabla f_2| \neq 0$! I DECIDED THAT EQ. (*) WAS WRONG. I TRACED THE ERROR TO

THE CALCULATION OF $\nabla \cdot \vec{F}$ AND THE FINAL RESULT GIVEN BELOW IS SATISFACTORY. CONSIDERING THE FIGURE ON P29 AGAIN AND ASSUMING THAT THE LOWER AND UPPER SURFACES OF THE BOX ARE ON $f=0$ AND $f=df$ RESPECTIVELY, WE HAVE



$$d(dn) = d\left[\frac{df}{|\nabla f|}\right]$$

$$= +df\,\frac{\partial}{\partial s}\left[\frac{1}{|\nabla f|}\right] ds$$

$$= -|\nabla f|\,\frac{\frac{\partial}{\partial s}|\nabla f|}{|\nabla f|^2}\,dn\,ds$$

$$= -\frac{\frac{\partial}{\partial s}|\nabla f|}{|\nabla f|}\,dn\,ds$$

$$\therefore \int \vec{F}\cdot d\vec{S} = \left(F + \frac{\partial F}{\partial s}ds\right)\left(R_1 + \frac{\partial R_1}{\partial s}ds\right)\left(dn - \frac{\frac{\partial}{\partial s}|\nabla f|}{|\nabla f|}dn\,ds\right)d\theta - F R_1\,d\theta\,dn$$

$$= \left[-\frac{F}{|\nabla f|}\frac{\partial}{\partial s}|\nabla f| + \frac{\partial F}{\partial s} + \kappa_1 F\frac{\partial R_1}{\partial s}\right] R_1\,d\theta\,dn\,ds$$

$$\nabla\cdot\vec{F} = -\frac{F}{|\nabla f|}\frac{\partial}{\partial s}|\nabla f| + \frac{\partial F}{\partial s} - R_1 F\frac{\partial \kappa_1}{\partial s}$$

$$= -K_t\cancel{\frac{\partial}{\partial s}|\nabla f|} + K_t\cancel{\frac{\partial}{\partial s}|\nabla f|} + |\nabla f|\frac{\partial K_t}{\partial s} - R_1 K_t|\nabla f|\frac{\partial \kappa_1}{\partial s}$$ (* SEE BELOW)

$\Rightarrow$

$$\phi\,\bar{\nabla}\cdot\left[\vec{K}\,|\nabla f|\,H(g)\,\delta(f)\right] = -|\nabla f|\,K_n\,H(g)\,\frac{\partial\phi}{\partial n}\,\delta(f)$$

$$+\,\phi\,H(g)\,|\nabla f|\left[\frac{\partial K_t}{\partial s} - R_1 K_t\frac{\partial \kappa_1}{\partial s}\right]\delta(f)$$

$$+\,\phi\,|\nabla f|\,|\nabla g|\,K_{t'}\sin\theta\,\delta(f)\,\delta(g)$$

(*) $\Rightarrow$ OR $\nabla\cdot[|\nabla f|\,\vec{t}\,] = -R_1|\nabla f|\frac{\partial \kappa_1}{\partial s}$ WHERE $\vec{t}$ IS THE UNIT TANGENT ALONG K_t.

$$I = -\int_{\substack{f=0\\ g>0}} K_n \frac{\partial \phi}{\partial n}\, dS$$

$$+ \int_{\substack{f=0\\ g>0}} \phi \left(\frac{\partial K_t}{\partial 6} - R_1 K_t \frac{\partial K_1}{\partial 6} \right] dS$$

$$+ \int_{\substack{f=0\\ g=0}} \phi K_{t'}\, d\Gamma$$ (SEE P 34-36)

EQ (**) ON PAGE 30 FOR A SPHERE IS CORRECT. IF WE LET $\phi = 1$, WE HAVE IN 3-D SPACE

$$\vec{\nabla} \cdot [\vec{K}\, |\nabla f|\, H(g)\, \delta(f)] = H(g) |\nabla f| \left[\frac{\partial K_t}{\partial 6} - R_1 K_t \frac{\partial K_1}{\partial 6} \right] \delta(f)$$
$$+ |\nabla f| |\nabla g| K_{t'} \sin\theta\, \delta(f)\, \delta(g)$$

NOTE THAT K_1 IS THE NORMAL CURVATURE ALONG THE DIRECTION PERPENDICULAR TO $\vec{K}_t$, $R_1 = \frac{1}{K_1}$ AND $K_{t'} = \vec{K}_t \cdot \vec{t}\,'$, $\vec{t}\,'$ IS THE INWARD TANGENT TO THE SURFACE WHICH IS PERPENDICULAR TO THE CURVE ON THE RIM OF THE SURFACE. THE ANGLE θ IS BET WEEN ∇f AND ∇g. I FEEL THERE IS A SIMPLER WAY THAN ABOVE TO GET I. THE FUNCTION $\vec{K}(\vec{y})\delta(f)$ IS THE SAME AS $\vec{\tilde{K}}(\vec{y})\delta(f)$ WHERE $\vec{\tilde{K}}(\vec{y})$ IS THE RESTRICTION OF $\vec{K}(\vec{y})$ TO THE SET $f = 0$. THEREFORE, IN FUNCTIONS INVOLVING THE TERM $\vec{K}(\vec{y})\,\delta(f)$, ONLY THE KNOWLEDGE OF $\vec{\tilde{K}}(\vec{y})$, AND NOT TERMS SUCH AS $\partial \vec{K}/\partial n$, SHOULD APPEAR IN THE FINAL RESULT WHEN THESE FUNCTIONS APPEAR UNDER AN INTEGRAL. THE FACT THAT I COULD NOT PROVE THIS BOTHERED ME FOR A LONG TIME. I FIRST DERIVED EQ. (*), P 30, IN SEPTEMBER 82, THINKING THAT THE RESULT WAS RIGHT.

IT WAS ONLY WHEN I COPIED THE RESULT IN THIS NOTEBOOK THAT I DISCOVERED THE ERROR.

ii) AFTER DERIVING FORMULATION (2) FOR SUPERSONIC PROPELLER NOISE CALCULATION WHICH I PUBLISHED IN MY VKI LECTURE NOTES, I TRIED TO EXTEND THE RESULT BY REMOVING THE THREE RESTRICTIONS WHICH I HAD USED, i.e., STEADY SURFACE PRESSURE, THIN AIRFOIL APPROXIMATION AND SPECIFICATION OF Δp, AND PROPELLER-LIKE MOTION OF THE BLADES. I WAS PARTICULARLY MOTIVATED WHEN I USED FORMULATION (2) ON A COMPUTER AND NOTICED A LOT OF IMPROVEMENTS IN THE ACOUSTIC PRESSURE SIGNATURE. I WORKED ABOUT A MONTH RELENTLESSLY PUTTING ASIDE EVERYTHING. JUST WHEN I THOUGHT I HAD ACHIEVED MY GOAL, I HIT UPON THE IDEA THAT RATHER THAN WORKING WITH THE SOLUTION OF THE FW-H EQ. AS I HAD DONE EARLIER, I MUST START WITH THE DIFFERENTIAL EQ. ITSELF. THINGS STARTED TO FALL IN THEIR PLACE IN MY MIND. I WROTE THE RESULTS FOR AIAA 8TH AEROACOUSTICS MEETING. BEFORE DERIVING THIS RESULT, WE WILL AGAIN LOOK AT THE FOLLOWING INTEGRAL IN 3-D:

$$I = \int \phi \, \nabla \cdot [\vec{K} |\nabla f| \delta(f)] \, d\vec{y}$$

LET US WRITE $\vec{K} = \vec{K}_t + \vec{K}_n$ WHERE $\vec{K}_t$ AND $\vec{K}_n$ ARE DECOMPOSITION OF $\vec{K}$ TANGENT AND NORMAL TO THE SURFACE $f = 0$. WE HAVE

$$\nabla \cdot [\vec{K} |\nabla f| \delta(f)] = \nabla \cdot [\vec{K}_t |\nabla f| \delta(f)] + \nabla \cdot [\vec{K}_n |\nabla f| \delta(f)]$$

WE HAVE
$$\nabla\cdot[\vec{K}_t|\nabla f|\delta(f)] = \nabla\cdot[\vec{K}_t|\nabla f|]\delta(f) + |\nabla f|\vec{K}_t\cdot\nabla f\,\delta'(f)$$
$$= \nabla\cdot[\vec{K}_t|\nabla f|]\delta(f)$$

WE CAN WRITE

$$\int \phi\,\nabla\cdot[\vec{K}_n|\nabla f|\delta(f)]\,dy = -\int_{f=0} K_n\frac{\partial\phi}{\partial n}\,dS$$

WE MUST INTERPRETE $\nabla\cdot[\vec{K}_t|\nabla f|]$. LET THE PROJECTION OF $\vec{K}$ ON $f=0$ GENERATE A SET OF CURVES WHOSE TANGENTS ARE ALONG LOCAL $\vec{K}_t$. OR BETTER YET, LET US HAVE A GENERAL CURVI-LINEAR COORDINATE NET u^1 AND u^2 ON $f=0$. THEN LET

$$\vec{t}_1 = \frac{\partial\vec{r}}{\partial u^1}, \quad \vec{t}_2 = \frac{\partial\vec{r}}{\partial u^2}$$

WHERE $\vec{r} = (y_1, y_2, y_3)$. THESE VECTORS ARE TANGENT TO COORDINATE CURVES. LET

$$\vec{K}_t = \lambda^1\vec{t}_1 + \lambda^2\vec{t}_2$$

LET US HAVE COORDINATE TRANSFORMATION $\vec{y} \to (u^1, u^2, f)$. THEN

$$\vec{t}_3 = \frac{\partial\vec{r}}{\partial n}\frac{\partial n}{\partial f}$$
$$= \vec{n}/|\nabla f|$$

WHERE $\vec{n}$ IS UNIT NORMAL TO $f=0$. WE HAVE $\vec{t}_1\cdot\vec{t}_3 = \vec{t}_2\cdot\vec{t}_3 = 0$. THE DETERMINANT OF COEFFICIENTS OF FIRST FUNDAMENTAL FORM IS

$$g_{(3)} = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & \frac{1}{|\nabla f|^2}\end{vmatrix} = \frac{g_{(2)}}{|\nabla f|^2}$$

WHERE $g_{(2)} = \begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix}$, $g_{ij} = \vec{t}_i \cdot \vec{t}_j$

WE HAVE $\nabla \cdot \vec{v} = \frac{1}{\sqrt{g_{(n)}}} \frac{\partial}{\partial u^k} (\sqrt{g_{(n)}}\, \lambda^k)$

$$\nabla \cdot [\vec{K}_t |\nabla f|] = \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^k} [\sqrt{g_{(2)}}\, \lambda^k]$$

$$\therefore I = \int \phi\, \nabla \cdot [\vec{K}_t |\nabla f|]\, \delta(f)\, d\vec{y}$$

$$= \int_{f=0} \frac{\phi}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^k} (\sqrt{g_{(2)}}\, \lambda^k)\, dS$$

$$= \int_{f=0} \phi \frac{\partial}{\partial u^k} (\sqrt{g_{(2)}}\, \lambda^k)\, du^1 du^2$$

IN PRACTICE TWO DIFFERENT MAPS OF A CLOSED SURFACE $f=0$ MUST BE USED. IF WE HAD INCLUDED $H(g)$ IN THE DIVERGENCE TERM, WE WOULD GET A LINE INTEGRAL AROUND THE RIM OF THE ~~OPEN~~ SURFACE: $f=0$, $g>0$. NOW THE ABOVE RESULT CAN BE INTEGRATED BY PARTS TO GIVE

$$I = -\int_{f=0} \lambda^k \frac{\partial \phi}{\partial u^k} \sqrt{g_{(2)}}\, du^1 du^2$$

$$= -\int_{f=0} K_t \cdot \nabla \phi\, d\vec{S} \qquad \text{AS EXPECTED.}$$

NOW I HAVE SOME DOUBTS ABOUT CORRECTNESS OF THE THE RESULT ON THE TOP OF P 33. THE DERIVATIVES OF $g_{(2)}$ ACTUALLY HAVE CURVATURE TERMS. I SHOULD

INVESTIGATE THIS FURTHER IN FUTURE. (* SEE BELOW)

iii) SUPERSONIC PROPELLER NOISE AND AERODYNAMICS

WE BEGIN BY WRITING THE FW-H EQUATION COMBINING THE THICKNESS SOURCE TERM WITH LOADING TERM USING ISOM'S RESULT. THE GOVERNING EQUATION IS THEREFORE:

$$\Box^2 p' = -\nabla \cdot [\tilde{p}\,\vec{n}\,|\nabla f|\,\delta(f)] \qquad (1)$$

WHERE $\tilde{p} = p_a + (\gamma - 1) p_0$. HERE p_a IS THE ABSOLUTE PRESSURE ON BLADE SURFACE, p_0 IS THE AMBIENT PRESSURE AND γ IS THE RATIO OF SPECIFIC HEATS. THE COMPLICATED FINAL RESULT IS OBTAINED IN SEVERAL STEPS AS FOLLOWS. THE PATH TO THE FINAL RESULT WAS FOUND AFTER A LOT OF WORK. IT IS EASIER THAN MANY OF THE OTHER METHODS WHICH I TRIED.

STEP 1

WE ARE WORKING IN FOUR DIMENSIONS. WE REMOVE THE ASYMMETRY OF ∇ OPERATOR, WHICH HAS ONLY SPATIAL VARIABLES, BY REPLACING IT WITH

$$\nabla_4 = \left(\nabla, \frac{1}{c}\frac{\partial}{\partial t}\right). \qquad (2)$$

WE THEREFORE WRITE EQ. (1) AS FOLLOWS:

$$\Box^2 p' = -\nabla_4 \cdot [\tilde{p}\,(\vec{n}, 0)\,|\nabla f|\,\delta(f)] \qquad (3)$$

NOW WE DECOMPOSE THE 4-VECTOR $(\vec{n}, 0)$ INTO TWO VECTORS WHICH ARE NORMAL AND TANGENT

(*) NOTE ADDED ON MAR. 6, 83 — I NOW HAVE LEARNED MORE ABOUT DIFFERENTIAL GEOMETRY AND DIFFERENTIABLE MANIFOLDS. I THINK I CAN CLEAN UP THE ABOVE NOW. (SEE PAGE 92, JUNE 83).

TO THE SURFACE $f=0$ IN 4-D. WE INTRODUCE THE UNIT NORMAL $\vec{N}$ TO SURFACE f AS

$$\vec{N} = \frac{\nabla_4 f}{|\nabla_4 f|}$$

$$= \frac{1}{\alpha_n}(\vec{n}, -M_n) \qquad (4)$$

WHERE $\vec{n}$ = UNIT NORMAL TO $f=0$ (TIME FROZEN)

$$M_n = v_n / c$$

$$\alpha_n = \sqrt{1+M_n^2}$$

<u>NOTATION</u>: WE WILL USE CAPITAL LETTERS FOR 4-VECTORS CONSISTENTLY.

WE ALSO DEFINE THE UNIT TANGENT TO THE SURFACE $f=0$ IN 4D ALONG THE PROJECTION OF $(\vec{n},0)$. THE PROJECTION VECTOR IS

$$\vec{P}_{(\vec{n},0)} = (\vec{n},0) - [\vec{N}\cdot(\vec{n},0)]\,\vec{N}$$

$$= (\vec{n},0) - \frac{1}{\alpha_n^2}(\vec{n}, -M_n)$$

$$= \frac{M_n}{\alpha_n^2}(M_n\vec{n}, 1)$$

THE UNIT VECTOR ALONG THIS VECTOR IS

$$\vec{T} = \frac{1}{\alpha_n}(M_n\vec{n}, 1) \qquad (5)$$

WE CAN THEREFORE WRITE UNIQUELY

$$(\vec{n}, 0) = \frac{1}{\alpha_n}(\vec{N} + M_n\vec{T}) \qquad (6)$$

THE SOURCE TERM ON THE RIGHT SIDE OF EQ. (3) BECOMES

$$-\nabla_4 \cdot [\tilde{p}(\vec{n},0)|\nabla f|\delta(f)] = \nabla_4 \cdot [-\frac{\tilde{p}}{\alpha_n}(\vec{N}+M_n\vec{T})|\nabla f|\delta(f)]$$

$$\nabla_4 \cdot [\frac{\tilde{p}}{\alpha_n}(\vec{N}+M_n\vec{T})|\nabla f|\delta(f)] = \nabla_4 \cdot [\frac{\tilde{p}}{\alpha_n}\vec{N}|\nabla f|\delta(f)] + \nabla_4 \cdot [\frac{\tilde{p}M_n}{\alpha_n}\vec{T}|\nabla f|\delta(f)]$$

$$\nabla_4 \cdot [\frac{\tilde{p}M_n}{\alpha_n}\vec{T}|\nabla f|\delta(f)] = \nabla_4 \cdot [\frac{\tilde{p}M_n|\nabla f|}{\alpha_n}\vec{T}]\delta(f) + \frac{\tilde{p}M_n|\nabla f|}{\alpha_n}\vec{T}\cdot\nabla_4 f\,\delta'(f)$$

$$= \nabla_4 \cdot [\frac{\tilde{p}M_n|\nabla f|}{\alpha_n}\vec{T}]\delta(f) \quad (7)$$

$\therefore$

$$\nabla_4 \cdot [\tilde{p}(\vec{n},0)|\nabla f|\delta(f)] = \nabla_4 \cdot [\frac{\tilde{p}}{\alpha_n}\vec{N}|\nabla f|\delta(f)] + \nabla_4 \cdot [\frac{\tilde{p}M_n|\nabla f|}{\alpha_n}\vec{T}]\delta(f) \quad (8)$$

IN EQ. (7), WE HAVE USED THE RESULT THAT $\vec{T}\cdot\nabla_4 f = |\nabla_4 f|\vec{N}\cdot\vec{T} = 0$. OUR WORK WITH THE SECOND TERM OF EQ. (8) IS FINISHED FOR NOW. WE WILL CONCENTRATE ON THE FIRST TERM IN OUR SECOND STEP. EQ. (3) IS THEREFORE WRITTEN AS FOLLOWS

$$\Box^2 p' \stackrel{\vee}{=} \nabla_4 \cdot [\frac{\tilde{p}}{\alpha_n}\vec{N}|\nabla f|\delta(f)] \stackrel{\vee}{-} \nabla_4 \cdot [\frac{\tilde{p}M_n|\nabla f|}{\alpha_n}\vec{T}]\delta(f) \quad (9)$$

$$\equiv Q_1 + Q_2$$

STEP 2

THE SOLUTION OF WAVE EQUATION

$$\Box^2 \phi = Q(\vec{x}, t)$$

IS

$$4\pi \phi(\vec{x}, t) = \int \frac{Q\,\delta(g)}{r}\, d\vec{y}\, d\tau$$

$$= \left\langle Q, \frac{\delta(g)}{r} \right\rangle$$

WHERE $g = \tau - t + r/c$.

WE CONSIDER EQ. (9) NOW. THE SOLUTION IS

$$4\pi p'(\vec{x}, t) = \left\langle Q_1, \frac{\delta(g)}{r} \right\rangle + \left\langle Q_2, \frac{\delta(g)}{r} \right\rangle \quad (10)$$

$$\left\langle Q_1, \frac{\delta(g)}{r} \right\rangle = -\left\langle \nabla_4 \cdot \left[\frac{\tilde{p}}{\alpha_n} \vec{N} |\nabla f| \delta(f)\right], \frac{\delta(g)}{r} \right\rangle$$

$$= \left\langle \frac{\tilde{p}}{\alpha_n} |\nabla f| \delta(f), \vec{N} \cdot \nabla_4 \left[\frac{\delta(g)}{r}\right] \right\rangle \quad (11)$$

(OBTAINED BY INTEGRATION BY PARTS)

NOTE THAT WE ARE NOW WORKING IN SOURCE COOR-DINATES $(\vec{y}, \tau)$; AND $(\vec{x}, t)$ IS KEPT FIXED.

WE HAVE

$$\vec{N} \cdot \nabla_4 \left(\frac{\delta(g)}{r}\right) = -\frac{\vec{N} \cdot \nabla_4 r}{r^2} \delta(g) + \frac{\vec{N} \cdot \nabla_4 g}{r} \delta'(g)$$

$\therefore$

$$\left\langle Q_1, \frac{\delta(g)}{r} \right\rangle = -\left\langle \frac{\tilde{p}}{\alpha_n} |\nabla f| \delta(f), \frac{\vec{N} \cdot \nabla_4 r}{r^2} \delta(g) \right\rangle$$

$$+ \left\langle \frac{\tilde{p}}{\alpha_n} |\nabla f| \delta(f), \frac{\vec{N} \cdot \nabla_4 g}{r} \delta'(g) \right\rangle$$

(12)

STEP. 3

WE NOW CONCENTRATE ON THE SECOND TERM OF EQ. (12). OUR INTENTION IS TO GET RID OF $\delta'(g)$.

WE HAVE

$$\text{2ND TERM OF EQ. (12)} = \int Q\,\delta(f)\,\delta'(g)\,d\vec{y}_4$$

WHERE

$$Q = \frac{\tilde{p}\,|\nabla f|\,\vec{N}\cdot\nabla_4 g}{\alpha_n r} \qquad (13)$$

WE CAN SHOW (APP. A, AIAA 8TH AEROACOUSTICS MEETING)

$$\int Q\,\delta(f)\,\delta'(g)\,d\vec{y}_4 = -\int \nabla_4\cdot(Q\vec{A})\,\delta(f)\,\delta(g) \qquad (14)$$

WHERE

$$\vec{A} = \frac{C[\alpha_n^2(-\hat{r},1) + (M_n + \cos\theta)(\vec{n}, -M_n)]}{\Lambda^2 + \sin^2\theta}, \qquad (15)$$

$$\Lambda^2 = 1 + M_n^2 - 2M_n\cos\theta \qquad (16)$$

$$\cos\theta = \vec{n}\cdot\hat{r}$$

$$\hat{r} = (\vec{x} - \vec{y})/r$$

WE HAVE

$$\nabla_4\cdot(Q\vec{A}) = \frac{1}{r}\nabla_4\cdot\left(\frac{\tilde{p}\,|\nabla f|\,\vec{N}\cdot\nabla_4 g}{\alpha_n}\vec{A}\right)$$

$$- \frac{\tilde{p}\,|\nabla f|\,\vec{N}\cdot\nabla_4 g}{\alpha_n r^2}\vec{A}\cdot\nabla_4 r \qquad (17)$$

$\Longrightarrow$

$$\left\langle Q_1, \frac{\delta(g)}{r}\right\rangle = \left\langle \frac{\tilde{p}}{\alpha_n}|\nabla f|\,\delta(f), \frac{(\vec{N}\cdot\nabla_4 g)\,\vec{A}\cdot\nabla_4 r - \vec{N}\cdot\nabla_4 r}{r^2}\,\delta(g)\right\rangle$$

$$- \left\langle \nabla_4\cdot\left(\frac{\tilde{p}\,|\nabla f|\,\vec{N}\cdot\nabla_4 g}{\alpha_n}\vec{A}\right)\delta(f), \frac{\delta(g)}{r}\right\rangle \qquad (18)$$

STEP 4 – THE NEAR-FIELD TERM

THE TERMS OF ORDER $1/r^2$ COME FROM $\langle Q_1, \frac{\delta(g)}{r} \rangle$ ONLY. WE MUST SIMPLIFY THE FOLLOWING EXPRESSION TO GET THE NEAR FIELD TERM:

$$E = (\vec{N} \cdot \nabla_4 g)\, \vec{A} \cdot \nabla_4 r - \vec{N} \cdot \nabla_4 r \qquad (19)$$

WE HAVE

$$\vec{N} = \frac{1}{\alpha_n} (\vec{n}, -M_n)$$

$$\nabla_4 g = \frac{1}{c} (-\vec{\hat{r}}, 1)$$

$$\nabla_4 r = (-\vec{\hat{r}}, 0) \qquad \nabla_4 = \left(\nabla_y, \frac{1}{c}\frac{\partial}{\partial \tau}\right)$$

$$\vec{N} \cdot \nabla_4 g = \frac{-1}{c\alpha_n} (\cos\theta + M_n) \qquad (20)$$

$$\vec{A} \cdot \nabla_4 r = \frac{c}{\Lambda^2 + \sin^2\theta} \left[+\alpha_n^2 - (M_n + \cos\theta)\cos\theta \right]$$

$$= \frac{c\,(M_n^2 - M_n \cos\theta + \sin^2\theta)}{\Lambda^2 + \sin^2\theta} \qquad (21)$$

$$\vec{N} \cdot \nabla_4 r = -\frac{\cos\theta}{\alpha_n} \qquad (22)$$

$\Longrightarrow$

$$E = \frac{1}{\alpha_n} \left[\frac{-(M_n + \cos\theta)(M_n^2 - M_n\cos\theta + \sin^2\theta)}{\Lambda^2 + \sin^2\theta} + \cos\theta \right]$$

$$= \frac{\alpha_n^2 (\cos\theta - M_n)}{\alpha_n (\Lambda^2 + \sin^2\theta)}$$

$$= \frac{\alpha_n (\cos\theta - M_n)}{\Lambda^2 + \sin^2\theta} \qquad (23)$$

WE WRITE $\rho' = \rho'_N + \rho'_F$ WHERE N AND F STAND FOR NEAR AND FAR-FIELD, RESPECTIVELY. WE HAVE SHOWN THAT

$$4\pi \rho'_N(\vec{x},t) = \left\langle \frac{\tilde{p}}{\alpha_n} |\nabla f| \delta(f), \frac{E}{r^2}\delta(g) \right\rangle$$

$$= \checkmark \int |\nabla f| \frac{\tilde{p}(\cos\theta - M_n)}{r^2(\Lambda^2 + \sin^2\theta)} \delta(f)\delta(g)\, d\vec{y}\, d\tau \tag{24}$$

AGAIN OUR TASK IS FINISHED FOR "NEAR-FIELD" TERM.

STEP 5 : THE FAR-FIELD TERM

WE HAVE BASICALLY TWO TERMS WHICH WE MUST ADD — ONE IS THE 2ND TERM IN EQ. (9) AND ANOTHER COMES FROM THE 2ND TERM IN EQ. (18). WE THUS GET

$$4\pi \rho'_F(\vec{x},t) = -\left\langle \delta(f) \nabla_4 \cdot \left\{ \frac{\tilde{p}|\nabla f|}{\alpha_n} \left[(\vec{N}\cdot\nabla_4 g)\vec{A} + M_n \vec{T} \right] \right\}, \frac{\delta(g)}{r} \right\rangle$$

$$= \checkmark - \int \nabla_4 \cdot (\tilde{p}|\nabla f|\vec{B}) \frac{\delta(f)\delta(g)}{r}\, d\vec{y}\, d\tau \tag{25}$$

WHERE

$$\vec{B} = \left[(\vec{N}\cdot\nabla_4 g)\vec{A} + M_n \vec{T} \right] / \alpha_n$$

$$= \frac{1}{\alpha_n} \left\{ \frac{-(M_n + \cos\theta)}{\alpha_n(\Lambda^2 + \sin^2\theta)} \left[\alpha_n^2(-\vec{\hat{r}}, 1) + (M_n + \cos\theta)(\vec{n}, -M_n) \right] + M_n \vec{T} \right\} \tag{26}$$

WE WANT TO SIMPLIFY THIS VECTOR. ONE EASY,

BUT LONG WAY, IS TO FIND COEFFICIENT OF $\vec{n}$ AND TIME COMPONENT

$$\text{COEFF. OF } \vec{n} = -(M_n + \cos\theta)^2 + M_n^2(\Lambda^2 + \sin^2\theta)$$

$$= \alpha_n^2(M_n^2 - 2M_n\cos\theta - \cos^2\theta)$$

$$= \alpha_n^2[-M_n(M_n - \cos\theta) - \cos\theta(M_n + \cos\theta)]$$

$$\text{COEFF. OF TIME COMP.} = -\alpha_n^2(M_n + \cos\theta) + M_n(M_n + \cos\theta)^2 + M_n(\Lambda^2 + \sin^2\theta)$$

$$= \alpha_n^2(M_n - \cos\theta)$$

$\therefore$

$$\vec{B} = \frac{1}{\Lambda^2 + \sin^2\theta}\left[+(\cos\theta + M_n)\hat{r} + M_n(M_n - \cos\theta)\vec{n} - \cos\theta(M_n + \cos\theta)\vec{n}, M_n - \cos\theta\right]$$

$$= \frac{1}{\Lambda^2 + \sin^2\theta}\left[(\cos\theta + M_n)(\hat{r} - \cos\theta\,\vec{n}, 0) + (M_n - \cos\theta)(M_n\vec{n}, 1)\right]$$

$$= \frac{1}{\Lambda^2 + \sin^2\theta}\left[(\cos\theta + M_n)\vec{T}_1 + \alpha_n(M_n - \cos\theta)\vec{T}\right] \tag{27}$$

WHERE

$$\vec{T}_1 = (\hat{r} - \cos\theta\,\vec{n}, 0) \tag{28}$$
$$\equiv (\vec{t}_1, 0)$$

WHERE $\vec{t}_1$ IS THE PROJECTION OF $\hat{r}$ ON THE TANGENT

PLANE OF $f=0$ (TIME FROZEN). NOTE THAT $\vec{t}_1$ IS NOT OF UNIT LENGTH. FOR THE NEXT STEP, WE WRITE

$$\vec{B} = \alpha \lambda_n \vec{T} + \lambda' \vec{T}_1 \quad \text{(SEE NOTE WRITTEN IN RED, BELOW)} \tag{29}$$

WHERE

$$\lambda = \frac{\dot{M}_n - \cos\theta}{\Lambda^2 + \sin^2\theta} \tag{30}$$

$$\lambda' = \frac{M_n + \cos\theta}{\Lambda^2 + \sin^2\theta} \tag{31}$$

(WE CHANGE THE SIGN OF λ IN STEP 9 REVISITED, P 70. ALSO WE USE SUBSCRIPT 1 FOR λ'.)

STEP 6 — FURTHER SIMPLIFICATION

WE NOW CALCULATE $\nabla_4 \cdot (\tilde{p} |\nabla f| \vec{B})$. DIRECT APPLICATION OF DIVERGENCE OPERATION IS NOT THE BEST WAY SINCE IT DOES NOT TAKE ADVANTAGE OF THE FACT THAT $\vec{B}$ IS IN THE TANGENT PLANE OF $f=0$ IN 4-D. THEREFORE, NEW CURVILINEAR COORDINATES MUST BE INTRODUCED. BEFORE DOING THIS, WE WRITE

$$\nabla_4 \cdot [\tilde{p} |\nabla f| \vec{B}] = |\nabla f| \vec{B} \cdot \nabla_4 \tilde{p} + \tilde{p} \nabla_4 \cdot (|\nabla f| \vec{B}) \tag{32}$$

WE FIRST CONCENTRATE ON $\vec{B} \cdot \nabla_4 \tilde{p}$. WE INTRODUCE (DEFINE) THE FUNCTION $\tilde{p}_B$ AS FOLLOWS

$$\tilde{p}_B(\vec{\eta}, \tau) = \tilde{p}(\vec{y}(\vec{\eta}, \tau), \tau) \tag{33}$$

WHERE $\vec{\eta}$ IS THE (LAGRANGIAN) COORDINATES OF POINTS ON THE BLADE IN A FRAME FIXED TO BLADE.

THE REASON WE DEFINE THIS FUNCTION IS THAT WE CAN NOW INTERPRETE PARTIAL DERIVATIVES UNAMBIGUOUSLY. WE HAVE

$$\vec{B} \cdot \nabla_4 \tilde{p} = \alpha_n \lambda \vec{T} \cdot \nabla_4 \tilde{p} + \lambda' \vec{T}_1 \cdot \nabla_4 \tilde{p} \qquad (34)$$

$$\vec{T} \cdot \nabla_4 \tilde{p} = \frac{1}{\alpha_n}\left(M_n \frac{\partial \tilde{p}}{\partial n} + \frac{1}{c}\frac{\partial \tilde{p}}{\partial \tau}\right) \qquad (35)$$

$$\frac{\partial \tilde{p}}{\partial \tau} = \frac{\partial \tilde{p}_B}{\partial \tau} + \left.\frac{\partial \vec{z}}{\partial \tau}\right|_{\vec{y}} \cdot \nabla_{\vec{z}} \tilde{p}_B$$

$$= \frac{\partial \tilde{p}_B}{\partial \tau} - \vec{V} \cdot \nabla \tilde{p}_B \qquad \left(\vec{M} = \frac{\vec{V}}{c}\right) \quad (36)$$

$$\frac{1}{c}\frac{\partial \tilde{p}}{\partial \tau} = \frac{1}{c}\frac{\partial \tilde{p}_B}{\partial \tau} - M_t \frac{\partial \tilde{p}_B}{\partial \sigma} - M_n \frac{\partial \tilde{p}_B}{\partial n} \qquad (37)$$

WHERE $M_t = v_t / c$, v_t IS THE PROJECTION OF $\vec{V}$ ON THE TANGENT PLANE OF $f=0$, FOR τ FIXED. WE THEREFORE HAVE, USING $\frac{\partial \tilde{p}}{\partial n} = \frac{\partial \tilde{p}_B}{\partial n}$,

$$\vec{T} \cdot \nabla_4 \tilde{p} = \frac{1}{\alpha_n}\left(\frac{1}{c}\frac{\partial \tilde{p}_B}{\partial \tau} - M_t \frac{\partial \tilde{p}_B}{\partial \sigma}\right) \qquad (38)$$

HERE $\frac{\partial \tilde{p}_B}{\partial \sigma}$ IS THE DIRECTIONAL DERIVATIVE OF $\tilde{p}_B$ ALONG v_t ON THE BLADE SURFACE. WE ALSO HAVE

$$\vec{T}_1 \cdot \nabla_4 \tilde{p} = \frac{\partial \tilde{p}_B}{\partial \sigma_1} \sin\theta = \vec{t}_1 \cdot \nabla \tilde{p}_B \qquad (39)$$

WHERE $\frac{\partial \tilde{p}_B}{\partial \sigma_1}$ THE DIRECTIONAL DERIVATIVE OF $\tilde{p}_B$

ALONG THE PROJECTION OF $\vec{\hat{F}}$ ON THE BLADE SURFACE. WE HAVE FOUND THAT

$$\vec{B}\cdot\nabla_4\tilde{p} = \lambda\left(\frac{1}{c}\frac{\partial\tilde{p}_B}{\partial\tau} - M_t\frac{\partial\tilde{p}_B}{\partial\sigma}\right) + \lambda'\sin\theta\frac{\partial\tilde{p}_B}{\partial\sigma_1} \qquad (40)$$

THIS IS VERY SATISFACTORY SINCE WE ONLY USE INFORMATION AVAILABLE ON THE BLADE SURFACE.

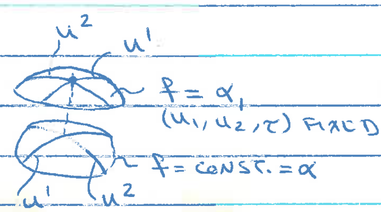
WE NOW CONCENTRATE ON $\nabla_4\cdot(|\nabla f|\vec{B})$. WE INTRODUCE VARIABLES THAT ARE "TANGIBLE", i.e. AVAILABLE TO US EASILY. IT APPEARS THAT SINCE $\vec{B}$ IS COMPOSED OF TWO VECTORS $\vec{T}$ AND $\vec{T}_1$ ON $f=0$ IN 4-D, WE SHOULD INTRODUCE VARIABLES WHICH WOULD GIVE NATURAL BASE VECTORS ALONG $\vec{T}$ AND $\vec{T}_1$. AFTER WORKING A WHILE ON THIS IDEA, I FOUND THAT I COULD NOT COME UP EASILY WITH COORDINATE VARIABLES WITH THIS PROPERTY. HAVING RECOGNIZED THIS, I USED A GENERAL NONORTHOGONAL CURVILINEAR COORDINATE SYSTEM WHICH STILL GIVES SIMPLE FINAL EXPRESSION FOR $\nabla_4\cdot(|\nabla f|\vec{B})$. OUR NEW VARIABLES ARE u^i, $i=1,2,3,4$ WHERE

(u^1, u^2) = ANY COORDINATE NET ON $\begin{cases} f = \text{CONST.}, \\ \tau \text{ FIXED} \end{cases}$

$u^3 = c\tau$

$u^4 = f$

$(\vec{y}, c\tau) \rightarrow (\vec{u})$



OUR NATURAL BASE VECTORS ARE :

$$\vec{A}_1 = \left(\frac{\partial \vec{y}}{\partial u^1}, 0\right) \equiv (\vec{a}_1, 0) \qquad (41)$$

$$\vec{A}_2 = \left(\frac{\partial \vec{y}}{\partial u^2}, 0\right) \equiv (\vec{a}_2, 0) \qquad (42)$$

$$\vec{A}_3 = \left(\frac{\partial \vec{y}}{\partial u^3}, c\frac{\partial \tau}{\partial u^3}\right)$$

$$= (\vec{M}, 1) \qquad (43)$$

WHERE $\vec{M}$ IS THE MACH NUMBER OF THE POINT WITH COORDINATES (u^1, u^2) FIXED TO THE BLADE.

$$\vec{A}_4 = \left(\frac{\partial \vec{y}}{\partial f}, 0\right)$$

$$= \left(\frac{\partial \vec{y}}{\partial n}\frac{\partial n}{\partial f}, 0\right)$$

$$= \left(\frac{\vec{n}}{|\nabla f|}, 0\right) \qquad (44)$$

NOTE THE $\vec{a}_1$ AND $\vec{a}_2$ ARE TANGENT TO SURFACE $f=0$, τ-FIXED, BUT ARE NOT NECESSARILY OF UNIT LENGTH. WE WILL RECTIFY THIS SHORTLY. WE FIRST NOTE THAT, SINCE $\vec{T}_1$ IS TANGENT TO $f=0$, τ-FIXED, IT CAN BE WRITTEN AS A LINEAR COMBINATION OF $\vec{A}_1$ AND $\vec{A}_2$

$$\vec{T}_1 = \alpha^1 \vec{A}_1 + \alpha^2 \vec{A}_2 \qquad (45)$$

ALSO

$$\vec{A}_3 = (\vec{M}_n + \vec{M}_t, 1)$$

$$= (\vec{M}_n, 1) + (\vec{M}_t, 0)$$

$$= \alpha_n \vec{T} + (\vec{M}_t, 0) \qquad (46)$$

ALSO $\vec{M}_t$ IS IN THE TANGENT PLANE OF $f=0$, τ FIXED, THUS

$$\vec{M}_t = \mu^1 \vec{A}_1 + \mu^2 \vec{A}_2 \qquad (45)$$

$$\Rightarrow \alpha_n \vec{T} = \vec{A}_3 - \mu^1 \vec{A}_1 - \mu^2 \vec{A}_2 \qquad (46)$$

(46)

$$\vec{B} = \alpha_n \lambda \vec{T} + \lambda^1 \vec{T}_1$$

$$= \lambda (\vec{A}_3 - \mu^1 \vec{A}_1 - \mu^2 \vec{A})$$

$$+ \lambda^1 (\alpha^1 \vec{A}_1 + \alpha^2 \vec{A}_2)$$

$$= (\lambda^1 \alpha^1 - \lambda \mu^1) \vec{A}_1$$

$$+ (\lambda^1 \alpha^2 - \lambda \mu^2) \vec{A}_2$$

$$+ \lambda \vec{A}_3$$

$$\equiv \beta^i \vec{A}_i \qquad (47)$$

$$\beta^i = (\lambda^1 \alpha^1 - \lambda \mu^1, \lambda^1 \alpha^2 - \lambda \mu^2, \lambda, 0) \qquad (48)$$

WE WILL INDICATE HOW α'S AND μ'S ARE CALCULATED SHORTLY. WE USE THE INVARIANT DEFINITION OF DIVERGENCE IN DIFFERENTIAL GEOMETRY:

$$\nabla_4 \cdot (|\nabla f| \vec{B}) = \frac{1}{\sqrt{g_{(4)}}} \frac{\partial}{\partial u^i} \left[|\nabla f| \sqrt{g_{(4)}}\, \beta^i \right] \qquad (49)$$

WHERE $g_{(4)}$ IS THE DETERMINANT OF THE COEFFICIENTS OF FIRST FUNDAMENTAL FORM. IT IS OBTAINED

AS FOLLOWS:

$$g_{(4)} = \det(g_{ij})$$

$$g_{ij} = \vec{A}_i \cdot \vec{A}_j$$

WE USE SOME SIMPLE GEOMETRIC ARGUMENT HERE.

$$g_{(4)} = \begin{vmatrix} g_{11} & g_{12} & \vec{a}_1\cdot\vec{M} & 0 \\ g_{12} & g_{22} & \vec{a}_2\cdot\vec{M} & 0 \\ \vec{a}_1\cdot\vec{M} & \vec{a}_2\cdot\vec{M} & (1+M^2) & \frac{M_n}{|\nabla f|} \\ 0 & 0 & \frac{M_n}{|\nabla f|} & \frac{1}{|\nabla f|^2} \end{vmatrix}$$

$$= \begin{vmatrix} g_{(3)} & \begin{matrix} 0 \\ 0 \\ M_n/|\nabla f| \end{matrix} \\ 0 \quad 0 \quad \frac{M_n}{|\nabla f|} & \frac{1}{|\nabla f|^2} \end{vmatrix}$$

$$= -\frac{M_n^2}{|\nabla f|^2}\, g_{(2)} + \frac{1}{|\nabla f|^2}\, g_{(3)} \qquad (50)$$

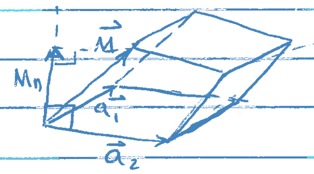
HERE $g_{(2)}$ IS THE COEFF. OF 1ST FUNDAMENTAL FORM FOR COORDINATE NET ON THE BLADE, τ FIXED.

$$g_{(3)} = \begin{vmatrix} g_{11} & g_{12} & \vec{a}_1\cdot\vec{M} \\ g_{12} & g_{22} & \vec{a}_2\cdot\vec{M} \\ \vec{a}_1\cdot\vec{M} & \vec{a}_2\cdot\vec{M} & M^2 \end{vmatrix} + \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ \vec{a}_1\cdot\vec{M} & \vec{a}_2\cdot\vec{M} & 1 \end{vmatrix}$$

$$= M_n^2\, g_{(2)} + g_{(2)} \qquad (51)$$

THE FIRST DETERMINANT IS FOUND GEOMETRICALLY

AS FOLLOWS:



1ST det = SQUARE OF VOLUME FORMED BY $(\vec{a}_1, \vec{a}_2, \vec{M})$

$$= [M_n \times (\text{AREA } (\vec{a}_1, \vec{a}_2)]^2$$

$$= M_n^2 [\text{AREA } (\vec{a}_1, \vec{a}_2)]^2$$

$$= M_n^2 g_{(2)}$$ SINCE $g_{(2)} = [\text{AREA}(\vec{a}_1, \vec{a}_2)]^2$

$\therefore$

$$g_{(4)} = \frac{g_{(2)}}{|\nabla f|^2} \qquad (52)$$

$\Rightarrow$

$$\nabla_4 \cdot (|\nabla f| \vec{B}) = \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \left[\sqrt{g_{(2)}}\, \beta^i \right] \qquad (53)$$

THIS IS HIGHLY SATISFACTORY SINCE f DOES NOT APPEAR EXPLICITLY HERE ($|\nabla f|$ WILL CANCEL OUT EVENTUALLY WHEN $\delta(f)$ IS INTEGRATED). SINCE THE BLADE IS ASSUMED RIGID $g_{(2)}$ DOES NOT DEPEND ON TIME τ, OR u^3. WE THUS HAVE

$$\frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\,\beta^i\right] = \frac{\partial}{\partial u^1}(\sqrt{g_{(2)}}\,\beta^1) + \frac{\partial}{\partial u^2}(\sqrt{g_{(2)}}\,\beta^2) + \sqrt{g_{(2)}}\frac{\partial \beta^3}{\partial u^3} \qquad (54)$$

HOWEVER, WE SHOULD REMEMBER THAT THESE ARE GENERALIZED DERIVATIVES AND β^i'S ARE DISCONTIN-

VOUS AT THE TRAILING EDGE. WE WILL TAKE CARE OF THIS IN THE NEXT STEP. FOR NOW, WE WILL SHOW HOW α'S AND μ'S ARE CALCULATED. WE HAVE

$$\vec{T}_1 = \alpha^1 \vec{A}_1 + \alpha^2 \vec{A}_2$$

$$\alpha^1 = \vec{T}_1 \cdot \vec{A}^1 = \vec{t}_1 \cdot \vec{a}^1 = g^{1i}\alpha_i \quad \text{(USE!)}$$

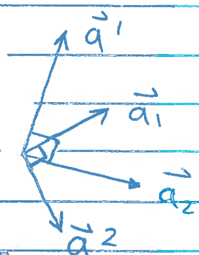
$$\alpha^2 = \vec{T}_1 \cdot \vec{A}^2 = \vec{t}_1 \cdot \vec{a}^2 = g^{2i}\alpha_i \quad \text{(USE!)}$$

WHERE $\vec{T}_1 = (\vec{t}_1, 0)$ AND $\vec{a}^1$ AND $\vec{a}^2$ ARE CONTRAVARIANT BASE VECTORS

$$\begin{cases} \vec{a}^1 \cdot \vec{a}_1 = 1 \\ \vec{a}^1 \cdot \vec{a}_2 = 0 \end{cases} \qquad \begin{cases} \vec{a}^2 \cdot \vec{a}_1 = 0 \\ \vec{a}^2 \cdot \vec{a}_2 = 1 \end{cases}$$

$$\alpha^i = g^{ij}\alpha_j$$

$$\alpha_j = \vec{t}_1 \cdot \vec{a}_j$$



$$(g^{ij}) = \frac{1}{g_{(2)}} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix}$$

LET $\vec{a}_{(j)}$ BE THE <u>UNIT VECTOR</u> ALONG $\vec{a}_j$, THEN

$$\alpha_j = \vec{t}_1 \cdot \left[\sqrt{g_{jj}}\, \vec{a}_{(j)}\right]$$

$$= \sqrt{g_{jj}}\; \vec{t}_1 \cdot \vec{a}_{(j)} \qquad \text{(NO SUM ON } j\text{)}$$

$$\therefore \quad \alpha^1 = g^{11}\alpha_1 + g^{12}\alpha_2$$

$$= g^{11}\sqrt{g_{11}}\ \vec{t}_1 \cdot \vec{a}_{(1)} + g^{12}\sqrt{g_{22}}\ \vec{t}_1 \cdot \vec{a}_{(2)}$$

$$= g_{22}\sqrt{g_{11}}\ \vec{t}_1 \cdot \vec{a}_{(1)} - g_{12}\sqrt{g_{22}}\ \vec{t}_1 \cdot \vec{a}_{(2)} \tag{55}$$

SIMILARLY

$$\alpha^2 = g^{21}\alpha_1 + g^{22}\alpha_2$$

$$= -g_{12}\sqrt{g_{11}}\ \vec{t}_1 \cdot \vec{a}_{(1)} + g_{11}\sqrt{g_{22}}\ \vec{t}_1 \cdot \vec{a}_{(2)} \tag{56}$$

IN A SIMILAR FASHION WE GET

$$\mu^1 = g_{22}\sqrt{g_{11}}\ \vec{M}_t \cdot \vec{a}_{(1)} - g_{12}\sqrt{g_{22}}\ \vec{M}_t \cdot \vec{a}_{(2)} \tag{57}$$

$$\mu^2 = -g_{12}\sqrt{g_{11}}\ \vec{M}_t \cdot \vec{a}_{(1)} + g_{11}\sqrt{g_{22}}\ \vec{M}_t \cdot \vec{a}_{(2)} \tag{58}$$

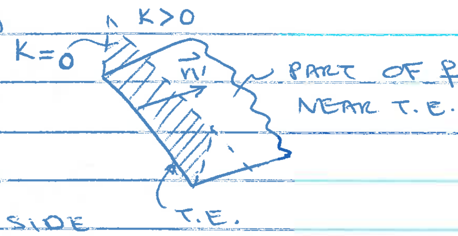
SUMMARIZING, WE HAVE SHOWN THAT

$$\nabla_4 \cdot (\tilde{p}\,|\nabla f|\,\vec{B}) = |\nabla f| \Big\{ \Big[\lambda \Big(\frac{1}{c}\frac{\partial \tilde{p}_B}{\partial \tau} - M_t \frac{\partial \tilde{p}_B}{\partial \sigma} \Big) + \lambda' \sin\theta \frac{\partial \tilde{p}_B}{\partial \sigma_1} \Big] + \frac{\tilde{p}}{\sqrt{g_{(2)}}} \frac{\bar{\partial}}{\partial u^i} \Big[\sqrt{g_{(2)}}\,\beta^i \Big] \Big\} \tag{59}$$

WE HAVE NOW USED THE NOTATION OF GENERALIZED DIFFERENTIATION $\bar{\partial}/\partial u^i$ TO PREPARE FOR THE NEXT STEP.

STEP 7 — DISCONTINUITY OF $\vec{B}$ AT T.E.

LET THE T.E. BE DESCRIBED BY THE INTERSECTION OF TWO SURFACES $f=0$ AND $K=0$. K IS DEFINED SUCH THAT IT IS POSITIVE ON THE SIDE THAT THE BLADE IS. WE WILL NOW CONSIDER ONLY THE UPPER SURFACE OF THE BLADE NEAR T.E.



$$\frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\,\beta^i\right] = \frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\,\beta^i\right] + \sqrt{g_{(2)}}\,\beta^i \frac{\partial K}{\partial u^i}\,\delta(K) \tag{60}$$

$$\beta^i \frac{\partial K}{\partial u^i} = \beta^1 \frac{\partial K}{\partial u^1} + \beta^2 \frac{\partial K}{\partial u^2} + \beta^3 \frac{\partial K}{\partial u^3}$$

$$\frac{\partial K}{\partial u^3} = \left.\frac{\partial K}{\partial u^3}\right|_{u_1,u_2} = 0 \quad \left\{ \begin{array}{l}\text{SURFACE K IS}\\ \text{ATTACHED TO}\\ \text{BLADE T.E.}\end{array}\right.$$

$$\beta^i \frac{\partial K}{\partial u^i} = \beta^1 \frac{\partial K}{\partial u^1} + \beta^2 \frac{\partial K}{\partial u^2}$$

$$= |\nabla K|\, \vec{B}_2 \cdot \vec{n}' \tag{61}$$

WHERE $\vec{B}_2 = (\beta^1, \beta^2)$ AND $\vec{n}'$ IS THE UNIT NORMAL TO THE SURFACE $K=0$ AND IS DEFINED SUCH THAT IT POINTS INTO THE REGION THAT THE BLADE SURFACE $f=0$ LIES (FIG. ABOVE). NOTE THAT $(\beta^1, \beta^2) = B_2$ IS WRITTEN IN TERMS OF SURFACE NATURAL BASE VECTORS ($\vec{a}_1$ AND $\vec{a}_2$) WHICH ARE

NOT IN GENERAL UNIT VECTORS. WE HAVE

$$\vec{B}_2 \cdot \vec{n}' = \beta^1 n'_1 + \beta^2 n'_2$$

$$= \beta^1 (g_{11} n'^1 + g_{12} n'^2)$$

$$+ \beta^2 (g_{21} n'^1 + g_{22} n'^2) \quad (62)$$

WHERE $\vec{n}' = (n'^1, n'^2)$ IN TERMS OF $\vec{a}_1$ AND $\vec{a}_2$ BASE VECTORS. IT IS BETTER TO USE

$$n'_1 = \vec{n}' \cdot \vec{a}_1 = \sqrt{g_{11}}\, \vec{n}' \cdot \vec{a}_{(1)} \quad (63\text{-a})$$

$$n'_2 = \vec{n}' \cdot \vec{a}_2 = \sqrt{g_{22}}\, \vec{n}' \cdot \vec{a}_{(2)} \quad (63\text{-b})$$

IN CALCULATING n'_1 AND n'_2 INSTEAD OF THE EXPRESSIONS IN PARENTHESES IN EQ (62).

WE THEREFORE HAVE

$$\frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\,\beta^i\right] = \frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\,\beta^i\right]$$

$$+ \sum_{u,l} \sqrt{g_{(2)}}\, |\nabla K|\, \vec{B}_2 \cdot \vec{n}'\, \delta(K) \quad (64)$$

THIS LAST TERM WILL GIVE US A LINE INTEGRAL ON THE TRAILING EDGE IN THE FINAL RESULT. WE HAVE A SIMILAR δ-FN FOR LOWER SURFACE WHICH WE WILL DISCUSS, AND IS ADDED TO EQ. (64) USING SUMMATION NOTATION.

STEP 8 PUTTING IT ALL TOGETHER

WE ARE NOW READY TO WRITE THE FINAL EXPRESSION FOR $p'(\vec{x},t)$. WE WRITE

$$4\pi p'(\vec{x},t) = I_N + I_{F1} + I_{F2} \qquad (65)$$

WHERE

$$I_N = \int \frac{\tilde{p}\,|\nabla f|\,(\cos\theta - M_n)}{r^2(\Lambda^2 + \sin^2\theta)}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau \qquad (66)$$

$$I_{F1} = \int \frac{|\nabla f|}{r}\Big[\lambda\Big(\frac{1}{c}\frac{\partial \tilde{p}_B}{\partial \tau} - M_t \frac{\partial \tilde{p}_B}{\partial \sigma}\Big) + \lambda' \frac{\partial \tilde{p}_B}{\partial \sigma_1}\sin\theta$$
$$+ \frac{\tilde{p}}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}\big[\sqrt{g_{(2)}}\,\beta^i\big]\Big]\delta(f)\,\delta(g)\,d\vec{y}\,d\tau \qquad (67)$$

$$I_{F2} = \int \sum_{u,l} \frac{\tilde{p}}{r}\,|\nabla f|\,|\nabla k|\,\vec{B}_2\cdot\vec{n}'\,\delta(f)\,\delta(k)\,\delta(g)\,d\vec{y}\,d\tau \qquad (68)$$

OUR IMMEDIATE TASK IS THE INTERPRETATION OF THE DELTA FUNCTIONS.

i) FOR INTERPRETATION OF EQS. (66) AND (67), WE CONSIDER

$$I = \int Q\,|\nabla f|\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau \qquad (69)$$

WE LET $(y_1, y_2) \rightarrow (f, g)$

$$d\vec{y} = \frac{df\,dg\,dy_3}{\left|\frac{\partial(f,g)}{\partial(y_1,y_2)}\right|}$$

$$= \frac{c\,df\,dg\,d\Gamma}{|\nabla f|\sin\theta}$$

$$\therefore \quad I = \int \frac{CQ}{\sin\theta} \, d\Gamma \, d\tau \qquad (70)$$

WHERE $d\Gamma$ IS THE ELEMENT OF THE CURVE OF INTERSECTION OF THE COLLAPSING SPHERE $g = 0$ WITH THE BLADE SURFACE.

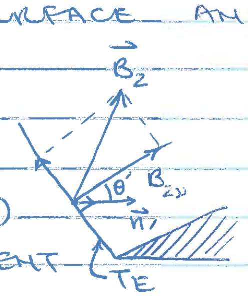
ii) FOR INTERPRETATION OF EQ. (68), WE FIRST WRITE THE BLADE SURFACE VECTOR AS

$$\vec{B}_2 = \vec{B}_{2t} + \vec{B}_{2\nu} \qquad (71)$$

WHERE $\vec{B}_{2t}$ IS PARALLEL TO THE T.E. AND $\vec{B}_{2\nu}$ IS NORMAL TO THE TRAILING EDGE. NOW LET $\vec{\nu}$ BE THE UNIT INWARD NORMAL TO THE T.E. LYING ON THE BLADE SURFACE AND LET $\vec{B}_{2\nu} = B_{2\nu} \vec{\nu}$, THEN

$$\vec{B}_2 \cdot \vec{n}' = B_{2\nu} \, \vec{\nu} \cdot \vec{n}'$$

$$= B_{2\nu} \cos\theta' \qquad (72)$$



WE NOTE THAT θ' IS THE COMPLEMENT OF THE ANGLE BETWEEN ∇f AND ∇K. THE REASON FOR ALL THIS BECOMES OBVIOUS BELOW. WE WRITE

$$I_{F2} = \int \sum_{u,l} \frac{\tilde{p} B_{2\nu}}{r} |\nabla f| |\nabla K| \cos\theta' \, \delta(f) \delta(K) \delta(g) \, d\vec{y} \, d\tau \qquad (73)$$

WE CONSIDER THE UPPER SURFACE NOW AND STUDY

$$I = \int Q |\nabla f| |\nabla K| \cos\theta' \, \delta(f) \delta(K) \delta(g) \, d\vec{y} \, d\tau \qquad (74)$$

WE LET $\vec{y} \rightarrow (f, g, K)$

WE HAVE

$$d\vec{y} = \frac{df\,dg\,dk}{\left|\frac{\partial(f,g,k)}{\partial(y_1,y_2,y_3)}\right|} \tag{75}$$

$$\left|\frac{\partial(f,g,k)}{\partial(y_1,y_2,y_3)}\right| = |\nabla f \times \nabla k \cdot \nabla g| \tag{76}$$

$$= \frac{1}{c}|\nabla f||\nabla k||\cos\theta' \cos\psi|$$

WHERE ψ IS THE ANGLE BETWEEN $\vec{F}$ AND THE T.E. WE THUS HAVE

$$I = \int \frac{CQ}{|\cos\psi|}\,d\tau \tag{77}$$

THE INTEGRATION IS OVER THE TIME INTERVAL THAT THE COLLAPSING SPHERE INTERSECTS THE TRAILING EDGE. WE NOTE THAT WE HAVE DEFINED $\vec{n}'$ SO THAT $\cos\theta' > 0$ ALWAYS BUT $\cos\psi$ CAN BE POSITIVE OR NEGATIVE. THE ABSOLUTE VALUE ON $\cos\psi$ IS, THEREFORE, NECESSARY. NOTE THAT THE FINAL FORM OF I IS INDEPENDENT OF THE WAY WE REPRESENT THE T.E AS THE INTERSECTION OF TWO SURFACES $f=0$ AND $k=0$. THIS IS AGAIN VERY SATISFACTORY. AS SEEN ABOVE, THERE IS NO DISTINCTION BETWEEN THE UPPER AND LOWER SURFACE CONTRIBUTIONS TO THE LINE INTEGRAL OTHER THAT THE FACT THAT B_{2D} FOR THE UPPER AND LOWER SURFACES MUST BE USED SEPARATELY. WE THUS HAVE

$$I = \int \frac{C(Q_u + Q_\ell)}{|\cos\psi|}\,d\tau \tag{78}$$

WHERE $Q_u = \dfrac{\tilde{p}(B_{2\nu})_u}{r}$, $Q_\ell = \dfrac{\tilde{p}(B_{2\nu})_\ell}{r}$.

OUR TASK IS NOT FINISHED YET SINCE THE CONDITION $\cos\psi = 0$ APPEARS OFTEN. THIS HAPPENS WHEN THE COLLAPSING SPHERE IS TANGENT TO THE TRAILING EDGE. WE SHOULD, THEREFORE, USE ANOTHER FORMULATION. TO INTEGRATE EQ. (74), WE FIRST LET $\tau \to g$, SINCE $\partial g/\partial\tau = 1$, WE HAVE

$$I = \int [Q\,|\nabla f|\,|\nabla K| \cos\theta']_{ret}\, \delta(F)\,\delta(K)\, d\vec{y} \qquad (79)$$

WHERE $F = [f]_{ret}$, $K = [K]_{ret}$. TO INTEGRATE THE PRODUCT OF THE DELTA FUNCTIONS, WE LET $(y_1, y_2) \to (F, K)$. WE HAVE

$$d\vec{y} = \frac{dF\,dK\,dy_3}{\dfrac{\partial(F,K)}{\partial(y_1,y_2)}}$$

$$= \frac{d\Gamma\,dK\,d\gamma}{|\nabla F \times \nabla K|} \qquad (80)$$

~~THE CURVE γ WILL BE EXPLAINED PHYSICALLY SOON.~~
WE HAVE

$$\nabla F = [|\nabla f|\,(\vec{n} - M_n \hat{r})]_{ret} \qquad (81\text{-}a)$$

$$\nabla K = [|\nabla K|\,(\vec{n}' - M_{n'} \hat{r})]_{ret} \qquad (81\text{-}b)$$

WHERE $M_{n'} = \vec{M}\cdot\vec{n}'$, $\vec{n}'$ BEING THE UNIT VECTOR ALONG ∇K. $\Rightarrow$

$$\nabla F \times \nabla K = [|\nabla f|\,|\nabla K|\,(\vec{n} - M_n \hat{r}) \times (\vec{n}' - M_n \hat{r})]_{ret} \qquad (82)$$

WE FIRST CONSIDER THE CROSS PRODUCT

$$\vec{E} = (\vec{n} - M_n\vec{\hat{r}}) \times (\vec{n}' - M_{n'}\vec{\hat{r}}) = \cos\theta'\,\vec{\Gamma} + (M_n\vec{n}' - M_{n'}\vec{n}) \times \vec{\hat{r}} \quad (83)$$

LET US DECOMPOSE $\vec{\hat{r}}$ INTO A VECTOR PARALLEL TO $\vec{\Gamma}$ (WHICH IS THE UNIT VECTOR TANGENT TO T.E. CURVE ALONG $\vec{n} \times \vec{n}'$) AND ONE IN THE PLANE NORMAL TO $\vec{\Gamma}$:

$$\vec{\hat{r}} = \cos\psi\,\vec{\Gamma} + \sin\psi\,\vec{\hat{r}}_p \quad (84)$$

WHERE $\vec{\hat{r}}_p$ IS THE UNIT VECTOR ALONG THE PROJECTION OF $\vec{\hat{r}}$ ON THE PLANE NORMAL TO THE T.E. WE NOW DEFINE THE COVARIANT BASE VECTORS $(\vec{e}_1, \vec{e}_2, \vec{e}_3) = (\vec{n}, \vec{n}', \vec{\Gamma})$. THE CONTRAVARIENT BASE VECTORS ARE $(\vec{e}^1, \vec{e}^2, \vec{e}^3)$

$$\vec{e}^1 = \frac{\vec{e}_2 \times \vec{e}_3}{(\vec{e}_1 \times \vec{e}_2 \cdot \vec{e}_3)} = \frac{\vec{n}' \times \vec{\Gamma}}{\cos\theta'}$$

$$\vec{e}^2 = \frac{\vec{\Gamma} \times \vec{n}}{\cos\theta'}$$

$$\vec{e}^3 = \frac{\vec{n} \times \vec{n}'}{\cos\theta'} = \vec{\Gamma}$$

LET $\vec{\hat{r}}_1 = \alpha^1\vec{e}_1 + \alpha^2\vec{e}_2 \therefore \vec{\hat{r}}$ CAN BE WRITTEN AS (COVARIENT COMPONENTS)

$$\vec{\hat{r}} = (\alpha^1\sin\psi, \alpha^2\sin\psi, \cos\psi)$$

$$M_n = \vec{M} \cdot \vec{n} \equiv M_1$$

$$M_{n'} = \vec{M} \cdot \vec{n}' \equiv M_2$$

$$\vec{E} = \cos\theta' \, \vec{e}^{\,3} + (M_1 \vec{e}_2 - M_2 \vec{e}_1) \times (\alpha^1 \sin\psi, \alpha^2 \sin\psi, \cos\psi)$$

$$= \cos\theta' \, \vec{e}^{\,3} + \cos\theta' \left[M_1 \cos\psi, M_2 \cos\psi, -(M_1\alpha^1 + M_2\alpha^2)\sin\psi \right]$$

(COVARIANT COMPONENTS) (*)

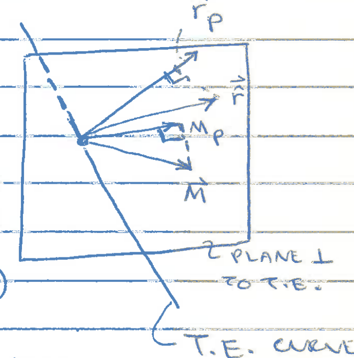
$$= \cos\theta' \left[M_1 \cos\psi, M_2 \cos\psi, 1 - \vec{M}_p \cdot \hat{\vec{r}}_p \sin\psi \right] \qquad (85)$$

WHERE

$$\vec{M}_p \cdot \hat{\vec{r}}_p = M_1 \alpha^1 + M_2 \alpha^2 \qquad (86)$$

PHYSICALLY, THIS IS THE DOT PRODUCT OF PROJECTION OF $\vec{M}$ ON THE PLANE (LOCAL) NORMAL TO THE T.E. WITH THE UNIT VECTOR ALONG THE PROJECTION OF $\hat{\vec{r}}$ ON THIS PLANE AS SHOWN BELOW.

WE HAVE



$$|\vec{E}|^2 = \vec{E} \cdot \vec{E}$$

$$= \cos^2\theta' \left[M_p^2 \cos^2\psi + (1 - \vec{M}_p \cdot \hat{\vec{r}} \sin\psi)^2 \right] \qquad (87)$$

(TO GET THIS, CONTRAVARIANT COMPONENTS OF $\vec{E}$ MUST BE CONSTRUCTED).

WE USE THIS IN EQ. (82) AND THEN (80), AND FINALLY EQ. (79) WHICH GIVES

$$I = \iint_{\substack{F=0 \\ K=0}} \left\{ \frac{Q}{\left[M_p^2 \cos^2\psi + (1 - \vec{M}_p \cdot \hat{\vec{r}}_p \sin\psi)^2 \right]^{1/2}} \right\}_{ret} d\gamma \qquad (88)$$

NOW WHEN $\cos\psi = 0 \Rightarrow \sin\psi = 1$, $\hat{\vec{r}}_p = \hat{\vec{r}}$. IF $\vec{M}_p \cdot \hat{\vec{r}} = 1$, WE HAVE A SINGULARITY. HOWEVER, THIS CONDITION RARELY OCCURS SINCE THE T.E. CURVE OF HIGH-SPEED PROPELLERS IS DESIGNED TO HAVE SUBSONIC SPEED NORMAL TO IT, i.e. $\vec{M}_p \cdot \hat{\vec{r}} < 1$ WHEN $\cos\psi = 0$.

(*) NOTE $\vec{M}_p \cdot \hat{\vec{r}}_p \sin\psi = \vec{M}_p \cdot \hat{\vec{r}}$

THE γ-CURVE IS THE CURVE GENERATED IN SPACE BY THE INTERSECTION OF THE T.E. CURVE WITH THE COLLAPSING SPHERE. WE SEE THAT EQ. (88) IS ONCE AGAIN INDEPENDENT OF THE WAY WE REPRESENT THE T.E. CURVE. IN FACT THE RESULT WOULD NOT BE ACCEPTABLE IF IT WAS NOT SO. AGAIN WE SHOULD WRITE EQ. (88) AS FOLLOWS:

$$I = \int_{\substack{F=0\\K=0}} \left\{ \frac{Q_u + Q_\ell}{\left[M_p^2 \cos^2\psi + (1 - \vec{M}_p \cdot \vec{r}_p \sin\psi)^2 \right]^{1/2}} \right\}_{ret} d\gamma \qquad (89)$$

STEP 9 – DETAILS OF CALCULATION OF I_{F1}

WE NOW GIVE THE DETAILS OF CALCULATION OF THE INTEGRAND OF I_{F1}. SPECIFICALLY, WE CONCENTRATE ON

$$E = \frac{1}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \left[\sqrt{g_{(2)}}\, \beta^i \right]$$

$$= \frac{\beta^i}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \sqrt{g_{(2)}} + \frac{\partial \beta^i}{\partial u^i} \qquad (90)$$

$$= \frac{1}{\sqrt{g_{(2)}}} \sum_{i=1}^{2} \beta^i \frac{\partial}{\partial u^i} \sqrt{g_{(2)}} + \frac{\partial \beta^i}{\partial u^i} \qquad (90)$$

FROM EQ. (48), β^i IS GIVEN AS

$$\beta^i = (\lambda^1 \alpha^1 - \lambda \mu^1, \lambda^1 \alpha^2 - \lambda \mu^2, \lambda, 0)$$

WE CAN RELATE $\frac{1}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \sqrt{g_{(2)}} \left(= \frac{1}{2} \frac{\partial}{\partial u^i} (\ln g_{(2)}) \right)$

SEE P 184, PROB. 7, A. GOETZ "INTRO. TO DIFF. GEOM."

TO CHRISTOFFEL SYMBOLS WHICH INVOLVE DERIVATIES OF g_{ij}. HOWEVER, IN THE GEOMETRY MODULE OF NASA PROPELLER PREDICTION PROGRAM, $g_{(2)}$ IS CALCULATED AS

$$g_{(2)} = \left| \frac{\partial \vec{r}}{\partial \xi_1} \times \frac{\partial \vec{r}}{\partial \xi_2} \right|^2 \qquad (91)$$

WHERE $\vec{r}$ IS THE POSITION VECTOR OF THE BLADE SURFACE AND $\xi_1 \equiv u^1$, $\xi_2 \equiv u^2$ ARE THE PARAMETERS TO REPRESENT THE SURFACE. THEREFORE, BECAUSE OF THE WELL-BEHAVED PROPERTY OF $g_{(2)}$, NUMERICAL EVALUATION OF $\frac{\partial}{\partial u^i} \sqrt{g_{(2)}}$ IS RECOMMENDED. WE NOTE THAT THESE QUANTITIES INVOLVE SURFACE CURVATURES. WE WOULD BYPASS EXPLICIT CALCULATION OF THESE CURVATURES BY NUMERICAL DIFFERENTIATION OF $\sqrt{g_{(2)}}$. WE SHOULD THEREFORE CONCENTRATE ON $\partial \beta^i / \partial u^i$. WE FIRST DISPOSE OF $\partial \beta^3 / \partial u^3 = \frac{1}{c} \partial \beta^3 / \partial \tau$ REMEMBERING THAT (u^1, u^2) ARE FIXED.

$$\frac{\partial \beta^3}{\partial \tau} = \frac{\partial}{\partial \tau} \left[\frac{M_n - \cos\theta}{\Lambda^2 + \sin^2\theta} \right] \qquad (92)$$

WE HAVE FROM $\beta_3 = \lambda$ (93)

$$\frac{\partial \beta^3}{\partial \tau} = \frac{\partial \lambda}{\partial M_n} \dot{M}_n + \frac{\partial \lambda}{\partial \theta} \dot{\theta} \qquad (94)$$

WE CALCULATE $\partial \lambda / \partial M_n$ AND $\partial \lambda / \partial \theta$ BELOW.

WE HAVE DEFINED $\dot{M}_n = \partial M_n / \partial \tau$ (FIXED u^1 AND u^2, i.e. WRT AN OBSERVER FIXED TO THE BLADE)

FROM $\cos\theta = \frac{\vec{r}\cdot\vec{n}}{r}$, WE GET

$$-\frac{1}{c}\sin\theta\,\dot{\theta} = \frac{-\vec{M}\cdot\vec{n}}{r} + \frac{\cos\theta\, M_r}{r}$$

$$= \frac{M_r\cos\theta - M_n}{r} \qquad (95)$$

WE GET FURTHER NEAR-FIELD TERMS FROM $\frac{\partial\beta^i}{\partial u^i}$ (FROM $\dot{\theta}$ TERM). WE NEED TO ADD THESE TERMS TO WHAT WE CALLED I_N. WE HAVE

$$\sin\theta\,\dot{\theta} = \frac{c\,(M_n - M_r\cos\theta)}{r} \qquad (96)$$

(NOTE: $M_n - M_r\cos\theta = (M_n\sin^2\theta - M_{tr}\cos\theta)$, $M_{tr} = \vec{M}_t\cdot\vec{\varepsilon}_1$)

FROM EQS (104) AND (105), WE GET

$$\frac{\partial\beta^3}{\partial u^3} = \frac{2\sin^2\theta - (M_n - \cos\theta)^2}{c(\Lambda^2 + \sin^2\theta)^2}\dot{M}_n$$

$$+ \frac{(M_n - M_r\cos\theta)\left[2(1-M_n^2) + (M_n - \cos\theta)^2\right]}{r(\Lambda^2 + \sin^2\theta)^2} \qquad (97)$$

OUR CHOICE OF GROUPING OF VARIABLES HERE IS FOR FUTURE ANALYSIS OF SINGULARITIES OCCURING AT POINTS WHERE $M_n = 1$ AND $\theta = 0$. AS WRITTEN ABOVE, EQ. (97) MAY NOT BE THE BEST FORM FOR USE ON A COMPUTER.

HERE, WE WRITE THE INTERMEDIATE ALGEBRAIC STEPS FOR FUTURE CHECKING.

NOW WE WORK ON $\partial\beta^1/\partial u^1$ AND $\partial\beta^2/\partial u^2$.

$$\beta^1 = \lambda'\alpha^1 - \lambda\mu^1 \qquad (98\text{-a})$$

$$\beta^2 = \lambda'\alpha^2 - \lambda\mu^2 \qquad (98\text{-b})$$

$$\lambda^1 = \frac{M_n + \cos\theta}{\Lambda^2 + \sin^2\theta} \qquad (99\text{-}a)$$

$$\lambda_1 = \frac{M_n - \cos\theta}{\Lambda^2 + \sin^2\theta} \qquad (99\text{-}b)$$

WE CAN EASILY SHOW THAT

$$\frac{\partial \beta^i}{\partial u^i} = \frac{\partial \lambda^1}{\partial u^i}\alpha^i + \lambda^1 \frac{\partial \alpha^i}{\partial u^i} - \frac{\partial \lambda}{\partial u^i}\mu^i - \lambda\frac{\partial \mu^i}{\partial u^i}$$

(NO SUM ON i) (100)

$$\begin{cases} \dfrac{\partial \lambda^1}{\partial u^i} = \dfrac{\partial \lambda^1}{\partial M_n}\dfrac{\partial M_n}{\partial u^i} + \dfrac{\partial \lambda^1}{\partial \theta}\dfrac{\partial \theta}{\partial u^i} & (101\text{-}a) \\ & i = 1, 2 \\ \dfrac{\partial \lambda}{\partial u^i} = \dfrac{\partial \lambda}{\partial M_n}\dfrac{\partial M_n}{\partial u^i} + \dfrac{\partial \lambda}{\partial \theta}\dfrac{\partial \theta}{\partial u^i} & (101\text{-}b) \end{cases}$$

WE THEREFORE REQUIRE :

GROUP 1 : $\dfrac{\partial \lambda^1}{\partial M_n}$, $\dfrac{\partial \lambda^1}{\partial \theta}$, $\dfrac{\partial \lambda}{\partial M_n}$, $\dfrac{\partial \lambda}{\partial \theta}$

GROUP 2 : $\dfrac{\partial M_n}{\partial u^i}$, $\dfrac{\partial \theta}{\partial u^i}$ $\quad i = 1, 2$

GROUP. 3 : $\dfrac{\partial \alpha^i}{\partial u^i}$, $\dfrac{\partial \mu^i}{\partial u^i}$ $\quad i = 1, 2$ (NO SUM ON i)

GROUP 1 QUANTITIES

$$\frac{\partial \lambda^1}{\partial M_n} = \frac{2(1 - M_n^2) + (M_n - \cos\theta)^2}{(\Lambda^2 + \sin^2\theta)^2} \checkmark \qquad (102)$$

$$\frac{\partial \lambda^1}{\partial \theta} = -\frac{[2\alpha_n^2 + (M_n + \cos\theta)^2]\sin\theta}{(\Lambda^2 + \sin^2\theta)^2} \checkmark \qquad (103)$$

MINUS SIGN

$$\frac{\partial \lambda}{\partial M_n} = \frac{3\sin^2\theta - \Lambda^2}{(\Lambda^2 + \sin^2\theta)^2} = \frac{2\sin^2\theta - (M_n - \cos\theta)^2}{(\Lambda^2+\sin^2\theta)^2} \qquad (104)$$

$$\frac{\partial \lambda}{\partial \theta} = \frac{\sin\theta\left[2(1-M_n^2) + (M_n - \cos\theta)^2\right]}{(\Lambda^2 + \sin^2\theta)^2} \checkmark \qquad (105)$$

GROUP 2 — $i = 1, 2$ THROUGHOUT

$$M_n = \vec{M}\cdot\vec{n}$$

$$\frac{\partial M_n}{\partial u^i} = \frac{\partial \vec{M}}{\partial u^i}\cdot\vec{n} + \vec{M}\cdot\frac{\partial \vec{n}}{\partial u^i}$$

$$\vec{M} = [\vec{V}_0(\tau) + \vec{\omega}\times\vec{\tau}\,]/c$$ ERROR CORRECTED ON P 90) (*)

$$\frac{\partial \vec{M}}{\partial u^i} = \vec{\omega}\times\frac{\partial\vec{\tau}}{\partial u^i} = \vec{\omega}\times\vec{a}_i \;/c \qquad (105\text{-}a)$$

$$\frac{\partial \vec{n}}{\partial u^i} = -\,b_{ij}\,g^{jk}\vec{a}_k$$ (WEINGARTEN'S FORMULA) (105-b)

WHERE

$$b_{ij} = \frac{\partial^2\vec{\tau}}{\partial u^i\partial u^j}\cdot\vec{n}$$ (COEFF. OF 2ND FUND. FORM)

$$\therefore \frac{\partial M_n}{\partial u^i} = \frac{1}{c}\,\vec{\omega}\times\vec{a}_i\cdot\vec{n} - b_{ij}\,g^{jk}M_k \checkmark \qquad (106)$$

WHERE $M_k = \vec{M}\cdot\vec{a}_k \equiv \mu_k$

$$\cos\theta = \vec{n}\cdot\hat{r} = \frac{\vec{n}\cdot\vec{r}}{r}$$

$$-\sin\theta\,\frac{\partial\theta}{\partial u^i} = \hat{r}\cdot\frac{\partial \vec{n}}{\partial u^i} + \vec{n}\cdot\frac{\partial\hat{r}}{\partial u^i}$$

$$\frac{\partial \hat{r}}{\partial u^i} = \frac{-\vec{a}_i}{r} + \frac{\hat{r}\cdot\vec{a}_i}{r}\hat{r} = \frac{(\hat{r}\cdot\vec{a}_i)\hat{r} - \vec{a}_i}{r} \qquad (107)$$

(*) MARK DUNN FOUND THIS ERROR WHEN HE CHECKED MY DERIVATION OF FORMULATION 3.

$$\frac{\partial \theta}{\partial u^i} = -\frac{1}{\sin\theta}\left[-b_{ij}\,g^{jk}\,\hat{r}_K + \frac{\vec{\hat{r}}\cdot\vec{a}_i\cos\theta}{r}\right]$$

$$= \frac{1}{\sin\theta}\left[b_{ij}\,g^{jk}\,\hat{r}_K - \frac{\hat{r}_i}{r}\cos\theta\right]$$

(SUM ON j & K)

NOTE: $\hat{r}_K = \vec{\hat{r}}\cdot\vec{a}_K = \alpha_K$, NOT CARTESIAN COMPONENTS! (108)

GROUP 3

$$\alpha^i = g^{ij}\,\alpha_j$$

$$\alpha_j = \vec{t}_1\cdot\vec{a}_j$$

$$\vec{t}_1 = \vec{\hat{r}} - \cos\theta\,\vec{n}$$

$$\frac{\partial \alpha^i}{\partial u^i} = \frac{\partial g^{ij}}{\partial u^i}\,\alpha_j + g^{ij}\frac{\partial \alpha_j}{\partial u^i} \qquad (109)$$

(NO SUM ON i)

$$\frac{\partial \alpha_j}{\partial u^i} = \frac{\partial \vec{t}_1}{\partial u^i}\cdot\vec{a}_j + \vec{t}_1\cdot\frac{\partial \vec{a}_j}{\partial u^i} \qquad (110)$$

FROM (108):

$$\frac{\partial \vec{t}_1}{\partial u^i} = \frac{\partial \vec{\hat{r}}}{\partial u^i} + \sin\theta\,\frac{\partial \theta}{\partial u^i}\,\vec{n} - \cos\theta\,\frac{\partial \vec{n}}{\partial u^i}$$

FROM THIS, BY TAKING DOT PRODUCT WITH $\vec{a}_j$, WE GET

$$\frac{\partial \vec{t}_1}{\partial u^i}\cdot\vec{a}_j = \frac{\partial \vec{\hat{r}}}{\partial u^i}\cdot\vec{a}_j - \cos\theta\,\frac{\partial \vec{n}}{\partial u^i}\cdot\vec{a}_j$$

USING WEINGARTEN'S EQUATION AND EQ. (107), WE GET

$$\frac{\partial \vec{t}_i}{\partial u^i} \cdot \vec{a}_j = \frac{\hat{r}_i \hat{r}_j - g_{ij}}{r} + \cos\theta \, b_{il} \delta^l_j \quad \text{(SUM ON } l\text{)}$$

$$= \frac{\hat{r}_i \hat{r}_j - g_{ij}}{r} + \cos\theta \, b_{ij} \qquad (112)$$

δ^l_j = KRONECKER DELTA

$$= \begin{cases} 1 & l = j \\ 0 & l \neq j \end{cases}$$

WE HAVE USED THE FACT THAT $g^{lk} g_{kj} = \delta^l_j$.

WE HAVE

$$\vec{t}_i \cdot \frac{\partial \vec{a}_j}{\partial u^i} = \alpha^k \vec{a}_k \cdot \frac{\partial \vec{a}_j}{\partial u^i}$$

$$= \alpha^k \Gamma_{ijk} \qquad \text{(SUM ON K)}$$

$$= \frac{1}{2} \alpha^k \left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right) \qquad (113)$$

(SUM ON K)

WE NOW HAVE ALL TERMS IN EQ. (109) TO CALCULATE $\partial \alpha^i / \partial u^i$. IN EQ. (113), Γ_{ijk} IS THE CHRISTOFFEL SYMBOL OF 1ST KIND. NOTE ALSO THAT IN EQ. (112)

TO CALCULATE $\partial \mu^i / \partial u^i$, WE AGAIN USE

$$\mu^i = g^{ij} \mu_j \qquad \text{(SUM ON } j\text{)}$$

$$\mu_j = \vec{M} \cdot \vec{a}_j$$

WE NEED TO DERIVE VERY FEW TERMS NOW. WE HAVE

$$\frac{\partial \mu^i}{\partial u^i} = \frac{\partial g^{ij}}{\partial u^i} \mu_j + g^{ij} \frac{\partial \mu_j}{\partial u^i} \qquad \text{(NO SUM ON } i\text{)}$$

$$\frac{\partial \mu_j}{\partial u^i} = \frac{\partial \vec{M}}{\partial u^i} \cdot \vec{a}_j + \vec{M} \cdot \frac{\partial \vec{a}_j}{\partial u^i}$$

WE HAVE CALCULATED $\partial \vec{M} / \partial u^i$ ON P.66. WE ALSO HAVE

$$\vec{M} \cdot \frac{\partial \vec{a}_j}{\partial u^i} = \mu^k \vec{a}_k \cdot \frac{\partial \vec{a}_j}{\partial u^i}$$

$$= \mu^k \Gamma_{ijk} \qquad \text{(SUM ON } k\text{)}$$

THEREFORE

$$\frac{\partial \mu^i}{\partial u^i} = \frac{\partial g^{ij}}{\partial u^i} \mu_j + \mu^k g^{ij} \Gamma_{ijk} + g^{ij} \vec{\omega} \times a_i \cdot \vec{a}_j \qquad (114)$$

(SUM ON j AND k ONLY)

(WRONG!) SEE P.

WE NOTE THAT WE NEED TO CALCULATE

i) COEFFICIENTS OF 1ST FUNDAMENTAL FORM g_{ij} ON THE SURFACE

ii) g^{ij} AND $g_{(2)} = \det(g_{ij})$ ON THE SURFACE

iii) COEFFICIENTS OF 2ND FUNDAMENTAL FORM ON THE SURFACE $b_{ij} = \frac{\partial^2 \vec{z}}{\partial u^i \partial u^j} \cdot \vec{n}$, $i = 1, 2$

iv) CHRISTOFFEL SYMBOLS OF 1ST KIND Γ_{ijk} WHICH ARE EIGHT DISTINCT TERMS.

WE CAN USE

$$\frac{1}{2 g_{(2)}} \frac{\partial g_{(2)}}{\partial u_i} = \Gamma_i{}^k{}_k = \Gamma_i{}^1{}_1 + \Gamma_i{}^2{}_2 \qquad (115)$$

(GOETZ, P 184)
BOOK RESULT IN ERROR !

JAN. 83
GPHM

STEP 9 REVISITED

AS A RESULT OF MESSY MANIPULATIONS IN STEP 9, MORE CARE IS NEEDED TO WRITE THE EXPRESSION FOR CALCULATION OF $p'(\vec{x},t)$ IN A SIMPLE FORM. WE MAKE A SIGN CHANGE IN DEFINITION OF λ. THIS MAKES THE EQUATIONS LOOK SOMEWHAT NICER. WE THEREFOR HAVE NOW

$$\lambda = \frac{\cos\theta - M_n}{\tilde{\Lambda}^2} \qquad (116-a)$$

$$\lambda_1 = \frac{\cos\theta + M_n}{\tilde{\Lambda}^2}$$ (WE MAKE SUPERSCRIPT ON λ' A SUBSCRIPT.)

(116-b)

$$\tilde{\Lambda}^2 = \Lambda^2 + \sin^2\theta \qquad (116-c)$$

THE REASON WE CHANGE SUPERSCRIPT OF λ' TO A SUBSCRIPT IS SO THAT WE WRITE $(\lambda')^2$ AS λ_1^2 WITHOUT PARENTHESES IN OUR EXPRESSIONS. ALSO λ' IS NOT USED AS A COMPONENT OF A CONTRAVARIANT VECTOR. WE SUMMARIZE OUR RESULTS SO FAR USING THE ABOVE NOTATION NOW:

$$4\pi p'(\vec{x},t) = \int \frac{\lambda |\nabla f| \tilde{p}}{r^2} \delta(f)\,\delta(g)\, d\vec{y}\, d\tau$$

$$+ \int \frac{|\nabla f|}{r}\left[\frac{\lambda}{c}\frac{\partial \tilde{p}_B}{\partial \tau} - \left(\lambda M_t \frac{\partial \tilde{p}_B}{\partial \sigma} + \lambda_1 \sin\theta \frac{\partial \tilde{p}_B}{\partial \sigma_1}\right)\right] \delta(f)\,\delta(g)\, d\vec{y}\, d\tau$$

$$\stackrel{\checkmark}{=} \int \frac{|\nabla f| \tilde{p}}{r\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\, \beta^i\right] \delta(f)\,\delta(g)\, d\vec{y}\, d\tau$$

$$\stackrel{\checkmark}{=} \int_{F=0,\,K=0} \left\{\frac{\tilde{p}\left[(B_{2u})_u + (B_{2v})_v\right]}{r\Lambda_0}\right\}_{ret} d\gamma \qquad (117)$$

WHERE NOW

$$\beta^i = (\lambda\mu^1 + \lambda_1\alpha^1, \lambda\mu^2 + \lambda_1\alpha^2, -\lambda, 0)$$ (118)

AND

$$\Lambda_0^2 = M_p^2\cos^2\psi + (1 - \vec{M}_p\cdot\hat{r}_p\sin\psi)^2$$ (119)

WE WILL ONCE AGAIN CONCENTRATE ON

$$E = \frac{1}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\,\beta^i\right]$$

$$= \frac{\beta_i}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}\left(\sqrt{g_{(2)}}\right) + \frac{\partial\beta^i}{\partial u^i} \qquad (i=1,2,3)$$ (120)

WE HAVE RIGID BLADES SO THAT $i=1,2$ IN THE FIRST TERM. ALSO WE WILL SHOW THAT

$$\frac{1}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}\left(\sqrt{g_{(2)}}\right) = \Gamma_{iK}^{K}$$ (115) $(K=1,2)$

(SEE NOTE 1, P 82). HERE Γ_{iK}^K IS CHRISTOFFEL SYMBOL OF 2ND KIND. THIS TERM WILL DROP OFF SOON. WE CONSIDER NOW $\partial\beta^i/\partial u^i$.

$$\sum_{i=1}^{3}\frac{\partial\beta^i}{\partial u^i} = \frac{1}{c}\frac{\partial\beta^3}{\partial\tau} + \frac{\partial\beta^i}{\partial u^i} \qquad i=1,2$$

$$\frac{1}{c}\frac{\partial\beta^3}{\partial\tau} = -\frac{1}{c}\frac{\partial\lambda}{\partial M_n}\dot{M}_n + \frac{1}{c}\frac{\partial\lambda}{\partial(\cos\theta)}\sin\theta\,\dot{\theta}$$

$$= -\frac{1}{c}\frac{\partial\lambda}{\partial M_n}\dot{M}_n + \frac{M_r\cos\theta - M_n}{cr}\frac{\partial\lambda}{\partial(\cos\theta)}$$

$$= -\frac{1}{c}\lambda_x\dot{M}_n + \frac{M_r\cos\theta - M_n}{cr}\lambda_y$$ (121)

HERE WE DEFINE $x = M_n$, $y = \cos\theta$ AND WRITE

$$\lambda_x = \frac{\partial \lambda}{\partial M_n}, \quad \lambda_y = \frac{\partial \lambda}{\partial(\cos\theta)} \qquad (122\text{-a})$$

ALSO
$$\lambda_{1x} = \frac{\partial \lambda_1}{\partial M_n}, \quad \lambda_{1y} = \frac{\partial \lambda_1}{\partial(\cos\theta)} \qquad (122\text{-b})$$

THIS NOTATION WILL HELP IN REDUCING CONFUSION. WE WILL DEVELOP RELATIONS BETWEEN THESE QUANTITIES AND λ AND λ_1. FOR NOW, WE WILL OBTAIN AN EXPRESSION FOR $\partial\beta^i/\partial u^i$, $i = 1, 2$

$$\frac{\partial \beta^i}{\partial u^i} = \frac{\partial}{\partial u^i}(-\lambda\mu^i + \lambda_1\alpha^i) \qquad \begin{matrix} i = 1, 2 \\ j = 1, 2 \end{matrix}$$

$$= \mu^i \frac{\partial \lambda}{\partial u^i} + \alpha^i \frac{\partial \lambda_1}{\partial u^i} + \lambda \frac{\partial \mu^i}{\partial u^i} + \lambda_1 \frac{\partial \alpha^i}{\partial u^i}$$

$$= \mu^i\left(-\lambda_x \frac{\partial M_n}{\partial u^i} - \lambda_y \sin\theta \frac{\partial \theta}{\partial u^i}\right)$$

$$+ \alpha^i\left(-\lambda_{1x} \frac{\partial M_n}{\partial u^i} - \lambda_{1y} \sin\theta \frac{\partial \theta}{\partial u^i}\right)$$

$$+ \lambda \frac{\partial \mu^i}{\partial u^i} + \lambda_1 \frac{\partial \alpha^i}{\partial u^i}$$

$$= \lambda_x\left(\tfrac{1}{c}\Omega_i\mu^i - M_t^2 K_n\right) \qquad \text{(NOTE 2, P 83)}$$

$$+ \lambda_y\left(\frac{M_{tr}}{r}\cos\theta - b_{ij}\alpha^i\mu^j\right) \qquad \text{(NOTE 3, P 84)}$$

$$+ \lambda_{1x}\left(\Omega_i\alpha^i - b_{ij}\alpha^i\mu^j\right) \qquad \text{(NOTE 2, P 83)}$$

$$+ \lambda_{1y}\left(\frac{\sin^2\theta\cos\theta}{r} - K_{n1}\sin^2\theta\right) \qquad \text{(NOTE 3, P 84)}$$

(CONT'D)

$$+ \lambda \left(2 M_n H - \mu^i \Gamma_i{}^j{}_j \right)$$

$$+ \lambda_1 \left(\frac{\sin^2\theta - 3}{r} + 2 H \cos\theta - \alpha^i \Gamma_i{}^k{}_k \right) \qquad (123)$$

(NOTE 4, P85)

WHERE $\Omega_i = (\vec{n} \times \vec{\omega}) \cdot \vec{a}_i / c$

$\Omega_i \alpha^i = \vec{\Omega} \cdot \vec{t}_1$

$\Omega_i \mu^i = \vec{\Omega} \cdot \vec{M}_t$

κ_n = NORMAL SURFACE CURVATURE ALONG $\vec{M}_t$

κ_{n1} = " " " " $\vec{t}_1$

H = MEAN CURVATURE

b_{ij} = COEFF. OF 2ND FUND. FORM

$M_{tr} = \vec{M}_t \cdot \vec{t}_1$

WE GROUP THE TERMS AS FOLLOWS. NOTE THAT $i, j = 1, 2$:

$$\sum_{i=1}^{3} \frac{\partial \beta^i}{\partial u^i} = \frac{1}{r} \left[(M_r \cos\theta - M_n + M_{tr}) \lambda_y + \lambda_{1y} \sin^2\theta \cos\theta + \lambda_1 (\sin^2\theta - 3) \right]$$

$$- \frac{1}{c} \lambda_x M_n + \lambda_x \Omega_i \mu^i + \lambda_{1x} \Omega_i \alpha^i$$

$$- \left(\lambda_x M_t^2 \kappa_n + \lambda_{1y} \kappa_{n1} \sin^2\theta \right)$$

$$- (-\lambda_y + \lambda_{1x}) b_{ij} \alpha^i \mu^j$$

$$- \beta^i \Gamma_i{}^j{}_j + 2H (\lambda M_n + \lambda_1 \cos\theta) \qquad (124)$$

WE NOTE THAT

$$M_r \cos\theta - M_n = (M_r - M_n \cos\theta)\cos\theta - M_n \sin^2\theta$$

ALSO $M_r - M_n \cos\theta = \vec{M} \cdot (\hat{r} - \vec{n} \cos\theta)$

$$= \vec{M}_t \cdot \vec{t}_1 = M_{tr}$$

$\therefore$ $M_r \cos\theta - M_n - M_{tr} = - M_n \sin^2\theta \qquad (125)$

WE CAN SHOW ALSO $\lambda_y = \lambda_{1x}$ SO THAT $\lambda_y + \lambda_{1x} = 2\lambda_y$.

WE NEED TO INTERPRET $b_{ij} \alpha^i \mu^j$. THIS IS DONE

IN NOTE 5, P88). THE RESULT IS

$$b_{ij}\alpha^i\mu^j = k_1\tilde{\alpha}^1\tilde{\mu}^1 + k_2\tilde{\alpha}^2\tilde{\mu}^2 \qquad (126)$$

WHERE k_1 AND k_2 ARE LOCAL PRINCIPAL CURVATURES AND

$$\tilde{\alpha}^i = \vec{t}_1 \cdot \tilde{a}^i \qquad (127\text{-a}) \qquad i=1,2$$

$$\tilde{\mu}^i = \vec{M}_t \cdot \tilde{a}^i . \qquad (127\text{-b})$$

HERE $\tilde{a}^i$, $i=1,2$ ARE UNIT BASE VECTORS ALONG PRINCIPAL DIRECTIONS, ORIENTATION IS IRRELEVANT BECAUSE OF APPEARANCE OF $\tilde{\alpha}^i\tilde{\mu}^i$. NOTE THAT ALL CURVATURES ARE SIGNED BY ASSUMING THAT THE NORMAL TO $f=0$ POINTS OUTWARD. WE HAVE SHOWN THAT

$$\sum_{i=1}^{3}\frac{\partial\beta^i}{\partial u^i} + \frac{\beta^i}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\,\beta^i\right]$$

$$= -\frac{1}{r}\left[(M_n\lambda_y - \lambda_{1y}\cos\theta - \lambda_1)\sin^2\theta + 3\lambda_1\right]$$

$$-\frac{1}{c}\lambda_x\dot{M}_n + \frac{1}{c}\Omega_i(\lambda_x\mu^i + \lambda_{1x}\alpha^i)$$

$$-(\lambda_x M_t^2 k_n + \lambda_{1y}k_{n1}\sin^2\theta)$$

$$-2\lambda_y(k_1\tilde{\alpha}^1\tilde{\mu}^1 + k_2\tilde{\alpha}^2\tilde{\mu}^2)$$

$$+2H(\lambda M_n + \lambda_1\cos\theta) \qquad (128)$$

NOW WE GO BACK TO EQ.(117) AND REWRITE IT AS FOLLOWS BY SEPARATING TERMS FOR FAR-FIELD AND NEAR FIELDS :

$$4\pi p'(\vec{x},t) = \int \frac{|\nabla f|\,\tilde{p}\,Q_N}{r^2}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

[SEE EQ. (145)]

$$+ \int \frac{|\nabla f|}{r}\left[\frac{\lambda}{c}\frac{\partial \tilde{p}_B}{\partial \tau} - \left(\lambda M_t \frac{\partial \tilde{p}_B}{\partial s} + \lambda_1 \sin\theta \frac{\partial \tilde{p}_B}{\partial s_1}\right)\right]$$

$$\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

[SEE EQ. (148)]

$$+ \int \frac{|\nabla f|\,\tilde{p}\,Q_F}{r}\,\delta(f)\,\delta(g)\,d\tau$$

$$- \int_{\substack{F=0\\K=0}} \left\{\frac{\tilde{p}\left[(B_{2p})_u + (B_{2p})_\ell\right]}{r\Lambda_0}\right\}_{ret} d\gamma \qquad (129)$$

WHERE ...

$$Q_N = \lambda + 3\lambda_1 + (M_n \lambda_y - \lambda_{1y}\cos\theta - \lambda_1)\sin^2\theta \qquad (130\text{-}a)$$

$$Q_F = \frac{1}{c}\left[\lambda_x \dot{M}_n - \Omega_i(\lambda_x \mu^i + \lambda_{1x}\alpha^i)\right]$$

$$+ (\lambda_x M_t^2 \kappa_n + \lambda_{1y}\kappa_{n1}\sin^2\theta)$$

$$+ 2\lambda_y(\kappa_1 \tilde{\alpha}^1 \tilde{\mu}^1 + \kappa_2 \tilde{\alpha}^2 \tilde{\mu}^2)$$

$$- 2H(\lambda M_n + \lambda_1 \cos\theta) \qquad (130\text{-}b)$$

WE NOW DEVELOP EXPRESSIONS FOR $\lambda_x, \lambda_y, \lambda_{1x}$ AND λ_{1y} IN TERMS OF λ AND λ_1.

$$\lambda_x = \lambda^2 - \frac{2\sin^2\theta}{\bar{\lambda}^4} \qquad \text{[FROM EQ. (104)]} \quad (131\text{-}a)$$

$$\lambda_y = \lambda^2 + \frac{2(1-M_n^2)}{\bar{\lambda}^4} \qquad \text{[FROM EQ. (105)]} \quad (131\text{-}b)$$

$$\lambda_{1x} = \lambda^2 + \frac{2(1-M_n^2)}{\bar{\lambda}^4} \qquad \text{[FROM (102)]} \quad (131\text{-}c)$$

$$\lambda_{1y} = \lambda_1^2 + \frac{2(1+M_n^2)}{\bar{\lambda}^4} \qquad \text{[FROM (103)]} \quad (131\text{-}d)$$

WE HAVE

$$\tilde{\Lambda}^2 = (M_n - \cos\theta)^2 + 2\sin^2\theta$$

$$\therefore \quad \frac{1}{\tilde{\Lambda}^2} = \lambda^2 + \frac{2\sin^2\theta}{\tilde{\Lambda}^4}$$

or
$$\frac{2\sin^2\theta}{\tilde{\Lambda}^4} = \frac{1}{\tilde{\Lambda}^2} - \lambda^2 \qquad (132)$$

WE FORM OTHER COMBINATIONS OF λ AND λ_1 TO GET THE TERMS WE NEED IN EQ. (131-a, d)

$$\lambda^2 + \lambda_1^2 = \frac{2(1+M_n^2)}{\tilde{\Lambda}^4} - \frac{2\sin^2\theta}{\tilde{\Lambda}^4}$$

$$\therefore \quad \frac{2(1+M_n^2)}{\tilde{\Lambda}^4} = \lambda^2 + \lambda_1^2 + \frac{2\sin^2\theta}{\tilde{\Lambda}^4}$$

$$= \lambda_1^2 + \frac{1}{\tilde{\Lambda}^2} \qquad (133)$$

ALSO
$$\lambda\lambda_1 = \frac{\cos^2\theta - M_n^2}{\tilde{\Lambda}^4}$$

$$= \frac{1 - M_n^2}{\tilde{\Lambda}^4} - \frac{\sin^2\theta}{\tilde{\Lambda}^4}$$

$$\therefore \quad \frac{2(1-M_n^2)}{\tilde{\Lambda}^4} = 2\lambda\lambda_1 - \lambda^2 + \frac{1}{\tilde{\Lambda}^2} \qquad (134)$$

USING EQS. (132-134) IN EQ. (131), WE GET

$$\lambda_x = 2\lambda^2 - \frac{1}{\tilde{\Lambda}^2} \qquad (135\text{-a})$$

$$\lambda_y = \lambda_{1x} = 2\lambda\lambda_1 + \frac{1}{\tilde{\Lambda}^2} \qquad (135\text{-b})$$

$$\lambda_{1y} = 2\lambda_1^2 + \frac{1}{\tilde{\Lambda}^2} \qquad (135\text{-c})$$

NOW WE GET A LOT OF SIMPLIFICATION IN Q_N

$$Q_N = \lambda + 3\lambda_1 + \left[M_n\left(2\lambda\lambda_1 + \frac{1}{\tilde{\Lambda}^2}\right) - \left(2\lambda_1^2 + \frac{1}{\tilde{\Lambda}^2}\right)\cos\theta - \lambda_1\right]\sin^2\theta \qquad (136\text{-a})$$

GO TO p 82 NOW

$$= \lambda + (\lambda_1 - \lambda_1 \sin^2\theta) + 2\lambda_1 + \left[2M_n\lambda\lambda_1 - 2\lambda_1^2\cos\theta - \lambda_1\right]\sin^2\theta$$

$$= (\lambda + \lambda_1)\cos^2\theta + 2\lambda_1\left[1 + (M_n\lambda - \lambda_1\cos\theta)\right]\sin^2\theta + 2\lambda_1\cos^2\theta$$

$$= \frac{2\cos^3\theta}{\tilde{\Lambda}^2} + 2\lambda_1\left[1 - \frac{M_n^2 + \cos^2\theta}{\tilde{\Lambda}^2}\right]\sin^2\theta + 2\lambda_1\cos^2\theta \qquad (136)$$

WE HAVE

$$1 - \frac{M_n^2 + \cos^2\theta}{\tilde{\Lambda}^2} = \frac{2 - 2M_n\cos\theta - 2\cos^2\theta}{\tilde{\Lambda}^2}$$

$$= 2\,\frac{1 + M_n^2}{\tilde{\Lambda}^2} - 2\,\frac{M_n^2 + M_n\cos\theta + \cos^2\theta}{\tilde{\Lambda}^2} \qquad (137)$$

$$Q_N = 4\,\frac{(1+M_n^2)}{\tilde{\Lambda}^2}\lambda_1\sin^2\theta + \frac{2}{\tilde{\Lambda}^4}\left[\tilde{\Lambda}^2\cos^3\theta - 2(M_n^3 + \cos^3\theta)\sin^2\theta\right] + 2\lambda_1\cos^2\theta \qquad (138)$$

IN SQ. BRACKET, USE

$$\tilde{\Lambda}^2 = (M_n - \cos\theta)^2 + 2\sin^2\theta$$

$$\therefore Q_N = 4\,\frac{(1+M_n^2)\sin^2\theta}{\tilde{\Lambda}^2}\lambda_1 + 2\lambda^2\cos^3\theta - \frac{4M_n^3\sin^2\theta}{\tilde{\Lambda}^4} + 2\lambda_1\cos^2\theta$$

(USE (132) FOR NEXT STEP)

$$= 2\lambda^2\cos^3\theta + 4(\alpha_n^2\cos\theta + M_n)\frac{\sin^2\theta}{\tilde{\Lambda}^4} + 2\lambda_1\cos^2\theta$$

$$= 2\lambda^2\cos^3\theta + 2(\alpha_n^2\cos\theta + M_n)\left(\frac{1}{\tilde{\Lambda}^2} - \lambda^2\right) + 2\lambda_1\cos^2\theta$$

$$Q_N = 2(\cos^3\theta - \alpha_n^2\cos\theta - M_n)\lambda^2 + 2\frac{\alpha_n^2\cos\theta + M_n}{\tilde{\Lambda}^2} + 2\lambda_1\cos^2\theta$$

$$= 2\frac{M_n^2\cos\theta}{\tilde{\Lambda}^2} + 2(1+\cos^2\theta)\lambda_1 + 2(\cos^3\theta - \alpha_n^2\cos\theta - M_n)\lambda^2 \qquad (139)$$

THIS IS THE SIMPLEST FORM I CAN GET Q_N INTO.
NOTE THAT $2\frac{M_n^2\cos\theta}{\tilde{\Lambda}^2} = M_n^2(\lambda+\lambda_1)$. (CORRECT(?) BUT DO NOT USE EQ. (139), SEE P 82(*))
TO CALCULATE Q_F, WE NEED JUST FEW MANIPULATIONS:

$$\lambda_x\mu^i + \lambda_{1x}\alpha^i = \left(2\lambda^2 - \frac{1}{\tilde{\Lambda}^2}\right)\mu^i + \left(2\lambda\lambda_1 + \frac{1}{\tilde{\Lambda}^2}\right)\alpha^i$$

$$= 2\lambda(\lambda\mu^i + \lambda_1\alpha^i) - \frac{\mu^i - \alpha^i}{\tilde{\Lambda}^2}$$

$$= 2\lambda\beta^i + \frac{\alpha^i - \mu^i}{\tilde{\Lambda}^2} \qquad i=1,2 \qquad (140)$$

ALSO WE HAVE

$$\lambda_x M_t^2 k_n + \lambda_{1y} k_{n1}\sin^2\theta = \left(2\lambda^2 - \frac{1}{\tilde{\Lambda}^2}\right)M_t^2 k_n + \left(2\lambda_1^2 + \frac{1}{\tilde{\Lambda}^2}\right)k_{n1}\sin^2\theta$$

$$= 2(\lambda^2 M_t^2 k_n + \lambda_1^2 k_{n1}\sin^2\theta) + \frac{1}{\tilde{\Lambda}^2}(k_{n1}\sin^2\theta - M_t^2 k_n) \qquad (141)$$

WE THUS GET

$$Q_F = \frac{1}{c}\lambda_x\dot{M}_n - 2\lambda\vec{B}_2\cdot\vec{\Omega} + \frac{1}{\tilde{\Lambda}^2}(\vec{M}_t - \vec{t}_1)\cdot\vec{\Omega}$$

$$+ 2(\lambda^2 M_t^2 k_n + \lambda_1^2 k_{n1}\sin^2\theta) + \frac{1}{\tilde{\Lambda}^2}(k_{n1}\sin^2\theta - M_t^2 k_n)$$

$$+ 2\left(2\lambda\lambda_1 + \frac{1}{\tilde{\Lambda}^2}\right)(k_1\tilde{\mu}^1\tilde{\alpha}^1 + k_2\tilde{\mu}^2\tilde{\alpha}^2)$$

$$- 2H(\lambda M_n + \lambda_1\cos\theta) \qquad (142)$$

WHERE $\vec{B}_2 = (\beta^1, \beta^2) = \lambda\vec{M}_t + \lambda_1\vec{t}_1$.

(*)

$$\vec{b} \equiv \vec{B}_2$$

WE NOTE THAT THE FOLLOWING VECTOR HAS A FUNDAMENTAL ROLE IN OUR THEORY

$$\vec{h} = \lambda \vec{M} + \lambda_1 \hat{r} \qquad (143)$$

THE PROJECTION OF THIS VECTOR ON THE TANGENT PLANE OF $f=0$, τ FIXED, IS $\vec{B}_2$. THE PROJECTION ON $\vec{n}$ IS $\lambda M_n + \cos\theta$, i.e.

$$\vec{h} = \vec{B}_2 + (\lambda M_n + \lambda_1 \cos\theta)\,\vec{n} \qquad (144)$$

IN MY AIAA PAPER, SINCE CAPITAL LETTERS ARE RESERVED FOR 4-VECTORS, WE USE $\vec{b}$ FOR $\vec{B}_2$. WE NOTE THAT

$$\lambda M_t \frac{\partial \tilde{p}_B}{\partial \sigma} + \lambda_1 \sin\theta \frac{\partial \tilde{p}_B}{\partial \sigma_1} = b \frac{\partial \tilde{p}_B}{\partial \sigma_b} \qquad (145)$$

WHERE $b = |\vec{b}|$ AND $\partial/\partial\sigma_b$ IS THE DIRECTIONAL DERIVATIVE IN THE DIRECTION OF $\vec{b}$. IT IS ALSO INTERESTING TO NOTE THAT

$$\lambda^2 M_t^2 \kappa_n + \lambda_1^2 \kappa_{n1} \sin^2\theta + 2\lambda\lambda_1(\kappa_1 \tilde{\mu}^1 \tilde{\alpha}^1 + \kappa_2 \tilde{\mu}^2 \tilde{\alpha}^2)$$
$$= \tilde{b}_1^2 \kappa_1 + \tilde{b}_2^2 \kappa_2 = b^2 \kappa_b \text{ (SEE NOTE 6, P89)} \qquad (146)$$

WHERE $\tilde{b}_1 = \vec{b} \cdot \vec{\tilde{a}}_1$ AND $\tilde{b}_2 = \vec{b} \cdot \vec{\tilde{a}}_2$, $\vec{\tilde{a}}_1$ AND $\vec{\tilde{a}}_2$ ARE UNIT VECTORS ALONG PRINCIPAL DIRECTIONS. WE ALSO NOTE THAT

$$\kappa_{n1} \sin^2\theta - M_t^2 \kappa_n + 2\kappa_1 \tilde{\mu}^1 \tilde{\alpha}^1 + 2\kappa_2 \tilde{\mu}^2 \tilde{\alpha}^2$$
$$= \tilde{\alpha}^1 \tilde{m}_+^1 \kappa_1 + \tilde{\alpha}^2 \tilde{m}_+^2 \kappa_2 + \tilde{\mu}^1 \tilde{m}_-^1 \kappa_1 + \tilde{\mu}^2 \tilde{m}_-^2 \kappa_2$$
$$\text{(SEE NOTE 6, P 89)} \qquad (147)$$

WHERE $\vec{m}_+ = \vec{t}_1 + \vec{M}_t$, $\vec{m}_- = \vec{t}_1 - \vec{M}_t$ ARE PROJECTIONS

OF VECTORS $\vec{M}_+ = \vec{\hat{r}} + \vec{M}$ AND $\vec{M}_- = \vec{\hat{r}} - \vec{M}$ ON THE TANGENT PLANE OF $f=0$, τ-FIXED. THE COMPONENTS OF $\vec{M}_+$ AND $\vec{M}_-$ ALONG PRINCIPAL DIRECTIONS ARE DENOTED BY $\tilde{M}^i_+$ AND $\tilde{M}^i_-$, RESPECTIVELY. WE NOTE THAT Q_F CAN NOW BE WRITTEN AS

$$Q_F = \frac{1}{c}\lambda_x \dot{M}_n - 2\lambda \vec{b}\cdot\vec{\Omega} - \frac{\vec{M}_-\cdot\vec{\Omega}}{\tilde{\Lambda}^2}$$

$$+2(\tilde{b}_1^2 \kappa_1 + \tilde{b}_2^2 \kappa_2) + \frac{1}{\tilde{\Lambda}^2}[\tilde{\alpha}^1\tilde{M}^1_+\kappa_1 + \tilde{\alpha}^2\tilde{M}^2_+\kappa_2$$

$$+ \tilde{\mu}^1\tilde{M}^1_-\kappa_1 + \tilde{\mu}^2\tilde{M}^2_-\kappa_2] - 2Hh_n$$

$$= \frac{1}{c}\lambda_x \dot{M}_n - 2\lambda\, \vec{b}\cdot\vec{\Omega} - \frac{\vec{M}_-\cdot\vec{\Omega}}{\tilde{\Lambda}^2}$$

$$+ 2b^2\kappa_b + \frac{1}{\tilde{\Lambda}^2}[(\tilde{\alpha}^1\tilde{M}^1_+ + \tilde{\mu}^1\tilde{M}^1_-)\kappa_1 + (\tilde{\alpha}^2\tilde{M}^2_+ + \tilde{\mu}^2\tilde{M}^2_-)\kappa_2]$$

$$-2Hh_n \qquad \text{(SEE NOTE 6, P.89)} \quad (148)$$

WHERE $b = |\vec{b}|$ AND $h_n = \vec{h}\cdot\vec{n} = \lambda M_n + \lambda_1 \cos\theta$. THIS IS AGAIN THE SIMPLEST FORM I CAN GET Q_F INTO. THE COLLAPSING OF TERMS IN SUCH A SYMMETRIC FORM GIVES AN INDICATION THAT OUR MANIPULATIONS ARE CORRECT. WE AGAIN NOTE THAT THE VECTORS $\vec{M}_+$ AND $\vec{M}_-$ PLAY FUNDAMENTAL ROLES IN NOISE GENERATION. IN PARTICULAR,

$$\lambda_1 = \frac{\vec{M}_+\cdot\vec{n}}{\tilde{\Lambda}^2}, \quad \lambda = \frac{\vec{M}_-\cdot\vec{n}}{\tilde{\Lambda}^2} \qquad (149\text{-a,b})$$

$$\lambda\vec{M} + \lambda_1\vec{\hat{r}} = \frac{1}{\tilde{\Lambda}^2}[\cos\theta\, \vec{M}_+ + M_n \vec{M}_-]$$

$$= \frac{\vec{n}\cdot}{\tilde{\Lambda}^2}[\vec{\hat{r}}\,\vec{M}_+ + \vec{M}\,\vec{M}_-]$$

$$\lambda \vec{M} + \lambda_1 \vec{r} = \frac{\vec{n}\cdot}{\tilde{\lambda}^2}\left[\vec{r}\vec{r} - \vec{M}\vec{M} + \vec{r}\vec{M} + \vec{M}\vec{r}\right] \quad (150)$$

WE SEE THAT $\vec{r}\vec{r} - \vec{M}\vec{M} + \vec{r}\vec{M} + \vec{M}\vec{r}$ IS A SYMMETRIC TENSOR WHICH AGAIN SEEMS TO BE FUNDAMENTAL. I DON'T KNOW WHY $\vec{M}\vec{M}$ APPEARS WITH NEGATIVE SIGN. IF WE USE THE PRINCIPAL DIRECTIONS AND NORMAL TO THE SURFACE AS DIRECTIONS FOR SETTING A LOCAL FRAME WITH UNIT BASE VECTORS, AND IF WE CALL THE ABOVE SYMMETRIC TENSOR DIVIDED BY $\tilde{\lambda}^2$ AS σ_{ij}, THEN

$$\frac{1}{\tilde{\lambda}^2}\left[(\tilde{\alpha}^1 m_+^1 + \tilde{\mu}^1 \tilde{m}_-^1)\kappa_1 + (\tilde{\alpha}^2 \tilde{m}_+^2 + \tilde{\mu}^2 \tilde{m}_-^2)\kappa_2\right]$$

$$= \sigma_{11}\kappa_1 + \sigma_{22}\kappa_2 \quad (151)$$

EQUATION (148) IS THE SIMPLEST FORM OF Q_F FOR NUMERICAL COMPUTATION. HOWEVER, USING THE ABOVE LOCAL FRAME, Q_F CAN BE WRITTEN IN THE FOLLOWING COMPACT FORM

$$Q_F = \frac{1}{c}\lambda_x \dot{M}_n - 2\lambda \vec{b}\cdot\vec{\Omega} - \frac{1}{\tilde{\lambda}^2}\vec{m}_-\cdot\vec{\Omega}$$
$$+ 2b^2\kappa_b + \sigma_{11}\kappa_1 + \sigma_{22}\kappa_2 - 2H h_n \quad (152)$$

SINCE $\vec{\Omega} = \vec{n}\times\vec{\omega}$, IT IS ALONG PITCH CHANGE AXIS WHICH, IN GENERAL, IS NORMAL TO $\vec{M}_t$. THEREFORE

$$2\lambda \vec{b}\cdot\vec{\Omega} + \frac{1}{\tilde{\lambda}^2}\vec{m}_-\cdot\vec{\Omega} \approx \left(2\lambda\lambda_1 + \frac{1}{\tilde{\lambda}^2}\right)\vec{t}_1\cdot\vec{\Omega}. \quad (153)$$

THIS ~~IS IMPORTANT WHEN PITCH CHANGE AXIS IS ALONG~~

THE RADIATION VECTOR $\vec{F}$.

THE EXPRESSION FOR Q_N, EQ.(139) IS STILL SOMEWHAT MESSY. MOST MANIPULATIONS ON P77 ARE NOT NECESSARY. WE, THEREFORE, START WITH EQ.(136-a) AND SIMPLIFY AS FOLLOWS:

$$Q_N = \lambda\cos^2\theta + \lambda_1\left[3 + 2(\lambda M_n - \lambda_1\cos\theta)\sin^2\theta - \sin^2\theta\right]$$

$$= \lambda\cos^2\theta + \lambda_1\left[2 + \cos^2\theta + 2(\lambda M_n - \lambda_1\cos\theta)\sin^2\theta\right]$$

$$= \lambda\cos^2\theta + (2+\cos^2\theta)\lambda_1 + 2\lambda_1(\lambda M_n - \lambda_1\cos\theta)\sin^2\theta \tag{154}$$

SEE P87 & P90

FOR NUMERICAL WORK, IT IS BETTER TO LEAVE THIS AS IT IS SINCE ALL QUANTITIES λ, λ_1, $\cos\theta$ AND $\sin\theta$ ARE KNOWN AT EACH STAGE. I SPENT A LOT OF TIME TO GET THIS IN A SIMPLE FORM OR A MORE SYMMETRIC FORM BUT I HAD NO SUCCESS. WE DO EXPECT THE NEAR-FIELD TERM TO BE MESSIER THAN THE FAR-FIELD TERM Q_F. WE WILL NOW DERIVE SOME INTERMEDIATE RESULTS LEADING TO EQUATIONS IN THIS STEP.

STEP 10 NOTES.

NOTE 1 — PROOF OF $\frac{1}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\right] = \frac{1}{2}\frac{\partial}{\partial u^i}\ln g_{(2)} = \Gamma^k_{ik}$

WE HAVE

$$\frac{\partial}{\partial u^i} g_{(2)} = g_{22}\frac{\partial g_{11}}{\partial u^i} + g_{11}\frac{\partial g_{22}}{\partial u^i} - 2g_{12}\frac{\partial g_{12}}{\partial u^i}$$

$$= 2g_{22}\Gamma_{1i1} + 2g_{11}\Gamma_{2i2} - 2g_{12}(\Gamma_{1i2} + \Gamma_{2i1})$$

$$= 2g_{(2)}\left[g^{11}\Gamma_{i11} + g^{22}\Gamma_{i22} + 2g^{12}\Gamma_{i12} + 2g^{12}\Gamma_{i21}\right]$$

USING THE RESULT THAT $\Gamma_{ij}^{\ \ k} = g^{kl}\Gamma_{ijl}$

WE CAN GROUP TERMS IN BRACKET AS

$$(g^{11}\Gamma_{i11} + g^{12}\Gamma_{i12}) + (g^{21}\Gamma_{i21} + g^{22}\Gamma_{i22})$$

$$= \Gamma_{i\ 1}^{\ 1} + \Gamma_{i\ 2}^{\ 2} = \Gamma_{i\ k}^{\ k}$$

$$\Rightarrow \frac{1}{2g_{(2)}}\frac{\partial}{\partial u^i}(g_{(2)}) = \frac{1}{2}\frac{\partial}{\partial u^i}\ln g_{(2)} = \Gamma_{i\ k}^{\ k}$$

NOTE 2 : CALCULATION OF $\mu^i \dfrac{\partial M_n}{\partial u^i}$

WE HAVE FROM EQ. (106), P 66

$$\frac{\partial M_n}{\partial u^i} = \frac{1}{c}\vec{n}\cdot\vec{\omega}\times\vec{a}_i - b_{ij}\,g^{jk}\mu_k$$

LET

$$\vec{\Omega} = \vec{n}\times\vec{\omega} \quad , \vec{\omega} \text{ ANG. VEL. VECTOR}$$

$\therefore$

$$\frac{\partial M_n}{\partial u^i} = \frac{1}{c}\Omega_i - b_{ij}\,g^{jk}\mu_k \qquad \mu_k = \vec{M}_t\cdot\vec{a}_i$$

$$\mu^i\frac{\partial M_n}{\partial u^i} = \mu^i\Omega_i - b_{ij}\,g^{jk}\mu_k\mu^i$$

$$= \mu^i\Omega_i - b_i^{\ k}\mu_k\mu^i \qquad (155)$$

NOW WE NOTE THAT $b_i^{\ k}\mu_k = -\mathcal{S}(\vec{M}_t)$ WHERE $\mathcal{S}(\vec{M}_t)$ IS THE SPHERICAL IMAGE OF $\vec{M}_t$ (SEE GOETZ, P 265). WE HAVE THE FOLLOWING RESULT FROM GOETZ, P 266, THAT THE NORMAL CURVATURE OF THE SURFACE IN THE DIRECTION OF VECTOR $\vec{a}$ (ARBITRARY VECTOR, NOT RELATED TO OUR a_i'S) IS

$$k_n = -\frac{\mathcal{S}(\vec{a})\cdot\vec{a}}{|\vec{a}|^2} = \frac{b_i^{\ k}a_k a^i}{|\vec{a}|^2}$$

$$\Rightarrow b_i^{\ k}\mu_k\mu^i = k_n M_t^2 \quad \text{, (CURVATURE ALONG } \vec{M}_t) \qquad (156)$$

$$\Rightarrow \mu^i \frac{\partial M_n}{\partial u^i} = \Omega_i \mu^i - \kappa_n M_t^2 \qquad (157)$$

WE NOTE THAT $\Omega_i \mu^i = \vec{\Omega} \cdot \vec{M}_t$. EVALUATION OF $\alpha^i \frac{\partial M_n}{\partial u^i}$ IS SIMILAR, BUT SIMPLER.

NOTE 3 – EVALUATION OF $-\sin\theta \frac{\partial \theta}{\partial u^i} \mu^i$

WE HAVE, FROM EQ. (108), P67, NOTING $\hat{r}_i \equiv \alpha_i$

$$-\sin\theta \frac{\partial\theta}{\partial u^i} = \frac{\alpha_i}{r}\cos\theta - b_{ij} g^{jk} \alpha_k$$

$$\therefore -\sin\theta \frac{\partial\theta}{\partial u^i}\mu^i = \frac{\alpha_i \mu^i}{r}\cos\theta - b_{ij}\alpha^j \mu^i \qquad (158)$$

WE DEFINE $\alpha_i \mu^i = \vec{t}_1 \cdot \vec{M}_t \equiv M_{t1}$. WE WILL LATER SIMPLIFY $b_{ij}\alpha^j\mu^i$.

FOR $-\sin\theta \frac{\partial\theta}{\partial u^i}\alpha^i = \frac{\alpha^i \alpha_i \cos\theta}{r} - b_{ij} g^{jk} \alpha_k \alpha^i$

$$= \frac{\sin^2\theta\cos\theta}{r} - b_i^{\ k}\alpha_k\alpha^i \qquad (159)$$

WE NOTE THAT $\alpha^i \alpha_i = |\vec{t}_1|^2 = \sin^2\theta$. ALSO WE USE SAME RESULT LEADING TO EQ. (156)

$$b_i^{\ k}\alpha_k\alpha^i = |\vec{t}_1|^2 \kappa_{n1}$$

$$= \kappa_{n1}\sin^2\theta \qquad (160)$$

WHERE NOW κ_{n1} IS THE NORMAL CURVATURE ALONG $\vec{t}_1$. $\Rightarrow$

$$-\sin\theta \frac{\partial\theta}{\partial u^i}\alpha^i = \frac{\sin^2\theta\cos\theta}{r} - \kappa_{n1}\sin^2\theta \qquad (161)$$

NOTE 4 – EVALUATION OF $\dfrac{\partial \mu^i}{\partial u^i}$

$$\frac{\partial \mu^i}{\partial u^i} = \frac{\partial g^{ij}}{\partial u^i}\mu_j + g^{ij}\frac{\partial \mu_j}{\partial u^i} \qquad (162)$$

WE HAVE (ALL REPEATED INDICES SUMMED OVER 1, 2)

$$g^{ij} g_{jk} = \delta^i_k$$

$$\frac{\partial g^{ij}}{\partial u^i} g_{jk} + g^{ij}\frac{\partial g_{jk}}{\partial u^i} = 0$$

$$\frac{\partial g^{ij}}{\partial u^i}\underbrace{g_{jk}\, g^{kl}}_{\delta_j^l} + g^{kl} g^{ij}\frac{\partial g_{jk}}{\partial u^i} = 0$$

$$\frac{\partial g^{il}}{\partial u^i} = - g^{kl} g^{ij}\frac{\partial g_{jk}}{\partial u^i} \qquad (163)$$

$$\therefore \quad \frac{\partial g^{ij}}{\partial u^i}\mu_j = - g^{kl} g^{ij}\mu_l \frac{\partial g_{jk}}{\partial u^i}$$

$$= - g^{ij}\mu^k \frac{\partial g_{jk}}{\partial u^i} \qquad (164)$$

$$\frac{\partial \mu_j}{\partial u^i} = \frac{\partial \vec{M}_t}{\partial u^i}\cdot \vec{a}_j + \vec{M}_t \cdot \frac{\partial \vec{a}_j}{\partial u^i}$$

$$\frac{\partial \vec{M}_t}{\partial u^i} = \frac{\partial}{\partial u^i}\left(\vec{M} - M_n \vec{n}\right)$$

$$= \frac{\partial \vec{M}}{\partial u^i} - \frac{\partial M_n}{\partial u^i}\vec{n} - M_n \frac{\partial \vec{n}}{\partial u^i}$$

$$\therefore \vec{a}_j \cdot \frac{\partial \vec{M}_t}{\partial u^i} = \vec{a}_j \cdot \frac{\partial \vec{M}}{\partial u^i} - M_n \vec{a}_j \cdot \frac{\partial \vec{n}}{\partial u^i}$$

$$= \underbrace{\vec{\omega} \times \vec{a}_i \cdot \vec{a}_j}_{\text{FROM EQ. (105-a)}} + M_n \vec{a}_j \underbrace{(-b_i^K \vec{a}_K)}_{\text{FROM EQ. (105-b)}}$$

$$= \vec{\omega} \cdot (\vec{a}_i \times \vec{a}_j) + M_n b_i^K g_{jK}$$

WE ALSO HAVE $\frac{\partial \vec{a}_j}{\partial u^i} = \Gamma_{ij}^K \vec{a}_K + b_{ij} \vec{n}$

$$\therefore \frac{\partial \mu_j}{\partial u^i} = \vec{\omega} \cdot (\vec{a}_i \times \vec{a}_j) + M_n b_i^K g_{jK} + \mu^l g_{lK} \Gamma_{ij}^K$$

$$g^{ij} \frac{\partial \mu_j}{\partial u^i} = \underbrace{g^{ij} \vec{\omega} \cdot (\vec{a}_i \times \vec{a}_j)}_{=0 \text{ SINCE } g^{ij} = g^{ji}} + M_n b_i^K \underbrace{g^{ij} g_{jK}}_{\delta_K^i} + \mu^l g^{ij} \underbrace{g_{lK} \Gamma_{ij}^K}_{\Gamma_{ijl}}$$

$$= M_n b_i^i + \mu^l g^{ij} \Gamma_{ijl} \qquad (165)$$

WE HAVE $b_i^i = 2H$ (GOETZ, P.267, EQ. 25.10) WHERE H IS THE LOCAL MEAN CURVATURE. WE THUS HAVE, FROM EQ. (164) AND (165)

$$\frac{\partial \mu^i}{\partial u^i} = -g^{ij} \mu^K \left[\frac{\partial g_{jK}}{\partial u^i} - \Gamma_{ijK} \right] + 2H M_n$$

$$\frac{\partial g_{jK}}{\partial u^i} - \Gamma_{ijK} = \frac{\partial g_{jK}}{\partial u^i} - \frac{1}{2} \left[\frac{\partial g_{iK}}{\partial u^j} + \frac{\partial g_{jK}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^K} \right]$$

$$= \Gamma_{iKj} \qquad (166)$$

$$\therefore \frac{\partial \mu^i}{\partial u^i} = -g^{ij} \mu^K \Gamma_{iKj} + 2 M_n H$$

$$= -\mu^K \Gamma_{Ki}^i + 2 M_n H \checkmark \qquad (167)$$

THERE PROBABLY IS A SIMPLER WAY OF DOING THIS BUT I DON'T KNOW ENOUGH DIFFERENTIAL GEOMETRY TO DO IT!

WE NOW FIND $\dfrac{\partial \alpha^i}{\partial u^i} = \alpha_j \dfrac{\partial g^{ij}}{\partial u^i} + g^{ij}\dfrac{\partial \alpha_j}{\partial u^i}$ (*)

FROM EQ. (163)

$$\alpha_j \frac{\partial g^{ij}}{\partial u^i} = - g^{ij}\alpha^k \frac{\partial g_{jk}}{\partial u^i}$$

$$\frac{\partial \alpha_j}{\partial u^i} = \frac{\partial \vec{t}_1}{\partial u^i}\cdot \vec{a}_j + \vec{t}_1 \cdot \frac{\partial \vec{a}_j}{\partial u^i}$$

$$= \underbrace{\frac{\alpha_i \alpha_j - g_{ij}}{r}}_{\text{EQ. (112)}: \hat{r}_i \equiv \alpha_i} + \cos\theta\, b_{ij} + \alpha^k g_{k\ell}\Gamma^{\ell}_{ij}$$

$$g^{ij}\frac{\partial \alpha_j}{\partial u^i} = \frac{\alpha^i\alpha_i - \delta^i_i}{r} + \cos\theta\, \underbrace{b_{ij}g^{ij}}_{b_i^{\,i} = 2H} + \alpha^k g^{ij}\Gamma_{ij,k}$$

$$= \frac{\sin^2\theta - \cancel{3}^{\,2}}{r} + 2H\cos\theta + \alpha^k g^{ij}\Gamma_{ij,k} \qquad (168)$$

ERROR! =2 SEE P 90

$$\frac{\partial \alpha^i}{\partial u^i} = - g^{ij}\alpha^k \Big[\underbrace{\frac{\partial g_{jk}}{\partial u^i} - \Gamma_{ij,k}}_{\Gamma_{ik,j} \text{ BY EQ. (166)}}\Big] + \frac{\sin^2\theta - 3}{r} + 2H\cos\theta$$

$$= \frac{\sin^2\theta - \cancel{3}^{\,2}}{r} + 2H\cos\theta - \alpha^k \Gamma_{ki}^{\,i} \qquad (169)$$

(*) THIS CAN BE DONE SIMPLER: $\alpha^i = \vec{t}_1 \cdot \vec{a}^i = \hat{\vec{r}}\cdot \vec{a}^i$

$\dfrac{\partial \alpha^i}{\partial u^i} = \dfrac{\partial \hat{r}}{\partial u^i}\cdot \vec{a}^i + \hat{\vec{r}} \cdot \dfrac{\partial \vec{a}^i}{\partial u^i}$, THE FIRST TERM IS GIVEN ABOVE AND

$\hat{\vec{r}} \cdot \dfrac{\partial \vec{a}^i}{\partial u^i} = (\alpha^j \vec{a}_j + \cos\theta\, \vec{n}) \cdot (-\Gamma^i_{ik}\vec{a}^k + b^i_i \vec{n}) = -\alpha^k \Gamma_{ki}^{\,i} + 2H\cos\theta .$

NOTE 5 — SIMPLIFICATION OF $b_{ij}\alpha^i\mu^j$.

WE CAN WRITE

$$b_{ij}\,\alpha^i\mu^j = b_{ij}\,g^{il}\,\alpha_l\,\mu^j$$
$$= b_j^{\,l}\,\alpha_l\,\mu^j$$
$$= \mathcal{S}(\vec{M}_t)\cdot\vec{t}_1 \qquad (170)$$

SINCE $\mathcal{S}(\vec{M}_t) = -b_j^{\,l}\mu^j$ (CONTRAVARIANT)

NOW LET US USE UNIT BASE VECTORS $\vec{\tilde{a}}_1$ AND $\vec{\tilde{a}}_2$ ALONG PRINCIPAL DIRECTIONS AND LET THE COMPONENTS OF $\vec{M}_t$ AND $\vec{t}_1$ BE $(\tilde{\mu}^1, \tilde{\mu}^2)$ AND $(\tilde{\alpha}^1, \tilde{\alpha}^2)$, RESPECTIVELY. THEN

$$\mathcal{S}(\vec{M}_t) = \tilde{\mu}^1\mathcal{S}(\vec{\tilde{a}}_1) + \tilde{\mu}^2\mathcal{S}(\vec{\tilde{a}}_2)$$

THIS FOLLOWS FROM LINEARITY PROPERTY OF SPHERICAL IMAGE (GOETZ, P.265, EQ. 25.5). NOW THE RODRIGUES THM SAYS THAT

$$\mathcal{S}(\vec{\tilde{a}}_1) = -k_1\,\vec{\tilde{a}}_1 \qquad \text{(GOETZ, P 285)}$$
$$\mathcal{S}(\vec{\tilde{a}}_2) = -k_2\,\vec{\tilde{a}}_2$$

WHERE k_1 AND k_2 ARE PRINCIPAL CURVATURES.

$$\Rightarrow \quad \mathcal{S}(\vec{M}_t) = -\tilde{\mu}^1 k_1\vec{\tilde{a}}_1 - \tilde{\mu}^2 k_2\vec{\tilde{a}}_2 \qquad (171)$$

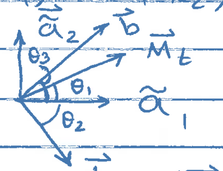
$$\therefore \quad b_{ij}\mu^j\alpha^i = k_1\,\tilde{\mu}^1\tilde{\alpha}^1 + k_2\,\tilde{\mu}^2\tilde{\alpha}^2 \qquad (172)$$

NOTE: $S(\vec{a}) = \alpha^i\dfrac{\partial\vec{n}}{\partial u^i} = \vec{a}\cdot\nabla_2\vec{n}$, i.e. IF $|\vec{a}|=1 \Rightarrow S(\vec{a})$ IS THE DIRECTIONAL DERIVATIVE OF $\vec{n}$ IN THE DIRECTION $\vec{a}$.

NOTE 6 : SIMPLIFICATION OF THE EXPRESSION

$E = \lambda^2 M_t^2 \kappa_n + \lambda_1^2 \kappa_{n_1} \sin^2\theta + 2\lambda\lambda_1(\kappa_1 \tilde{\mu}^1 \tilde{\alpha}^1 + \kappa_2 \tilde{\mu}^2 \tilde{\alpha}^2)$

BY EULER'S THM, WE HAVE



$|\vec{t}_1| = \sin\theta$

$$(173\,a\text{-}d)\quad \begin{cases} \kappa_n = \kappa_1 \cos^2\theta_1 + \kappa_2 \sin^2\theta_2 \\ \kappa_{n_1} = \kappa_1 \cos^2\theta_2 + \kappa_2 \sin^2\theta_2 \\ M_t\cos\theta_1 = \tilde{\mu}^1 \;,\; M_t \sin\theta_1 = \tilde{\mu}^2 \\ \sin\theta\cos\theta_2 = \tilde{\alpha}^1 \;,\; \sin\theta\sin\theta_2 = \tilde{\alpha}^2 \end{cases}$$

$$\Rightarrow \quad E = (\lambda\tilde{\mu}^1 + \lambda_1\tilde{\alpha}^1)^2 \kappa_1 + (\lambda\tilde{\mu}^2 + \lambda_1\tilde{\alpha}^2)^2 \kappa_2$$

$$= \tilde{b}_1^2 \kappa_1 + \tilde{b}_2^2 \kappa_2$$

$$= b^2 (\cos^2\theta_3\, \kappa_1 + \sin^2\theta_3\, \kappa_2) \qquad \text{(SEE FIG.)}$$

$$= b^2 \kappa_b \qquad (173)$$

WHERE $\vec{b} = \lambda\vec{M}_t + \lambda_1 \vec{t}_1$ AND κ_b IS THE NORMAL CURVATURE ALONG $\vec{b}$. ALSO $b = |\vec{b}|$

SIMPLIFICATION OF THE EXPRESSION

$E = \kappa_{n_1} \sin^2\theta - M_t^2 \kappa_n + 2\kappa_1 \tilde{\mu}^1 \tilde{\alpha}^1 + 2\kappa_2 \tilde{\mu}^2 \tilde{\alpha}^2$

USING THE ABOVE RELATIONS (173 a-d), WE HAVE

$$E = \kappa_1(\tilde{\alpha}^1)^2 + \kappa_2(\tilde{\alpha}^2)^2 - \kappa_1(\tilde{\mu}^1)^2 - \kappa_2(\tilde{\mu}^2)^2 + 2\kappa_1 \tilde{\mu}^1\tilde{\alpha}^1 + 2\kappa_2 \tilde{\mu}^2 \tilde{\alpha}^2$$

$$= \kappa_1 \tilde{\alpha}^1 (\tilde{\alpha}^1 + \tilde{\mu}^1) + \kappa_1 \tilde{\mu}^1 (\tilde{\alpha}^1 - \tilde{\mu}^1) + \kappa_2 \tilde{\alpha}^2 (\tilde{\alpha}^2 + \tilde{\mu}^2) + \kappa_2 \tilde{\mu}^2 (\tilde{\alpha}^2 - \tilde{\mu}^2)$$

$$= \kappa_1(\tilde{\alpha}^1 \tilde{m}_+^1 + \tilde{\mu}^1 \tilde{m}_-^1) + \kappa_2(\tilde{\alpha}^2 \tilde{m}_+^2 + \tilde{\mu}^2 \tilde{m}_-^2) \qquad (174)$$

WHERE $\vec{m}_+ = \vec{t}_1 + \vec{M}_t = (\tilde{m}_+^1, \tilde{m}_+^2)$, $\vec{m}_- = \vec{t}_1 - \vec{M}_t = (\tilde{m}_-^1, \tilde{m}_-^2)$ IN BASE $(\vec{\tilde{a}}_1, \vec{\tilde{a}}_2)$

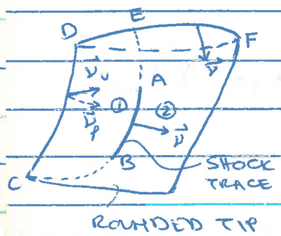
$$\sigma_{11} = [(\tilde{\alpha}^1)^2 - (\tilde{\mu}^1)^2 + 2\tilde{\alpha}^1\tilde{\mu}^1]/\tilde{\lambda}^2 \,,\quad \sigma_{22} = [(\tilde{\alpha}^2)^2 - (\tilde{\mu}^2)^2 + 2\tilde{\alpha}^2\tilde{\mu}^2]/\tilde{\Lambda}^2 \qquad (174\text{-}a)$$

SUMMARY

WE SUMMARIZE OUR FINAL RESULTS TO BE USED FOR NUMERICAL COMPUTATIONS:

$$4\pi p'(\vec{x},t) = \int \frac{\tilde{p}\,|\nabla f|\,Q_N}{r^2}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

$$+\int \frac{|\nabla f|}{r}\left[\frac{\lambda}{c}\frac{\partial \tilde{p}_B}{\partial \tau} - b\,\frac{\partial \tilde{p}_B}{\partial \sigma_b}\right]\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$



$$+\int \frac{\tilde{p}\,|\nabla f|\,Q_F}{r}\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

$$-\int_{T.E.}\frac{1}{r}\left\{\frac{\tilde{p}\,[(b_\nu)_u+(b_\nu)_l]}{\Lambda_0}\right\}_{ret} d\gamma - \int_{DEFD}\frac{1}{r}\left[\frac{\tilde{p}\,b_\nu}{\Lambda_0}\right]_{ret} d\gamma$$

$$-\int_{SHOCK}\frac{1}{r}\left[\frac{\Delta p\, b_\nu}{\Lambda_0}\right]d\Sigma\,, \quad \Delta p = p_2 - p_1$$

WHERE

$$Q_N = -2\lambda_1\,(\lambda M_n + \lambda_1\cos\theta)\sin^2\theta + \lambda_1\cos^2\theta$$

$$= \lambda_1\,(\cos^2\theta - 2\,\lambda_1 h_n \sin^2\theta) + \lambda$$

$$\cancel{= \lambda\,[2(\cos\theta - M_n)\lambda_1 + 1]}$$

(CORRECTED, SEE P87) 6/3/83 F.F.

$$Q_F = \frac{1}{c}\left[\left(2\lambda^2 - \frac{1}{\tilde{\Lambda}^2}\right)\dot{M}_n - 2\lambda\,\vec{b}\cdot\vec{\Omega} - \frac{1}{\tilde{\Lambda}^2}\,(\vec{t}_1 - \vec{M}_t)\cdot\vec{\Omega}\right] + 2b^2\kappa_b$$ (CORRECTED, SEE P66)

$$+\,\sigma_{11}\kappa_1 + \sigma_{22}\kappa_2 - 2H h_n$$

$$\lambda = \frac{\cos\theta - M_n}{\tilde{\Lambda}^2}$$

$$\lambda_1 = \frac{\cos\theta + M_n}{\tilde{\Lambda}^2} \qquad (\equiv \lambda' \text{ EARLIER})$$

NOTE ADDED ON JUNE 28, 83: I HAVE NOW CLEANED UP THE DERIVATION OF THE ABOVE RESULT. ALSO I HAVE DERIVED AERODYNAMIC FORMULA. FROM CANCELATIONS AND THE FINAL RESULT, I AM NOW CONVINCED THE ABOVE IS CORRECT.

$$\tilde{\Lambda}^2 = \Lambda^2 + \sin^2\theta$$

$$\Lambda^2 = 1 + M_n^2 - 2 M_n \cos\theta$$

$\vec{b} = \lambda \vec{M}_t + \lambda_1 \vec{t}_1$ $(\equiv \vec{B}_2$ EARLIER$)$

$\vec{t}_1 = \vec{r}_1 - \vec{n}\cos\theta$ PROJ. OF $\vec{r}$ ON $\xi = 0$, τ FIXED.

$\vec{\Omega} = \vec{n} \times \vec{\omega}$

$\vec{\omega}$ = ANGULAR VELOCITY OF PROPELLER

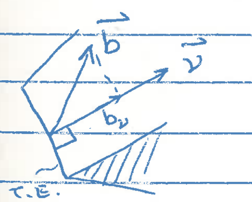
κ_1, κ_2 = PRINCIPAL CURVATURES

κ_b = NORMAL CURVATURE ALONG $\vec{b}$

H = MEAN CURVATURE

σ_{11}, σ_{22} = 2 COMPONENTS OF TENSOR $(\vec{t}_1\vec{t}_1 - \vec{M}_t\vec{M}_t + \vec{t}_1\vec{M}_t + \vec{M}_t\vec{t}_1)/\tilde{\Lambda}^2$ IN $(\tilde{a}_1, \tilde{a}_2)$ LOCAL FRAME SEE EQ. (174-a), P. 89.

$(\tilde{a}_1, \tilde{a}_2)$ = UNIT VECTORS ALONG PRINCIPAL DIRECTIONS



$b_\nu = \vec{b} \cdot \vec{\nu}$ COMPONENT OF $\vec{b}$ NORMAL TO THE EDGE CURVE (eg. T.E.) $(\equiv B_{2\nu}$ EARLIER$)$

$\vec{\nu}$ = UNIT INWARD VECTOR LYING ON THE SURFACE $\xi = 0$, τ FIXED, AND NORMAL TO T.E. (ALSO SEE FIG. ON P. 90)

$\vec{\nu}$ IS KNOWN AS THE GEODESIC NORMAL (GOETZ)

$$\Lambda_0^2 = M_p^2 \cos^2\psi + (1 - \vec{M}_p \cdot \hat{r}_p \sin\psi)^2$$ {SEE P139-140

$\vec{M}_p$ = PROJECTION OF $\vec{M}$ ON LOCAL PLANE NORMAL TO EDGE CURVE. $|\vec{M}_p| = M_p$

$\hat{r}_p$ = UNIT VECTOR ALONG PROJECTION OF $\vec{r}$ ON LOCAL PLANE NORMAL TO EDGE CURVE

ψ = THE ANGLE BETWEEN $\hat{r}$ AND T.E.

$h_n = \lambda M_n + \lambda \cos\theta$

NOTE ADDED ON SEP. 26, 83: WHEN SHARON PADULA PROGRAMMED MY EQ., WE COULD NOT GET SATISFACTORY RESULTS. SEE EXPLANATION ON P110.

F.F. NOTE ADDED ON 6/9/91: THE FORMULATION IS O.K., WE HAD BUGS IN THE CODE!

* A NOTE ON GENERALIZED FUNCTIONS

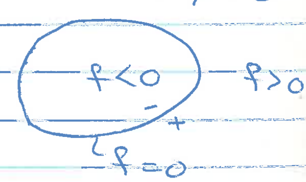
WHEN A FUNCTION IS MULTIPLIED BY $\delta(f)$, WE OFTEN DO NOT IMMEDIATELY RESTRICT THE FUNCTION TO THE SURFACE $f=0$. THAT IS, WE ASSUME THAT THE FUNCTION IS DEFINED IN THE VICINITY OF THE SURFACE. PERHAPS THIS PROCEDURE CAN LENGTHEN MANIPULATIONS. THIS CAN EASILY BE SEEN IN ONE-DIM'L CASE:

$$\frac{d}{dx}[\phi(x)\,\delta(x)] = \phi'(x)\,\delta(x) + \phi(x)\,\delta'(x)$$
$$= \phi'(0)\,\delta(x) + \phi(x)\,\delta'(x)$$
$$\text{ALSO} = \frac{d}{dx}[\phi(0)\,\delta(x)]$$
$$= \phi(0)\,\delta'(x)$$

I OFTEN WONDERED IF RESTRICTING ϕ TO $f=0$ HELPS IN OTHER DIMENSIONS. IN THREE DIMENSIONS, ONE MAY HAVE TO USE SOME DIFFERENTIAL GEOMETRY AS SHOWN BELOW.

$\nabla^2\phi = 0$

CONSIDER THE DERIVATION OF GREEN'S THM USING G.F.'S.



$$\text{LET } \tilde{\phi} = \begin{cases} \phi & f>0 \\ 0 & f<0 \end{cases}$$

$$\bar{\nabla}\tilde{\phi} = \nabla\tilde{\phi} + \tilde{\phi}\,\nabla f\,\delta(f)$$

$$\bar{\nabla}^2\tilde{\phi} = \underbrace{\nabla^2\tilde{\phi}}_{=0} + \frac{\partial\tilde{\phi}}{\partial n}|\nabla f|\,\delta(f) + \bar{\nabla}\cdot[\tilde{\phi}\,\nabla f\,\delta(f)]$$

NOTE $\tilde{\phi} \equiv \tilde{\phi}_+ = \phi$, $\frac{\partial\tilde{\phi}}{\partial n} = \frac{\partial\tilde{\phi}_+}{\partial n} = \frac{\partial\phi}{\partial n}$ (IN LAST STEP)

i) WE DO NOT RESTRICT ϕ TO $f=0$ IN THIS STEP.

$$\bar{\nabla}\cdot[\phi\,\nabla f\,\delta(f)] = \bar{\nabla}\cdot(\phi\,\nabla f)\,\delta(f) + \phi\,|\nabla f|^2\,\delta'(f) \qquad (*)$$

$$I = -\int \frac{\phi\,|\nabla f|^2}{r}\,\delta'(f)\,d\vec{y} \qquad \{\text{USING GREEN'S FN OF } \nabla^2\}$$

$$= +\int \nabla\cdot\left[\frac{\phi\,\nabla f}{r}\right]\delta(f)\,d\vec{y}$$

$$= \int \left[\frac{\nabla\cdot(\phi\,\nabla f)}{r} + \frac{\phi\,|\nabla f|\cos\theta}{r^2}\right]\delta(f)\,d\vec{y}$$

$$= \int \frac{\nabla\cdot(\phi\,\nabla f)}{r}\,\delta(f)\,d\vec{y} + \int_{f=0} \frac{\phi\cos\theta}{r^2}\,dS$$

THE FIRST INTEGRAL IN I CANCELS THE FIRST TERM IN (*) SO THAT

$$4\pi\,\tilde{\phi}(\vec{x}) = \int\left[\frac{\phi\cos\theta}{r^2} - \frac{1}{r}\frac{\partial\phi}{\partial n}\right]dS$$

ii) IN THE FIRST STEP, WE COULD WRITE

$$\bar{\nabla}\cdot[\phi\,\nabla f\,\delta(f)] = \bar{\nabla}\cdot[\langle\phi\,\nabla f\rangle\,\delta(f)]$$

WHERE $\langle\phi\,\nabla f\rangle$ IS THE RESTRICTION OF $\phi\,\nabla f$ TO $f=0$. NOW LET $\vec{y}\rightarrow(u^1,u^2,f) \Rightarrow$

$$\langle\phi\,\nabla f\rangle = \text{FN}(u^1,u^2)$$

$$\bar{\nabla}\cdot[\langle\phi\,\nabla f\rangle\,\delta(f)] = \frac{|\nabla f|}{\sqrt{g_{(2)}}}\,\frac{\partial}{\partial f}\left[\langle\phi|\nabla f|^2\rangle\frac{\sqrt{g_{(2)}}}{|\nabla f|}\,\delta(f)\right]$$

$$= \frac{|\nabla f|}{\sqrt{g_{(2)}}}\,\langle\phi|\nabla f|\sqrt{g_{(2)}}\rangle\,\delta'(f)$$

HERE WE HAVE USED THE INVARIANT DEFINITION OF DIVERGENCE ($\vec{V} = v^i$)

$$\nabla \cdot \vec{V} = \frac{1}{\sqrt{g_{(3)}}} \frac{\partial}{\partial u^i} \left[\sqrt{g_{(3)}}\, v^i \right]$$

WHERE $g_{(3)}$ IS THE DETERMINANT OF THE FIRST FUNDAMENTAL FORM. WE HAVE

$$g_{(3)} = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & \frac{1}{|\nabla f|^2} \end{vmatrix} = \frac{g_{(2)}}{|\nabla f|}$$

WHERE $g_{(2)}$ IS THE DET. OF 1ST FUND. FORM OF THE SURFACE. NOTE THAT THE NATURAL BASE VECTOR $\vec{a}_3$ IS $\vec{n} / |\nabla f|$. IN THE EXPRESSION

$$\frac{|\nabla f|}{\sqrt{g_{(2)}}} \langle \phi \, |\nabla f| \sqrt{g_{(2)}} \rangle \, \delta'(f)$$

WE MAY NOT CANCEL $\sqrt{g_{(2)}}$ IN $\langle \cdots \rangle$ WITH THAT IN THE DENOMINATOR SINCE THE FORMER IS RESTRICTION OF $\sqrt{g_{(2)}}$ TO THE SURFACE. NOW WE HAVE

$$I = - \int \frac{|\nabla f|}{r \sqrt{g_{(2)}}} \langle \phi \, |\nabla f| \sqrt{g_{(2)}} \rangle \, \delta'(f) \, d\vec{y}$$

$$= - \int \frac{\cancel{|\nabla f|}}{r \sqrt{\cancel{g_{(2)}}}} \langle \phi \, |\nabla f| \sqrt{g_{(2)}} \rangle \, \delta'(f) \, \frac{\sqrt{\cancel{g_{(2)}}}\, du^1 du^2 df}{\cancel{|\nabla f|}}$$

$$= \int \langle \phi \, |\nabla f| \sqrt{g_{(2)}} \rangle \underbrace{\frac{1}{|\nabla f|} \frac{\partial}{\partial n}}_{=\partial/\partial f} \left(\frac{1}{r} \right) \delta(f) \, df \, du^1 du^2 \quad \left[\begin{array}{l} |\nabla f| \text{ IN DENOMINATOR} \\ \text{IS NOW RESTRICTED TO } f=0 \end{array} \right]$$

$$= \int \phi \, \frac{\cos\theta}{r^2} \, dS \qquad \text{AS EXPECTED.}$$

WE NOW GO BACK AND WORK ON WHAT I TRIED TO DERIVE EARLIER (P34, OCT. 82). WE NOW CONSIDER

$$I = \int \vec{\nabla} \cdot [\vec{K} |\nabla f| \delta(f)] G(\vec{y})\, d\vec{y}$$

WHERE $\vec{K}$ HAS A COMPONENT TANGENT TO THE SURFACE $f=0$. LET $\vec{K} = \vec{K}_t + \vec{K}_n$, i.e. $\vec{K}$ IS DECOMPOSED INTO ITS NORMAL AND TANGENTIAL COMPONENTS. AS BEFORE, WE LET $\vec{y} \to (u^1, u^2, f)$.

$$\vec{\nabla} \cdot [\vec{K} |\nabla f| \delta(f)] = \vec{\nabla} \cdot [\vec{K}_t |\nabla f| \delta(f)] + \nabla \cdot [\vec{K}_n |\nabla f| \delta(f)]$$

$$= \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \left[\langle K_t^i \cancel{|\nabla f|} \rangle \frac{\sqrt{g_{(2)}}}{\cancel{|\nabla f|}} \delta(f) \right]$$

$$+ \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial f} \left[\langle K_n |\nabla f|^2 \rangle \frac{\sqrt{g_{(2)}}}{|\nabla f|} \delta(f) \right]$$

$$= \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \langle K_t^i \sqrt{g_{(2)}} \rangle \delta(f)$$

$$+ \frac{|\nabla f|}{\sqrt{g_{(2)}}} \langle K_n |\nabla f| \sqrt{g_{(2)}} \rangle \delta'(f)$$

$$\Rightarrow I = \int \frac{\cancel{|\nabla f|}}{\cancel{\sqrt{g_{(2)}}}} \frac{\partial}{\partial u^i} \langle K_t^i \sqrt{g_{(2)}} \rangle G(\vec{y}) \delta(f) \frac{\cancel{\sqrt{g_{(2)}}}\, du^1 du^2 df}{\cancel{|\nabla f|}}$$

$$+ \int \frac{\cancel{|\nabla f|}}{\cancel{\sqrt{g_{(2)}}}} \langle K_n |\nabla f| \sqrt{g_{(2)}} \rangle G(\vec{y}) \delta'(f) \frac{\cancel{\sqrt{g_{(2)}}}\, du^1 du^2 df}{\cancel{|\nabla f|}}$$

$$= \int_{f=0} \frac{1}{\langle \sqrt{g_{(2)}} \rangle} \frac{\partial}{\partial u^i} \langle K_t^i \sqrt{g_{(2)}} \rangle G(\vec{y}) \langle \sqrt{g_{(2)}} \rangle\, du^1 du^2$$

$$- \int_{f=0} \langle K_n \cancel{|\nabla f|} \sqrt{g_{(2)}} \rangle \frac{1}{\langle \cancel{|\nabla f|} \rangle} \frac{\partial G}{\partial n}\, du^1 du^2$$

$$I = \int_{f=0} G(\vec{y})\, \nabla_2 \cdot \vec{K}_t \, dS$$

$$- \int_{f=0} K_n \frac{\partial G}{\partial n} \, dS$$

WHERE

$$\nabla_2 \cdot \vec{K}_t = \frac{1}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \left(\sqrt{g_{(2)}} \, K_t^i \right) \qquad i = 1, 2.$$

FOR A CLOSED SURFACE $f=0$, IS IT POSSIBLE THAT THE FIRST INTEGRAL IS ZERO FOR $G = 1$(*)? I SUSPECT IT IS, UNLESS THERE ARE SINGULARITIES ON THE SURFACE. THIS TYPE OF QUESTION HAS MADE ME TO FEEL I NEED TO LEARN MORE ABOUT MANIFOLD THEORY AND TOPOLOGY. I NOW FEEL THAT THE ABOVE DERIVATION IS SATISFACTORY.

DIGRESSION

THE QUESTION OF EXTENDING $g_{(2)}$ BEYOND THE SURFACE IS HIGHLY INTERESTING. WE ONLY NEED TO DEFINE IT IN THE VICINITY OF THE SURFACE. FORTUNATELY SUCH A RESULT IS AVAILABLE IN A. GOETZ (PROB. 3, P 268). IF WE MOVE A SURFACE PARALLEL TO ITSELF BY A CONSTANT DISTANCE a ALONG LOCAL NORMAL, THEN

$$\tilde{g}_{(2)} = (1 - 2Ha + Ka^2)^2 g_{(2)}$$

WHERE $\tilde{g}_{(2)}$ AND $g_{(2)}$ ARE THE DETERMINANTS OF 1ST FUND. FORM FOR THE NEW AND ORIGINAL SURFACES, RESPECTIVELY. ALSO H AND K ARE THE MEAN AND GAUSSIAN CURVATURES RESPECTIVELY. THUS

(*) SEE NEXT PAGE!

$g_{(2)}$ IS DEFINED IN THE VICINITY OF $f=0$. NOW WE NOTE THAT

$$\left.\frac{\partial\sqrt{g_{(2)}}}{\partial n}\right|_{f=0} = \left.\frac{\partial\sqrt{g_2}}{\partial a}\right|_{f=0} = -2H\sqrt{g_{(2)}}$$

$$\Rightarrow \quad \nabla\cdot\vec{n}\Big|_{f=0} = \frac{|\nabla f|}{\sqrt{g_{(2)}}}\frac{\partial}{\partial f}\left(\cancel{|\nabla f|}\frac{\sqrt{g_{(2)}}}{\cancel{|\nabla f|}}\right)\Bigg|_{f=0}$$

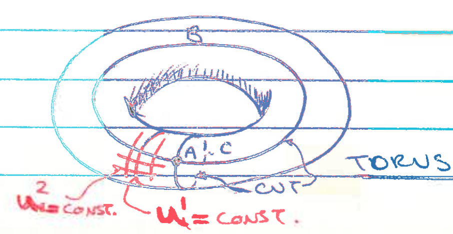
$$= \frac{\cancel{|\nabla f|}}{\sqrt{g_{(2)}}}\frac{1}{\cancel{|\nabla f|}}\frac{\partial\sqrt{g_{(2)}}}{\partial n}\Bigg|_{f=0}$$

$$= -2H \qquad \text{AS EXPECTED.}$$

NOTE THAT $\vec{n} = (0, 0, |\nabla f|)$ IN THE NATURAL BASE VECTORS $(\vec{a}_1, \vec{a}_2, \vec{n}/|\nabla f|)$.

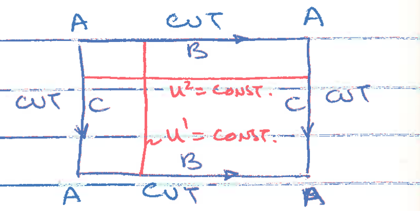
I AM STARTING A SEPARATE NOTEBOOK ON DIFFERENTIAL GEOMETRY.

IS $\int_{f=0} \nabla_2\cdot\vec{K}_t \, dS = 0$ FOR A FINITE CLOSED (COMPACT) SURFACE? PERHAPS! THE PROOF BELOW WAS FOUND AFTER SOME EXAMPLES WERE CONSTRUCTED FOR A "SPHERE"! HOWEVER, I THINK THAT THE GENUS OF THE SURFACE HAS SOMETHING TO DO WITH THE ANSWER. I CAN PROVE THE ABOVE EASILY FOR A TORUS (A SURFACE OF GENUS 1) AS FOLLOWS.



CUT THE SURFACE AS SHOWN AND MAKE u^1 = CONST. u^2 = CONST. LINES AS SHOWN.

WE CAN ALWAYS DO THIS FOR ANY SURFACE OF GENUS 1.



NOW

$$\int \nabla_2 \cdot \vec{K}_t \, dS = \int \frac{\partial K_t^i}{\partial u^i} \, du^1 du^2$$

$$= \int_c^d \int_a^b \frac{\partial K_t^1}{\partial u^1} \, du^1 du^2 + \int_a^b \int_c^d \frac{\partial K_t^2}{\partial u^2} \, du^2 du^1$$

$$= \int_c^d [K_t^1]_a^b \, du^2 + \int_a^b [K_t^2]_c^d \, du^1$$

$$= 0$$

WE ASSUME $\vec{K}_t \in C^1$ HERE. THEREFORE

$$[K_t^1]_a^b = K_t^1(b, u^2) - K_t^1(a, u^2) = 0$$

$$[K_t^2]_c^d = K_t^2(u^1, d) - K_t^2(u^1, c) = 0$$

NOTE THAT THE POINTS (b, u^2) AND (a, u^2) ARE THE SAME POINTS OF THE SURFACE. (SIMILARLY POINTS (u^1, d) AND (u^1, c)). SINCE WE NEED THE ASSUMPTION OF $\vec{K}_t \in C^1$, WE EXPECT THAT THE GENUS OF THE SURFACE ENTER THE PROBLEM. THUS HAIRY BALL THM SAYS THAT $\nexists \vec{K}_t \in C^1$ ON A SPHERE AND OUR RESULT SHOULD NOT HOLD FOR A SPHERE IN GENERAL. I NEED TO LOOK FURTHER INTO THIS. IT IS INTERESTING TO NOTE HOW TOPOLOGY CREPT INTO THE PROBLEM. END OF DIGRESSION.

COMPARING DERIVATIONS (i) AND (ii) ON P 93, WE SEE THAT, UNLIKE THE 1-DIMENSIONAL CASE ON P 92, THE RESTRICTION OF ϕ TO $\xi = 0$ HAS COMPLICATED THE PROBLEM(*). HOWEVER, DERIVATION (ii) HAS CLEARED A LOT OF MYSTERIES IN MY MIND. I KNOW THAT IF I USE THE NOTATION OF MODERN DIFFERENTIAL GEOMETRY, I CAN CLEAN UP MY DERIVATIONS A LOT. HOWEVER, A CONDENSED NOTATION HAS THE DRAWBACK THAT THE DERIVATION IS HARD TO FOLLOW AND IS A STRAIN ON (MY) BRAIN. THE BEST EXAMPLE OF THIS IS THE NOTATION FOR PARTIAL DIFFERENTIATION WHICH EVERYBODY THINKS IT IS EASY, EXCEPT ME. I FINALLY FOUND A BOOK JUST RECENTLY WHICH SUPPORTS MY VIEW VERY NICELY. THE BOOK IS BY H. A. THURSTON AND IS CALLED "PARTIAL DIFFERENTIATION". THE PREFACE STARTS AS FOLLOWS:

"PARTIAL DIFFERENTIATION IS NOTORIOUSLY A DIFFICULT SUBJECT."

AGAIN IN THE INTRODUCTION THURSTON SAYS:

"PART OF THE DIFFICULTY OF PARTIAL DIFFERENTIATION LIES IN THE MERE EXTRA COMPLICATION OF HAVING MORE THAN ONE VARIABLE (......) AND BECOMES LESS WITH PRACTICE. BUT A MORE SERIOUS DIFFICULTY REMAINS. MANY STUDENTS, EVEN GOOD ONES, NEVER BECOME CONFIDENT IN THE SUBJECT (AND SOME WHO ARE TURN OUT ON EXAMINATION TO BE FALSELY SO); OTHERS, THOUGH COMPETENT IN THE TECHNIQUE, ARE NOT CLEAR JUST WHAT THE TECHNIQUE IS BEING USED FOR."

(*) HOWEVER, THIS NEW APPROACH OPENS UP NEW POSSIBILITIES SUCH AS DERIVING MY NEW SUPERSONIC FORMULATION EASILY.

SEP. 83
GPHM

* SOURCE DISTRIBUTIONS ON OPEN SURFACES – ELLIPTIC PROBLEM

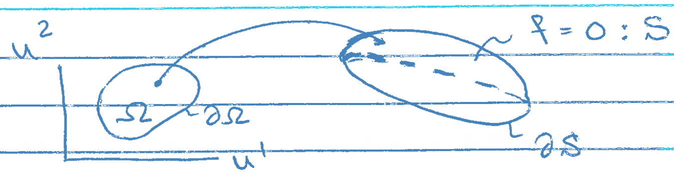
I AM STILL PUZZLED ABOUT THE APPEARANCE OF LINE INTEGRALS IN MY SUPERSONIC PROPELLER ACOUSTICS FORMULATION. I THINK I CAN CLARIFY SOME OF THE MYSTRY NOW. CONSIDER

$$\nabla^2 \phi = - \nabla \cdot \left[\vec{Q} \, |\nabla f| \, \delta(f) \right]$$

WE FIRST LOOK AT THIS PROBLEM BY USING THE SOLUTION

$$4\pi \phi(\vec{x}) = \nabla \cdot \int_{f=0} \frac{\vec{Q}}{r} \, dS$$

WE ASSUME THAT $f=0$ IS AN OPEN PIECE OF A SURFACE IN 3-D WITH PARAMETRIZATION (u^1, u^2).



WE ALSO ASSUME $f=0$ IS INDEPENDENT OF $\vec{x}$ SO THAT

$$4\pi \phi(\vec{x}) = \int_{f=0} \vec{Q} \cdot \nabla_x \left(\frac{1}{r}\right) dS$$

$$= - \int_{f=0} \vec{Q} \cdot \nabla_y \left(\frac{1}{r}\right) dS$$

NOW LET $\vec{y} \rightarrow (u^1, u^2, u^3)$ WHERE u^3 IS DISTANCE ALONG NORMAL TO $f=$ CONST. SURFACE.

WE HAVE

$$\vec{Q}\cdot\nabla_y\left(\frac{1}{r}\right) = Q^i\frac{\partial}{\partial u^i}\left(\frac{1}{r}\right) \qquad i=1\text{-}3$$

$$= Q^i\frac{\partial}{\partial u^i}\left(\frac{1}{r}\right) + Q_n\frac{\partial}{\partial n}\left(\frac{1}{r}\right) \qquad i=1,2$$

$$= Q^i\frac{\partial}{\partial u^i}\left(\frac{1}{r}\right) + \frac{Q_n\cos\theta}{r^2} \qquad i=1,2$$

$$\Rightarrow$$

$$4\pi\phi(\vec{x}) = -\int_{f=0}\frac{Q_n\cos\theta}{r^2}\,dS - \int_{\Omega} Q^i\frac{\partial}{\partial u^i}\left(\frac{1}{r}\right)\sqrt{g_{(2)}}\,du^1du^2 \qquad i=1,2$$

NOW WE MUST USE SOMETHING LIKE DIVERGENCE THM.
WE NOTE THAT

$$I = \int_{\Omega}\frac{\partial}{\partial u^i}\left[Q^i\sqrt{g_{(2)}}\cdot\frac{1}{r}\right]du^1du^2 = \int_{\Omega}\frac{1}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}\left[\frac{\sqrt{g_{(2)}}\,Q_i}{r}\right]dS$$

$$= \int\nabla_s\cdot\left(\frac{\vec{Q}_t}{r}\right)dS$$

WHERE $\nabla_s\cdot(\cdot)$ IS THE SURFACE DIVERGENCE AND $\vec{Q}_t$ IS THE TANGENTIAL COMPONENT OF $\vec{Q}$ ALONG SURFACE S. WE HAVE

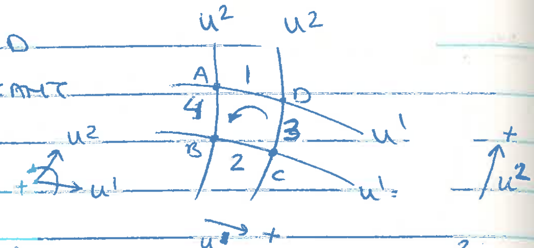
$$I = \int_{\Omega}\frac{1}{r}\frac{\partial}{\partial u^i}\left(Q^i\sqrt{g_2}\right)du^1du^2 + \int_{\Omega}\sqrt{g_{(2)}}\,Q_i\frac{\partial}{\partial u^i}\left(\frac{1}{r}\right)du^1du^2$$

$$\therefore \int\sqrt{g_{(2)}}\,Q_i\frac{\partial}{\partial u^i}\left(\frac{1}{r}\right)du^1du^2 = I - \int_{\Omega}\frac{1}{r}\nabla_s\cdot(\vec{Q}_t)\,dS$$

NOW WE MUST BE ABLE TO RELATE I TO THE LINE INTEGRAL OVER $\partial\Omega$ (STOKES THM). LET US CALL $\vec{Q}_t/r$ AS $\vec{K}$. NOW $\vec{K}$ IS A SURFACE VECTOR. WE WANT TO INTERPRET

$$I = \int_{\Omega}\frac{\partial}{\partial u^i}\left[K^i\sqrt{g_{(2)}}\right]du^1du^2$$

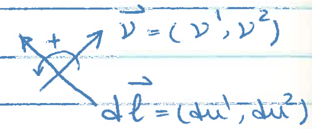
CONSIDER THE REGION ABCD FORMED BY LINES OF CONSTANT u^1 AND u^2 AS SHOWN. THEN



$$I_{ABCD} = (\sqrt{g_{(2)}}\,K^1)_3\, du^2 - (\sqrt{g_{(2)}}\,K^1)_4\, du^2 + (\sqrt{g_{(2)}}\,K^2)_1\, du^1 - (\sqrt{g_{(2)}}\,K^2)_2\, du^1$$

NOTE THAT WE ARE TAKING du^1 AND du^2 AS POSITIVE HERE.

WE NOW CONSIDER THE EDGE OF THE OPEN SURFACE AND THE UNIT OUTWARD NORMAL $\vec{\nu}$ WHICH LIES ON THIS SURFACE. THE RELATION BETWEEN $d\vec{l}$, THE VECTOR TANGENT TO THE EDGE AND $\vec{\nu}$ IS SHOWN IN THE FIGURE.



WE HAVE

$$\vec{\nu}\cdot d\vec{l} = \nu_1\, du^1 + \nu_2\, du^2 = 0 \qquad (*)$$

$$\vec{\nu}\cdot\vec{\nu} = \nu_1 \nu^1 + \nu_2 \nu^2 = 1$$

$$= \nu_1\left[g^{11}\nu_1 + g^{12}\nu_2\right] + \nu_2\left[g^{21}\nu_1 + g^{22}\nu_2\right]$$

$$= \frac{1}{g_{(2)}}\left[g_{22}\nu_1^2 - 2g_{12}\nu_1\nu_2 + g_{11}\nu_2^2\right]$$

USING $(*)$:

$$= \frac{\nu_1^2}{g_{(2)}(du^2)^2}\left[g_{11}(du^1)^2 + 2g_{12}\,du^1 du^2 + g_{22}(du^2)^2\right]$$

$$= \frac{\nu_1^2\, dl^2}{g_{(2)}\,(du^2)^2}$$

$$= \frac{\nu_2^2\, dl^2}{g_{(2)}\,(du^1)^2}$$

$$\therefore \quad \nu_1\, dl = \pm\sqrt{g_{(2)}}\, du^2$$

$$\nu_2\, dl = \pm\sqrt{g_{(2)}}\, du^1$$

FROM (*), WE KNOW THAT THE SIGNS OF ν_1 AND ν_2 MUST BE OPPOSITE. WE MUST TAKE ν_1 WITH POSITIVE SIGN SO THAT

$$J = \nu^1 du^2 - \nu^2 du^1 > 0$$

$$J = (g^{11}\nu_1 + g^{12}\nu_2)\, du^2 - (g^{21}\nu_1 + g^{22}\nu_2)\, du^2$$

IF WE TAKE

$$\nu_1 = \sqrt{g_{(2)}}\, \frac{du^2}{dl}, \quad \nu_2 = \sqrt{g_{(2)}}\, \frac{du^1}{dl}$$

THEN

$$J = \frac{1}{\sqrt{g_{(2)}}\, dl} \left[g_{11}(du^1)^2 + 2 g_{12}\, du^1 du^2 + g_{22}(du^2)^2 \right]$$

$$= \frac{dl}{\sqrt{g_{(2)}}} > 0$$

NOW WE HAVE

$$K^i \nu_i\, dl = K_\nu\, dl$$

$$= K^1 \sqrt{g_{(2)}}\, du^2 - K^2 \sqrt{g_{(2)}}\, du^1 \quad (**)$$

NOW WE WRITE I_{ABCD} WITH du^1 AND du^2 AS SIGNED QUANTITIES

$$I_{ABCD} = (\sqrt{g_{(2)}}\, K^1)_3\, du^2 + (\sqrt{g_{(2)}}\, K^1)_4\, du^2$$

$$- (\sqrt{g_{(2)}}\, K^2)_1\, du^1 - (\sqrt{g_{(2)}}\, K^2)_2\, du^1$$

$$= \oint_{ABCD} K_\nu\, dl \qquad \text{USING } (**) \text{ ABOVE}$$

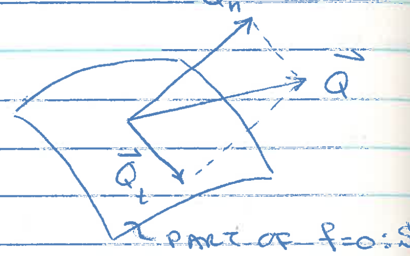
$$\therefore \quad I = \oint_{\partial S} K_\nu\, dl \qquad (\text{WE USE } \partial S \text{ AS BOUNDARY OF } f=0 : S)$$

$$= \oint_{\partial S} \frac{\vec{Q}_f \cdot \vec{\nu}}{r}\, dl$$

THEREFORE, WE HAVE

$$4\pi\phi(\vec{x}) = -\int_{f=0} \frac{Q_n \cos\theta}{r^2}\, dS + \int_{f=0} \frac{1}{r} \nabla_s \cdot (\vec{Q}_t)\, dS$$

$$- \oint_{\partial S} \frac{1}{r} \vec{Q}_t \cdot \vec{\nu}\, d\ell$$



LET US NOW LOOK AT THE SOURCE TERM ALONE AND MANIPULATE IT:

$$\nabla \cdot [\vec{Q}\, |\nabla f|\, \delta(f)] = \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} [Q^i \sqrt{g_{(2)}}]\, \delta(f)$$

$$+ \frac{|\nabla f|}{\sqrt{g_{(2)}}} \langle \sqrt{g_{(2)}} |\nabla f| Q_n \rangle \delta'(f) \qquad i=1,2$$

HERE, AS BEFORE $\langle \cdot \rangle$ STANDS FOR RESTRICTION OF THE FUNCTION TO $f=0$. LET US WORK WITH THE SECOND EXPRESSION FIRST:

$$I_0 = \int \frac{|\nabla f|}{r\sqrt{g_{(2)}}} \langle \sqrt{g_{(2)}}\, Q_n \rangle\, \delta'(f)\, dy$$

$$= \int \frac{|\cancel{\nabla f}|}{r\cancel{\sqrt{g_{(2)}}}} \left\langle \sqrt{g_{(2)}}\, Q_n \right\rangle \delta'(f)\, \frac{\cancel{\sqrt{g_{(2)}}}\, du^1 du^2 df}{|\cancel{\nabla f}|}$$

$$= -\int_{f=0} \langle \sqrt{g_{(2)}} |\cancel{\nabla f}| Q_n \rangle \frac{1}{|\cancel{\nabla f}|} \frac{\partial}{\partial n}\left(\frac{1}{r}\right) du^1 du^2$$

$$= -\int_{f=0} \frac{Q_n \cos\theta}{r^2}\, dS$$

NOTE THAT THE SIGN OF THE CONTRIBUTION OF THIS INTEGRAL TO $4\pi\phi(\vec{x})$ IS CORRECT SINCE WE MUST PUT A MINUS SIGN IN FRONT OF $\nabla\cdot[\vec{Q}|\nabla f|\delta(f)]$ AND THE GREEN'S FUNCTION HAS ANOTHER NEGATIVE SIGN. NOW WE CONSIDER THE FIRST TERM. WE NOTE THAT BECAUSE OF THE FACT THAT THE SURFACE $f=0$ IS FINITE WE WILL HAVE A LINE INTEGRAL AROUND THE BOUNDARY. WE ACTUALLY CAN DO THIS IN A RATHER ELEGANT WAY WHICH SHOWS THE POWER OF GENERALIZED FUNCTIONS. ON SURFACE $f=0$ WE HAVE PARAMETRIZATION (u^1,u^2). LET THE BOUNDARY IN R^2 BE SPECIFIED BY $K(u^1,u^2)=0 \Longrightarrow$

$$\frac{\bar\partial}{\partial u^i}(\sqrt{g_{(2)}}\,Q^i) = \frac{\partial}{\partial u^i}(\sqrt{g_{(2)}}\,Q^i) + \langle\sqrt{g_{(2)}}\,Q^i\rangle\frac{\partial K}{\partial u^i}\,\delta(K)$$

NOW, WE NOTE THAT (ASSUMING $K>0$ ON S)

$$Q^i\frac{\partial K}{\partial u^i} = -\vec{Q}_t\cdot\vec{\nu}\,|\nabla K| \qquad (\vec{\nu} \text{ IS OUTWARD UNIT NORMAL})$$

$\therefore$

$$\frac{|\nabla f|}{\sqrt{g_{(2)}}}\frac{\bar\partial}{\partial u^i}[Q^i\sqrt{g_{(2)}}]\,\delta(f) = \frac{|\nabla f|}{\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}(\sqrt{g_{(2)}}\,Q^i)\,\delta(f)$$

$$-\langle\frac{|\nabla f|}{\sqrt{\cancel{g_{(2)}}}}\rangle\langle\sqrt{\cancel{g_{(2)}}}\,\vec{Q}_t\cdot\vec{\nu}\rangle|\nabla K|\,\delta(f)\,\delta(K)$$

$$I_1 = \int\frac{|\cancel{\nabla f}|}{r\sqrt{g_{(2)}}}\frac{\partial}{\partial u^i}(\sqrt{g_{(2)}}\,Q^i)\,\delta(f)\,\frac{dS\,df}{|\cancel{\nabla f}|}$$

$$= \int_{f=0}\frac{1}{r}\,\nabla_s\cdot\vec{Q}_t\,dS \qquad [\text{SIGN IS CORRECT IN } 4\pi\phi(\vec{x})]$$

$$I_2 = -\int\frac{1}{r}\langle|\nabla f|\rangle|\nabla K|\langle\vec{Q}_t\cdot\vec{\nu}\rangle\,\delta(f)\,\delta(K)\,d\vec{y}$$

WE HAVE

$$d\vec{y} = \frac{df\,ds}{|\nabla f|}$$

$$= \frac{df\,dK\,dl}{|\nabla f| \times |\nabla K|}$$

$$\Longrightarrow \quad I_2 = -\int \langle |\cancel{\nabla f}| \rangle |\cancel{\nabla K}| \langle \vec{Q}_t \cdot \vec{\nu} \rangle \, \delta(f)\,\delta(K) \frac{df\,dK\,dl}{r|\cancel{\nabla f}|\,|\cancel{\nabla K}|}$$

$$= -\oint_{\partial S} \frac{1}{r} \vec{Q}_t \cdot \vec{\nu}\, dl$$

[THE SIGN IS CORRECT IN $4\pi\phi(\vec{x})$]

$$4\pi\phi(\vec{x}) = I_0 + I_1 + I_2$$

I THINK THIS SECOND METHOD IS MORE ELEGANT THAN THE FIRST. AGAIN WE CONFIRM THAT THE LINE INTEGRAL MUST EXIST. WE SHOULD BE ABLE TO USE SIMILAR TECHNIQUE FOR THE PROBLEM

$$\Box^2 \phi = \nabla_4 \cdot [\vec{Q}\, |\nabla f|\, \delta(f)]$$

WHERE $\vec{Q}$ IS A 4-VECTOR. I HAVE HAD SOME PROBLEMS IN OBTAINING THE FINAL FORM OF THE LINE INTEGRAL. MY PROBLEMS ARE RELATED TO THE CHOICE OF COORDINATES. I AM SOMEWHAT CONFUSED. THE SURFACE $f=0$ IS AGAIN ASSUMED OPEN.

* NOTE ON AN INTEGRAL CONTAINING $\delta'(f)$

IN INTEGRATING

$$I = \int q\, \delta'(f)\, d\vec{y} \qquad (*)$$

IN 3-D, WE START BY USING THE FOLLOWING RESULT

$$I_1 = \int \nabla \cdot [q \vec{A}\, \delta(f)]\, d\vec{y} = 0$$

$$= \int \nabla \cdot (q\vec{A})\, \delta(f)\, d\vec{y} + \int q \vec{A} \cdot \nabla f\, \delta'(f)\, d\vec{y}$$

NOW WE TAKE $\vec{A} \cdot \nabla f = 1$ OR $\vec{A} = \frac{\nabla f}{|\nabla f|^2}$.
FROM THIS WE GET

$$I = -\int \nabla \cdot [q \frac{\nabla f}{|\nabla f|^2}]\, \delta(f)\, d\vec{y} \qquad (**)$$

SINCE IN $(*)$, WE CAN REPLACE $\vec{A}$ BY $\langle \vec{A} \rangle$, THE RESTRICTION OF $\vec{A}$ TO $f=0$, DO WE GET A DIFFERENT RESULT THAN $(**)$? THE ANSWER IS NO. WE CAN SEE THIS IF WE USE (u^1, u^2, f) AS OUR NEW COORDINATES
u^3

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & \frac{1}{|\nabla f|^2} \end{bmatrix}$$

$$A_i = (0, 0, \frac{1}{|\nabla f|}) \quad \text{IN} \quad \vec{a}_1, \vec{a}_2, \vec{a}_3 = \frac{\vec{n}}{|\nabla f|}$$

$$g_{(3)} = \frac{g_{(2)}}{|\nabla f|^2}$$

$$\nabla \cdot (q\vec{A}) = \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i} \left[q A^i \frac{\sqrt{g_{(2)}}}{|\nabla f|} \right] \qquad i = 1-3$$

WE NOTE $A^3 = g^{33} A_3 = 1$ SO THAT $\frac{\partial A^3}{\partial u^3} = 0$. THIS IS EQUIVALENT TO TAKING $\langle \vec{A} \rangle$ INSTEAD OF $\vec{A}$.

WE NOTICE THAT A SIMILAR THING HAPPENS FOR

$$I = \int q\,\delta'(f)\,\delta(g)\,d\vec{y}$$

WE START WITH

$$I_1 = \int \nabla \cdot [q\vec{A}\,\delta(f)\,\delta(g)]\,d\vec{y}$$

$$= \int \nabla\cdot(q\vec{A})\,\delta(f)\,\delta(g)\,d\vec{y} + \int q\vec{A}\cdot\nabla f\,\delta'(f)\,\delta(g)\,d\vec{y}$$

$$+ \int q\vec{A}\cdot\nabla g\,\delta(f)\,\delta'(g)\,d\vec{y}$$

$$= 0$$

WE TAKE $\vec{A} \ni \begin{cases} \vec{A}\cdot\nabla f = 1 \\ \vec{A}\cdot\nabla g = 0 \end{cases}$

AND WE GET

$$\int q\,\delta'(f)\,\delta(g)\,d\vec{y} = -\int \nabla\cdot(q\vec{A})\,\delta(f)\,\delta(g)\,d\vec{y}$$

WE FIND THAT

$$\vec{A} = \frac{\vec{n} - \cos\theta\,\vec{n}'}{|\nabla f|\sin^2\theta}$$

WHERE

$$\vec{n} = \frac{\nabla f}{|\nabla f|}, \quad \vec{n}' = \frac{\nabla g}{|\nabla g|}$$

AGAIN WE CAN SHOW THAT IN TAKING THE DIVERGENCE, THE VECTOR $\vec{A}$ CAN BE REPLACED BY $\langle\vec{A}\rangle$, THE RESTRICTION OF $\vec{A}$ TO $f = g = 0$. PROOF IS AS FOLLOWS. INTRODUCE VARIABLES

$$u^1 = f = \phi(\vec{y}),\ u^2 = g = \psi(\vec{y}),\ u^3 = h = \gamma(\vec{y})$$

THEN WE CAN SHOW THAT THE NATURAL

BASE VECTORS ARE

$$\vec{a}_1 = \frac{\nabla\psi \times \nabla\gamma}{J}$$

$$\vec{a}_2 = \frac{\nabla\gamma \times \nabla\phi}{J}$$

$$\vec{a}_3 = \frac{\nabla\phi \times \nabla\psi}{J}$$

WHERE $J = \frac{\partial(\phi,\psi,\gamma)}{\partial(y_1,y_2,y_3)}$. FROM THESE, IT FOLLOWS THAT

$$\vec{a}^1 = \nabla\phi$$

$$\vec{a}^2 = \nabla\psi$$

$$\vec{a}^3 = \nabla\gamma$$

WE HAVE

$$\nabla\cdot(q\vec{A}) = \frac{1}{|J|}\frac{\partial}{\partial u^i}\left[|J|\,q\,A^i\right]$$

HOWEVER, WE HAVE

$$\begin{cases} A^1 = \vec{A}\cdot\nabla\phi = \vec{A}\cdot\nabla f = 1 \\ A^2 = \vec{A}\cdot\nabla\psi = \vec{A}\cdot\nabla g = 0 \end{cases}$$

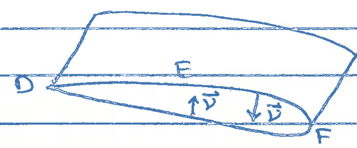
$$\Rightarrow \quad \frac{\partial A^1}{\partial u^1} = \frac{\partial A^2}{\partial u^2} = 0$$

$\therefore$ ONLY $\frac{\partial A^3}{\partial u^3} = \frac{\partial A^3}{\partial h}$ APPEARS IN THE DIVERGENCE TERM, i.e. WE COULD TAKE $\langle\vec{A}\rangle = \vec{A}(0,0,h)$ TO BEGIN WITH.

* SUPERSONIC PROPELLER NOISE THEORY REVISITED

THE EQ. I HAD DERIVED EARLIER FOR SUPERSONIC PROPELLER NOISE CALCULATION WAS PROGRAMMED BY SHARON PADULA. THE PROGRAM WAS CAREFULLY CHECKED AND DEBUGGED. HOWEVER, WE HAVE NOT BEEN GETTING THE EXPECTED RESULTS. SOMETHING SEEMED TO BE WRONG WITH THE LINE INTEGRALS. WORKING ON THIS PROBLEM, I REMEMBERED THAT THE LINE INTEGRAL FROM THE AIRFOIL SHAPED SURFACE HAS BEEN NEGLECTED. THE LINE INTEGRAL $\int_{DEFD}$ MUST ALSO INCLUDE THE FOLLOWING

$$\int_{DEFD} \frac{1}{r}\left[\frac{\rho_0 c^2 b_\nu}{\Lambda_0}\right]_{ret} d\gamma$$



(THIS HAS TO BE ADDED TO EQ. ON P90)

WHERE $\vec{\nu}$ IS THE UNIT VECTOR TANGENT TO THE AIRFOIL SURFACE AND NORMAL TO THE PERIMETER CURVE. THIS CONTRIBUTION IS FOR THICKNESS NOISE ONLY.

BECAUSE OF SUCH PROBLEMS WITH ISOM'S THICKNESS NOISE RESULT, I DECIDED TO GO BACK TO OUR EARLIER FORMULATION WITH v_n. IN FACT, AS WILL BE SHOWN BELOW, WE HAVE ALREADY DONE MOST OF THE HARD WORK.

WE START WITH THE FOLLOWING EQUATION

$$\Box^2 p' = \nabla_4 \cdot \left[\vec{Q}\,|\nabla f|\,\delta(f)\right]$$

$$\vec{Q} = (-p\,\vec{n},\ \rho_0 c^2 M_n)$$

WE NON-DIMENSIONALIZE p & p' WRT $\rho_0 c^2$:

$$\Box^2 p' = \nabla_4 \cdot [\vec{Q}\, |\nabla f|\, \delta(f)]$$
$$\vec{Q} = (-p\vec{n}\,, M_n)$$

WE DO NOT USE SPECIAL SYMBOLS FOR NONDIMENSIONAL PARAMETERS.

LET $\vec{Q} = \vec{Q}_T + \vec{Q}_N$,

$$\vec{N} = \frac{1}{\alpha_n}(\vec{n}, -M_n)$$

WE HAVE $\vec{Q}_T = \vec{Q} - \vec{Q}_N$

$$= \vec{Q} - (\vec{Q}\cdot\vec{N})\vec{N}$$

$$= \frac{M_n}{\alpha_n^2}(1-p)(\vec{M}_n, 1),$$

$$\vec{Q}_N = -\frac{1}{\alpha_n^2}(p + M_n^2)(\vec{n}, -M_n)$$

NOW WE FOLLOW THE PROCEDURE WE USED EARLIER.

$$\Box^2 p' = \nabla_4 \cdot [(\vec{Q}_T + \vec{Q}_N)|\nabla f|\, \delta(f)]$$
$$= \nabla_4 \cdot [\vec{Q}_T |\nabla f|]\, \delta(f) + \nabla_4 \cdot [\vec{Q}_N |\nabla f|\, \delta(f)]$$

$$-4\pi p'(\vec{x}, t) = \int \frac{1}{r} \nabla_4 \cdot [\vec{Q}_T |\nabla f|]\, \delta(f)\, \delta(g)\, d\vec{y}\, d\tau$$
$$+ \int \frac{1}{r} \nabla_4 \cdot [\vec{Q}_N |\nabla f|\, \delta(f)]\, \delta(g)\, d\vec{y}\, d\tau$$
$$= I_1 - \int \frac{1}{r} |\nabla f|\, \vec{Q}_N \cdot \nabla_4 g\, \delta(f)\, \delta'(g)\, d\vec{y}\, d\tau$$
$$\underbrace{+ \int \frac{1}{r^2} |\nabla f|\, \vec{Q}_N \cdot \nabla_4 r\, \delta(f)\, \delta(g)\, d\vec{y}\, d\tau}_{I_2}$$
$$= I_1 + I_2 + \int \nabla_4 \cdot \left[\frac{A}{r} |\nabla f|\, \vec{Q}_N \cdot \nabla_4 g\right] \delta(f)\, \delta(g)\, d\vec{y}\, d\tau$$

$$4\pi p'(\vec{x},t) = I_1 + I_2 - \int \frac{\vec{A}\cdot\nabla_4 r}{r^2}(|\nabla f|\,\vec{Q}_N\cdot\nabla_4 g)\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$
$$+ \int \frac{1}{r}\nabla_4\cdot[|\nabla f|\,\vec{Q}_N\cdot\nabla_4 g\,\vec{A}\,]\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$
$$\equiv I_1 + I_2 - I_3 + I_4$$

HERE I_1 AND I_4 DEPEND ON $\frac{1}{r}$, AND I_2 AND I_3 DEPEND ON $\frac{1}{r^2}$. WE WRITE

$$4\pi p'(\vec{x},t) = I_1' + I_2'$$
$$I_1' = I_1 + I_4$$
$$I_2' = I_2 - I_3$$

$$I_1' = \int \frac{1}{r}\nabla_4\cdot\{|\nabla f|[\vec{Q}_\tau + (\vec{Q}_N\cdot\nabla_4 g)\vec{A}]\}\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$

$$I_2' = \int \frac{|\nabla f|}{r^2}[\vec{Q}_N\cdot\nabla_4 r - (Q_N\cdot\nabla_4 g)\,\vec{A}\cdot\nabla_4 r\,]\,\delta(f)\delta(g)\,d\vec{y}\,d\tau$$

$$\vec{A} = c\alpha_n\left[\frac{\alpha_n(-\vec{\hat{r}},1)}{\tilde{\Lambda}^2} + \lambda_1\vec{N}\right]$$
$$= \frac{c}{\tilde{\Lambda}^2}\left[-\alpha_n^2(\vec{t}_1,0) + (1-M_n\cos\theta)(\vec{M}_n,1)\right]$$

$$\vec{E}_1 \equiv \vec{Q}_\tau + \vec{Q}_N\cdot\nabla_4 g\,\vec{A} = \frac{M_n(1-p)}{\alpha_n^2}(\vec{M}_n,1)$$
$$- \frac{p+M_n^2}{c\,\alpha_n^2}(\vec{n},-M_n)\cdot(\vec{\hat{r}},1)\,\vec{A}$$

$$1-p = 1+M_n^2 - p - M_n^2$$

$$\vec{E}_1 = M_n(\vec{M}_n,1) + \frac{p+M_n^2}{\alpha_n^2}\underbrace{\left[-M_n(\vec{M}_n,1) + \frac{1}{c}(\cos\theta+M_n)\,\vec{A}\right]}_{E_2}$$

$$\vec{E}_2 \equiv -M_n(\vec{M}_n,1) + \frac{1}{c}(\cos\theta + M_n)\vec{A}$$

$$= -M_n(\vec{M}_n,1) + \frac{\cos\theta + M_n}{\tilde{\lambda}^2}\left[-\alpha_n^2(\vec{t}_1,0) + (1 - M_n\cos\theta)(\vec{M}_n,1)\right]$$

$$= -\alpha_n^2\lambda_1(\vec{t}_1,0) + \frac{1}{\tilde{\lambda}^2}\Big[\underbrace{-M_n\tilde{\lambda}^2 + (\cos\theta + M_n)(1 - M_n\cos\theta)}_{E}\Big](\vec{M}_n,1)$$

$$E = -M_n\left[2\alpha_n^2 - (\cos\theta + M_n)^2\right] + (\cos\theta + M_n)(1 - M_n\cos\theta)$$

$$= (\cos\theta + M_n)\left[\cancel{M_n\cos\theta} + M_n^2 + 1 - \cancel{M_n\cos\theta}\right] - 2\alpha_n^2 M_n$$

$$= \alpha_n^2\left[\cos\theta - M_n\right]$$

$$\therefore \vec{E}_2 = -\alpha_n^2\lambda_1(\vec{t}_1,0) + \alpha_n^2\lambda(\vec{M}_n,1)$$

$$\vec{E}_1 = (p + M_n^2)\left[\lambda(\vec{M}_n,1) - \lambda_1(\vec{t}_1,0)\right] + M_n(\vec{M}_n,1)$$

$$\lambda(\vec{M}_n,1) - \lambda_1(\vec{t}_1,0) = \lambda\left[\vec{A}_3 - (\vec{M}_t,0)\right] - \lambda_1(\vec{t}_1,0)$$

$$= -\left[\lambda(\vec{M}_t,0) + \lambda_1(\vec{t}_1,0)\right] + \lambda\vec{A}_3$$

$$\equiv -\vec{B}$$ (SEE P45, P70)

$$M_n(\vec{M}_n,1) = M_n(\vec{A}_3 - (\vec{M}_t,0))$$

$$= -\vec{B}'$$

$$\vec{B}' = (M_n\mu^1, M_n\mu^2, -M_n, 0) = \beta'^i$$

IN NATURAL BASE VECTORS $(\vec{A}_1, \vec{A}_2, \vec{A}_3, \vec{A}_4)$.

WE CAN NOW WRITE I_1 AS FOLLOWS :

$$I_1' = \checkmark \int \frac{1}{r} \nabla_4 \cdot [|\nabla f|(p+M_n^2)\vec{B}]\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

$$\checkmark \int \frac{1}{r} \nabla_4 \cdot [|\nabla f|\,\vec{B}']\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau$$

THE FIRST INTEGRAL IS IN FACT SIMILAR TO EQ.(25), P, 43. IT CAN BE FOUND BY REPLACING $\tilde{p}$ BY $p+M_n^2$, REMEMBERING THAT IT ALSO HAS NEAR-FIELD CONTRIBUTION. FORTUNATELY, WE HAVE

$$E' = \vec{Q}_N \cdot \nabla_4 r - (Q_N \cdot \nabla_4 g)(\vec{A} \cdot \nabla_4 r)$$

$$= -\frac{p+M_n^2}{\alpha_n^2}\left\{(\vec{n},-M_n)\cdot(-\vec{r},0) - [(\vec{n},-M_n)\cdot \underbrace{(-\vec{r},1)}_{C}](\vec{A}\cdot\nabla_4 r)\right\}$$

$$\vec{A}\cdot\nabla_4 r = \frac{C}{\tilde{\lambda}^2}\left[-\alpha_n^2(\vec{t}_1,0) + (1-M_n\cos\theta)(\vec{M}_n,1)\right]\cdot(-\hat{r},0)$$

$$= \frac{C}{\tilde{\lambda}^2}\left[+\alpha_n^2\sin^2\theta + M_n^2\cos^2\theta - M_n\cos\theta\right]$$

$$= \frac{C}{\tilde{\lambda}^2}\left[\tilde{\lambda}^2 - (1-M_n\cos\theta)\right]$$

$$= C - \frac{C(1-M_n\cos\theta)}{\tilde{\lambda}^2}$$

$$E' = -\frac{p+M_n^2}{\alpha_n^2}\left\{\underbrace{-\cos\theta + (\cos\theta+M_n)\left(1-\frac{1-M_n\cos\theta}{\tilde{\lambda}^2}\right)}_{E,\ P113}\right\}$$

$$= -\frac{p+M_n^2}{\alpha_n^2}\,\frac{\overbrace{M_n\tilde{\lambda}^2 - (\cos\theta+M_n)(1-M_n\cos\theta)}}{\tilde{\lambda}^2}$$

$$= (p+M_n^2)\frac{\cos\theta - M_n}{\tilde{\lambda}^2}$$

$$= \lambda\,(p+M_n^2)\ \checkmark$$

$\therefore$ I_2' IS ALSO SIMILAR TO EQ (24), P43 IF WE REPLACE $\tilde{p}$ BY $p + M_n^2$. $\therefore$ ALL OF OUR PREVIOUS RESULTS ARE VALID BUT WE MUST ADD THE EFFECT OF 2ND INTEGRAL IN I_1', PREVIOUS PAGE. THIS INTEGRAL WILL INTRODUCE A SURFACE AND ALSO SOME ADDITIONAL LINE INTEGRALS.

$$\bar{\nabla}_4 \cdot [|\nabla f| \vec{B}'] = \nabla_4 \cdot [|\nabla f| \vec{B}'] + |\nabla f| \vec{B}' \cdot \nabla_4 K \, \delta(K)$$

$\Rightarrow$ ADDITIONAL LINE INTEGRALS, ASSUMING THAT THE SHOCK IS NOT OSCILLATING, HAS THE FORM

$$I_\ell = \checkmark \int \frac{1}{r}\left[\frac{M_n \vec{M}_t \cdot \vec{\nu}}{\Lambda_o}\right] d\gamma$$

WE HAVE THE SAME LINE INTEGRALS OVER AB, CD & DEFO. ALSO

$$\nabla_4 \cdot [|\nabla f| B'] = \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\, \beta'^i\right] \quad i = 1-4$$

$$= \frac{|\nabla f|}{\sqrt{g_{(2)}}} \frac{\partial}{\partial u^i}\left[\sqrt{g_{(2)}}\, \beta'^i\right] \quad i = 1-2$$

$$\checkmark -\frac{1}{c}|\nabla f| \dot{M}_n$$

$$= |\nabla f| \left[-\frac{1}{c}\dot{M}_n + \frac{\beta'^i}{\sqrt{g_{(2)}}} \frac{\partial \sqrt{g_{(2)}}}{\partial u^i} + \frac{\partial \beta'^i}{\partial u^i}\right] \quad i = 1-2$$

$$\beta'^i = M_n u^i$$

$$\frac{\beta'^{i}}{\sqrt{g_{(2)}}}\frac{\partial\sqrt{g_{(2)}}}{\partial u^{i}} + \frac{\partial \beta'^{i}}{\partial u^{i}} = M_n \mu^i \Gamma_{ij}^{j} + \mu^i \frac{\partial M_n}{\partial u^i} + M_n \frac{\partial \mu^i}{\partial u^i}$$

$$= M_n \mu^i \Gamma_{ij}^{j} + \mu^i \left[\frac{1}{c}\Omega_i - \mu^k b_{ik}\right]$$

$$+ M_n \left(-\mu^j \Gamma_{ji}^{i} + M_n b_i^{\,i}\right)$$

$$= \frac{1}{c}\mu^i \Omega_i - \mu^i \mu^j b_{ij} + M_n^2 b_i^{\,i}$$

$$= \frac{1}{c}\vec{M}_t \cdot \vec{\Omega} - M_t^2 \kappa_m + 2 M_n^2 H$$

κ_m = NORMAL CURVATURE IN DIRECTION OF $\vec{M}_t$.

WE DENOTE $\tilde{Q}'_F = -\nabla_4 \cdot \left[|\nabla f| \vec{B}'\right] / |\nabla f|$ {THE MINUS SIGN COMES FROM 2ND INTEGRAL OF I'_1, P114

$$= \mp \frac{1}{c}\dot{M}_n - \frac{1}{c}\vec{M}_t \cdot \vec{\Omega}$$

$$+ M_t^2 \kappa_m - 2 M_n^2 H$$

~~WE ALSO NEED~~

(κ_m IS DENOTED κ_t IN AIAA TECH. NOTE)

$$\frac{\partial M_n^2}{\partial \sigma_b} = 2 M_n \frac{\partial M_n}{\partial \sigma_b}$$

$$= 2 M_n \frac{\vec{b}}{b} \cdot \nabla_2 M_n$$

$$= \frac{2 M_n}{b}\left(\lambda \mu^i + \lambda_1 \alpha^i\right) \frac{\partial M_n}{\partial u^i}$$

$$= \frac{2 M_n}{b}\left(\lambda \mu^i + \lambda_1 \alpha^i\right)\left(\frac{1}{c}\Omega_i - \mu^j b_{ij}\right)$$

$$= \frac{2 M_n}{b}\left[\frac{1}{c}\vec{b}\cdot\vec{\Omega} - \underbrace{\lambda M_t^2 \kappa_m - \lambda_1 \alpha^i \mu^j b_{ij}}_{\text{THIS CAN BE SIMPLIFIED. SEE P118.}}\right]$$

$$\alpha^i \mu^j b_{ij} = \tilde{\alpha}^1 \tilde{\mu}^1 \kappa_1 + \tilde{\alpha}^2 \tilde{\mu}^2 \kappa_2$$

$$b = |\vec{b}| = |\lambda \vec{M}_t + \lambda_1 \vec{t}_1|$$

WE SUMMARIZE OUR RESULTS FOR

THICKNESS NOISE

$$\frac{4\pi}{\rho_0 c^2}\, p'_T(\vec{x},t) = \int_{F=0} \frac{1}{r}\left\{\frac{1}{\Lambda}\left(M_n^2 Q_F + \tilde{Q}'_F + \frac{2\lambda}{c} M_n \dot{M}_n - \tilde{Q}''_F\right\}_{ret} d\Sigma\right.$$

$$+ \int_{F=0} \frac{1}{r^2}\left[\frac{M_n^2 Q_N}{\Lambda}\right]_{ret} d\Sigma$$

$$- \int_{TE} \frac{1}{r}\left\{\frac{(M_n b'_\nu)_u + (M_n b'_\nu)_\ell}{\Lambda_0}\right\}_{ret} d\gamma$$

$$- \int_{DEFD} \frac{1}{r}\left[\frac{M_n b'_\nu}{\Lambda_0}\right]_{ret} d\gamma$$

$$\tilde{Q}'_F = M_t^2 \kappa_m + \frac{1}{c}\dot{M}_n - \frac{1}{c}\vec{M}_t\cdot\vec{\Omega} - 2M_n^2 H$$

$$\tilde{Q}''_F = b\,\frac{\partial M_n^2}{\partial \sigma_b}$$

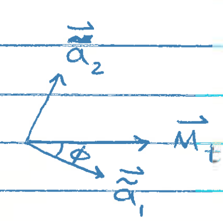
$$= 2M_n\left[\frac{1}{c}\vec{b}\cdot\vec{\Omega} - \lambda M_t^2 \kappa_m - \lambda_1(\tilde{\alpha}^1\tilde{\mu}^1\kappa_1 + \tilde{\alpha}^2\tilde{\mu}^2\kappa_2)\right]$$

$$\vec{b} = \lambda \vec{M}_t + \lambda_1 \vec{t}_1$$

$$\vec{b}' = M_n \vec{b} + \vec{M}_t$$

$$\kappa_m = \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi$$

NORMAL CURVATURE IN THE DIRECTION OF $\vec{M}_t$.



WE SIMPLIFY FURTHER THE INTEGRAND OF THE FIRST SURFACE INTEGRAL:

$$E = \frac{2\lambda}{c} M_n \dot{M}_n + \tilde{Q}'_F - \tilde{Q}''_F$$

$$= \frac{1}{c}(2\lambda M_n + 1)\dot{M}_n - \frac{1}{c}(\vec{M}_t + 2M_n\vec{b})\cdot\vec{\Omega}$$

$$+ M_t^2 \kappa_m + 2M_n\left[\lambda M_t^2 \kappa_m + \lambda_1(\tilde{\alpha}^1\tilde{\mu}^1\kappa_1 + \tilde{\alpha}^2\tilde{\mu}^2\kappa_2)\right]$$

$$- 2M_n^2 H$$

$$\lambda M_t^2 \kappa_m + \lambda_1(\tilde{\alpha}^1\tilde{\mu}^1\kappa_1 + \tilde{\alpha}^2\tilde{\mu}^2\kappa_2) = \lambda \mu^i\mu^j b_{ij} + \lambda_1 \mu^i\alpha^j b_{ij}$$

$$= \mu^i(\lambda\mu^j + \lambda_1\alpha^j) b_{ij}$$

$$= \tilde{\mu}^1\tilde{b}^1\kappa_1 + \tilde{\mu}^2\tilde{b}^2\kappa_2$$

$$E = \frac{1}{c}(2\lambda M_n + 1)\dot{M}_n - \frac{1}{c}(\vec{M}_t + 2M_n\vec{b})\cdot\vec{\Omega}$$

$$+ M_t^2\kappa_m + 2M_n(\tilde{\mu}^1\tilde{b}^1\kappa_1 + \tilde{\mu}^2\tilde{b}^2\kappa_2) - 2M_n^2 H$$

$$\equiv \frac{1}{c}(2\lambda M_n + 1)\dot{M}_n + Q'_F$$

$$Q'_F = -\frac{1}{c}(\vec{M}_t + 2M_n\vec{b})\cdot\vec{\Omega} + M_t^2\kappa_m - 2M_n^2 H$$

$$+ 2M_n(\tilde{\mu}^1\tilde{b}^1\kappa_1 + \tilde{\mu}^2\tilde{b}^2\kappa_2)$$

THICKNESS NOISE

$$4\pi \frac{p_T'(\vec{x},t)}{\rho_0 c^2} = \int_{F=0} \frac{1}{r}\left\{\frac{1}{\Lambda}\left(M_n^2 Q_F + \frac{1}{c}(2\lambda M_n + 1)\dot{M}_n + Q_F'\right)\right\}_{ret} d\Sigma$$

$$+ \int_{F=0} \frac{1}{r^2}\left[\frac{M_n^2 Q_N}{\Lambda}\right]_{ret} d\Sigma$$

$$- \int_{TE} \frac{1}{r}\left\{\frac{(M_n b_\nu')_u + (M_n b_\nu')_\ell}{\Lambda_0}\right\} d\gamma$$

$$- \int_{DEFD} \frac{1}{r}\left[\frac{M_n b_\nu'}{\Lambda_0}\right] d\gamma$$

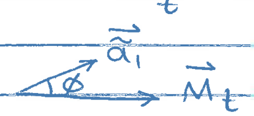
$$Q_F' = -\frac{1}{c}(\vec{M}_t + 2 M_n \vec{b})\cdot\vec{\Omega} + M_t^2 \kappa_m - 2 M_n^2 H$$
$$+ 2 M_n (\tilde{u}^1 \tilde{b}^1 \kappa_1 + \tilde{u}^2 \tilde{b}^2 \kappa_2) \quad (*)$$

$$b_\nu' = \vec{b}' \cdot \vec{\nu} = M_n b_\nu + \vec{M}_t \cdot \vec{\nu}$$

$$\vec{b}' = M_n \vec{b} + \vec{M}_t$$

κ_m = NORMAL CURVATURE ALONG $\vec{M}_t$

$$= \kappa_1 \cos^2\phi + \kappa_2 \sin^2\phi$$



(*) I INTEND TO CALL $\frac{1}{c}(2\lambda M_n + 1)\dot{M}_n + Q_F'$ AS Q_F' IN MY PUBLICATIONS.

NOTE WRITTEN ON JUNE 9, 1991

IT HAS BEEN EIGHT YEARS I HAVE NEGLECTED TO WRITE MY RESEARCH PROGRESS IN THIS BOOK. MY PUBLICATIONS DOCUMENT THE PROGRESS I HAVE MADE IN THESE YEARS. I KEPT THE MATHEMATICAL DERIVATIONS ON LOOSE LEAF PAPERS AND CANNOT COMPLETELY REMEMBER MY MENTAL PROCESSES. IN THESE YEARS, MY MAIN ACHIEVEMENTS ARE:

i) MAKING MY SUPERSONIC ACOUSTIC FORMULATION WORK ON A COMPUTER. THE CODE WAS ORIGINALLY CALLED DFP-ATP (DFP: DUNN-FARASSAT-PADULA). IT IS NOW CALLED ASSPIN (ADVANCED SUBSONIC AND SUPERSONIC PROPELLER INDUCED NOISE, A NAME SUGGESTED BY PADULA).

ii) DEVELOPING A CLEAR MATHEMATICAL UNDERSTANDING OF THE FW-H EQUATION. I GAVE 10 HOURS OF LECTURES ON THIS AT UNIVERSITY OF ROME "LA SAPIENZA", SUMMER 1989. THE ROLES OF DIFFERENTIAL GEOMETRY AND GENERAL TENSOR ANALYSIS ARE SIGNIFICANT IN FINDING SOLUTIONS. (SEE P.123)

iii) APPLYING ACOUSTIC EQUATIONS TO AERODYNAMIC PROBLEMS. BOB MILLIKEN CODED THE STEADY SOLUTION IN UNSTEADY GROUND-FIXED FRAME. MIKE MYERS AND I HAD AN AIAA PAPER ON THE THEORY. MARK DUNN IS WRITING A PH.D. THESIS ON THIS SUBJECT FOR ROTATING BLADES. I FOUND LUIGI MORINO VERY INTERESTED IN THIS SUBJECT. IN FACT, HE IS ACTIVELY WORKING ALONG

THE LINES OF MY LECTURES IN ROME. ALSO YUNG-JANG LEE, WHO WAS MY NRC POST-DOCTORAL FELLOW (MAY 90 - APR 91) HAS FOLLOWED OUR WORK CLOSELY APPLYING OUR METHOD TO VELOCITY POTENTIAL SUCCESSFULLY. LEE WAS FULLY EDUCATED IN TAIWAN BUT HIS PH.D. WORK WAS BASED ON MY WORK WITHOUT MY INVOLVEMENT.

iv) REDERIVING W.R. MORGANS' KIRCHHOFF FORMULA FOR MOVING SURFACES BY GENERALIZED FUNCTION THEORY. MIKE MYERS AND I PUBLISHED A JSV ARTICLE ON THIS. OUR DERIVATION IS VERY CLEAN SHOWING ONCE MORE THE POWER OF GENERALIZED FUNCTION THEORY. I FEEL THAT MUCH FUTURE APPLICATIONS WILL FOLLOW.

v.) WORKING ON QUADRUPOLE SOURCE TERM OF THE FW-H EQ. I SHOWED THAT SHOCKS ARE POTENT NOISE GENERATORS. ALSO THE CONTRIBUTIONS OF SURFACE DIPOLE AND MONOPOLE, EXTRACTED FROM QUADRUPOLE TERM OF FW-H EQ., TO THE NOISE OF ROTATING BLADES CAN BE SHOWN TO BE SMALL COMPARED TO THICKNESS AND LOADING TERMS. (*)

vi.) UNDERSTANDING SINGULARITIES OF THE ACOUSTIC FIELD OF ROTATING BLADES. ENRICO DE BERNARDIS DID A PH.D. THESIS ON THIS WORKING WITH ME. MORE RECENTLY THE ROLE OF QUADRUPOLES IN REMOVAL OF SINGULARITIES OF THICKNESS AND LOADING NOISE WAS SHOWN BY ME. MIKE MYERS AND I WILL PRESENT A PAPER

(*) THE QUADRUPOLES IN BL OF A VISCOUS MODEL WAS SHOWN TO REMOVE THE LARGE CONTRIBUTIONS OF SURFACE SOURCES.

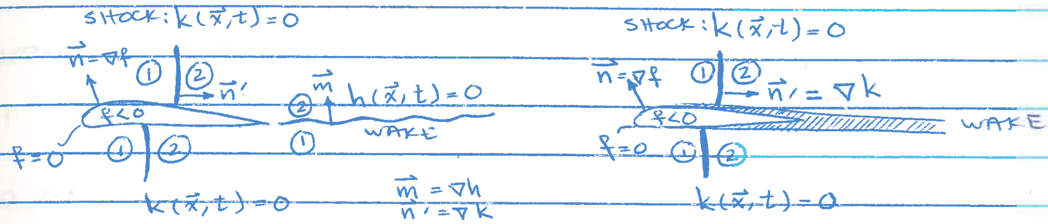
ON THIS SUBJECT IN AIAA 14TH AEROACOUSTICS MEETING IN AACHEN, GERMANY, MAY 1992. (*)

Parts of these pages have been removed by Steve Miller of NASA Langley.
The text removed is of a personal nature and is non-technical.

(*) I DIDN'T WRITE THIS PAPER EITHER!

* THE MATHEMATICAL ASPECTS OF THE FW-H EQ.

DEVELOPED MOSTLY IN ROME, ITALY, JULY 1989, U. OF ROME, LA SAPIENZA.



INVISCID MODEL VISCOUS MODEL

WE START WITH INVISCID MODEL FIRST. THE VISCOUS MODEL DIFFERS IN THE WAKE MODELLING WHICH, INSTEAD OF BEING A VORTEX SHEET, IS A REGION WITH HIGH VELOCITY GRADIENT. WE WILL CONCENTRATE ON THE QUADRUPOLE TERM FIRST. TAKING FIRST DERIVATIVE OF $T_{ij}H(f)$, WE GET

$$\frac{\partial}{\partial x_j}\left[T_{ij}H(f)\right] = \frac{\partial T_{ij}}{\partial x_j}H(f) + T_{ij}n_j\delta(f)$$

$$+\Delta T_{ij}n'_j\delta(k) + \Delta T_{ij}m_j\delta(h) \qquad (1)$$

WE TAKE A 2ND DERIVATIVE AND ADD THICKNESS AND LOADING TERMS. WE THUS WRITE FW-H EQ. AS

$$\Box^2 p' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}H(f) \qquad \text{PURE QUADRUPOLE TERM}$$

$$\left.\begin{aligned}&+\frac{\partial}{\partial t}\left[\rho_0 v_n \delta(f)\right] - \frac{\partial}{\partial x_i}\left[p n_i \delta(f)\right]\\ &+\frac{\partial T_{ij}}{\partial x_j}n_i\delta(f) + \frac{\partial}{\partial x_i}\left[T_{ij}n_j\delta(f)\right]\end{aligned}\right\} \text{BLADE SURFACE TERMS}$$

$$+\Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right)n'_i\delta(k) + \frac{\partial}{\partial x_i}\left[\Delta T_{ij}n'_j\delta(k)\right] \qquad \text{SHOCK SURFACE TERMS}$$

$$+\Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right)m_i\delta(h) + \frac{\partial}{\partial x_i}\left[\Delta T_{ij}m_j\delta(h)\right] \qquad \text{WAKE SURFACE TERMS}$$

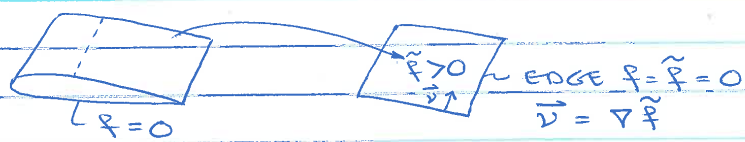
(2)

HERE WE HAVE DEFINED $\Delta = [\;]_2 - [\;]_1$ WHERE SUBSCRIPTS REFER TO REGIONS 1 AND 2 IN THE FIGS ON PREVIOUS PAGE. THIS EQ. SHOWS THAT WAKES ARE POTENTIALLY SOURCES OF SOUND. THIS DOES NOT SEEM TO BE A WELL-KNOWN FACT.

WE MANIPULATE FURTHER SOME OF THE TERMS OF EQ.(2). WE CONSIDER OPEN SURFACES EVEN FOR THE BLADE SURFACE BECAUSE WE INTEND TO USE OUR RESULTS FOR SUPERSONICALLY MOVING SOURCES. WE CONSIDER TERMS OF THE FOLLOWING FORMS:

$$E_1 = \frac{\partial}{\partial t}\left[\rho_0 v_n H(\tilde{f})\delta(f)\right] \qquad (3-a)$$

$$E_2 = \frac{\partial}{\partial x_i}\left[p n_i H(\tilde{f})\delta(f)\right] \qquad (3-b)$$



WE DEFINE THE EDGE OF THE OPEN SURFACE BY $f = \tilde{f} = 0$. IN GENERAL, WE PUT A COORDINATE NET (u^1, u^2) ON $f=0$, $\tilde{f}>0$. THEN WE DEFINE $\tilde{f}(u^1, u^2, t) = 0$ IN SUCH A WAY THAT $\tilde{f}>0$ ON THE OPEN PIECE OF SURFACE AND $\vec{\nu} = \nabla\tilde{f}$ WHERE $\vec{\nu}$ IS THE UNIT INWARD GEODESIC NORMAL OF THE SURFACE, i.e. $\vec{\nu}\cdot\nabla f = 0$, $|\vec{\nu}| = 1$.

EVALUATION OF E_1

WE ASSUME $v_n = v_n(u^1, u^2, t)$. HERE (u^1, u^2) CAN BE VIEWED AS LAGRANGIAN VARIABLES, WE HAVE

ASSUMED HERE THAT $v_n(u^1, u^2, t) = \hat{v}_n$ WHERE THE HAT (∧) STANDS FOR THE RESTRICTION OF v_n TO $f = 0$. INDEED, WE MUST CARRY ALONG (∧) WHEN THE TERM $\delta'(f)$ APPEARS TO REMIND US OF THE RESTRICTION PROCESS. WE HAVE

$$E_1 = \rho_0 \hat{v}_n H(\tilde{f})\delta(f) - \rho_0 v_n v_\nu \delta(\tilde{f})\delta(f) - \rho_0 \hat{v}_n^2 H(\tilde{f})\delta'(f) \qquad (4)$$

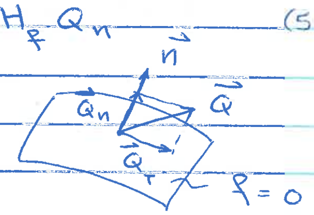
HERE, WE HAVE USED $\partial\tilde{f}/\partial t = -v_\nu$ WHERE v_ν IS THE PROJECTION OF VELOCITY OF THE EDGE ALONG $\vec{\nu}$.

EVALUATION OF E_2

WE USE THE FOLLOWING RESULT:

$$\nabla\cdot\vec{Q} = \nabla_2\cdot\vec{Q}_T + \frac{\partial Q_n}{\partial n} - 2H_f Q_n \qquad (5)$$

WHERE $\vec{Q} = \vec{Q}_T + Q_n\vec{n}$ AND H_f IS THE MEAN CURVATURE OF THE SURFACE $f = 0$. IN E_2 WE HAVE



$$\vec{Q} = \vec{Q}_n = \hat{p}H(\tilde{f})\vec{n}\delta(f)$$

WHERE AGAIN $\hat{p} = p|_{f=0}$ AS BEFORE. THEREFORE

$$E_2 = \frac{\partial}{\partial n}[\hat{p}H(\tilde{f})\delta(f)] - 2H_f\hat{p}H(\tilde{f})\delta(f)$$

$$= \hat{p}H(\tilde{f})\delta'(f) - 2H_f\hat{p}H(\tilde{f})\delta(f) \qquad (6)$$

HERE WE HAVE USED THE FACT THAT $\partial\hat{p}/\partial n = 0$. WE MUST RETAIN ∧ ON $\hat{p}$ IN THE TERM INVOLVING $\delta'(f)$.

THE FW-H EQ. IN A NEW FORM

WE NOW ASSUME THAT THE EDGES OF THE SHOCKS AND THE WAKE ARE DESCRIBED BY $k = \tilde{k} = 0$ AND $h = \tilde{h} = 0$, RESPECTIVELY. THESE SURFACES ARE DEFINED SUCH THAT $\tilde{k} > 0$ ON THE SHOCK SURFACES AND $\tilde{h} > 0$ ON THE WAKE SURFACE. LET US DEFINE

$$Q_i = T_{ij} n_j \quad \text{(BODY)} \quad (7\text{-a})$$

$$q_i = \Delta T_{ij} n'_j \quad \text{(SHOCK)} \quad (7\text{-b})$$

$$q'_i = \Delta T_{ij} m_j \quad \text{(WAKE)} \quad (7\text{-c})$$

WE ALSO DENOTE THE GEODESIC NORMALS (UNIT, INWARD) TO THE SHOCK AND WAKE SURFACES AS $\vec{\nu}'$ AND $\vec{\mu}$, RESPECTIVELY. WE NOW DEFINE THE EXPRESSIONS

$$E_3 = \frac{\partial}{\partial x_i}\left[\hat{Q}_i H(\tilde{f})\delta(f)\right] = \nabla\cdot\left[\hat{\vec{Q}}\, H(\tilde{f})\,\delta(f)\right] \quad (8a)$$

$$E_4 = \frac{\partial}{\partial x_i}\left[\hat{q}_i\, H(\tilde{k})\delta(k)\right] = \nabla\cdot\left[\hat{\vec{q}}\, H(\tilde{k})\delta(k)\right] \quad (8\text{-b})$$

$$E_5 = \frac{\partial}{\partial x_i}\left[\hat{q}'_i\, H(\tilde{h})\delta(h)\right] = \nabla\cdot\left[\hat{\vec{q}}'\, H(\tilde{h})\delta(h)\right] \quad (8\text{-c})$$

USING EQ.(5), WE GET

$$E_3 = \nabla_2\cdot\left[\vec{Q}_T H(\tilde{f})\right]\delta(f) - 2H_f\hat{Q}_n H(\tilde{f})\delta(f) + \hat{Q}_n H(\tilde{f})\delta'(f)$$

$$= \nabla_2\cdot\vec{Q}_T\, H(\tilde{f})\delta(f) + Q_\nu\,\delta(\tilde{f})\delta(f)$$

$$- 2H_f\hat{Q}_n H(\tilde{f})\delta(f) + \hat{Q}_n H(\tilde{f})\delta'(f) \quad (9)$$

SIMILARLY, WE HAVE

$$E_4 = \nabla_2 \cdot \vec{q}_T H(\tilde{k})\delta(k) + q_{\nu'}\delta(\tilde{k})\delta(k) - 2H_k \hat{q}_{n'} H(\tilde{k})\delta(k) + \hat{q}_{n'} H(\tilde{k})\delta'(k) \qquad (10)$$

$$E_5 = \nabla_2 \cdot \vec{q}\,'_T H(\tilde{h})\delta(h) + q'_\mu \delta(\tilde{h})\delta(k) - 2H_h q'_m H(\tilde{h})\delta(h) + \hat{q}'_m H(\tilde{h})\delta'(h) \qquad (11)$$

SUBSTITUTING E_1 TO E_5 IN THE FW-H EQ. (2), WE GET

$$\Box^2 p' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} H(f) \qquad \text{PURE QUADRUPOLE TERM}$$

$$+ \sum_{\alpha=1}^{N} \left\{ (\rho_o v_n - 2H_f \hat{p}) H(\tilde{f}_\alpha)\delta(f) - (\rho_o \hat{v}_n^2 + \hat{p}) H(\tilde{f}_\alpha)\delta'(f) - \rho_o v_n v_\nu \delta(\tilde{f}_\alpha)\delta(f) \right\} \qquad \text{THICKNESS + LOADING TERMS}$$

$$\sum_{\alpha=1}^{N} \left\{ \left(\nabla_2 \cdot \vec{Q}_T - 2H_f Q_n + \frac{\partial T_{ij}}{\partial x_j} n_i \right) H(\tilde{f}_\alpha)\delta(f) + \hat{Q}_n H(\tilde{f}_\alpha)\delta'(f) + Q_\nu \delta(\tilde{f}_\alpha)\delta(f) \right\} \qquad \text{BLADE SURFACE TERMS FROM QUAD. SOURCE}$$

$$+ \sum_{\alpha=1}^{N'} \left\{ \left[\nabla_2 \cdot \vec{q}_T - 2H_k q_{n'} + \Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right) n'_i \right] H(\tilde{k}_\alpha)\delta(k) + \hat{q}_{n'} H(\tilde{k}_\alpha)\delta'(k) + q_{\nu'}\delta(\tilde{k}_\alpha)\delta(k) \right\} \qquad \text{SHOCK SURFACE TERMS}$$

$$+ \sum_{\alpha=1}^{N''} \left\{ \left[\nabla_2 \cdot \vec{q}\,'_T - 2H_h q'_m + \Delta\left(\frac{\partial T_{ij}}{\partial x_j}\right) m_i \right] H(\tilde{h}_\alpha)\delta(h) + \hat{q}'_m H(\tilde{h}_\alpha)\delta'(h) + q'_\mu \delta(\tilde{h}_\alpha)\delta(h) \right\} \qquad \text{WAKE SURFACE TERMS}$$

(12)

IN THIS EQUATION, WE HAVE ASSUMED THAT THE BLADE SURFACE, SHOCK SURFACE AND THE WAKE SURFACE HAVE BEEN DIVIDED INTO N, N' AND N'' PANELS, RESPECTIVELY.

FOR VISCOUS MODEL, WE DEFINE

$$P_i = P_{ij} n_j$$

AND FOR LOADING NOISE WE HAVE

$$\nabla \cdot [\vec{\hat{P}} H(\tilde{f}) \delta(f)] = \nabla_2 \cdot \vec{P}_T H(\tilde{f}) \delta(f) + P_\nu \delta(\tilde{f}) \delta(f) - 2 P_n H_f H(\tilde{f}) \delta(f) + \hat{P}_n H(\tilde{f}) \delta'(f) \quad (13)$$

IN EQ. (2), WE REPLACE $p n_i$ WITH P_i AND IN EQ. (12), THE THICKNESS AND LOADING TERMS BECOME

$$\sum_{\alpha=1}^{N} \Big\{ (\rho_0 \dot{v}_n - \nabla_2 \cdot P_T + 2 P_n H_f) H(\tilde{f}_\alpha) \delta(f) - (\rho_0 \hat{v}_n^2 + \hat{P}_n) H(\tilde{f}_\alpha) \delta'(f) - (P_\nu + \rho_0 v_n v_\nu) \delta(\tilde{f}_\alpha) \delta(f) \Big\} \quad (14)$$

THE SOURCE ORDERING

WE NOW STUDY QUALITATIVELY THE ORDERING OF SOURCE TERMS OF THE FW-H EQUATION. THIS STUDY IS NECESSARILY APPROXIMATE AND CAN ONLY BE USED AS A GUIDE IN DECIDING WHICH TERMS SHOULD BE CODED FIRST. I AM NOT FOND OF THIS KIND OF ARGUMENTS WHICH SOME PEOPLE CONSIDER THE END PRODUCT OF THEIR RESEARCH. IN REALITY, THE QUALITATIVE ARGUMENTS FORM THE INITIAL STAGE OF RESEARCH. ULTIMATELY, IT IS THE NUMERICAL ANALYSIS WHICH ANSWERS THE SOURCE ORDERING QUESTION.

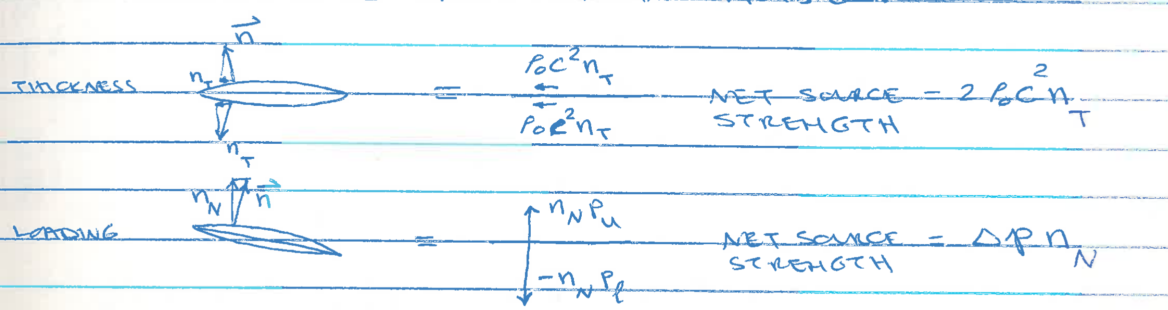
WE START WITH THICKNESS AND LOADING NOISE. WE WORK WITH EQ. (2). WE COMPARE THICKNESS AND LOADING IN DIVERGENCE FORM. FOR THICKNESS WE

USE ISOM'S FORMULA, i.e., THICKNESS NOISE IS EQUIVALENT TO LOADING NOISE WITH UNIFORM PRESSURE $\rho_0 c^2$. WE WANT TO COMPARE

$$E_{TH} = -\frac{\partial}{\partial x_i}\left[\rho_0 c^2 n_i \delta(f)\right] \qquad (15-a)$$

$$E_L = -\frac{\partial}{\partial x_i}\left[p n_i \delta(f)\right] \qquad (15-b)$$

WE CONSIDER THE FOLLOWING THIN AIRFOIL MODELS FOR THICKNESS AND LOADING NOISE:



IN PRACTICE $2\rho_0 c^2 n_T = O(\Delta p n_N)$, i.e. THICKNESS AND LOADING NOISE ARE OF THE SAME ORDER OF MAGNITUDE WITH DIFFERENT DIRECTIVITY PATTERNS. THICKNESS NOISE HAS IN PLANE DIRECTIVITY WHILE LOADING NOISE HAS DIRECTIVITY MORE OR LESS NORMAL TO DIRECTION OF MOTION.

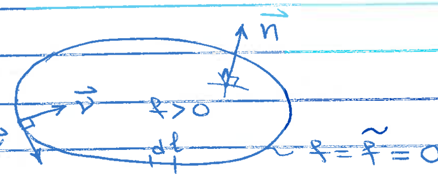
NOTE ADDED IN DEC. 91: IN A PAPER CO-AUTHORED WITH KEN BRENTNER AND PRESENTED AT THE AHS SPECIALIST MEETING IN PHILADELPHIA (OCT. 91), WE HAVE SHOWN THAT RANK ORDERING ARGUMENT FOR SURFACE AND VOLUME SOURCES OF FW-H EQ. CAN BE VERY TRICKY AND CANCELLATION AMONG THESE SOURCES CAN RESULT IN WRONG CONCLUSIONS.

* SOLUTION OF WAVE EQUATION WITH SOURCE TERMS WHICH ARE GENERALIZED FUNCTIONS

LET $f(\vec{x},t)=0$ THE EQUATION OF A MOVING SURFACE. LET $\tilde{f}(\vec{x},t)=0$ BE DEFINED IN SUCH A WAY THAT $f>0$, $\tilde{f}=0$ DEFINE AN OPEN SURFACE WITH THE EDGE $f=\tilde{f}=0$. IN PRACTICE, WE ASSUME THAT f IS DEFINED BY PARAMETERS (u^1,u^2) AND $\tilde{f}=\tilde{f}(u^1,u^2,t)$. IN ADDITION, WE ASSUME

$$\nabla f=\vec{n}, \quad \nabla\tilde{f}=\vec{\nu}$$

WHERE $|\vec{n}|=|\vec{\nu}|=1$ AND $\vec{\nu}$ IS THE GEODESIC UNIT NORMAL AT THE EDGE AS SHOWN. WE ARE INTERESTED IN INTERPRETING SEVERAL TERMS APPEARING IN FW-H EQ. WE ALSO GIVE A NEW INTERPRETATION OF Δ_0.



WE START WITH INTERPRETING THE FOLLOWING INTEGRALS FIRST

$$I_1=\int Q\,\delta(F)\,\delta(\tilde{F})\,d\vec{y} \qquad (1)$$

$$I_2=\int Q\,\delta'(F)\,d\vec{y} \qquad (2)$$

$$I_3=\int Q\,\delta'(F)\,H(\tilde{F})\,d\vec{y} \qquad (3)$$

HERE, F AND $\tilde{F}$ ARE ANY FUNCTION OF $\vec{y}$ BUT

THEY WILL EVENTUALLY BECOME $[\rho]_{ret}$ AND $[\tilde{\rho}]_{ret}$ RESPECTIVELY. FOR THIS REASON, WE WILL ASSUME THAT $|\nabla F| \neq 1$, $|\nabla \tilde{F}| \neq 1$. WE WILL USE THE NOTATIONS WHICH ARE LATER USED IN THE SOLUTION OF THE WAVE EQUATION. IN PARTICULAR, WE WILL USE THE FOLLOWING NOTATIONS:

NOTE: THIS $\tilde{\Lambda}$ IS NOT THE SAME $\tilde{\Lambda}$ IN FORM. 3

$$\vec{N} = \frac{\nabla F}{|\nabla F|}, \quad \tilde{N} = \frac{\nabla \tilde{F}}{|\nabla \tilde{F}|} \qquad (4\text{-}a,b)$$

$$\Lambda = |\nabla F|, \quad \tilde{\Lambda} = |\nabla \tilde{F}| \qquad (4\text{-}c,d)$$

$$\Lambda_0 = |\nabla F \times \nabla \tilde{F}| \qquad (4\text{-}e)$$

(SEE P 139, IN PARTICULAR THE FOOTNOTE)

i) INTERPRETATION OF I_1

LET dL BE THE ELEMENT OF LENGTH OF $F = \tilde{F} = 0 \Rightarrow$

$$d\vec{y} = \frac{dF\, d\tilde{F}\, dy_3}{|\partial(F,\tilde{F})/\partial(y_1,y_2)|}$$

$$= \frac{dy_3}{|\partial(F,\tilde{F})/\partial(y_1,y_2)|\,/\,|\nabla F \times \nabla \tilde{F}|} \; \frac{dF\, d\tilde{F}}{|\nabla F \times \nabla \tilde{F}|}$$

$$= \frac{dy_3}{|\Gamma_3|} \; \frac{dF\, d\tilde{F}}{\Lambda_0}$$

$$= \frac{dF\, d\tilde{F}\, dL}{\Lambda_0} \qquad (5)$$

WHERE $\vec{\Gamma} = \nabla F \times \nabla \tilde{F} / \Lambda_0$ IS UNIT TANGENT VECTOR TO THE EDGE CURVE $F = \tilde{F} = 0$.

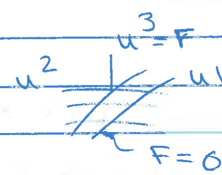
SUBSTITUTING IN EQ (1) AND INTEGRATING WRT F AND $\tilde{F}$, WE GET

$$\boxed{\int Q\,\delta(F)\,\delta(\tilde{F})\,d\vec{y} = \int_{F=\tilde{F}=0} \frac{Q}{\Lambda_0}\,dL} \qquad (6)$$

(ii) INTERPRETATION OF I_2

$$I_2 = \int Q\,\delta'(F)\,d\vec{y}$$

$$d\vec{y} = \frac{dy_1\,dy_2\,dF}{|\partial F/\partial y_3|}$$



$$= \frac{dy_1\,dy_2}{|\partial F/\partial y_3|/|\nabla F|}\;\frac{dF}{|\nabla F|}$$

$$= \frac{dy_1\,dy_2}{|N_3|}\;\frac{dF}{\Lambda} = \frac{dF\,d\Sigma}{\Lambda} \qquad (7)$$

$$d\Sigma = \sqrt{g_{(2)}}\;du^1\,du^2 \qquad (8)$$

$\sqrt{g_{(2)}}$ IS A FUNCTION OF $u^3 = F$. WE HAVE

$$\frac{\partial\sqrt{g_{(2)}}}{\partial N} = -2H_F\sqrt{g_{(2)}} \qquad (9)$$

WHERE H_F IS THE LOCAL MEAN CURVATURE OF $F=0$. THEREFORE

$$I_2 = \int \frac{\sqrt{g_{(2)}}\,Q}{\Lambda}\,\delta'(F)\,dF\,du^1\,du^2$$

$$= -\int_{F=0} \frac{\partial}{\partial F}\left[\frac{Q\sqrt{g_{(2)}}}{\Lambda}\right] du^1 du^2$$

$$= -\int_{F=0} \frac{1}{\Lambda}\frac{\partial}{\partial N}\left[\frac{Q\sqrt{g_{(2)}}}{\Lambda}\right] du^1 du^2$$

$$\boxed{\int Q\,\delta'(F)\,d\vec{y} = -\int_{F=0} \frac{1}{\Lambda}\frac{\partial}{\partial N}\left[\frac{Q}{\Lambda}\right] d\Sigma + \int_{F=0} \frac{2H_F Q}{\Lambda^2}\,d\Sigma} \quad (10)$$

WE HAVE USED EQ. (9) IN THE LAST STEP.

iii) INTERPRETATION OF I_3

USING EQ. (10), WE HAVE

$$I_3 = -\int_{F=0} \frac{1}{\Lambda}\frac{\partial}{\partial N}\left[\frac{Q\,H(\tilde{F})}{\Lambda}\right] d\Sigma + \int_{\substack{F=0\\ \tilde{F}>0}} \frac{2H_F Q}{\Lambda^2}\,d\Sigma$$

$$= -\int_{\substack{F=0\\ \tilde{F}>0}} \frac{1}{\Lambda}\frac{\partial}{\partial N}\left[\frac{Q}{\Lambda}\right] d\Sigma + \int_{\substack{F=0\\ \tilde{F}>0}} \frac{2H_F Q}{\Lambda^2}\,d\Sigma$$

$$- \int_{F=0} \frac{\vec{N}\cdot\vec{\tilde{N}}\,Q}{\Lambda^2}\,|\nabla\tilde{F}|\,\delta(\tilde{F})\,d\Sigma \quad (11)$$

WE WILL NOW CONSIDER INTEGRALS OF THE FORM

$$I_4 = \int_{F=0} \tilde{Q}\,|\nabla\tilde{F}|\,\delta(\tilde{F})\,d\Sigma \quad (12)$$

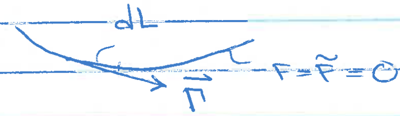
$$I_4 = \int_{F=0} \tilde{Q}\,|\nabla\tilde{F}|\,\delta(\tilde{F})\sqrt{g_{(2)}}\,du^1 du^2 \quad (13)$$

WHERE WE ASSUME $\nabla\tilde{F}$ IS NOT NECESSARILY

PARALLEL TO THE GEODESIC NORMAL AT THE EDGE $F = \tilde{F} = 0$. WE HAVE

$$du^1 du^2 = \frac{d\tilde{F}\, du^2}{|\partial F/\partial u^1|} \qquad (14)$$

$$d\vec{L} = \vec{\Gamma}\, dL$$

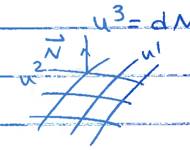


$$= \Gamma^1 dL\, \vec{e}_1 + \Gamma^2 dL\, \vec{e}_2$$

$$= du^1 \vec{e}_1 + du^2 \vec{e}_2 \qquad (15)$$

$$dL = \frac{du^1}{\Gamma^1} = \frac{du^2}{\Gamma^2} \qquad \text{(ORIENTED)}$$

$$= \frac{du^1}{|\Gamma^1|} = \frac{du^2}{|\Gamma^2|} \qquad \text{(NONORIENTED)} \quad (16)$$



$$\sin\tilde{\theta}\, |\nabla\tilde{F}|\, \vec{\Gamma} = \vec{N} \times \nabla\tilde{F}$$

$$= \vec{N} \times \frac{\partial\tilde{F}}{\partial u^i} \vec{e}^{\,i}$$

$$= (\vec{e}^{\,3} \times \vec{e}^{\,i}) \frac{\partial\tilde{F}}{\partial u^i} \qquad (17)$$

$$\boxed{\sin\tilde{\theta} = \frac{\Lambda_0}{\Lambda\tilde{\Lambda}}}$$

$$\vec{e}^{\,3} \times \vec{e}^{\,1} = \alpha\, \vec{e}_2$$

$$(\vec{e}^{\,3} \times \vec{e}^{\,1}) \cdot e^2 = \frac{1}{\sqrt{g_{(2)}}} = \alpha\, \vec{e}_2 \cdot \vec{e}^{\,2} = \alpha$$

$$\therefore \vec{e}^{\,3} \times \vec{e}^{\,1} = \frac{1}{\sqrt{g_{(2)}}} \vec{e}_2 \qquad (18)$$

ALSO $$\vec{e}^{\,3} \times \vec{e}^{\,2} = -\frac{1}{\sqrt{g_{(2)}}} \vec{e}_1 \qquad (19)$$

$\therefore$ FROM EQ. (17),

$$\sin\tilde{\theta}\,\vec{\Gamma} = \frac{-1}{|\nabla\tilde{F}|\sqrt{g_{(2)}}}\frac{\partial\tilde{F}}{\partial u^2}\vec{e}_1 + \frac{1}{|\nabla\tilde{F}|\sqrt{g_{(2)}}}\frac{\partial\tilde{F}}{\partial u^1}\vec{e}_2 \quad (20)$$

$$\therefore\quad \Gamma^1 = -\frac{\partial\tilde{F}/\partial u^2}{|\nabla\tilde{F}|\sqrt{g_{(2)}}}\frac{1}{\sin\tilde{\theta}} \quad (21\text{-}a)$$

$$\Gamma^2 = \frac{\partial\tilde{F}/\partial u^1}{|\nabla\tilde{F}|\sqrt{g_{(2)}}}\frac{1}{\sin\tilde{\theta}} \quad (21\text{-}b)$$

SO WE GET

$$dL = \frac{|\nabla\tilde{F}|\sqrt{g_{(2)}}\sin\tilde{\theta}}{|\partial\tilde{F}/\partial u^2|}du^1 \quad (22\text{-}a)$$

$$= \frac{|\nabla\tilde{F}|\sqrt{g_{(2)}}\sin\tilde{\theta}}{|\partial\tilde{F}/\partial u^1|}du^2 \quad (22\text{-}b)$$

FROM EQS. (14) AND (22-b), WE HAVE

$$|\nabla\tilde{F}|\sqrt{g_{(2)}}\,du^1du^2 = \frac{1}{|\partial F/\partial u^1|}|\nabla\tilde{F}|\sqrt{g_{(2)}}\,du^2\,dF$$

$$= dF\,dL/\sin\tilde{\theta} \quad (23)$$

SUBSTITUTE IN EQ. (13) AND INTEGRATE WRT $\tilde{F}$. THE FINAL RESULT IS

(✱)

$$\boxed{\int_{F=0}\tilde{Q}\,|\nabla\tilde{F}|\,\delta(\tilde{F})\,d\Sigma = \int_{F=\tilde{F}=0}\frac{\tilde{Q}}{\sin\tilde{\theta}}dL = \int_{F=\tilde{F}=0}\Lambda\tilde{\Lambda}\tilde{Q}\frac{dL}{\Lambda_0}} \quad (24)$$

TAKING $\tilde{Q} = \frac{\tilde{N}\cdot\vec{N}}{\Lambda^2}Q$ IN EQ. (11), WE GET THE FOLLOWING RESULT FOR I_3:

(✱) SEE P.143 FOR A SIMPLER DERIVATION OF THIS EQ.

$$\int\limits_{\substack{F=0\\ \tilde F>0}} Q\,\delta'(F)\,H(\tilde F)\,d\vec y = \int \left[\frac{-1}{\Lambda}\frac{\partial}{\partial N}\left(\frac{Q}{\Lambda}\right) + \frac{2H_F Q}{\Lambda^2}\right] d\Sigma - \int\limits_{F=\tilde F=0} \frac{\tilde\Lambda Q\,\vec N\cdot\vec{\tilde N}}{\Lambda\Lambda_0}\,dL \qquad (25)$$

WE EMPHASIZE THAT UPTO THIS POINT F AND $\tilde F$ ARE ARBITRARY FUNCTIONS OF $\vec y$.

WE NOW SOLVE THE WAVE EQUATION

$$\Box^2\phi = q(\vec x,t)H(\tilde f)\,\delta'(f) \qquad (26)$$

WHERE, AS MENTIONED EARLIER $\nabla f = \vec n$, $\nabla\tilde f = \vec\nu$. USING THE GREEN'S FUNCTION SOLUTION OF EQ. (26), WE HAVE

$$4\pi\phi(\vec x,t) = \int \frac{1}{r} q\,H(\tilde f)\,\delta'(f)\,\delta(g)\,d\vec y\,d\tau$$

$$= \int \frac{1}{r} Q\,H(\tilde F)\,\delta'(F)\,d\vec y \qquad (27)$$

WHERE

$$Q = [q]_{ret}, \quad \tilde F = [\tilde f]_{ret}, \quad F = [f]_{ret} \qquad (28\text{-}a,b,c)$$

USING EQ. (25), WE WRITE THE SOLUTION OF EQ. (26) AS FOLLOWS

$$4\pi\phi(\vec x,t) = \int\limits_{\substack{F=0\\ \tilde F>0}} \left[\frac{-1}{\Lambda}\frac{\partial}{\partial N}\left(\frac{Q}{r\Lambda}\right) + \frac{2H_F Q}{r\Lambda^2}\right] d\Sigma - \int\limits_{F=\tilde F=0} \frac{\tilde\Lambda Q\,\vec N\cdot\vec{\tilde N}}{r\Lambda\Lambda_0}\,dL \qquad (29)$$

WE SEE FROM THIS EQUATION THAT THE SOURCE TERM OF THE TYPE IN EQ. (26) HAS A LINE INTEGRAL AROUND THE EDGE OF THE SURFACE $\tilde{F} > 0$, $F = 0$. FOR OUR SINGULARITY ANALYSIS OF THE SOLUTION OF FW-H EQUATION, WE WANT TO EXTRACT THIS LINE SOURCE FROM THE SOURCE TERM OF EQ. (26). WE USE EQ. (6) IN REVERSE ORDER TO GET THE FOLLOWING IMPORTANT RESULT

$$\int_{F=\tilde{F}=0} \frac{Q\, \vec{N}\cdot\vec{\tilde{N}}\tilde{\Lambda}}{r\Lambda\Lambda_0}\, dL = \int_{F=\tilde{F}=0} \frac{Q\, \vec{N}\cdot\vec{\tilde{N}}\tilde{\Lambda}}{r\Lambda} \frac{dL}{\Lambda_0}$$

$$= \int \frac{Q\, \vec{N}\cdot\vec{\tilde{N}}\tilde{\Lambda}}{r\Lambda}\, \delta(F)\delta(\tilde{F})\, d\vec{y}$$

$$= \int \frac{1}{r} \frac{q\, \vec{N}\cdot\vec{\tilde{N}}\tilde{\Lambda}}{\Lambda}\, \delta(f)\delta(\tilde{f})\delta(g)\, d\vec{y}\, d\tau \qquad (30)$$

NOW LET US DEFINE A NEW DISTRIBUTION $\delta_c'(f)$ WHICH IS "BLIND" TO THE EDGE OF THE SURFACE $F > 0$, $\tilde{F} = 0$, i.e.

$$\int \frac{1}{r} q\, H(\tilde{f})\delta_c'(f)\delta(g)\, d\vec{y}\, d\tau = \int \frac{Q}{r} H(\tilde{F})\delta_c'(F)\, d\vec{y}$$

$$\equiv \int_{\substack{F=0\\ \tilde{F}>0}} \left[-\frac{1}{\Lambda}\frac{\partial}{\partial N}\left(\frac{Q}{r\Lambda}\right) + \frac{2H_F Q}{r\Lambda^2} \right] d\Sigma \qquad (31)$$

THEN EQ. (29) CAN BE WRITTEN, USING EQS. (30) AND (31) AS FOLLOWS:

$$4\pi\phi(\vec{x},t) = \int \frac{q}{r}\left[H(\tilde{f})\delta_c'(f) - \frac{\vec{N}\cdot\tilde{\vec{N}}\tilde{\Lambda}}{\Lambda}\delta(\tilde{f})\delta(f)\right]\delta(g)\,d\vec{y}\,d\tau$$

$$\equiv \int \frac{q}{r} H(\tilde{f})\,\delta'(f)\,\delta(g)\,d\vec{y}\,d\tau \qquad (32)$$

WE THUS HAVE OBTAINED THE SIGNIFICANT RESULT

$$\boxed{H(\tilde{f})\delta'(f) = H(\tilde{f})\delta_c'(f) - \frac{\vec{N}\cdot\tilde{\vec{N}}\tilde{\Lambda}}{\Lambda}\delta(\tilde{f})\delta(f)} \qquad (33)$$

THERE IS SOMETHING UNIQUE AND INTERESTING ABOUT THIS RESULT. WE NOW HAVE THE OBSERVER POSITION ON THE RIGHT SIDE OF THIS. WE HAVE

$$\vec{N} = \frac{\vec{n} - M_n\hat{r}}{\Lambda}, \quad \tilde{\vec{N}} = \frac{\vec{\nu} - M_\nu\hat{r}}{\tilde{\Lambda}} \qquad (34\text{-a,b})$$

$$\Lambda^2 = 1 + M_n^2 - 2M_n\cos\theta \qquad (34\text{-c})$$

$$\tilde{\Lambda}^2 = 1 + M_\nu^2 - 2M_\nu\cos(\vec{\nu},\hat{r}) \qquad (34\text{-d})$$

$$\Lambda_0^2 = (1 - \vec{M}_p\cdot\hat{r})^2 + \cos^2\psi\, M_p^2 \qquad (34\text{-e})$$

IN THE INTEGRAND OF GREEN'S FUNCTION SOLUTION, WE THUS HAVE

$$\frac{\vec{N}\cdot\tilde{\vec{N}}}{\Lambda}\tilde{\Lambda} = FH(\vec{y},\tau;\vec{x})$$

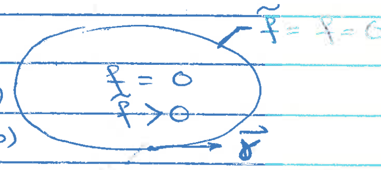
WE NOTE THAT NO OBSERVER TIME IS INVOLVED. OUR PURPOSE IN DERIVING EQ.(33) WILL BECOME OBVIOUS LATER (SEE P.144-149)

THE INTERPRETATION OF Λ_0

WE WILL FIRST SHOW THAT

$$\Lambda_0 = |\nabla\tilde{F} \times \nabla F| \quad (*) \qquad (35)$$

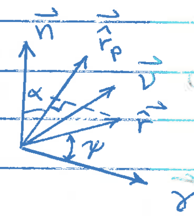
THEN WE WILL DERIVE THE RELATION BETWEEN dL AND $d\ell$. AS BEFORE, $|\nabla\tilde{f}| = |\nabla f| = 1$, AND $F = [f]_{ret}$, $\tilde{F} = [\tilde{f}]_{ret}$.



PROOF : $\nabla F = \vec{n} - M_n \hat{r}$ (36-a)

$\nabla\tilde{F} = \vec{\nu} - M_\nu \hat{r}$ (36-b)

$$\nabla\tilde{F} \times \nabla F = (\vec{\nu} - M_\nu \hat{r}) \times (\vec{n} - M_n \hat{r})$$
$$= \vec{\gamma} + \hat{r} \times (M_n \vec{\nu} - M_\nu \vec{n}) \qquad (37)$$



$\hat{r}_p$ IS IN $(\vec{n}, \vec{\nu})$-PLANE

$$\hat{r} = \cos\psi\, \vec{\gamma} + \sin\psi\, \hat{r}_p$$

(SEE FIGURE). THE UNIT VECTOR $\hat{r}_p$ IS IN THE PLANE FORMED BY $\vec{n}$ AND $\vec{\nu}$ MAKING AN ANGLE α WITH $\vec{n}$: $\hat{r}_p = \cos\alpha\, \vec{n} + \sin\alpha\, \vec{\nu}$ (38)

$$\therefore \quad \hat{r} = \cos\psi\, \vec{\gamma} + \sin\alpha \sin\psi\, \vec{\nu} + \cos\alpha \sin\psi\, \vec{n}$$
$$= (\cos\psi, \sin\alpha \sin\psi, \cos\alpha \sin\psi) \qquad (39)$$

IN $(\vec{\gamma}, \vec{\nu}, \vec{n})$-FRAME.

$$\hat{r} \times (M_n \vec{\nu} - M_\nu \vec{n}) = \begin{vmatrix} \vec{\gamma} & \vec{\nu} & \vec{n} \\ \cos\psi & \sin\psi \sin\alpha & \sin\psi \cos\alpha \\ 0 & M_n & -M_\nu \end{vmatrix}$$

(*) ALTHOUGH WE HAVE DEFINED Λ_0 ON PAGE 131 AS $|\nabla\tilde{F} \times \nabla F|$, HERE BY Λ_0 WE MEAN THE SYMBOL THAT WE HAVE USED IN OUR PAPERS, i.e.

$$\Lambda_0^2 = (1 - \vec{M}_p \cdot \hat{r})^2 + \cos^2\psi\, M_p^2$$

THIS MEANS THAT Λ_0 ON P131 IS CONSISTENT WITH OUR PREVIOUS USAGE OF THIS SYMBOL.

$$= -\sin\psi\,(M_\nu \sin\alpha + M_n \cos\alpha)\,\vec{\gamma}$$
$$+ M_\nu \cos\psi\,\vec{\nu} + M_n \cos\psi\,\vec{n}$$
$$= -\vec{M}_p \cdot \vec{r}\,\vec{\gamma} + \cos\psi\,\vec{M}_p \qquad (40)$$

HERE $\vec{M}_p = M_\nu \vec{\nu} + M_n \vec{n}$ (41)

$\vec{M}_p$ IS THE PROJECTION OF $\vec{M}$ IN THE PLANE FORMED BY $\vec{n}$ AND $\vec{\nu}$. SINCE $\vec{\gamma} \perp \vec{M}_p$, WE HAVE

$$\boxed{\begin{aligned} \nabla\tilde{F} \times \nabla F &= (1 - \vec{M}_p \cdot \hat{\vec{r}})\,\vec{\gamma} + \cos\psi\,\vec{M}_p \qquad (42) \\ \Lambda_0^2 &= (1 - \vec{M}_p \cdot \vec{r})^2 + \cos^2\psi\, M_p^2 \qquad (43) \end{aligned}}$$

$$\vec{\Gamma} = \frac{\nabla\tilde{F} \times \nabla F}{|\nabla\tilde{F} \times \nabla F|}$$

$$\boxed{\vec{\Gamma} = \frac{1 - \vec{M}_p \cdot \hat{\vec{r}}}{\Lambda_0}\,\vec{\gamma} + \frac{\cos\psi}{\Lambda_0}\,\vec{M}_p} \qquad (44)$$

HERE $\vec{\Gamma}$ IS THE UNIT TANGENT VECTOR TO THE EDGE OF THE Σ SURFACE GIVEN BY $\tilde{F} = F = 0$

WE NOTE THAT

$$\vec{M}_p \cdot \vec{r} = \vec{M}_p \cdot \hat{\vec{r}}_p \sin\psi \qquad (45)$$

ALSO NOTE THAT $\vec{\Gamma}$ IS IN THE PLANE FORMED BY $\vec{\gamma}$ AND $\vec{M}_p$.

THE RELATION BETWEEN dl AND dL

THE ELEMENT OF LENGTH OF THE EDGE $\tilde{f} = f = 0$ IS DENOTED dl WHILE THE ELEMENT OF LENGTH OF THE EDGE $\tilde{F} = F = 0$ IS DENOTED dL. PREVIOUSLY, WE HAVE USED $d\gamma$ FOR dL IN OUR PAPERS. LET $\vec{y}(l, \tau)$ DESCRIBE THE EDGE $\tilde{f} = f = 0$. THEN THE EDGE $\tilde{F} = F = 0$ IS DESCRIBED BY $\vec{Y}(l; \vec{x}, t) = \vec{y}(l, \tau^*)$ WHERE τ^* IS THE SOLUTION OF

$$\tau^* - t + |\vec{x} - y(l, \tau^*)|/c = 0 \qquad (46)$$

THAT IS $\tau^* = \tau^*(l; \vec{x}, t)$. THE TANGENT TO THE CURVE $\tilde{F} = F = 0$ IS GIVEN BY

$$\frac{d\vec{Y}}{dl} = \frac{\partial \vec{y}}{\partial l} + \frac{\partial \vec{y}}{\partial \tau^*} \frac{\partial \tau^*}{\partial l}$$

$$= \vec{\gamma} + \frac{\partial \tau^*}{\partial l} \vec{V} \qquad (47)$$

WHERE $\vec{V} = \partial \vec{y} / \partial \tau^*$ IS THE LOCAL VELOCITY OF THE EDGE $\tilde{f} = f = 0$ AND $\vec{\gamma}$ IS THE UNIT TANGENT TO THIS CURVE. FROM EQ. (46)

$$\frac{\partial \tau^*}{\partial l} - \frac{1}{c} \hat{r}_i \left[\gamma_i + v_i \frac{\partial \tau^*}{\partial l} \right] = 0 \qquad (48\text{-a})$$

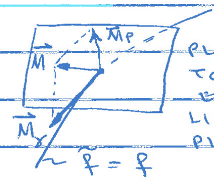
$$\frac{\partial \tau^*}{\partial l} = \frac{\cos \psi}{(1 - M_r) c} \qquad (48\text{-b})$$

WHERE v_i IS THE COMPONENT OF $\vec{V}$ AND $\cos \psi = \gamma_i \hat{r}_i = \vec{\gamma} \cdot \hat{r}$. DEFINING $\vec{M} = \vec{V}/c$, AND USING EQ. (48-b) IN EQ. (47), WE GET

$$\frac{d\vec{Y}}{d\ell} = \vec{\gamma} + \frac{\cos\psi}{1-M_r}\vec{M}$$

$$= \frac{1}{1-M_r}\left[(1-M_r)\vec{\gamma} + \cos\psi\,\vec{M}\right] \qquad (49)$$

$$M_r = \vec{M}\cdot\hat{\vec{r}}$$
$$= (\vec{M}_p + \vec{M}_\gamma)\cdot\hat{\vec{r}}$$
$$= \vec{M}_p\cdot\hat{\vec{r}} + \vec{M}_\gamma\cdot\hat{\vec{r}}$$



PLANE $\perp$ TO THE EDGE. $\vec{M}_p$ LIES IN THIS PLANE

$$(1-M_r)\vec{\gamma} + \cos\psi\,\vec{M} = (1-\vec{M}_p\cdot\hat{\vec{r}})\vec{\gamma} + \cos\psi\,\vec{M}_p + \cos\psi\,\vec{M}_\gamma - \vec{M}_\gamma\cdot\hat{\vec{r}}\,\vec{\gamma}$$

BUT

$$\cos\psi\,\vec{M}_\gamma - \vec{M}_\gamma\cdot\hat{\vec{r}}\,\vec{\gamma} = \hat{\vec{r}}\cdot\vec{\gamma}\,\vec{M}_\gamma - \vec{M}_\gamma\cdot\hat{\vec{r}}\,\vec{\gamma}$$
$$= \hat{\vec{r}}\times(\vec{M}_\gamma\times\vec{\gamma})$$
$$= 0$$

$$\therefore \frac{d\vec{Y}}{d\ell} = \frac{1}{1-M_r}\left[(1-\vec{M}_p\cdot\hat{\vec{r}})\vec{\gamma} + \cos\psi\,\vec{M}_p\right]$$

$$= \frac{\Lambda_0}{1-M_r}\vec{\Gamma} \qquad (50)$$

SINCE $|d\vec{Y}| = dL$, WE HAVE FROM THIS

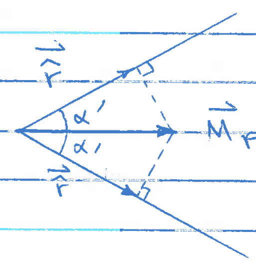
$$\boxed{\frac{d\ell}{1-M_r} = \frac{dL}{\Lambda_0}} \qquad (51)$$

CONDITIONS FOR $\Lambda_0 = 0$

FROM EQ. (43), WE MUST HAVE SIMULTANEOUSLY

$$\begin{cases} \cos\psi = \hat{\vec{r}}\cdot\vec{\gamma} = 0 & (52\text{-a}) \\ \vec{M}_p \cdot \hat{\vec{r}} = 1 & (52\text{-b}) \end{cases}$$

THE FIRST CONDITION REQUIRES $\hat{\vec{r}} \perp \vec{\gamma}$, i.e. THE OBSERVER DIRECTION PERPENDICULAR TO THE EDGE $\tilde{f} = f = 0$. THE SECOND CONDITION REQUIRES $|\vec{M}_p| \geq 1$. LET $M_p = |\vec{M}_p| \geq 1$, THEN THERE ARE TWO DIRECTIONS ON EITHER SIDES OF $\vec{M}_p$ WHERE THE SECOND CONDITION IS SATISFIED AS SHOWN BELOW:



$$\cos\alpha' = \frac{1}{M_p} \qquad (53)$$

THIS FIGURE IS DRAWN IN THE PLANE NORMAL TO THE EDGE $\tilde{f} = f = 0$ CONTAINING $\vec{n}$ AND $\hat{\vec{r}}$

A QUICK DERIVATION OF EQ. (24) (P135)

$$\int \tilde{Q}\,|\nabla\tilde{F}|\,\delta(\tilde{F})\,d\Sigma = \int \tilde{Q}\,|\nabla\tilde{F}|\,|\nabla F|\,\delta(\tilde{F})\,\delta(F)\,d\vec{y}$$

$$= \int \tilde{Q}\,|\nabla\tilde{F}|\,|\nabla F|\,\frac{dL}{\Lambda_0} \qquad \text{BY EQ.(6)}$$

$$= \int \tilde{Q}\,\frac{dL}{\sin\tilde{\theta}} = \int \frac{\Lambda\tilde{\Lambda}\tilde{Q}}{\Lambda_0}\,dL$$

SINCE $\Lambda_0 = |\nabla\tilde{F}\times\nabla F| = |\nabla\tilde{F}|\,|\nabla F|\sin\tilde{\theta} = \Lambda\tilde{\Lambda}\sin\tilde{\theta}$ (54)

LINE INTEGRAL SINGULARITY

WHEN $\Lambda_0 = 0$, i.e. $\psi = 90°$ AND $\vec{M}_p \cdot \vec{r} = 1$, WE GET (LOGARITHMIC) SINGULARITY IN THE ACOUSTIC PRESSURE FIELD WHEN ONLY THICKNESS AND LOADING TERMS OF THE FW-H EQ. IS USED IN NOISE PREDICTION. WE WILL NOW ADD THE CONTRIBUTION OF QUADRUPOLE SOURCES TO THE TE OF AIRFOIL AND SHOW THAT UNDER REALISTIC PHYSICAL CONDITIONS, THIS SINGULARITY DISAPPEARS. WE FIRST COLLECT ALL SOURCE TERMS CONTRIBUTING TO TE LINE INTEGRAL. WE START WITH EQ.(12), P127.

COEFF. OF $H(\tilde{f}_\alpha)\delta'(f)$:

$$\underbrace{-(\rho_0 v_n^2 + p)}_{\text{TH. + LOAD.}} + \underbrace{Q_n}_{\text{QUAD.}} = -\rho_0 v_n^2 - \cancel{p} + \rho v_n^2 + \cancel{p} - p' c^2$$

$$= p' c^2 (M_n^2 - 1) \qquad (55)$$

COEFF. OF $\delta(\tilde{f}_\alpha)\delta(f)$:

$$\underbrace{-\rho_0 v_n v_\nu}_{\text{THICK.}} + \underbrace{Q_\nu}_{\text{QUAD.}} = -\rho_0 v_n v_\nu + \rho v_n v_\nu$$

$$= \rho' c^2 M_n M_\nu \qquad (56)$$

ASSUMPTION 1: THE FLUID STICKS TO THE BODY, i.e. WE HAVE A VISCOUS FLUID

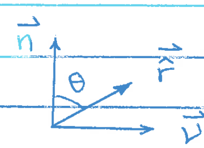
THIS ASSUMPTION IS ONLY USED IN Q_ν WHERE WE HAVE TAKEN $v_\nu = u_\nu$ i.e. BODY VELOCITY ALONG $\vec{\nu}$ = FLUID VELOCITY $\vec{u} \cdot \vec{\nu}$ ALONG $\vec{\nu}$. WE NOW USE EQ. (33) TO FIND LINE INTEGRAL CONTRIBUTION OF $H(\tilde{f}_\alpha)\delta'(f)$. WE DROP $\rho' c^2$ FROM BOTH EQS (55) AND (56). THE COEFF. OF $\delta(\tilde{f}_\alpha)\delta(f)$ IS, THEREFORE,

$$E = -(M_n^2 - 1)\,\frac{\vec{N}\cdot\vec{\tilde{N}}\,\tilde{\Lambda}}{\Lambda} + M_n M_\nu \qquad (57)$$

WE KNOW THAT WHEN $\Lambda_0 = |\nabla F \times \nabla \tilde{F}| = 0$, WE HAVE $\nabla F \parallel \nabla \tilde{F}$. WE FIND THIS RELATION IN TERMS OF OTHER KNOWN VECTORS BELOW:

$$\nabla F = \Lambda \vec{N} = \vec{n} - M_n \vec{\hat{r}}$$
$$= (1 - M_n \cos\theta)\,\vec{n} - M_n \sin\theta\,\vec{\nu} \qquad (58)$$

BUT $\vec{M_p}\cdot\vec{\hat{r}} = 1$



$\vec{\hat{r}}$ IS IN THE PLANE OF $\vec{n}$ & $\vec{\nu}$

$$(M_n \vec{n} + M_\nu \vec{\nu})\cdot(\cos\theta\,\vec{n} + \sin\theta\,\vec{\nu})$$
$$= M_n \cos\theta + M_\nu \sin\theta = 1 \qquad (59)$$

$$\Longrightarrow\ 1 - M_n \cos\theta = M_\nu \sin\theta$$

SUBSTITUTING IN EQ. (58), WE GET

$$\Lambda \vec{N} = (M_\nu \vec{n} - M_n \vec{\nu})\,\sin\theta \qquad (60)$$

SIMILARLY

$$\tilde{\Lambda}\,\vec{\tilde{N}} = \vec{\nu} - M_\nu \vec{\hat{r}}$$
$$= -M_\nu \cos\theta\,\vec{n} + (\underbrace{1 - M_\nu \sin\theta}_{M_n \cos\theta})\,\vec{\nu}$$
$$= -(M_\nu \vec{n} - M_n \vec{\nu})\cos\theta \qquad (61)\ (*)$$

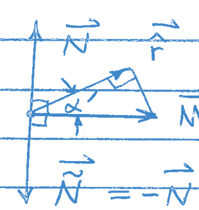
WE SEE CLEARLY THAT $\vec{N} \parallel \vec{\tilde{N}}$, AND THAT

$$\vec{N}\cdot\vec{M_p} = \vec{N}\cdot(M_n \vec{n} + M_\nu \vec{\nu})$$
$$= 0\ ! \qquad (62)$$

THAT IS, CONDITION $\Lambda_0 = 0$ IS EQUIVALENT TO $\vec{N} \perp \vec{M_p}$ AND $\vec{\tilde{N}} \perp \vec{M_p}$. WE HAVE DRAWN THE GEOMETRY BELOW.

FURTHERMORE, FROM EQS. (60) AND (61), WE HAVE, WHEN $\Lambda_0 = 0$

$$\Lambda = M_p \sin\theta \qquad (63)$$



$\vec{M_p}$ IS THE PROJECTION OF $\vec{M}$ ON $(\vec{n},\vec{\nu})$-PLANE

(*) SEE P 151. WE HAVE $\vec{\tilde{N}} = \mp\vec{N}$ DEPENDING ON SIGN OF $\cos\theta$!

AND $\tilde{\lambda} = M_p |\cos\theta|$ (64)

ALSO $\vec{N}$ AND $\vec{\tilde{N}}$ ARE IN $(\vec{n}, \vec{v})$-PLANE.

WE NOW SIMPLIFY EQ. (57). WE HAVE

$$\frac{\vec{N} \cdot \vec{\tilde{N}}\, \tilde{\Lambda}}{\Lambda} = -\cot\theta = \frac{[\mathrm{sig}(\cos\theta)] M_p \,\mathrm{sig}(\cos\theta)\cos\theta}{M_p \sin\theta} \qquad \text{EQS. (60), (61) \& (63)} \quad (65)$$

$$\therefore \quad E = \frac{1}{\sin\theta}\left[(M_n^2 - 1)\cos\theta + M_n M_v \sin\theta\right]$$

$$= \frac{1}{\sin\theta}\Big[-\cos\theta + M_n \underbrace{(M_n\cos\theta + M_v \sin\theta)}_{=1,\ \text{EQ. 59}}\Big]$$

$$= \frac{\cos\theta - M_n}{\sin\theta}$$

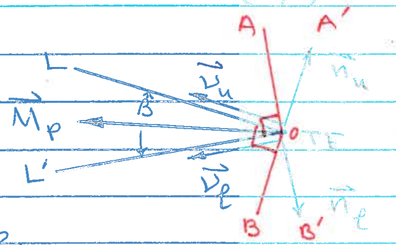
$$= -M_p \frac{(\vec{n} - M_n\hat{r})\cdot\vec{\hat{r}}}{M_p\sin\theta}$$

$$= -M_p \vec{N} \cdot \vec{\hat{r}}$$

$$= -M_p \sin\alpha' \quad (66)$$

NOW WE NOTICE SOMETHING VERY INTERESTING. FOR TWO SURFACES MEETING AT THE TRAILING EDGE AND AT THE POINT ON THIS TE WHERE $\Lambda_0 = 0$, WE HAVE

$$\vec{N}_\ell = -\vec{N}_v \quad (67)$$



THIS RELATION IS VALID FOR FINITE TE ANGLE β. WE ALREADY KNOW THAT $\vec{N}_v \cdot \vec{M}_p = \vec{N}_\ell \cdot \vec{M}_p = 0$, i.e.

$\angle AOL' = \angle BOL = 90^\circ$

$\angle AOB = \pi - \beta = \angle A'OB'$

$\vec{N}_u \perp \vec{M}_\rho$, $\vec{N}_\ell \perp \vec{M}_\rho$ $\therefore$ $\vec{N}_u \parallel \vec{N}_\ell$.

$$\Lambda_u \vec{N}_u = M_{\nu u}\vec{n}_u - M_{nu}\vec{\nu}_u \qquad (68\text{-}a)$$

$$\Lambda_\ell \vec{N}_\ell = M_{\nu\ell}\vec{n}_\ell - M_{n\ell}\vec{\nu}_\ell \qquad (68\text{-}b)$$

IF $\vec{M}_\rho$ IS WITHIN THE LARGE ANGLE $\angle$ AOB THEN $M_{\nu u} > 0$, $M_{\nu\ell} > 0$. BUT $\vec{n}_u$ AND $\vec{n}_\ell$ ARE POINTING TO EITHER SIDES OF $\vec{M}_\rho$. THE ONLY WAY THAT $\vec{N}_u \parallel \vec{N}_\ell$ AND STILL SATISFY (68-a,b) IS THAT $\vec{N}_\ell = -\vec{N}_u$. SAME CONCLUSION HOLDS IF $\vec{M}_\rho$ LIES WITHIN THE $\angle$ A'OB' ($= \pi - \beta$). UNDER THESE CONDITIONS

$$\vec{N}_u \cdot \hat{r} = -\vec{N}_\ell \cdot \hat{r} \qquad (69)$$

WE HAVE THUS SHOWN THAT FOR ONE SURFACE

$$\left\{\begin{array}{l}\text{THE NET LINE INTEGRAL}\\ \text{CONTRIBUTION FROM THICK.,}\\ \text{LOAD + QUAD SOURCES}\end{array}\right\} = -\rho' c^2 M_\rho \vec{\mathbf{N}} \cdot \hat{r} \qquad (70)$$

(FROM EQS. (55), (56) AND (66).)

WE NOW INTRODUCE TWO MORE ASSUMPTIONS:

(NOT NEEDED!)
ASSUMPTION 2: $\vec{M}_\rho$ IS WITHIN THE ANGLE $\angle$ AOB OR $\angle$ A'OB' (BOTH ANGLES $= \pi - \beta$). (✱)

(ASSUMPTION 2, SEE P 148-149)
ASSUMPTION 3: AT THE TE $\rho'_u = \rho'_\ell$, i.e. $\rho_u = \rho_\ell$, OR ALTERNATIVELY IF ENTROPIC CHANGES CAN BE NEGLECTED, $p_u = p_\ell$

UNDER THESE THREE ASSUMPTIONS, AT THE POINT

(✱) SEE NEXT PAGE

ON THE TE WITH $|\vec{M}_p| > 1$, AND ALONG $\hat{r}$ FOR WHICH $\vec{M}_p \cdot \hat{r} = 1$, WE HAVE

$$-\rho_u' c^2 M_p \vec{N}_u \cdot \hat{r} - \rho_\ell' c^2 M_p \vec{N}_\ell \cdot \hat{r}$$

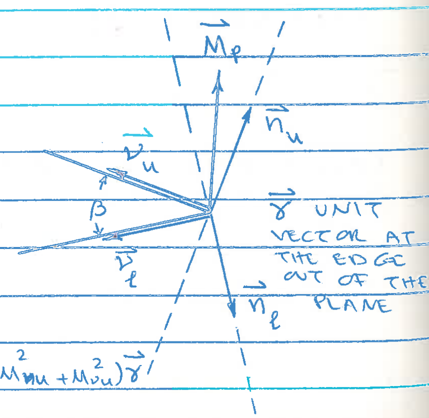
$$= -\rho_u' c^2 M_p \hat{r} \cdot (\vec{N}_u + \vec{N}_\ell) = 0 \ ! \quad \text{BY EQ. (69)} \quad (71)$$

THIS STUDY IS NOT COMPLETE. WE HAVE ONLY SHOWN THAT, THE NUMERATOR OF THE INTEGRAND IN THE LINE INTEGRAL (THICK + LOAD. + QUAD) IS ZERO WHEN THE DENOMINATOR Λ_0 IS ZERO. TO COMPLETE THIS WORK, WE MUST SHOW THAT THE INTEGRAL ITSELF REMAINS FINITE WHEN THE COLLAPSING SPHERE EXITS THE EDGE WHEN $\hat{r}$ IS SUCH THAT $\vec{M}_p \cdot \hat{r} = 1$ AT THAT MOMENT. BUT THIS ANALYSIS IS EXACTLY WHAT ENRICO DE BERNARDIS DID IN HIS PH.D. THESIS (U. OF ROME, LA SAPIENZA, 1989). DE BERNARDIS ONLY CONSIDERED THICKNESS AND LOADING PROBLEM ONLY. WE WILL DO THIS LATER. WE WILL ALSO JUSTIFY THE THIRD ASSUMPTION.

(*) ASSUMPTION 2 IS NOT NECESSARY. IF $\vec{M}_p$ IS WITHIN THE ANGLE SHOWN, THEN

$M_{nu} > 0$, $M_{\nu u} > 0$

$M_{n\ell} < 0$, $M_{\nu\ell} < 0$



$$\Lambda_u \vec{M}_p \times \vec{N}_u = \begin{vmatrix} \vec{n}_u & \vec{\nu}_u & \vec{\gamma} \\ M_{nu} & M_{\nu u} & 0 \\ M_{\nu u} & -M_{nu} & 0 \end{vmatrix} = -(M_{nu}^2 + M_{\nu u}^2)\vec{\gamma}$$

$$= -M_p^2 \vec{\gamma} \qquad (72)$$

NOW FOR LOWER SURFACE VECTORS $(\vec{\nu}_\ell, \vec{n}_\ell, \vec{\gamma})$ FORMS A RIGHT-HANDED SYSTEM. ∴

$$\Lambda_\ell \vec{M_p} \times \vec{N_\ell} = \begin{vmatrix} \vec{\nu}_\ell & \vec{n}_\ell & \vec{\gamma} \\ M_{\nu\ell} & M_{n\ell} & 0 \\ -M_{n\ell} & M_{\nu\ell} & 0 \end{vmatrix} = M_p^2 \vec{\gamma} \qquad (73)$$

$$\therefore \quad \vec{N}_\ell = -\vec{N}_u \quad \text{ALWAYS!}$$

WE THEREFORE HAVE ONLY TWO ASSUMPTIONS:

ASSUMPTION 1: VISCOUS FLUID
ASSUMPTION 2: AT THE EDGE, $p_u' = p_\ell'$ OR, EQUIVALENTLY, $p_u = p_\ell$.

(*)

NOTE ADDED ON NOV. 15, 1993: I WROTE A PAPER ON THIS PROBLEM BASED ON MY TALK IN THE INTERNATIONAL CONFERENCE ON THEORETICAL AND COMPUTATIONAL ACOUSTICS, JULY 5-9, 1993 IN MYSTIC, CONNECTICUT. THE WRITTEN VERSION OF MY TALK HAS THE TITLE: LINE SOURCE SINGULARITY IN THE WAVE EQUATION AND ITS REMOVAL BY QUADRUPOLE SOURCES — A SUPERSONIC PROPELLER NOISE PROBLEM. IT WILL APPEAR IN THE PROCEEDINGS OF THE CONFERENCE EDITED BY FFOWCS WILLIAMS AND LEE.

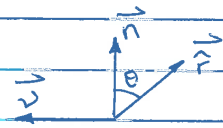
* NOTE ADDED ON NOVEMBER 15, 1993

Part of these pages have been removed by Steve Miller of NASA Langley.
The text removed is of a personal nature and is non-technical.

I TRIED TO FINISH A LONG PAPER ON GENERALIZED FUNCTIONS AND IT WILL APPEAR IN PROCEEDINGS OF THE SYMPOSIUM ON UNSTEADY AERODYNAMICS AND AEROACOUSTICS IN HONOR OF 80TH BIRTHDAY OF W.R. SEARS, MARCH 1-2, 1993, TUCSON, AZ. IT WILL ALSO APPEAR AS A NASA TP.

* NOTE ON THE DIRECTION OF $\vec{N}$ AND $\vec{\tilde N}$ (SEE P 145)

THE CONDITION $\Lambda_0 = |\nabla F \times \nabla \tilde F| = \Lambda \tilde\Lambda \sin\tilde\theta$ IMPLIES $\tilde\theta = 0$ OR π. INDEED BOTH VALUES OF $\tilde\theta = 0$ AND π ARE POSSIBLE, i.e. $\vec{\tilde N} = \pm \vec N$. IT WAS SHOWN THAT IF $\vec{\hat r}$ IS IN THE QUADRANT BETWEEN $\vec\nu$ AND $\vec n$, THEN $\vec{\tilde N} = -\vec N$ (*). WE NOW TAKE $\vec{\hat r}$ IN 2ND QUADRANT AS SHOWN ON THE RIGHT. WE HAVE



$$\vec{\hat r} = \vec n \cos\theta - \vec\nu \sin\theta$$

$$\vec M_p \cdot \vec{\hat r} = 1$$

$$(M_n \vec n + M_\nu \vec\nu)\cdot(\vec n\cos\theta - \vec\nu\sin\theta) = M_n\cos\theta - M_\nu \sin\theta = 1 \quad (*)$$

$$\vec n - M_n \vec{\hat r} = (1 - M_n\cos\theta)\,\vec n + M_n \sin\theta\, \vec\nu \qquad \text{USE } (*) \text{ HERE}$$

$$= -\sin\theta\,(M_\nu \vec n - M_n \vec\nu)$$

$$\vec\nu - M_\nu \vec{\hat r} = (1 + M_\nu\sin\theta)\,\vec\nu - M_\nu \vec n \cos\theta$$

$$= -\cos\theta\,(M_\nu \vec n - M_n \vec\nu)$$

$\therefore\ \vec{\tilde N} = \vec N$, i.e. $\tilde\theta = 0$ IN THIS CASE IF $0 < \theta < \frac{\pi}{2}$ AND $\vec{\tilde N} = -\vec N$ FOR $\frac{\pi}{2} < \theta < \pi$.

IN PRACTICE, THIS SITUATION IS SIMILAR TO RADIATION FROM TE LINE SOURCES. THE CASE $\vec{\tilde N} = -\vec N$ IS FOR TE LINE SOURCES. LET US SEE WHAT HAPPENS TO E (P 145):

$$E = M_n M_\nu + \frac{\tilde\Lambda \cos\tilde\theta}{\Lambda}(1 - M_n^2)$$

$$= M_n M_\nu + \frac{M_p|\cos\theta|\cos\tilde\theta}{M_p\sin\theta}(1 - M_n^2)$$

BUT $\cos\tilde\theta = \mathrm{sig}(\cos\theta)$ AND $|\cos\theta| = \mathrm{sig}(\cos\theta)\cos\theta$

$$\therefore\ E = M_n M_\nu + \cot\theta\,(1 - M_n^2)$$

$$= \frac{1}{\sin\theta}\left[M_n \underbrace{(-M\cos\theta + M_\nu \sin\theta)}_{-1} + \cos\theta\right]$$

(*) BECAUSE I MISTAKENLY ASSUMED $\cos\theta > 0$. SEE FOOTNOTE P 145.

$$E = \frac{-M_n + \cos\theta}{\sin\theta}$$

$$= M_p \frac{(\vec{n} - M_n\hat{r})\cdot\vec{r}}{\Lambda} \quad (\text{SINCE } \Lambda = M_p \sin\theta)$$

$$= M_p \vec{N} \cdot \hat{r}$$

SO THE ANALYSIS WILL GO THROUGH BECAUSE $\vec{N}_\ell = -\vec{N}_u$ IS ALWAYS TRUE.

NOTE: IN MY PAPER WITH MIKE MYERS ON THIS SUBJECT I HAVE DEFINED $\vec{N}\cdot\vec{\tilde{N}} = \cos\theta'$ INSTEAD OF $\cos\tilde{\theta}$. THE ANGLE BETWEEN $\vec{\hat{r}}$ AND $\vec{M}_p$ WHEN $\Lambda_0 = 0$ APPEARS IS CALLED μ IN OUR PAPER INSTEAD OF α' HERE.

* THE NASA LANGLEY TECHNICAL PAPER ON GENERALIZED FUNCTIONS WAS EDITED FOR ENGLISH BY NANCY SHEHEEN THIS MONTH AND IT LOOKS GOOD! I SPENT MY TIME READING ON NON-STANDARD ANALYSIS, ALGEBRA, TOPOLOGY AND GENERALIZED FUNCTIONS. I AM ABOUT TO UNDERSTAND THE NEW KIND OF GF'S ALLOWING MULTIPLICATION. OUR WORK ON DUCTED FAN NOISE PREDICTION IS NOT GOING WELL. WE RELIED HEAVILY ON ALLISON TURBINE CODE ADPAC WHICH DOES NOT GIVE WAVES ON THE DUCT WALL(*).

LANGLEY REORGANIZATION IS TAKING PLACE BUT I DON'T THINK IT IS GOING TO AFFECT OUR GROUP. I AM GOING TO FINISH MY RP IN JANUARY. THE RP IS ON DETAILED DERIVATION OF FORMULATION 3 AND IS COAUTHORED WITH ENRICO DE BERNARDIS OF CIRA. HOWEVER, WHEN I STARTED WRITING (REVISING!) THE RP WHICH WENT THROUGH EDITORIAL IN FEB. 1992, I DISCOVERED THAT I CAN MAKE TREMENDOUS SIMPLIFICATION BY TAKING THE 4VECTOR $\vec{A}$ IN THE PLANE OF $\vec{Q}=(-p\vec{n}, M_n)$ INSTEAD OF THE PLANE $\vec{N}=\nabla_4 f/|\nabla_4 f|$ AND $\nabla_4 g$. WHAT I DID WAS VERY NATURAL. THE TRICK IS TO THINK OF THE UNNATURAL! I WILL WRITE THE NEW DERIVATION NEXT.(**)

(*) CHARLIE FINALLY MANAGED TO RUN ADPAC CORRECTLY BY MID-1994! ADPAC IS BETTER THAN WE THOUGHT!

(**) THERE IS EVEN A BETTER WAY! 1/29/95
IT IS BASED ON THE SAME TECHNIQUE AS THE SUPERSONIC KIRCHHOFF FORMULA.

* FORMULATION 3 REVISITED

FORMULATION 3 WAS DERIVED EARLIER IN THIS NOTEBOOK. AS I WAS REVISING MY RP WITH ENRICO, I DISCOVERED A SURPRISING SIMPLIFICATION IN THE FINAL RESULT. HERE IS THE FULL DERIVATION. WE START WITH FW-H EQ. IN THE FORM USED TO DERIVE FORM. 3:

$$\Box^2 p' = \nabla_4 \cdot [\vec{Q}\,\delta(f)] \qquad (1)$$

$$\vec{Q} = (-p\vec{n}, M_n)$$

AS BEFORE p AND p' ARE NORMALIZED WRT $\rho_0 c^2$. ALSO, WE HAVE TAKEN $|\nabla f| = 1$ OVER $f = 0$. THE SOLUTION IS GIVEN BY

$$4\pi p'(\vec{x},t) = \int \nabla_4 \cdot [\vec{Q}\,\delta(f)] \frac{\delta(g)}{r}\, d\vec{y}\, d\tau$$

$$= -\int \vec{Q} \cdot \nabla_4 \left[\frac{\delta(g)}{r}\right] \delta(f)\, d\vec{y}\, d\tau$$

$$= -\int \frac{\vec{Q} \cdot \nabla_4 g}{r}\, \delta(f)\, \delta'(g)\, d\vec{y}\, d\tau$$

$$+ \int \frac{\vec{Q} \cdot \nabla_4 r}{r^2}\, \delta(f)\, \delta(g)\, d\vec{y}\, d\tau \qquad (2)$$

NOW WE DERIVE AN OLD IDENTITY AGAIN. WE WANT TO FIND THE IDENTITY OF GF FOR

$$I = \int q(\vec{y},\tau)\, \delta(f)\, \delta'(g)\, d\vec{y}\, d\tau \qquad (3)$$

WE ALREADY HAVE A RESULT BASED ON MY LATER WORK BUT WE ASSUME f AND g ARE ARBITRARY FUNCTIONS HERE IN $\mathbb{R}^4 \to \mathbb{R}$. WE START WITH

$$\int \nabla_4 \cdot [q\vec{A}\,\delta(f)\,\delta(g)]\, d\vec{y}\, d\tau = 0 \qquad (4)$$

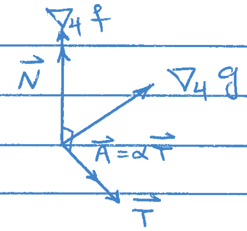
WHERE $\vec{A}$ IS AN ARBITRARY 4-VECTOR. WE HAVE

$$\nabla_4\cdot[q\vec{A}\,\delta(f)\,\delta(g)] = \nabla_4\cdot[q\vec{A}]\,\delta(f)\,\delta(g) + q\vec{A}\cdot\nabla_4 f\,\delta'(f)\,\delta(g) + q\vec{A}\cdot\nabla_4 g\,\delta(f)\,\delta'(g) \qquad (5)$$

NOW LET US TAKE

$$\vec{A}\cdot\nabla_4 f = 0, \qquad \vec{A}\cdot\nabla_4 g = 1 \qquad (6a,b)$$

IN OUR OLD DERIVATION, WE NATURALLY TOOK $\vec{A}$ TO BE IN THE PLANE OF $\nabla_4 f$ AND $\nabla_4 g$. BUT WE HAVE ANOTHER "ALMOST" NATURAL CHOICE IN 4D. THE VECTOR $\vec{Q} = (-p\vec{n}, M_n)$ IS IN THE PLANE $\vec{N}$ AND $\vec{T}$, WHERE



$$\vec{N} = \frac{\nabla_4 f}{|\nabla_4 f|} = \frac{1}{\alpha_n}(\vec{n}, -M_n)$$

$$\vec{T} = \frac{1}{\alpha_n}(M_n\vec{n}, 1)$$

$$\alpha_n^2 = 1 + M_n^2$$

LET US NOW TAKE $\vec{A} = \alpha\vec{T}$, WE ALREADY SATISFY $\vec{A}\cdot\nabla_4 f = 0$. WE FIND α FROM THE SECOND CONDITION:

$$\alpha\vec{T}\cdot\nabla_4 g = \frac{\alpha}{c\alpha_n}(M_n\vec{n}, 1)\cdot(-\hat{r}, 1) = 1$$

or $$\frac{\alpha}{c\alpha_n}(1 - M_n\cos\theta) = 1 \Rightarrow \alpha = \frac{c\alpha_n}{1 - M_n\cos\theta}$$

NOW $1 - M_n\cos\theta$ HAS THE SAME SINGULARITY AS $\tilde{A}$, i.e. $M_n = 1$, $\cos\theta = 1$. BUT IT IS SEEN THAT THE NEW $\vec{A}$ IS MUCH SIMPLER THAN PREVIOUS $\vec{A}$. OUR NEW $\vec{A}$ IS

$$\vec{A} = \frac{c\alpha_n\,(M_n\vec{n}, 1)}{1 - M_n\cos\theta} \qquad (5)$$

$\therefore$

$$I = -\int \nabla_4\cdot[q\vec{A}]\,\delta(f)\,\delta(g)\,d\vec{y}\,d\tau \qquad (6)$$

WE HAVE ONE MORE ITEM TO SORT OUT. IT IS THE QUESTION OF RESTRICTION OF FUNCTIONS MULTIPLYING THE DELTA FUNCTIONS. THE SYMBOL q IN OUR WORK STANDS FOR $\nabla_4 g \cdot \vec{Q}/r$. ALL THESE ARE RESTRICTED TO $f=0$. THIS IS CLEAR FROM EQS. (1) AND (2). WE HAVE

$$q = \frac{1}{cr}(-\hat{r},1)\cdot(-\rho\vec{n},M_n)$$
$$= \frac{1}{cr}(M_n + \rho\cos\theta)$$

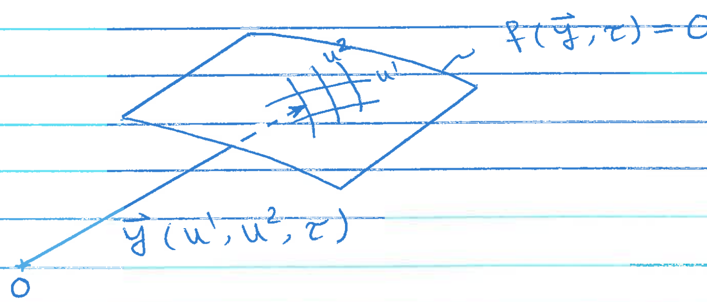
$$\nabla_4\cdot(q\vec{A}) = -\frac{\vec{A}\cdot\nabla_4 r}{c\,r^2}(M_n+\rho\cos\theta)$$
$$+\frac{1}{cr}\nabla_4\cdot\left[(M_n+\rho\cos\theta)\vec{A}\right]$$

NOTE ADDED IN APRIL 96

I ABANDONED THIS METHOD AFTER I FOUND THE METHOD THAT I USED IN THE SUPERSONIC KIRCHHOFF FORMULA. SURPRISINGLY, THE SUPERSONIC PROPELLER NOISE THEORY HAS A LOT LESS TERMS IN THE FINAL RESULT THAN I THOUGHT. ONE MAIN PARAMETER IS THE MEAN CURVATURE OF Σ-SURFACE WHICH I DENOTED H_F. THIS PARAMETER APPEARS ALSO IN THE SUPERSONIC KIRCHHOFF FORMULA. AFTER SOME ATTEMPTS, AND MUCH DELAY, I FINALLY FOUND A METHOD TO DERIVE THE EXPRESSION FOR H_F. THIS IS WRITTEN IN DETAIL STARTING P158.

* THE MEAN CURVATURE OF Σ-SURFACE ($f=0$ RIGID) IN THE NEW FORMULATION OF SUPERSONIC PROPELLER THEORY AND THE SUPERSONIC KIRCHHOFF FORMULA, WE NEED THE MEAN CURVATURE H_F OF THE SIGMA SURFACE: $F(\vec{y};\vec{x},t)=[f(\vec{y},\tau)]_{ret} = 0$. THIS MEAN CURVATURE IS VERY COMPLICATED. UNLESS THE RIGHT VARIABLES ARE SELECTED, THE DERIVATION OF THIS QUANTITY BECOMES INTRACTABLE.

SELECTION OF VARIABLES



WE DESCRIBE THE SURFACE $f(\vec{y},\tau)=0$ (THE MOVING BODY) BY THE VECTOR $\vec{y}(u^1,u^2,\tau)$. THE Σ-SURFACE IS DESCRIBED BY

$$\vec{y}' = \vec{y}\,[u^1,u^2,\tau(u^1,u^2)]$$
$$= \vec{y}'(u^1,u^2) \qquad (1)$$

WHERE τ IS IMPLICITLY DEFINED BY

$$|\vec{x} - \vec{y}(u^1,u^2,\tau)| + c(\tau - t) = 0. \qquad (2)$$

COEFFICIENTS OF FIRST AND SECOND FUNDAMENTAL FORMS

LET $(\vec{y}_1, \vec{y}_2)$ AND $(\vec{y}_1', \vec{y}_2')$ BE THE NATURAL BASIS VECTORS ON $f=0$ AND $F=0$, RESPECTIVELY. THIS MEANS THAT

$$\vec{y}_i = \frac{\partial \vec{y}}{\partial u^i} \quad (\tau \text{ FIXED})$$

$$\vec{y}_i' = \frac{\partial \vec{y}'}{\partial u^i}$$

WE HAVE

$$\vec{y}_1' = \vec{y}_1 + \frac{\partial \vec{y}}{\partial \tau} \frac{\partial \tau}{\partial u^1} \qquad (3)$$

WE GET $\partial \tau / \partial u^1$ FROM EQ. (2) AS FOLLOWS

$$\frac{\partial \tau}{\partial u^1} = \frac{\hat{r} \cdot \vec{y}_1}{c(1-M_r)} = \frac{\hat{r}_1}{c(1-M_r)} \qquad (4)$$

NOTING THAT $\partial \vec{y}/\partial \tau = \vec{v}$ (LOCAL VELOCITY OF SURFACE $f=0$), AND USING $\vec{M} = \vec{v}/c$, WE GET

$$\vec{y}_1' = \vec{y}_1 + \frac{\hat{r}_1}{1-M_r} \vec{M} \qquad (5)$$

AS ALWAYS, $\hat{r}$ IS UNIT RADIATION VECTOR

SIMILARLY

$$\vec{y}_2' = \vec{y}_2 + \frac{\hat{r}_2}{1-M_r} \vec{M} \qquad (6)$$

FROM (5) AND (6), THE COEFS OF 1ST FUNDAMENTAL FORMS ON $F=0$ AND $f=0$ ARE RELATED AS FOLLOWS

$$g_{11}' = \vec{y}_1' \cdot \vec{y}_1' = g_{11} + \frac{2M_1 \hat{r}_1}{1-M_r} + \frac{\hat{r}_1^2 M^2}{(1-M_r)^2} \qquad (7\text{-}a)$$

WHERE $g_{11} = \vec{y}_1 \cdot \vec{y}_1$, $M_1 = \vec{M} \cdot \vec{y}_1$.

$$g'_{12} = \vec{y}'_1 \cdot \vec{y}'_2 = g'_{21}$$

$$= g_{12} + \frac{M_1 \hat{r}_2 + M_2 \hat{r}_1}{1 - M_r} + \frac{\hat{r}_1 \hat{r}_2 M^2}{(1 - M_r)^2} \qquad (7\text{-}b)$$

WHERE $g_{12} = \vec{y}_1 \cdot \vec{y}_2 = g_{21}$ AND $M_2 = \vec{M} \cdot \vec{y}_2$

$$g'_{22} = \vec{y}'_2 \cdot \vec{y}'_2 = g_{22} + \frac{2 M_2 \hat{r}_2}{1 - M_r} + \frac{\hat{r}_2^2 M^2}{(1 - M_r)^2} \qquad (7\text{-}c)$$

WHERE $g_{22} = \vec{y}_2 \cdot \vec{y}_2$. FROM THESE WE FIND THE DETERMINANT OF THE COEFFS. OF THE FIRST FUNDAMENTAL FORM ON Σ SURFACE IN TERMS OF THE SIMILAR QUANTITY ON $f = 0$:

$$g'_{(2)} = \begin{vmatrix} g'_{11} & g'_{12} \\ g'_{21} & g'_{22} \end{vmatrix} = g'_{11} g'_{22} - g'^{2}_{12}$$

$$= g_{(2)} + g_{11} \left[\frac{2 M_2 \hat{r}_2}{1 - M_r} + \frac{\hat{r}_2^2 M^2}{(1 - M_r)^2} \right]$$

$$+ g_{22} \left[\frac{2 M_1 \hat{r}_1}{1 - M_r} + \frac{\hat{r}_1^2 M^2}{(1 - M_r)^2} \right]$$

$$- 2 g_{12} \left[\frac{M_1 \hat{r}_2 + M_2 \hat{r}_1}{1 - M_r} + \frac{\hat{r}_1 \hat{r}_2 M^2}{(1 - M_r)^2} \right]$$

$$- \frac{(M_1 \hat{r}_2 - M_2 \hat{r}_1)^2}{(1 - M_r)^2} \qquad (8)$$

NOW WE USE THE FACT THAT

$$g^{11} = \frac{g_{22}}{g_{(2)}}, \quad g^{12} = -\frac{g_{12}}{g_{(2)}}, \quad g^{22} = \frac{g_{11}}{g_{(2)}}$$

AND

$$\vec{a} \cdot \vec{b} = g^{ij} a_i b_j,$$

TO WRITE EQ. (8) AS FOLLOWS

$$g'_{(2)} = g_{(2)}\left[1 + \frac{2\vec{M}_t \cdot \vec{\hat{r}}_t}{1-M_r} + \frac{M^2 r_t^2}{(1-M_r)^2} - \frac{(M_1\hat{r}_2 - M_2\hat{r}_1)^2}{g_{(2)}(1-M_r)^2}\right] \quad (9)$$

WHERE $\vec{M}_t$ AND $\vec{\hat{r}}_t$ ARE PROJECTIONS OF $\vec{M}$ AND $\vec{\hat{r}}$ ON THE LOCAL TANGENT PLANE ON $f = 0$ (i.e. THE ACTUAL BODY IN MOTION). NOW WE WORK ON THE LAST TERM INSIDE SQ. BRACKETS IN EQ. (9). NOTE THAT

$$\vec{M}_t = M_1\vec{y}^1 + M_2\vec{y}^2 \quad (10\text{-}a)$$

$$\vec{\hat{r}}_t = \hat{r}_1\vec{y}^1 + \hat{r}_2\vec{y}^2 \quad (10\text{-}b)$$

WHERE $\vec{y}^1$ AND $\vec{y}^2$ ARE THE DUAL BASIS VECTORS

$$\vec{y}^i = g^{ij}\vec{y}_j$$

WE HAVE

$$\vec{M}_t \times \vec{\hat{r}}_t = (M_1\hat{r}_2 - M_2\hat{r}_1)\,\vec{y}^1 \times \vec{y}^2$$

$$\Rightarrow \quad |\vec{M}_t \times \vec{\hat{r}}_t|^2 = (M_1\hat{r}_2 - M_2\hat{r}_1)^2\,|\vec{y}^1 \times \vec{y}^2|^2$$

$$= \frac{(M_1\hat{r}_2 - M_2\hat{r}_1)^2}{g_{(2)}} \quad (11)$$

SINCE $|\vec{y}^1 \times \vec{y}^2| = \dfrac{1}{\sqrt{g_{(2)}}}$.

EQUATION (9) BECOMES

$$g'_{(2)} = g_{(2)}\left[1 + \frac{2\vec{M}_t \cdot \vec{\hat{r}}_t}{1-M_r} + \frac{M^2 r_t^2 - |\vec{M}_t \times \vec{\hat{r}}_t|^2}{(1-M_r)^2}\right]$$

$$= \frac{g_{(2)}}{(1-M_r)^2}\Big[(1-M_r)^2 + 2\vec{M}_t \cdot \vec{\hat{r}}_t(1-M_r) + M^2 r_t^2 - |\vec{M}_t \times \vec{\hat{r}}_t|^2\Big] \quad (12)$$

NEXT WE WRITE $\hat{r} = \vec{r}_t + \cos\theta\, \vec{n}$, $\vec{M} = \vec{M}_t + \vec{M}_n$

$\therefore \quad M_r = \vec{M}\cdot\hat{r} = \vec{M}_t\cdot\hat{r}_t + M_n \cos\theta$, $\hat{r}\cdot\vec{n} = \cos\theta$

LET ψ BE THE ANGLE BETWEEN $\vec{r}_t$ AND $\vec{M}_t$.

THEN

$$E = (1-M_r)^2 + 2\vec{M}_t\cdot\hat{r}_t(1-M_r) + M^2\hat{r}_t^2 - |\vec{M}_t\times\hat{r}_t|^2$$

$$= 1 - 2M_r + M_r^2 + 2M_t\hat{r}_t\cos\psi - 2M_t\hat{r}_t M_r\cos\psi$$

$$+ (M_n^2 + M_t^2)\hat{r}_t^2 - M_t^2\hat{r}_t^2\sin^2\psi \qquad (13)$$

NOW USE

$$M_r = M_t\hat{r}_t\cos\psi + M_n\cos\theta$$

IN EQ. (13), SIMPLIFY TO GET

$$E = 1 + M_n^2 - 2M_n\cos\theta \equiv \Lambda^2 \qquad (14)$$

$$\therefore \quad \boxed{g'_{(2)} = \left[\frac{\Lambda}{1-M_r}\right]^2 g_{(2)}} \qquad (15)$$

NOTE: THIS IS EXPECTED SINCE

$$dS = \sqrt{g_{(2)}}\, du^1 du^2$$

$$d\Sigma = \sqrt{g'_{(2)}}\, du^1 du^2$$

EQ. (15) GIVES $\dfrac{dS}{|1-M_r|} = \dfrac{d\Sigma}{\Lambda}$ WHICH WAS OBTAINED GEOMETRICALLY AND PUBLISHED IN MY TR-R451 IN 1970'S.

NOTE: g'_{ij} CAN BE WRITTEN AS:

$$\boxed{g'_{ij} = g_{ij} + \frac{M_i\hat{r}_j + M_j\hat{r}_i}{1-M_r} + \frac{M^2\hat{r}_i\hat{r}_j}{(1-M_r)^2}}$$

COEFFICIENTS OF 2ND FUNDAMENTAL FORM ON Σ-SURFACE

$$\vec{y}_i' = \vec{y}_i + \frac{\hat{r}_i}{1-M_r}\vec{M}, \quad \tau_i = \frac{\hat{r}_i}{c(1-M_r)}$$

$$b_{ij}' = \vec{y}_{ij}' \cdot \vec{N}, \quad \vec{N} = \frac{\vec{n} - M_n\hat{r}}{\Lambda}$$

$$\vec{y}_{ij}' = \vec{y}_{ij} + \overbrace{\tau_j \frac{\partial y_i}{\partial \tau}}^{(1)} + \vec{M}\cdot\left[\overbrace{\frac{\partial}{\partial u^j}\left(\frac{\hat{r}_i}{1-M_r}\right)}^{(2)} + \overbrace{\tau_j\frac{\partial}{\partial\tau}\left(\frac{\hat{r}_i}{1-M_r}\right)}^{(3)}\right]$$

$$+ \frac{\hat{r}_i}{1-M_r}\left[\overbrace{\frac{\partial \vec{M}}{\partial u^j}}^{(4)} + \tau_j\frac{\partial \vec{M}}{\partial \tau}\right]$$

(1) $$\frac{\partial \vec{y}_i}{\partial \tau} = \frac{\partial^2 \vec{y}}{\partial u^i \partial \tau} = \frac{\partial \vec{v}}{\partial u^i} = c\frac{\partial \vec{M}}{\partial u^i}$$ (SEE 4)

$$\frac{\partial}{\partial u^j}\left(\frac{\hat{r}_i}{1-M_r}\right) = \frac{1}{1-M_r}\frac{\partial \hat{r}_i}{\partial u^j} + \frac{\hat{r}_i}{(1-M_r)^2}\frac{\partial M_r}{\partial u^j}$$

$$\frac{\partial \hat{r}_i}{\partial u^j} = \frac{\partial}{\partial u^j}\frac{(\vec{x}-\vec{y})\cdot\vec{y}_i}{r} = \checkmark\ \frac{g_{ij}}{r} + \vec{\hat{r}}\cdot\vec{y}_{ij} + \checkmark\ \frac{\hat{r}_i\hat{r}_j}{r}$$

$$= \frac{1}{r}(\hat{r}_i\hat{r}_j - g_{ij}) + \vec{\hat{r}}\cdot(\Gamma_{ij}^k \vec{y}_k + b_{ij}\vec{n}) \checkmark$$

$$= \frac{1}{r}(\hat{r}_i\hat{r}_j - g_{ij}) + \Gamma_{ij}^k\hat{r}_k + b_{ij}\cos\theta \checkmark$$

$$\frac{\partial M_r}{\partial u^j} = \frac{\partial}{\partial u^j}\vec{M}\cdot\vec{\hat{r}} = \frac{\partial \vec{M}}{\partial u^j}\cdot\vec{\hat{r}} + \vec{M}\cdot\frac{\partial}{\partial u^j}\frac{\vec{x}-\vec{y}}{r}$$

$$= \overbrace{\vec{\hat{r}}\cdot\frac{\partial \vec{M}}{\partial u^j}}^{(5)} + \frac{1}{r}(M_r\hat{r}_j \overset{\checkmark}{-} M_j) \checkmark$$

(2) $$\frac{\partial}{\partial u^j}\left(\frac{\hat{r}_i}{1-M_r}\right) = \frac{1}{1-M_r}(\Gamma_{ij}^k\hat{r}_k + b_{ij}\cos\theta) \checkmark$$

$$+ \frac{\hat{r}_i}{(1-M_r)^2}\vec{\hat{r}}\cdot\frac{\partial\vec{M}}{\partial u_j} + \frac{1}{r(1-M_r)}(\hat{r}_i\hat{r}_j - g_{ij})$$

$$+ \frac{\hat{r}_i}{r(1-M_r)^2}(M_r\hat{r}_j - M_j) \checkmark$$

$$\frac{\partial}{\partial\tau}\left(\frac{\hat{r}_i}{1-M_r}\right) = \frac{1}{1-M_r}\frac{\partial\hat{r}_i}{\partial\tau} + \frac{\hat{r}_i}{(1-M_r)^2}\frac{\partial M_r}{\partial\tau}$$

$$\frac{\partial\hat{r}_i}{\partial\tau} = \frac{\partial}{\partial\tau}\frac{(\vec{x}-\vec{y})\cdot\vec{y}_i}{r}$$

$$= -\frac{v_i}{r} + \hat{\vec{r}}\cdot\frac{\partial\vec{v}}{\partial u^i} + \frac{\hat{r}_i v_r}{r} \checkmark$$

$$= c\,\hat{\vec{r}}\cdot\frac{\partial\vec{M}}{\partial u^i} + \frac{c}{r}(M_r\hat{r}_i - M_i) \checkmark$$

$$\frac{\partial M_r}{\partial\tau} = \frac{\partial}{\partial\tau}\vec{M}\cdot\hat{\vec{r}} = \frac{\partial\vec{M}}{\partial\tau}\cdot\hat{\vec{r}} + \vec{M}\cdot\frac{\partial}{\partial\tau}\left(\frac{\vec{x}-\vec{y}}{r}\right)$$

$$= \hat{\vec{r}}\cdot\frac{\partial\vec{M}}{\partial\tau} + \frac{c}{r}(M_r^2 - M^2) \checkmark$$

(3) $$\frac{\partial}{\partial\tau}\left(\frac{\hat{r}_i}{1-M_r}\right) = \frac{c}{1-M_r}\hat{\vec{r}}\cdot\frac{\partial\vec{M}}{\partial u^i} + \frac{\hat{r}_i}{(1-M_r)^2}\hat{\vec{r}}\cdot\frac{\partial\vec{M}}{\partial\tau}$$

$$+ \frac{c}{r(1-M_r)}(M_r\hat{r}_i - M_i) + \frac{c\,\hat{r}_i}{r(1-M_r)^2}(M_r^2 - M^2)$$

$$\vec{M} = \frac{1}{c}\left[\vec{v}_0(\tau) + \vec{\omega}\times\vec{y}\right] \qquad \vec{\omega}: \text{ANGULAR VELOCITY}$$

(4) $$\frac{\partial\vec{M}}{\partial u^i} = \frac{1}{c}\vec{\omega}\times\vec{y}_i$$

$$\vec{y}\,'_{ij} = \vec{y}_{ij} + \frac{\hat{r}_j}{1-M_r}\frac{\partial\vec{M}}{\partial u^i} \checkmark$$

$$+ \left\{\frac{1}{1-M_r}\left[\Gamma^k_{ij}\hat{r}_k + b_{ij}\cos\theta + \frac{\hat{r}_i\hat{r}_j - g_{ij}}{r}\right]\right.$$

$$+ \frac{\hat{r}_i}{(1-M_r)^2}\left[\hat{\vec{r}}\cdot\frac{\partial\vec{M}}{\partial u^j} + \frac{M_r\hat{r}_j - M_j}{r}\right]$$

$$+ \frac{\hat{r}_j}{(1-M_r)^2}\left[\hat{\vec{r}}\cdot\frac{\partial\vec{M}}{\partial u^i} + \frac{M_r\hat{r}_i - M_i}{r}\right]$$

$$\left.+ \frac{\hat{r}_i\hat{r}_j}{(1-M_r)^3}\left[\frac{1}{c}\hat{\vec{r}}\cdot\frac{\partial\vec{M}}{\partial\tau} + \frac{1}{r}(M_r^2 - M^2)\right]\right\}\vec{M} +$$

(CONT'D)

$$+ \frac{\hat{r}_i}{1-M_r}\left[\frac{\partial \vec{M}}{\partial u^j} + \frac{\hat{r}_j}{c(1-M_r)}\frac{\partial \vec{M}}{\partial \tau}\right]$$

SIMPLIFICATIONS

1 - COEFF. OF $\vec{M}/r$ TERMS $= E_1$

$$E_1 = \frac{1}{1-M_r}\left[\hat{r}_i \hat{r}_j - g_{ij}\right]$$

$$+ \frac{1}{(1-M_r)^2}\left[2 M_r \hat{r}_i \hat{r}_j - \hat{r}_i M_j - M_i \hat{r}_j\right]$$

$$+ \frac{1}{(1-M_r)^3}(M_r^2 - M^2)\,\hat{r}_i \hat{r}_j$$

WE HAVE SHOWN EARLIER THAT

$$g'_{ij} = g_{ij} + \frac{M_i \hat{r}_j + M_j \hat{r}_i}{1-M_r} + \frac{M^2 \hat{r}_i \hat{r}_j}{(1-M_r)^2}$$

SUBSTITUTE FOR g_{ij} IN E_1

$$E_1 = \frac{1}{1-M_r}\left[\hat{r}_i \hat{r}_j - g'_{ij} + \frac{M_i \hat{r}_j + M_j \hat{r}_i}{1-M_r} + \frac{M^2 \hat{r}_i \hat{r}_j}{(1-M_r)^2}\right]$$

$$+ \frac{1}{(1-M_r)^2}\left[2 M_r \hat{r}_i \hat{r}_j - \hat{r}_i M_j - M_i \hat{r}_j\right]$$

$$+ \frac{1}{(1-M_r)^3}(M_r^2 - M^2)\,\hat{r}_i \hat{r}_j$$

$$= -\frac{g'_{ij}}{1-M_r} + \frac{\hat{r}_i \hat{r}_j}{(1-M_r)^3} \checkmark$$

WE KEEP g'_{ij} SINCE WE NEED $g'^{ij} g'_{ij} = 2$ IN H_r !

2 - ONCE MORE WE COLLECT TERMS IN PREPARATION FOR MULTIPLYING $\vec{y}'_{ij}$ BY $\vec{N}$. WE GROUP THE TERMS AS FOLLOWS:

$$\vec{y}'_{ij} = \vec{y}_{ij} + \frac{1}{1-M_r}\left[\hat{r}_j \cdot \frac{\partial \vec{M}}{\partial u^i} + \hat{r}_i \cdot \frac{\partial \vec{M}}{\partial u^j}\right] \checkmark$$

$$+ \frac{1}{c(1-M_r)^2}\hat{r}_i \cdot \hat{r}_j \, \dot{\vec{M}} \checkmark$$

$$+ \frac{1}{1-M_r}\left(\Gamma^k_{ij}\hat{r}_k + b_{ij}\cos\theta\right)\vec{M} \checkmark$$

$$+ \frac{1}{(1-M_r)^2}\left(\hat{r}_i \, \vec{\hat{r}} \cdot \frac{\partial \vec{M}}{\partial u^j} + \hat{r}_j \, \vec{\hat{r}} \cdot \frac{\partial \vec{M}}{\partial u^i}\right)\vec{M} \checkmark$$

$$+ \frac{1}{c(1-M_r)^3}\dot{M}_r \, \hat{r}_i \cdot \hat{r}_j \, \vec{M} \checkmark$$

$$- \frac{1}{r(1-M_r)} g'_{ij}\vec{M} + \frac{1}{r(1-M_r)^3}\hat{r}_i \cdot \hat{r}_j \vec{M}$$

WHERE $\dot{M}_r = \vec{\hat{r}} \cdot \dot{\vec{M}}$.

$$\vec{y}_{ij} \cdot \vec{N} = \left(\Gamma^k_{ij}\vec{y}_k + b_{ij}\vec{n}\right) \cdot \frac{\vec{n} - M_n\vec{\hat{r}}}{\Lambda}$$

$$= \frac{1}{\Lambda}\left[b_{ij}(1 - M_n\cos\theta) - M_n\Gamma^k_{ij}\hat{r}_k\right] \checkmark$$

$$\vec{N} \cdot \left(\hat{r}_j \frac{\partial \vec{M}}{\partial u^i} + \hat{r}_i \frac{\partial \vec{M}}{\partial u^j}\right) = \frac{\vec{n} - M_n\vec{\hat{r}}}{c\Lambda} \cdot \left(\hat{r}_j \, \vec{\omega} \times \vec{y}_i + \hat{r}_i \, \vec{\omega} \times \vec{y}_j\right)$$

$$= \frac{1}{\Lambda}\left[\hat{r}_j q_i + \hat{r}_i q_j - M_n(\hat{r}_i w_j + \hat{r}_j w_i)\right] \checkmark$$

WHERE $\vec{q} = \frac{1}{c}\vec{n} \times \vec{\omega}$, $\vec{w} = \frac{1}{c}\vec{\hat{r}} \times \vec{\omega}$

ALSO $\vec{\hat{r}} \cdot \frac{\partial \vec{M}}{\partial u^j} = \frac{1}{c}\vec{\hat{r}} \cdot (\vec{\omega} \times \vec{y}_j) = w_j$

$$\vec{\hat{r}} \cdot \frac{\partial \vec{M}}{\partial u^i} = w_i \checkmark$$

$$\vec{M} \cdot \vec{N} = \frac{M_n(1-M_r)}{\Lambda}$$

$$\therefore \; b'_{ij} = \vec{y}'_{ij} \cdot \vec{N}$$

$$= \frac{1}{\Lambda}\left[(1-M_n\cos\theta)\, b_{ij} - M_n \Gamma^k_{ij}\, \hat{r}_k\right]$$

$$+ \frac{1}{\Lambda(1-M_r)}\left[\hat{r}_j\cdot q_i + \hat{r}_i\cdot q_j - M_n(\hat{r}_i\cdot w_j + \hat{r}_j\cdot w_i)\right]$$

$$+ \frac{M_n}{\Lambda}\left[\Gamma^k_{ij}\,\hat{r}_k + b_{ij}\cos\theta\right]$$

$$+ \frac{M_n}{\Lambda(1-M_r)}\left[\hat{r}_i\cdot w_j + \hat{r}_j\cdot w_i\right]$$

$$+ \frac{1}{c\Lambda(1-M_r)^2}\left[\dot{M}_n - \cancel{M_n\dot{M}_r} + \cancel{M_n\dot{M}_r}\right]\hat{r}_i\cdot\hat{r}_j$$

$$- \frac{M_n g'_{ij}}{r\Lambda} + \frac{M_n}{r(1-M_r)^2\Lambda}\,\hat{r}_i\cdot\hat{r}_j$$

$$\boxed{\begin{aligned} b'_{ij} = {} & \frac{b_{ij}}{\Lambda} + \frac{1}{\Lambda(1-M_r)}\left[\hat{r}_i\cdot q_j + \hat{r}_j\cdot q_i\right] \\ & + \frac{\dot{M}_n}{c\Lambda(1-M_r)^2}\,\hat{r}_i\cdot\hat{r}_j - \frac{M_n}{r\Lambda}\, g'_{ij} \\ & + \frac{M_n}{r(1-M_r)^2\Lambda}\,\hat{r}_i\cdot\hat{r}_j \end{aligned}}$$

WE HAVE LEFT g'_{ij} IN THIS FORMULA SINCE THE MEAN CURVATURE IS DEFINED BY THE FORMULA $H_F = \frac{1}{2} b'^i_i = \frac{1}{2} g'^{ij} b'_{ij}$ AND $g'^{ij} g'_{ij} = \delta^i_i = 2$.

$$g'_{ij} = g_{ij} + \frac{1}{1-M_r}(M_i \hat{r}_j + M_j \hat{r}_i) + \frac{M^2}{(1-M_r)^2}\hat{r}_i \hat{r}_j$$

$$g'^{11} = \frac{(1-M_r)^2}{\Lambda^2}\frac{g'_{22}}{g_{(2)}}$$

$$g'^{12} = \frac{(1-M_r)^2}{\Lambda^2}\frac{g'_{12}}{g_{(2)}}$$

$$g'^{22} = \frac{(1-M_r)^2}{\Lambda^2}\frac{g'_{11}}{g_{(2)}}$$

$$g'^{ij} = \frac{(1-M_r)^2}{\Lambda^2} g^{ij} + \frac{1-M_r}{\Lambda^2} h^{ij} + \frac{M^2}{\Lambda^2} m^{ij}$$

(SEE P173)

WHERE

$$\begin{cases} h^{11} = \dfrac{2}{g_{(2)}} M_2 \hat{r}_2 \\ h^{12} = h^{21} = -\dfrac{1}{g_{(2)}}(M_1 \hat{r}_2 + M_2 \hat{r}_1) \\ h^{22} = \dfrac{2}{g_{(2)}} M_1 \hat{r}_1 \end{cases}$$

$$\begin{cases} m^{11} = \dfrac{1}{g_{(2)}} \hat{r}_2^{\,2} \\ m^{12} = m^{21} = -\dfrac{1}{g_{(2)}} \hat{r}_1 \hat{r}_2 \\ m^{22} = \dfrac{1}{g_{(2)}} \hat{r}_1^{\,2} \end{cases}$$

WE WILL GEOMETRICALLY INTERPRET h^{ij} AND m^{ij} AS PRODUCT OF TWO CONTRAVARIANT VECTORS.

MEAN CURVATURE H_F OF THE Σ-SURFACE

$$2H_F = b'^{i}_{i} = g'^{ij} b'_{ij}$$

$$= -\frac{M_n}{r\Lambda} \underbrace{g'^{ij} g'_{ij}}_{=2}$$

$$+ \frac{1}{\Lambda^3}\left[(1-M_r)^2 g^{ij} + (1-M_r)h^{ij} + M^2 m^{ij}\right]$$

$$E_{ij} \begin{cases} \left[b_{ij} + \frac{1}{1-M_r}(\hat{r}_i q_j + \hat{r}_j q_i) \right. \\ \left. + \frac{1}{(1-M_r)^2}\left(\frac{1}{c}\dot{M}_n + \frac{1}{r}M_n\right)\hat{r}_i \hat{r}_j \right] \end{cases}$$

$$\boxed{g^{ij}E_{ij} = 2H_f + \frac{2}{1-M_r}\vec{\hat{r}}_t \cdot \vec{q} + \frac{1}{(1-M_r)^2}\left(\frac{1}{c}\dot{M}_n + \frac{1}{r}M_n\right)\hat{r}_t^2}$$

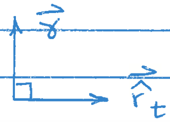
WHERE $2H_f = g^{ij} b_{ij} = b^i_i$ IS TWICE THE MEAN CURVATURE H_f OF $f=0$. WE NEXT CALCULATE $h^{ij}E_{ij}$:

$$h^{ij}E_{ij} = h^{ij}b_{ij} + \frac{1}{1-M_r}h^{ij}(\hat{r}_i q_j + \hat{r}_j q_i)$$
$$+ \frac{1}{(1-M_r)^2}\left(\frac{1}{c}\dot{M}_n + \frac{1}{r}M_n\right)h^{ij}\hat{r}_i\hat{r}_j$$

WHAT IS THE GEOMETRICAL MEANING OF h^{ij}?
WE KNOW $\vec{\hat{r}}_t = \hat{r}_1 \vec{y}^1 + \hat{r}_2 \vec{y}^2$, $\vec{M}_t = M_1 \vec{y}^1 + M_2 \vec{y}^2$.
WHAT IS THE VECTOR WITH COVARIANT COMPONENTS

$$\gamma^1 = -\frac{\hat{r}_2}{\sqrt{g(2)}}, \quad \gamma^2 = \frac{\hat{r}_1}{\sqrt{g(2)}} \ ?$$

WE HAVE $|\vec{\gamma}|^2 = g_{ij}\gamma^i\gamma^j =$

$\boxed{\vec{\gamma} = \vec{n}\times\hat{\vec{r}}}$ 

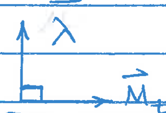
$$= \frac{1}{g_{(2)}}\left[g_{11}\hat{r}_2^2 - 2g_{12}\hat{r}_1\hat{r}_2 + g_{22}\hat{r}_1^2\right]$$

$$= g^{ij}\hat{r}_i\hat{r}_j = \hat{r}_t^2 = \sin^2\theta$$

ALSO $\vec{\gamma}\cdot\hat{\vec{r}}_t = t^i\hat{r}_i = 0$, i.e. $\vec{\gamma} \perp \hat{\vec{r}}_t$

$\therefore |\vec{\gamma}| = \sin\theta$ & $\vec{\gamma}$ IS TANGENT TO Γ-CURVE, i.e. THE CURVE OF INTERSECTION OF $\xi = 0$ AND THE COLLAPSING SPHERE. THE INTERPRETATION OF

$$\lambda^1 = -\frac{M_2}{\sqrt{g_{(2)}}}, \quad \lambda^2 = \frac{M_1}{\sqrt{g_{(2)}}}$$



IS SIMILAR: $\boxed{\vec{\lambda} = \vec{n}\times\vec{M}_t}$

$$|\vec{\lambda}|^2 = g^{ij}M_iM_j = M_t^2$$

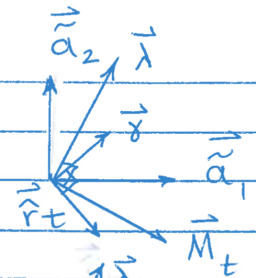
$$\vec{\lambda}\cdot\vec{M}_t = 0 \quad \text{i.e. } \vec{\lambda}\perp\vec{M}_t, \ |\vec{\lambda}| = M_t$$

WE THUS HAVE $\boxed{h^{ij} = \gamma^i\lambda^j + \gamma^j\lambda^i}$

$$h^{ij}b_{ij} = 2\gamma^i\lambda^j b_{ij} \qquad \text{(SEE NOTE 5, P 88)}$$

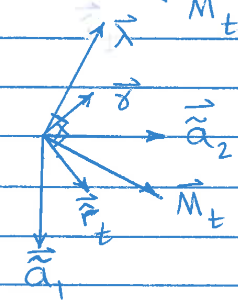
$$= 2(K_1\tilde{\lambda}^1\tilde{\gamma}^1 + K_2\tilde{\lambda}^2\tilde{\gamma}^2)$$

WHERE $(\tilde{\gamma}^1, \tilde{\gamma}^2)$ AND $(\tilde{\lambda}^1, \tilde{\lambda}^2)$ ARE COMPONENTS OF $\vec{\gamma}$ AND $\vec{\lambda}$ ALONG PRINCIPAL DIRECTIONS, RESPECTIVELY, WRT <u>UNIT</u> VECTORS IN THESE DIRECTIONS. NOTE THAT THE VECTOR $\vec{\lambda}$ HERE IS NOT THE SAME AS $\vec{\mu}$ ON P 88. NOTE ALSO THAT THE PRINCIPAL DIRECTIONS ARE ORTHOGONAL, AND THE NAMING OF DIRECTIONS 1 AND 2 IS ARBITRARY, WE CAN SHOW THAT $h^{ij}b_{ij}$ DOES NOT CHANGE-



$\vec{\tilde{a}}_1, \vec{\tilde{a}}_2$ UNIT VECTORS IN PRINCIPAL DIRECTIONS

$(\vec{\tilde{a}}_1, \vec{\tilde{a}}_2, \vec{n})$ RIGHT-HANDED



HERE $(\vec{\tilde{a}}_1, \vec{\tilde{a}}_2, \vec{n})$ IS RIGHT HANDED: $K_1 \to K_2$, $K_2 \to K_1$
$\tilde{\lambda}^1 \to \tilde{\lambda}^2$, $\tilde{\gamma}^1 \to \tilde{\gamma}^2$
$\tilde{\lambda}^2 \to -\tilde{\lambda}^1$, $\tilde{\gamma}^2 \to -\tilde{\gamma}^1$

$\therefore\ b_{ij} h^{ij}$ REMAINS THE SAME !

$$h^{ij}(\hat{r}_i q_j + \hat{r}_j q_i) = 2\gamma^i \lambda^j (\hat{r}_i q_j + \hat{r}_j q_i)$$
$$= \underbrace{2\gamma^i \hat{r}_i}_{=0} \lambda^j q_j + 2\gamma^i q_i \lambda^j \hat{r}_j$$
$$= 2(\vec{\gamma} \cdot \vec{q})(\vec{\lambda} \cdot \hat{r}_t)$$

$$h^{ij} \hat{r}_i \hat{r}_j = \gamma^i \lambda^j \hat{r}_i \hat{r}_j = 0$$

$$\boxed{h^{ij} E_{ij} = 2(K_1 \tilde{\lambda}^1 \tilde{\gamma}^1 + K_2 \tilde{\lambda}^2 \tilde{\gamma}^2) + \frac{2}{1-M_r}(\vec{\gamma} \cdot \vec{q})(\vec{\lambda} \cdot \hat{r}_t)}$$ ✓

WHAT IS m^{ij} ? FROM THE DEFINITION OF $\vec{\lambda}$, WE SEE THAT

$$\boxed{m^{ij} = \gamma^i \gamma^j}$$

$$m^{ij} b_{ij} = \gamma^i \gamma^j b_{ij} = \hat{r}_t^2 K_\gamma = K_\gamma \sin^2\theta$$

WHERE K_γ IS THE NORMAL CURVATURE ALONG $\vec{\gamma}$.

$$m^{ij}(\hat{r}_i q_j + \hat{r}_j q_i) = \gamma^i \gamma^j (\hat{r}_i q_j + \hat{r}_j q_i) = 0$$

$$m^{ij}\hat{r}_i \hat{r}_j = \gamma^i \gamma^j \hat{r}_i \hat{r}_j = 0$$

$$\therefore \quad \boxed{m^{ij} E_{ij} = K_\gamma \sin^2\theta}$$

$$2H_F = -\frac{2M_n}{r\Lambda} + \frac{1}{\Lambda^3}\Big[(1-M_r)^2 g^{ij} E_{ij} + (1-M_r) h^{ij} E_{ij} + M^2 m^{ij} E_{ij}\Big]$$

$$= -\frac{2M_n}{r\Lambda} + \frac{1}{\Lambda^3}\Big[2(1-M_r)^2 H_F + 2(1-M_r)\vec{\hat{r}}_t \cdot \vec{q} + \left(\frac{1}{c}\dot{M}_n + \frac{1}{r}M_n\right)\sin^2\theta\Big]$$

$$+ \frac{2}{\Lambda^3}\Big[(1-M_r)(K_1 \tilde{\lambda}' \tilde{\gamma}' + K_2 \tilde{\lambda}' \tilde{\gamma}') + (\vec{\gamma}\cdot\vec{q})(\vec{\lambda}\cdot\vec{\hat{r}}_t)\Big]$$

$$+ \frac{1}{\Lambda^3} M^2 K_\gamma \sin^2\theta$$

WE HAVE USED $\hat{r}_t^2 = \sin^2\theta$. WE REWRITE THIS RESULT AS FOLLOWS :

$$\boxed{\begin{aligned} H_F = {} & \frac{M_n}{r\Lambda}\left[1 - \frac{\sin^2\theta}{2\Lambda^2}\right] + \frac{(1-M_r)^2}{\Lambda^3} H_f \\ & + \frac{1-M_r}{\Lambda^3}\left[\vec{\hat{r}}_t \cdot \vec{q} + K_1 \tilde{\lambda}^1 \tilde{\gamma}^1 + K_2 \tilde{\lambda}^2 \tilde{\gamma}^2\right] \\ & + \frac{\sin^2\theta}{2\Lambda^3}\left[\frac{1}{c}\dot{M}_n + M^2 K_\gamma\right] + \frac{1}{\Lambda^3}(\vec{\gamma}\cdot\vec{q})(\vec{\lambda}\cdot\vec{\hat{r}}_t) \end{aligned}}$$

$\Lambda = (1 + M_n^2 - 2M_n \cos\theta)^{1/2}$

$\cos\theta = \vec{n}\cdot\vec{\hat{r}}$

$\vec{\hat{r}}_t = \vec{\hat{r}} - \vec{n}\cos\theta \quad , \quad |\vec{\hat{r}}_t| = \sin\theta$

K_1, K_2 PRINCIPAL CURVATURES ON $f=0$

H_f MEAN CURVATURE OF $f=0$

$\vec{q} = \frac{1}{c}\vec{n}\times\vec{\omega} \quad , \quad \vec{\omega}$ ANGULAR VELOCITY

$\vec{\gamma} = \vec{n}\times\vec{\hat{r}}_t = \vec{n}\times\vec{\hat{r}} \, ; \, \vec{\lambda} = \vec{n}\times\vec{M}_t = \vec{n}\times\vec{M}$

$\vec{M}_t = \vec{M} - M_n\vec{n} \quad , \quad M = |\vec{M}|$

K_γ MEAN CURVATURE IN DIRECTION $\vec{\gamma}$

$(\tilde{\lambda}^1, \tilde{\lambda}^2), (\tilde{\gamma}^1, \tilde{\gamma}^2)$ COMPONENTS OF $\vec{\lambda}$ AND $\vec{\gamma}$ IN PRINCIPAL DIRECTIONS WRT UNIT BASIS VECTORS, RESPECTIVELY

$\dot{M}_n = \vec{n}\cdot\dot{\vec{M}} \quad \left(\neq \frac{\partial M_n}{\partial\tau} \; !\right)$

$\vec{\hat{r}}_t$ AND $\vec{M}_t$ ARE PROJECTIONS OF $\vec{\hat{r}}$ AND $\vec{M}$ ON THE LOCAL TANGENT PLANE TO $f=0$.

WE NOTE THAT IF $\vec{M}=0 \Rightarrow H_F = H_f$ AS EXPECTED. WE HAVE ALSO SHOWN THAT:

$$\boxed{g'^{ij} = \frac{(1-M_r)^2}{\Lambda^2} g^{ij} + \frac{1-M_r}{\Lambda^2}(\gamma^i\lambda^j + \gamma^j\lambda^i) + \frac{M^2}{\Lambda^2}\gamma^i\gamma^j}$$

* THE MEAN CURVATURE OF THE Σ - SURFACE ($f = 0$ DEFORMABLE)

I RECENTLY EXTENDED H_F TO THE CASE WHERE $f = 0$ IS DEFORMABLE. OUR SUBSONIC KIRCHHOFF FORMULA IS FOR DEFORMABLE SURFACES AND ALTHOUGH IN HELICOPTER ROTOR APPLICATIONS, THE KIRCHHOFF SURFACE IS RIGID, I WANTED TO HAVE A SUPERSONIC FORMULA FOR DEFORMABLE SURFACES FOR COMPLETENESS. AS IT WILL BE SEEN BELOW, THE EXTENSION IS EASILY DONE.

WE FIRST NOTE THAT FOR A DEFORMABLE SURFACE GIVEN BY $\vec{y} = \vec{y}(u^1, u^2, \tau)$, THE VELOCITY $\vec{V}$ IS ALWAYS DEFINED BY THE RELATION:

$$\vec{V} = \frac{\partial \vec{y}}{\partial \tau} \quad (u^1, u^2) \text{ FIXED}$$

THERE IS A PROBLEM ABOUT WHETHER THE FINAL RESULT WE OBTAIN IS INVARIANT UNDER CHANGE OF COORDINATES. I WILL RETURN TO THIS PROBLEM LATER.

SUMMARY OF THE RESULTS FROM RIGID CASE

$$\vec{y}'_i = \vec{y}_i + \frac{\hat{r}_i}{1 - M_r} \vec{M} \qquad (1)$$

$$g'_{ij} = g_{ij} + \frac{M_i \hat{r}_j + M_j \hat{r}_i}{1 - M_r} + \frac{M^2 \hat{r}_i \hat{r}_j}{(1 - M_r)^2} \qquad (2)$$

$$g'^{ij} = \frac{(1 - M_r)^2}{\Lambda^2} g^{ij} + \frac{1 - M_r}{\Lambda^2} (\gamma^i \lambda^j + \gamma^j \lambda^i) + \frac{M^2}{\Lambda^2} \gamma^i \gamma^j \qquad (3)$$

$$\vec{\gamma} = \vec{n} \times \hat{r}, \quad \vec{\lambda} = \vec{n} \times \vec{M} \quad \text{(SURFACE VECTORS)}$$

$M_i = \vec{M} \cdot \vec{y}_i$, $\hat{r}_i = \hat{r} \cdot \vec{y}_i$ (i.e. M_i AND $\hat{r}_i$ ARE COVARIANT COMPONENTS OF $\vec{M}$ AND $\hat{r}$)

TO BE CONSISTENT WITH THE PAPER I AM WRITING WITH MIKE MYERS, I WILL USE $\psi(u^1, u^2; \vec{x}, t)$ FOR THE SOLUTION OF RETARDED TIME EQUATION

$$c(\psi - t) + |\vec{x} - \vec{y}(u^1, u^2, \psi)| = 0$$

THIS WILL ALLOW US TO USE τ FOR SOURCE TIME AS INDEPENDENT VARIABLE. WE HAVE

$$\frac{\partial \psi}{\partial u^i} = \frac{\hat{r}_i}{c(1 - M_r)} \qquad (4) \qquad (P159; EQ.4)$$

WE HAVE

$$y'_{i;j} = \vec{y}_{i;j} + \frac{\partial \vec{y}_i}{\partial \tau} \frac{\partial \psi}{\partial u^j} + \vec{M}\left[\frac{\partial}{\partial u^j}\left(\frac{\hat{r}_i}{1 - M_r}\right) + \frac{\partial}{\partial \tau}\left(\frac{\hat{r}_i}{1 - M_r}\right)\frac{\partial \psi}{\partial u^j}\right]$$
$$+ \frac{\hat{r}_i}{1 - M_r}\left(\frac{\partial \vec{M}}{\partial u^j} + \frac{\partial \vec{M}}{\partial \tau}\frac{\partial \psi}{\partial u^j}\right) \qquad (5)$$

<u>NOTATION</u>: $\vec{M}_{,j} = \dfrac{\partial \vec{M}}{\partial u^j}$, $\dot{\vec{M}} = \dfrac{\partial \vec{M}}{\partial \tau}$

WE SHOWED $\dfrac{\partial \vec{y}_i}{\partial \tau} = c\dfrac{\partial \vec{M}}{\partial u^i} = c\vec{M}_{,i}$

NOW WE TACKLE EACH TERMS IN SQUARE BRACKETS IN EQ.(5). WE USE OUR PREVIOUS RESULTS AS MUCH AS POSSIBLE. WE WILL NOT ASSUME THAT $\vec{M}$ IS GIVEN BY A RELATION SUCH AS $\vec{M} = \frac{1}{c}\left[\vec{v}_0(\tau) + \vec{\omega} \times \vec{y}\right]$.

$$\frac{\partial}{\partial u^j}\left(\frac{\hat{r}_i}{1-M_r}\right) = \frac{1}{1-M_r}\left[\frac{1}{r}(\hat{r}_i\hat{r}_j - g_{ij}) + \Gamma^k_{ij}\hat{r}_k + b_{ij}\cos\theta\right] + \frac{\hat{r}_i}{(1-M_r)^2}\underbrace{\left[\hat{r}\cdot\vec{M}_{,j} + \frac{1}{r}(M_r\hat{r}_j - M_j)\right]}_{\frac{\partial M_r}{\partial u^j}}$$

$$= \frac{1}{r(1-M_r)}(\hat{r}_i\hat{r}_j - g_{ij}) + \frac{\hat{r}_i}{r(1-M_r)^2}(M_r\hat{r}_j - M_j) + \frac{1}{1-M_r}(\Gamma^k_{ij}\hat{r}_k + b_{ij}\cos\theta) + \frac{\hat{r}\cdot\vec{M}_{,j}\hat{r}_i}{(1-M_r)^2} \quad (6)$$

$$\frac{\partial}{\partial \tau}\left(\frac{\hat{r}_i}{1-M_r}\right) = \frac{c}{r(1-M_r)}(M_r\hat{r}_i - M_i) + \frac{c\,\hat{r}_i}{r(1-M_r)^2}(M_r^2 - M^2) + \frac{c}{1-M_r}\vec{M}_{,i}\cdot\hat{r} + \frac{\dot{\vec{M}}\cdot\hat{r}\ \hat{r}_i}{(1-M_r)^2} \quad (7)$$

NOW WE PUT EVERYTHING TOGETHER USING FIRST GAUSS' FORMULA $\vec{y}_{ij} = \Gamma^k_{ij}\vec{y}_k + b_{ij}\vec{n}$

$$\vec{y}\,'_{ij} = \Gamma^k_{ij}\vec{y}_k + b_{ij}\vec{n} + \frac{1}{1-M_r}\hat{r}_j\vec{M}_{,i}$$
$$+ \vec{M}\left[\frac{1}{r(1-M_r)}(\hat{r}_i\hat{r}_j - g_{ij}) + \frac{\hat{r}_i}{r(1-M_r)^2}(M_r\hat{r}_j - M_j) + \frac{1}{1-M_r}(\Gamma^k_{ij}\hat{r}_k + b_{ij}\cos\theta) + \frac{\hat{r}\cdot\vec{M}_{,j}\hat{r}_i}{(1-M_r)^2}\right]$$
$$+ \frac{\vec{M}\hat{r}_j}{1-M_r}\left[\frac{1}{r(1-M_r)}(M_r\hat{r}_i - M_i) + \frac{(M_r^2 - M^2)\hat{r}_i}{r(1-M_r)^2} + \frac{1}{1-M_r}\vec{M}_{,i}\cdot\hat{r} + \frac{\dot{\vec{M}}\cdot\hat{r}\ \hat{r}_i}{c(1-M_r)^2}\right]$$
$$+ \frac{\hat{r}_i}{1-M_r}\left[\vec{M}_{,j} + \frac{\dot{\vec{M}}\hat{r}_j}{c(1-M_r)}\right]$$

$$\vec{y}'_{ij} = \frac{\vec{M}}{r(1-M_r)}\left[\hat{r}_i\hat{r}_j - g_{ij} + \frac{1}{1-M_r}(M_r\hat{r}_i\hat{r}_j - \hat{r}_i M_j + M_r\hat{r}_i M_j - \hat{r}_j M_i) + \frac{M_r^2 - M^2}{(1-M_r)^2}\hat{r}_i\hat{r}_j\right]$$

$$+ \Gamma^k_{ij}\vec{y}_k + b_{ij}\vec{n} + \frac{1}{1-M_r}(\hat{r}_j\vec{M}_{,i} + \hat{r}_i\vec{M}_{,j})$$

$$+ \frac{\vec{M}}{1-M_r}(\Gamma^k_{ij}\hat{r}_k + b_{ij}\cos\theta)$$

$$+ \frac{\vec{M}}{(1-M_r)^2}(\hat{r}\cdot\vec{M}_{,j}\hat{r}_i + \hat{r}\cdot\vec{M}_{,i}\hat{r}_j)$$

$$+ \frac{\dot{\vec{M}}}{c(1-M_r)^2}\hat{r}_i\hat{r}_j + \frac{\dot{\vec{M}}\cdot\vec{r}\ \vec{M}}{c(1-M_r)^3}\hat{r}_i\hat{r}_j$$

$$= \frac{\vec{M}}{r(1-M_r)}\left[\frac{\hat{r}_i\hat{r}_j}{(1-M_r)^2} - g'_{ij}\right] + \Gamma^k_{ij}\vec{y}_k + b_{ij}\vec{n}$$

$$+ \frac{\vec{M}}{1-M_r}(\Gamma^k_{ij}\hat{r}_k + b_{ij}\cos\theta)$$

$$+ \frac{1}{1-M_r}(\hat{r}_j\vec{M}_{,i} + \hat{r}_i\vec{M}_{,j})$$

$$+ \frac{\vec{M}}{(1-M_r)^2}(\hat{r}_i\vec{M}_{,j} + \hat{r}_j\vec{M}_{,i})\cdot\hat{r}$$

$$+ \frac{\dot{\vec{M}}}{c(1-M_r)^2}\hat{r}_i\hat{r}_j + \frac{\dot{\vec{M}}\cdot\vec{r}\ \vec{M}}{c(1-M_r)^3}\hat{r}_i\hat{r}_j$$

WE KNOW THAT $\vec{N} = \dfrac{\vec{n} - M_n\hat{r}}{\Lambda}$. WE HAVE

$\vec{N}\cdot\vec{M} = \dfrac{M_n(1-M_r)}{\Lambda}$ WHICH HELPS IN

SIMPLIFICATION OF $b'_{ij} = \vec{N}\cdot\vec{y}'_{ij}$ WHICH IS WRITTEN NEXT PAGE.

$$b'_{ij} = \frac{M_n}{r\Lambda}\left[\frac{\hat{r}_i \hat{r}_j}{(1-M_r)^2} - g'_{ij}\right]$$
$$+ \frac{1}{\Lambda}\left[-\Gamma^k_{ij} M_n \hat{r}_k + b_{ij}(1 - M_n \cos\theta)\right]$$
$$+ \frac{M_n}{\Lambda}\left[\Gamma^k_{ij}\hat{r}_k + b_{ij}\cos\theta\right]$$
$$+ \frac{1}{(1-M_r)\Lambda}(\hat{r}_i \vec{M}_{,j} + \hat{r}_j \vec{M}_{,i})\cdot\vec{n}$$
$$+ \frac{\dot{\vec{M}}\cdot\vec{n}}{c(1-M_r)^2\Lambda}\hat{r}_i\hat{r}_j$$
$$= \frac{M_n}{r\Lambda}\left[\frac{\hat{r}_i\hat{r}_j}{(1-M_r)^2} - g'_{ij}\right] + \frac{b_{ij}}{\Lambda}$$
$$+ \frac{1}{\Lambda(1-M_r)}(\hat{r}_i\vec{M}_{,j} + \hat{r}_j\vec{M}_{,i})\cdot\vec{n}$$
$$+ \frac{\dot{\vec{M}}\cdot\vec{n}}{c\Lambda(1-M_r)^2}\hat{r}_i\hat{r}_j$$

WE CAN GO FARTHER HERE. WE HAVE

$$\vec{M}_{,i} = \frac{\partial\vec{M}}{\partial u^i} = \frac{\partial}{\partial u^i}(M^k\vec{y}_k + M_n\vec{n})$$
$$= M^k_{,i}\vec{y}_k + M^k(\Gamma^l_{ik}\vec{y}_l + b_{ik}\vec{n})$$
$$+ M_{n,i}\vec{n} - M_n b^k_i \vec{y}_k$$
$$= (M^k_{,i} + \Gamma^l_{ik} M^k - M_n b^k_i)\vec{y}_k$$
$$+ (b_{ik}M^k - M_{n,i})\vec{n}$$

$\therefore$

$$(\hat{r}_i M_{,j} + \hat{r}_j M_{,i})\cdot\vec{n} = (\hat{r}_i b_{jk} + \hat{r}_j b_{ik})M^k$$
$$- (\hat{r}_i M_{n,j} + \hat{r}_j M_{n,i})$$

$$\vec{\dot{M}} = \frac{\partial}{\partial \tau}\left(M^k \vec{y}_k + M_n \vec{n}\right)$$

$$= \dot{M}^k \vec{y}_k + c M^k \vec{M}_{,k} + \dot{M}_n \vec{n} + M_n \frac{\partial \vec{n}}{\partial \tau}$$

$$\vec{\dot{M}} \cdot \vec{n} = c M^k \left(b_{kl} M^l \quad M_{n,k}\right) + \dot{M}_n$$

NOTE THAT

$$\vec{n} \cdot \frac{\partial \vec{n}}{\partial \tau} = \frac{\partial}{\partial \tau} |\vec{n}|^2 = 0$$

$$\vec{\dot{M}} \cdot \vec{n} = c\, b_{kl} M^k M^l - c M_{n,k} M^k + \dot{M}_n$$

$$= c M_t^2 K_M - c M_{n,k} M^k + \dot{M}_n$$

WHERE $\vec{M}_t = \vec{M} - M_n \vec{n}$ AND K_M IS THE NORMAL CURVATURE OF $f = 0$ IN THE DIRECTION OF $\vec{M}_t$. FOR NOW WE KEEP THIS TERM $\vec{\dot{M}} \cdot \vec{n}$ AS IS.

$$b'_{ij} = \frac{M_n}{r\Lambda}\left[\frac{\hat{r}_i \hat{r}_j}{(1-M_r)^2} - g'_{ij}\right] + \frac{b_{ij}}{\Lambda}$$

$$+ \frac{1}{\Lambda(1-M_r)}\left[(\hat{r}_i b_{jk} + \hat{r}_j b_{ik}) M^k - (\hat{r}_i M_{n,j} + \hat{r}_j M_{n,i})\right]$$

$$+ \frac{\vec{\dot{M}} \cdot \vec{n}}{c\Lambda(1-M_r)^2}\, \hat{r}_i \hat{r}_j$$

WE HAVE

$$g'^{ij} \hat{r}_i \hat{r}_j = \frac{(1-M_r)^2}{\Lambda^2} \hat{r}_t^2 + \frac{1-M_r}{\Lambda^2} \overbrace{(\gamma^i \lambda^j + \gamma^j \lambda^i) \hat{r}_i \hat{r}_j}^{=0}$$

$$+ \frac{M^2}{\Lambda^2} \underbrace{\gamma^i \gamma^j \hat{r}_i \hat{r}_j}_{=0} = \frac{(1-M_r)^2}{\Lambda^2} \sin^2\theta$$

$$g'^{ij} g'_{ij} = 2$$

$$g'^{ij} b_{ij} = 2\frac{(1-M_r)^2}{\Lambda^2} H_f + \frac{1-M_r}{\Lambda^2} b_{ij}(\gamma^i\lambda^j + \gamma^j\lambda^i)$$

$$+ \frac{M^2}{\Lambda^2} b_{ij}\gamma^i\gamma^j$$

$$= 2\frac{(1-M_r)^2}{\Lambda^2} H_f + 2\frac{1-M_r}{\Lambda^2}(K_1\tilde{\gamma}_1\tilde{\lambda}_1 + K_2\tilde{\gamma}_2\tilde{\lambda}_2)$$

$$+ \frac{M^2}{\Lambda^2} K_\gamma \sin^2\theta$$

$$g'^{ij}(\hat{r}_i b_{jk} + \hat{r}_j b_{ik}) M^k = 2\frac{(1-M_r)^2}{\Lambda^2} b_{ij}\hat{r}^i M^j$$

$$+ \frac{1-M_r}{\Lambda^2}(\gamma^i\lambda^j + \gamma^j\lambda^i)(\hat{r}_i b_{jk} + \hat{r}_j b_{ik}) M^k$$

$$+ \frac{M^2}{\Lambda^2} \underbrace{\gamma^i\gamma^j(\hat{r}_i b_{jk} + \hat{r}_j b_{ik}) M^k}_{=0}$$

$$= 2\frac{(1-M_r)^2}{\Lambda^2}(K_1\tilde{\hat{r}}_1\tilde{M}_1 + K_2\tilde{\hat{r}}_2\tilde{M}_2)$$

$$+ 2\frac{1-M_r}{\Lambda^2}\hat{r}_t\cdot\vec{\lambda}\;(K_1\tilde{\gamma}_1\tilde{M}_1 + K_2\tilde{\gamma}_1\tilde{M}_2)$$

$$g'^{ij}(\hat{r}_i M_{n,j} + \hat{r}_j M_{n,i}) = 2\frac{(1-M_r)^2}{\Lambda^2}\vec{\hat{r}}_t\cdot\nabla_2 M_n$$

$$+ \frac{1-M_r}{\Lambda^2}(\gamma^i\lambda^j + \gamma^j\lambda^i)(\hat{r}_i M_{n,j} + \hat{r}_j M_{n,i})$$

$$+ \frac{M^2}{\Lambda^2}\underbrace{\gamma^i\gamma^j(\hat{r}_i M_{n,j} + \hat{r}_j M_{n,i})}_{=0}$$

$$= 2\frac{(1-M_r)^2}{\Lambda^2}\vec{\hat{r}}_t\cdot\nabla_2 M_n$$

$$+ 2\frac{(1-M_r)}{\Lambda^2}\vec{\hat{r}}_t\cdot\vec{\lambda}\;\vec{\gamma}\cdot\nabla_2 M_n$$

NOTE ADDED ON MARCH 23, 1998

PROFESSOR MARK FARRIS WORKED IN SUMMER 1997 ON THIS PROBLEM FOR DEFORMABLE $f(\vec{y},\tau)=0$. HE ESSENTIALLY PROVED THAT MY FORMULA FOR $f=0$ RIGID IS CORRECT AND FOUND ADDITIONAL TERMS FOR DEFORMABLE SURFACE. A REPORT ON THE SUPERSONIC FW-H AND KIRCHHOFF FORMULA WILL BE PUBLISHED LATER.

I HAVE BEEN WRITING MY RESEARCH PROGRESS IN OTHER NOTEBOOKS.

MAR 98

CORRECTIONS TO THE SOLUTION OF INHOMOGENEOUS WAVE EQUATION INVOLVING $\delta'(f)$ [SOME OF THESE RESULTS ARE WRONG! SEE P 184] 5/5/98

CONSIDER THE WAVE EQUATION

$$\square^2 \phi = \underset{\sim}{Q}(\vec{x},t)\, h(\tilde{f})\, \delta'(f) \qquad (1)$$

THIS IS EQ. (4.23e) IN MY NASA TP 3428 (1994)

THE FORMAL SOLUTION OF THIS EQUATION IS

$$4\pi \phi(\vec{x},t) = \int \frac{1}{r} \underset{\sim}{Q}(\vec{y},\tau)\, h(\tilde{f})\, \delta'(f)\, \delta(g)\, d\vec{y}\, d\tau$$

$$= \int \frac{1}{r} [\underset{\sim}{Q}(\vec{y},\tau)]_{ret}\, h(\tilde{F}) \frac{\partial}{\partial n} \delta[F]\, \delta(g)\, d\vec{y}$$

$$= \int \frac{1}{r} [\underset{\sim}{Q}(1 - M_n \cos\theta]_{ret}\, h(\tilde{F})\, \delta'(F)\, d\vec{y} \qquad (2)$$

THE REASON IS

$$\frac{\partial}{\partial n} \delta(F) = \vec{n} \cdot \nabla F\, \delta'(F)$$

$$= (1 - M_n \cos\theta)_{ret}\, \delta'(F) \qquad (3)$$

$\Rightarrow$ EQ. (4.37) NASA TP 3428 IS WRONG! NO!

EQUATION (4.39) IN TP-3428 IS CORRECT.

WE REDERIVE THE CORRECT IDENTITY BELOW.

LET $|\nabla f| \neq 1$ ON $f = 0$. WE WANT TO FIND

$$I = \int \phi(\vec{x})\, \delta'(f)\, d\vec{x} \qquad (4)$$

WE HAVE $\nabla\left(\frac{f}{|\nabla f|}\right)\Big|_{f=0} = 1 \Rightarrow$

$$\delta\left(\frac{f}{|\nabla f|}\right) = |\underset{\sim}{\nabla f}|\, \delta(f) \Rightarrow \frac{\partial}{\partial f}\delta(f) = \delta'(f) = \frac{1}{|\nabla f|} \frac{\partial}{\partial n}\left[\frac{1}{|\underset{\sim}{\nabla f}|} \delta\left(\frac{f}{|\nabla f|}\right)\right] \qquad (5)$$

$$\Rightarrow \delta'(f) = \frac{1}{|\nabla f||\tilde{\nabla f}|}\frac{\partial}{\partial n}\delta\left(\frac{f}{|\nabla f|}\right), \quad u^3 = \frac{f}{|\nabla f|}$$

$$\therefore \quad I = \int \frac{\phi(\vec{x})}{|\nabla f||\tilde{\nabla f}|}\frac{\partial}{\partial u^3}\delta(u^3)\sqrt{g_{(2)}}\,du^1du^2du^3$$

$$= \int_{f=0} -\frac{\partial}{\partial u^3}\left[\frac{\phi(\vec{x})\sqrt{g_{(2)}}}{|\nabla f||\tilde{\nabla f}|}\right]_{u^3=0} du^1du^2$$

$$= \int_{f=0}\left\{-\frac{\partial}{\partial u^3}\left(\frac{\phi(\vec{x})}{|\nabla f||\tilde{\nabla f}|}\right) + 2H_f\frac{\phi(\vec{x})}{|\nabla f|^2}\right\}\sqrt{g_{(2)}}\,du^1du^2$$

$$= \iint_{f=0}\left\{\frac{-1}{|\nabla f|}\frac{\partial}{\partial n}\left[\frac{\phi(\vec{x})}{|\nabla f|}\right] + \frac{2H_f\,\phi(\vec{x})}{|\nabla f|^2}\right\}dS$$

THIS IS CORRECT!

$\Rightarrow$ EQ. (2) BECOMES

$$4\pi\phi(\vec{x},t) = \int_{F=0}\left\{\frac{-1}{\Lambda}\frac{\partial}{\partial N}\left\{\frac{[\underset{\sim}{Q}(1-M_n\cos\theta)]_{ret}}{r\Lambda}h(\tilde{F})\right\}\right.$$
$$\left. + \frac{2[\underset{\sim}{Q}(1-M_n\cos\theta)]_{ret}H_F h(\tilde{F})}{\Lambda^2}\right\}d\Sigma$$

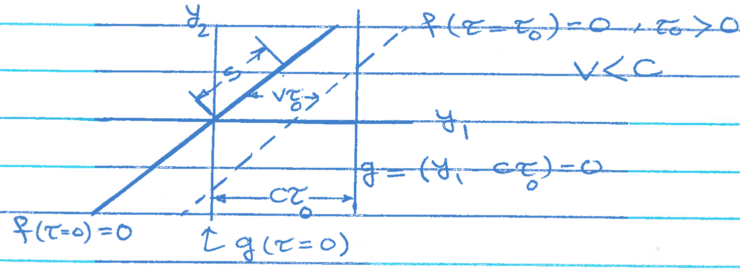
THIS MEANS THAT EQ. (4.40), NASA TP-3428 IS WRONG! NO!
EQ. (4.43) OF NASA TP-3428 IS, THEREFORE, WRONG ALSO. THE CORRECT RESULT IS
NO!

$$4\pi\phi(\vec{x},t) = \int_{\substack{F=0\\ \tilde{F}>0}}\left\{-\frac{1}{\Lambda}\frac{\partial}{\partial N}\left\{\frac{[\underset{\sim}{Q}(1-M_n\cos\theta)]_{ret}}{r\Lambda}\right\}\right.$$
$$\left. + \frac{2H_F[\underset{\sim}{Q}(1-M_n\cos\theta)]_{ret}}{r\Lambda^2}\right\}d\Sigma$$
$$-\int_{\substack{F=0\\ \tilde{F}=0}}\frac{[\underset{\sim}{Q}(1-M_n\cos\theta)]_{ret}\cot\theta'}{r\Lambda^2}dL$$

NO!

NOTE ON THE RESULTS ON PAGE 182-183

I WAS LED TO THE RESULTS ON PAGE 182-183 BECAUSE I NEEDED A FACTOR IN THE NUMERATOR OF ALL INTEGRALS TO GET RID OF A 1/Λ SINGULARITY. MIKE MYERS OF GWU WAS SURE I WAS RIGHT. MARK FARRIS OF TEXAS MIDWESTERN U. HAD HIS DOUBTS. I HAD REASONS TO BELIEVE I WAS WRONG. SO I SOLVED THE FOLLOWING PROBLEM IN THREE WAYS ANALYTICALLY:



$$f=\frac{1}{\sqrt{2}}\left[y_1-v\tau-y_2\right],\quad |\nabla f|=1$$

$$g=y_1-c\tau$$

$$I=\int q(s,\tau)\,\delta'(f)\,\delta(g)\,d\vec{y}\,d\tau$$

$$q(s,\tau)=q\left(\frac{y_2}{\sqrt{2}},\tau\right)=\sin\pi s\,e^{i\omega\tau}$$

WE ASSUME $|s|\leq 1 \quad \therefore \quad q(\pm 1,\tau)=0$

I SHOWED THAT WHAT I HAD DONE IN MY PREVIOUS PAPERS WAS CORRECT, i.e.

$$I_1=\int \frac{q(\vec{y},\tau)}{r}\,\delta'(f)\,\delta(g)\,d\vec{y}\,d\tau$$

$$=\int\frac{1}{r}\left[q\right]_{ret}\,\delta'(F)\,d\vec{y}$$

$$=\int_{F=0}\left\{-\frac{\partial}{\partial N}\left[\frac{[q]_{ret}}{r\Lambda}\right]+\frac{2H_F[q]_{ret}}{r\Lambda}\right\}\frac{d\Sigma}{\Lambda}$$

IS CORRECT. I HAD PROVEN THIS MUCH BY THE TIME OF AST WORKSHOP IN CLEVELAND (APR. 21-23, 98). I WORKED EVERY EVENING IN THE HOTEL TO FIND A WAY OUT OF THE DILEMA OF THE SINGULARITY IN MY SUPERSONIC FORMULATION. I HAVE ALWAYS LIKED THE ELEGANCE OF THE Σ-SURFACE FORMULATION. I WAS FORCED TO LOOK AT OTHER FORMS OF THE SOLUTION TO FW-H & KIRCHHOFF EQUATIONS. I FINALLY FOUND WHAT I BELIEVE TO BE THE SIMPLEST FORM OF THE SOLUTION OF

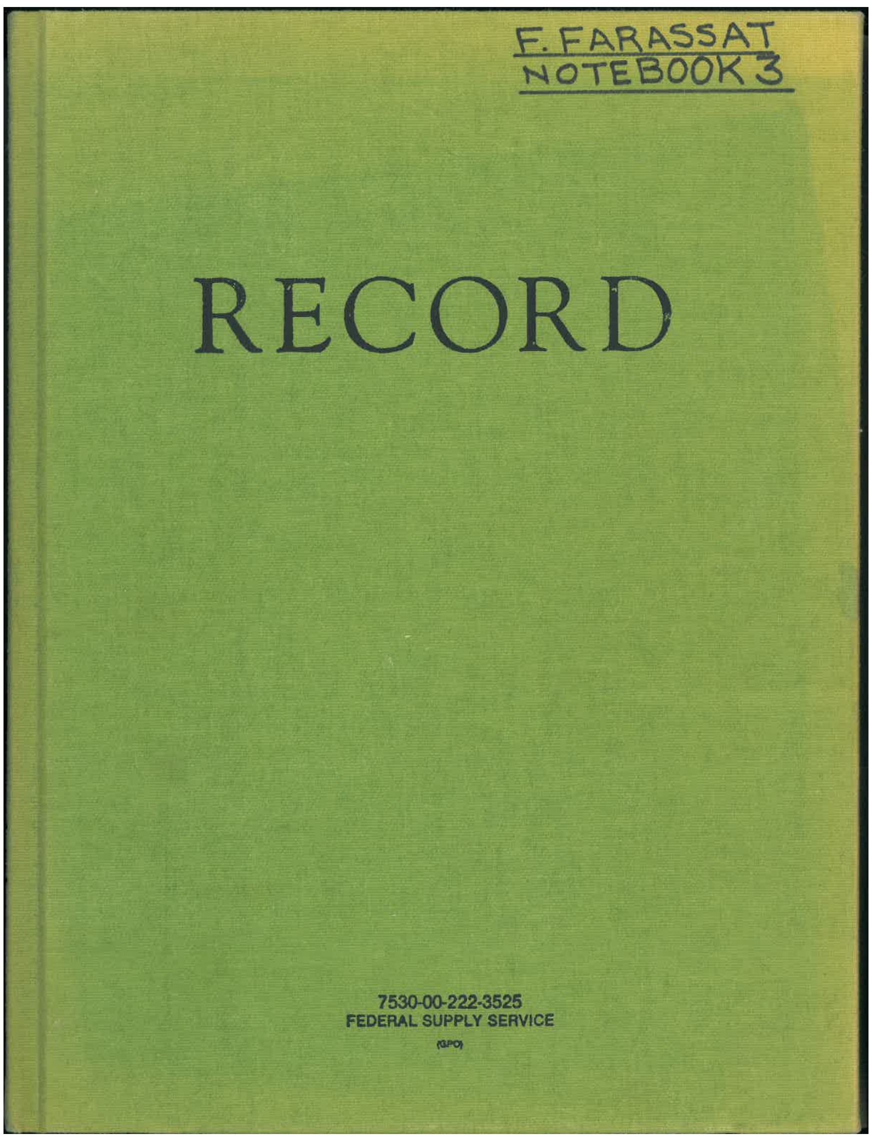
$$\square^2 p' = \underset{\sim}{q}(\vec{y},\tau)\,\delta'(f)$$

THE PIECES OF THE PUZZLE OF THE SINGULARITY FELL IN THE RIGHT PLACE. THE STUDY OF THE SINGULARITY IS MUCH SIMPLER NOW. I HAVE SHOWN THAT THERE ARE NO SINGULARITIES IN THE SUPERSONIC SOLUTION OF THE FW-H & K-EQUATION.

I AM PUBLISHING THE RESULTS IN AIAA PAPER NO. 98-2375 FOR THE AEROACOUSTICS CONFERENCE IN TOULOUSE, FRANCE. THE PAPER IS COAUTHORED WITH KEN BRENTNER AND MARK DUNN. I LEARNED ONE THING FROM ALMOST FOUR MONTHS OF VERY HARD WORK AND MANY SLEEPLESS NIGHTS WHEN I THOUGHT ABOUT THE PROBLEM OF SINGULARITIES IN THE SOLUTION OF THE WAVE EQUATION. I LEARNED THAT I CAN TRUST MY INTUITION! THE FINAL RESOLUTION OF THE PUZZLE IS EXACTLY WHAT I FELT TO BE THE ANSWER ALL ALONG.

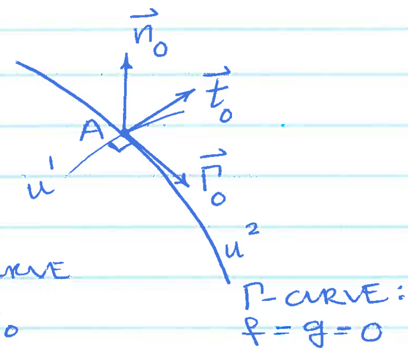
MAY 5, 1998
ON VACTION IN
DUCK, N.C.

12 Notebook Three



CHRISTOFFEL SYMBOL Γ_{1i}^{i} IN FORMULATION 4

IN DERIVATION OF FORMULATION 4 (AIAA-98-2375, 4TH AIAA/CEAS AEROACOUSTICS CONFERENCE, JUNE 2-4, 1998, TOULOUSE, FRANCE), WE LEFT OUT THE GEOMETRICAL MEANING OF Γ_{1i}^{i} (SUM ON i). WE SHOW HERE THAT Γ_{1i}^{i} IS THE NEGATIVE OF GEODESIC CURVATURE OF THE Γ-CURVE.



LET THE Γ-CURVE BE DESCRIBED BY $\vec{r}_0(u^2)$ WITH $\vec{\Gamma}_0$ THE UNIT TANGENT TO THE $\vec{\Gamma}$-CURVE. THE UNIT NORMAL TO $f=0$ ALONG Γ-CURVE IS DENOTED BY $\vec{n}_0$ AND $\vec{t}_0$ IS THE UNIT GEODESIC NORMAL OF Γ-CURVE ALONG THE PROJECTION OF $\vec{r}$ ON THE LOCAL TANGENT PLANE. LET (u^1, u^2) BE THE GAUSSIAN COORDINATE SYSTEM ON $f=0$, LOCALLY ORTHOGONAL WITH u^1 AND u^2 AS LENGTH VARIABLES. IN THE VICINITY OF A POINT A ON Γ-CURVE, THE SURFACE $f=0$ CAN BE DESCRIBED BY POSITION VECTOR $\vec{r}(u^1, u^2)$ AS FOLLOWS

(** SEE P3)

$$\vec{r} = \vec{r}_0(u^2) + u^1\left[\vec{t}_0(u^2) + \frac{1}{2}K_1 u^1 \vec{n}_0(u^2)\right]$$
$$= \vec{r}_0(u^2) + u^1\vec{t}_0(u^2) + \frac{1}{2}K_1(u^2)(u^1)^2\vec{n}_0(u^2) \checkmark$$

WHERE K_1 IS THE LOCAL NORMAL CURVATURE OF $f=0$ ALONG $\vec{t}_0$ ON THE $\vec{\Gamma}$-CURVE. LET

$$\vec{r}_i = \frac{\partial \vec{r}}{\partial u^i}, \quad \vec{r}_{ij} = \frac{\partial^2 \vec{r}}{\partial u^i \partial u^j}$$

$$\Rightarrow \begin{cases} \vec{r}_1 = \vec{t}_0 + K_1 u' \vec{n}_0 \\ \vec{r}_2 = \vec{\Gamma}_0(u^2) + u' \vec{t}_0{}' + \frac{1}{2}(u')^2[K_1' \vec{n}_0 + K_1 \vec{n}_0{}'] \end{cases}$$

$$\begin{cases} \vec{r}_{11} = K_1 \vec{n}_0 \\ \vec{r}_{12} = \vec{t}_0{}' + o(u') \\ \vec{r}_{22} = \vec{\Gamma}_0{}' + o(u') \end{cases}$$

WE HAVE THE FOLLOWING RESULT (GOETZ, P. 229, INTRO. TO DIFF. GEOM.) KNOWN AS KOVALEVSKY FORMULAS (*, P3)

$$\vec{\Gamma}_0{}' = K_g \vec{t}_0 + K_2 \vec{n}_0 \quad , K_2 \text{ NOR. CURV. ALONG } \vec{\Gamma}_0$$

$$\vec{t}_0{}' = -K_g \vec{\Gamma}_0 + \tau_g \vec{n}_0$$

WHERE K_g IS THE GEODESIC CURVATURE AND τ_g IS THE GEODESIC TORSION. WE THUS HAVE, ON Γ-CURVE

$$\begin{cases} \vec{r}_{11} = K_1 \vec{n}_0 \\ r_{12} = -K_g \vec{\Gamma}_0 + \tau_g \vec{n}_0 \\ \vec{r}_{22} = K_g \vec{t}_0 + K_2 \vec{n}_0 \end{cases} \qquad \begin{cases} \vec{r}_1 = \vec{t}_0 \\ \vec{r}_2 = \vec{\Gamma}_0 \end{cases}$$

LET $g_{ij} = \vec{r}_i \cdot \vec{r}_j$ AND $G = [g_{ij}]$, $G^{-1} = [g^{ij}]$.

ON Γ-CURVE, WE HAVE

$$g_{11} = g_{22} = 1, \quad g_{12} = 0$$

$$g^{11} = g^{22} = 1, \quad g^{12} = 0$$

$$\Gamma_{ijk} = \vec{r}_{ij} \cdot \vec{r}_K \Rightarrow$$

$$\Gamma_{111} = \Gamma_{112} = \Gamma_{121} = \Gamma_{222} = 0, \quad \Gamma_{122} = -K_g, \quad \Gamma_{221} = K_g$$

$$\Gamma_{1i}^{\ i} = \Gamma_{11}^{\ 1} + \Gamma_{12}^{\ 2}$$

$$\begin{cases}\Gamma_{11}^{\ 1} = g^{1i}\,\Gamma_{11i} = g^{11}\,\Gamma_{111} + g^{12}\,\Gamma_{112} = 0 \\ \Gamma_{12}^{\ 2} = g^{2i}\,\Gamma_{12i} = g^{21}\,\Gamma_{121} + g^{22}\,\Gamma_{122} = -k_g\end{cases}$$

$$\therefore \quad \boxed{\Gamma_{1i}^{\ i} = -k_g}$$

(P2) (*) WE DID NOT NEED TO USE THE THIRD KOVALEVSKY FORMULA

$$\vec{n}_0' = -k_2\,\vec{\Gamma}_0 - \tau_g\,\vec{t}_0$$

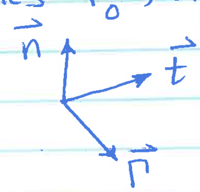
(**) P1

WE HAVE $\vec{r} = \vec{r}_0 + \left.\dfrac{\partial \vec{r}}{\partial u^1}\right|_{u^1=0} u^1 + \dfrac{1}{2}\left.\dfrac{\partial^2 \vec{r}}{\partial (u^1)^2}\right|_{u^1=0} (u^1)^2 + O(u^1)^3$

$$= \vec{r}_0(u^2) + u^1\,\vec{t}_0(u^2) + \frac{1}{2}\,k_1 (u^1)^2\,\vec{n}_0 + O(u^1)^3$$

— CONVENTION FOR VECTORS $\vec{\Gamma}_0$, $\vec{n}_0$ AND $\vec{t}_0$

$$\vec{t}_0 = \vec{n}_0 \times \vec{\Gamma}_0$$

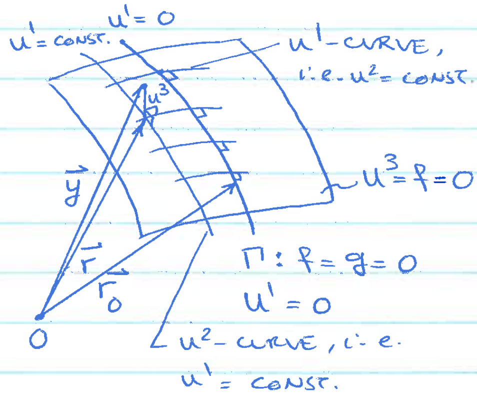


JUNE 98

CORRECTIONS TO MY AIAA-98-2375 PAPER

THERE ARE TWO CORRECTIONS TO THIS PAPER AS FOLLOWS.

1. PARAMETRIZATION OF SPACE AROUND THE SURFACE $f = 0$: $\vec{y} \rightarrow \vec{u}$, $\vec{u} = (u^1, u^2, u^3)$



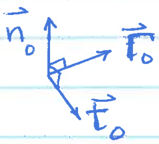
DEFINE $\vec{r}_0 = \vec{r}_0(u^2)$ TO BE THE Γ-CURVE WITH u^2 AS LENGTH VARIABLE ALONG $\vec{r}_0$. USING PLANES NORMAL TO Γ, DEFINE u^1-CURVES WITH u^1 AS LENGTH VARIABLE ALONG u^1-CURVE $\Rightarrow$ ALONG u^2-CURVES, u^2 IS NOT THE LENGTH VARIABLE UNLESS $u^1 = 0$. TAKING u^3 AS THE DISTANCE OF A POINT FROM $f = 0$, WE HAVE

$$\vec{y}(u^1, u^2, u^3) = \vec{r}(u^1, u^2) + u^3 \vec{n}(u^1, u^2)$$

WHERE $\vec{n}$ IS THE LOCAL UNIT NORMAL TO $f = 0$:

$$\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \quad (\text{SEE P2 FOR DEFN OF } \vec{r}_1 \,\&\, \vec{r}_2)$$

WE HAVE



$$\vec{r}_1 = \vec{t}_0(u^2) + K_1 u^1 \vec{n}_0(u^2)$$

$$\vec{r}_2 = \vec{\Gamma}_0(u^2) + u^1 \vec{t}_0{}'(u^2)$$

$$= (1 - K_g u^1)\, \vec{\Gamma}_0(u^2) + \tau_g u^1 \vec{n}_0(u^2)$$

NOTE: K_1, K_g AND τ_g ARE FUNCTIONS OF u^2 AND ARE DEFINED ON Γ-CURVE.

$$\vec{r}_1 \times \vec{r}_2 = (1 - K_g u^1)\vec{n}_0 - K_1 u^1 (1 - K_g u^1)\vec{t}_0$$

$$- \tau_g u^1 \vec{\Gamma}_0 = -\tau_g u^1 \vec{\Gamma}_0 - K_1 u^1 \vec{t}_0 + (1 - K_g u^1)\vec{n}_0 + O(u^1$$

$$\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \vec{n}_0 - (\kappa_1 \vec{t}_0 + \tau_g \vec{\Gamma}_0) u^1$$
$$- \tfrac{1}{2}(u^1)^2 [2\kappa_g \tau_g \vec{\Gamma}_0 + (\kappa_1^2 + \tau_g^2)\vec{n}_0] + \cdots$$

I HAVE USED MATHEMATICA TO GET THIS EXPANSION.

$$\frac{\partial g}{\partial u^1} = \vec{y}_1 \cdot \nabla g \quad \left\{ \begin{array}{l} \text{NOTE: } \hat{\vec{r}} = \hat{\vec{r}}_0(u^2) + \frac{\partial \hat{\vec{r}}_0}{\partial \vec{y}} d\vec{y} \\ d\vec{y} = \vec{t}_0 u^1 + \vec{n}_0 u^3, \quad \frac{\partial \hat{\vec{r}}_0}{\partial \vec{y}} d\vec{y} = -\frac{d\vec{y}}{r} \\ \Rightarrow \hat{\vec{r}} = \hat{\vec{r}}_0 - \frac{d\vec{y}}{r}, \quad d\vec{y} \cdot \vec{y}_1 = 0! \end{array} \right.$$

$$\vec{y}_1 = \vec{r}_1 + u^3 \vec{n}_1$$
$$= \vec{t}_0 + \kappa_1 u^1 \vec{n}_0 - u^3(\kappa_1 \vec{t}_0 + \tau_g \vec{\Gamma}_0) + \cdots$$

$$c\frac{\partial g}{\partial u^1} = \sin\theta + \kappa_1(u^1\cos\theta - u^3\sin\theta) + \cdots$$

NOW WE CONSIDER EQ. (29) OF MY PAPER

$$4\pi \phi_2(\vec{x},t) = \int \frac{\underset{\sim}{q}_2(\vec{y},\tau)}{r} H(\tilde{f})\, \delta_s'(u^3)\, \delta(g) \sqrt{g_{(2)}}\, du^1 du^2 du^3 d\tau$$
$$= c\iint \left[\frac{\sqrt{g_{(2)}}\, H(\tilde{f})\, \underset{\sim}{q}_2}{r\,|\sin\theta + \kappa_1(u^1\cos\theta - u^3\sin\theta)|}\right]_{g=0} \delta_s'(u^3)\, du^2 du^3 d\tau$$
$$= -c\int \left\{\frac{\partial}{\partial u^3}\left[\frac{\underset{\sim}{q}_2\sqrt{g_{(2)}}\, H(\tilde{f})}{r\,|\sin\theta + \kappa_1(u^1\cos\theta - u^3\sin\theta)|}\right]_{g=0}\right\}_{u^3=0} du^2 d\tau$$

WE NEED SOME DERIVATIVES AS FOLLOWS

$$r = |\vec{x} - \vec{y}|, \quad r|_{g=0} = c(t-\tau) = \text{CONST.}$$

$$\left.\frac{\partial r}{\partial u^3}\right|_{g=0} = 0 = \left[\frac{\partial r}{\partial u^3} + \frac{\partial r}{\partial u^1}\frac{du^1}{du^3}\right]_{g=0}$$

$$\frac{\partial r}{\partial u^3} = \underbrace{\frac{\partial r}{\partial y_i}}_{-\hat{\vec{r}}} \cdot \underbrace{\frac{\partial y_i}{\partial u^3}}_{\vec{y}_3}, \quad \frac{\partial r}{\partial u^1} = \underbrace{\frac{\partial r}{\partial y_i}}_{-\hat{\vec{r}}} \cdot \underbrace{\frac{\partial y_i}{\partial u^1}}_{\vec{y}_1}$$

$$\Rightarrow \quad \left.\frac{\partial u^1}{\partial u^3}\right|_{g=0} = -\frac{\partial r/\partial u^3}{\partial r/\partial u^1} = -\frac{\vec{\hat{r}}\cdot\vec{n}_0}{\vec{\hat{r}}\cdot\vec{t}_0} = -\cot\theta \checkmark$$

$$\sin\theta\big|_{g=0} = \vec{t}_0 \cdot \vec{\hat{r}}$$

$$\left.\frac{\partial \sin\theta}{\partial u^3}\right|_{g=0} = \vec{t}_0 \cdot \left[\frac{\partial \vec{\hat{r}}}{\partial u^3} + \frac{\partial \vec{\hat{r}}}{\partial u^1}\frac{\partial u^1}{\partial u^3}\right]$$

$$\frac{\partial \vec{\hat{r}}}{\partial u^3} = \vec{y}_3 \cdot \nabla \frac{\vec{r}}{r}$$

$$= -\frac{\vec{y}_3}{r} - \frac{\vec{y}_3\cdot\nabla r}{r^2}\vec{r}$$

$$= -\frac{\vec{y}_3}{r} + \frac{\vec{y}_3\cdot\vec{\hat{r}}}{r}\vec{\hat{r}}$$

$$\vec{y}_3 = \vec{n}_0 + o(u^1, u^2)$$

$$\frac{\partial \vec{\hat{r}}}{\partial u^1} = \vec{y}_1 \cdot \nabla \frac{\vec{r}}{r}$$

$$= -\frac{\vec{y}_1}{r} - \frac{\vec{y}_1\cdot\nabla r}{r^2}\vec{r}$$

$$= -\frac{\vec{y}_1}{r} - \frac{\vec{y}_1\cdot\vec{\hat{r}}}{r}\vec{\hat{r}}$$

$$\vec{y}_1 = \vec{t}_0 + o(u^1, u^3)$$

$$\left.\frac{\partial \sin\theta}{\partial u^3}\right|_{g=0} = \frac{\cos\theta\sin\theta}{r} + \left[-\frac{1}{r} + \frac{\sin^2\theta}{r}\right](-\cot\theta)$$

$$= \frac{\cos\theta\sin\theta}{r} + \frac{\cos^3\theta}{r\sin\theta}$$

$$= \frac{\cot\theta}{r}$$

WE NEED TO FIND

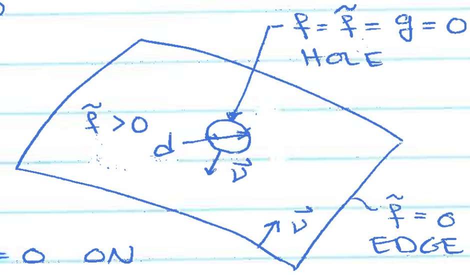
$$E = \frac{\partial}{\partial u^3}\left\{\left[\frac{1}{r|\partial g/\partial u^1|}\right]_{g=0}\right\}_{u^3=0} = \frac{\partial}{\partial u^3}\left\{\left[\frac{c}{r|\sin\theta + K_1(u^1\cos\theta - u^3\sin\theta)|}\right]_{g=0}\right\}_{u^3}$$

WE NOTE THAT $\left.\frac{\partial r}{\partial u^3}\right|_{g=0} = 0$ BECAUSE $r = c(t-\tau)$ $=$ CONST.

$$E = c\left[-\frac{1}{r\sin^2\theta}\,\frac{\partial \sin\theta}{\partial u^3} - \frac{K_1\cos\theta}{r\sin^2\theta}\,\frac{\partial u^1}{\partial u^3} + \frac{K_1 \cot\theta}{r\sin^2\theta}\right]$$

$$= c\left(-\frac{\cos\theta}{r^2\sin^3\theta} + \frac{K_1}{r\sin^3\theta}\right) \checkmark \qquad \text{NOTE } g = u^3 = 0 \Rightarrow u^1 = 0!$$

WE MUST NOW CONSIDER THE CONDITION $\theta = 0$ WHICH GIVES AN EXTRA TERM WHICH I DISCOVERED BY SOLVING A PROBLEM FOR WHICH I FOUND THE ANALYTIC SOLUTION. ASSUME $g = 0$ GETS TANGENT TO $f = 0$ AT A SINGLE POINT ONLY. AS THE Γ-CURVE SHRINKS TO A POINT, TAKE ONE OF THESE Γ-CURVES AS $\tilde{f} = 0$ ON THE PANEL SUCH THAT $\tilde{f} > 0$ ON THE PANEL. THEN



$$\frac{\partial}{\partial u^3} H(\tilde{f}) = -\left[\vec{\nu}\cdot\vec{t}_1 \cot\theta\, \delta(\tilde{f})\right]_{EDGE} - \left[\vec{\nu}\cdot\vec{t}_1 \cot\theta\, \delta(\tilde{f})\right]_{HOLE}$$

$$4\pi\phi_2(\vec{x},t) = c\int\left[\frac{\cos\theta}{r\sin^2\theta}\,\vec{t}_1\cdot\nabla_2 q_2 + \frac{\cos\theta}{r^2\sin^3\theta} q_2\right] d\Gamma\, d\tau$$

$$- c\int\left[\frac{K_1}{r\sin^3\theta} + \frac{K_g\cos\theta}{r\sin^2\theta}\right] q_2\, d\Gamma\, d\tau$$

$$+ \int_{EDGE}\left[\frac{\cos\tilde{\theta}\,\cos\theta}{r\Lambda_0\sin^2\theta} q_2\right]_{ret} dL$$

$$+ \lim_{d\to 0}\int_{HOLE}\left[\frac{\cos\tilde{\theta}\cos\theta}{r\Lambda_0\sin^2\theta} q_2\right]_{ret} dL$$

LET
$$I = \int \frac{\cos\tilde{\theta}\cos\theta\, q_2}{r\Lambda_0 \sin^2\theta}\, dL$$

WE HAVE
$$\cos\tilde{\theta} = \vec{\nu}\cdot\hat{r} = \vec{\nu}\cdot\vec{t}_1 \sin\theta$$

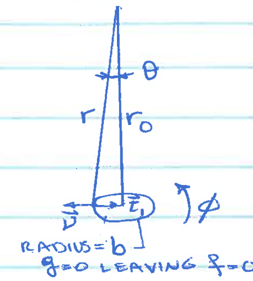
$$\frac{dL}{\Lambda_0} = \frac{d\ell}{|1-M_r|}$$

NOTE ADDED ON AUG. 22, 99: I FOUND OUT TODAY THAT $\lim_{d\to 0} I$ DERIVED HERE IS ONLY VALID FOR A FLAT SURFACE! IT NEEDS TO BE GENERALIZED. F.F.

AS $d \to 0$, $\vec{\nu}\cdot\vec{t}_1 \to -1$

$$I = -\int \frac{\cos\theta\, q_2\, d\ell}{r|1-M_r|\sin\theta}$$

$$= -\int_0^{2\pi} \frac{\pm(r_0/r)\, q_2\, b\, d\phi}{r|1-M_r|(b/r)}$$



$$\therefore \lim_{d\to 0} I = \mp\left[\frac{2\pi\, q_2}{|1-M_n|}\right]_T \qquad T: \text{TANGENCY POINT}$$

NOTE THAT q_2 HAS A FACTOR OF $M_n^2 - 1$ FOR BOTH FW-H EQ. AND K EQ. WE HAVE THUS FOUND THE FOLLOWING SOLUTION FOR $\phi_2(\vec{x},t)$:

$$\boxed{\begin{aligned} 4\pi\phi_2(\vec{x},t) = &\int \left[\frac{1}{r}\cot\theta\, \vec{t}_1\cdot\nabla_2 q_2 + \frac{\cot\theta}{r^2\sin\theta}\, q_2\right]_{ret} \frac{d\Sigma}{\Lambda} \\ &- \int \left[\left(\frac{K_1}{r\sin^2\theta} + \frac{K_g\cot\theta}{r}\right) q_2\right]_{ret} \frac{d\Sigma}{\Lambda} \\ &+ \int_{EDGE} \left[\frac{\cos\tilde{\theta}\cot\theta}{r\Lambda_0\sin\theta}\, q_2\right]_{ret} dL \\ &\mp \left[\frac{2\pi\, q_2}{|1-M_n|}\right]_T \qquad \begin{cases} + : \vec{n}\cdot\hat{r} \to -1 & \text{ENTERING } f=0 \\ - : \vec{n}\cdot\hat{r} \to +1 & \text{LEAVING } f=0 \end{cases} \end{aligned}}$$

SEE P15 FOR ADDITIONAL INFORMATION. THE LAST TERM IS ONLY VALID FOR A FLAT SURFACE!

CONVENTION: $K_g > 0$ IF THE CENTER OF CURVATURE FOR K_g IS ON THE SIDE $\vec{t}_1$ POINTS INTO.

[signature] 8/13/98

VALIDATION OF $\phi_2(\vec{x},t)$

WE CONSIDER THE FOLLOWING PROBLEM FOR WHICH WE HAVE ANALYTIC RESULT FROM CLASSICAL AND ONE FROM FORMULATION 4:

$$\Box^2 \phi = +\frac{\partial}{\partial x_3}\left[q_2(x_1,x_2,t)\,\delta(x_3)\right] = q_2\,\delta'(x_3)$$

$$q_2 = -(1+\rho^2)\,e^{i\omega t}, \quad \rho \leq 1, \quad \rho^2 = x_1^2 + x_2^2$$

CLASSICAL RESULT:

$$4\pi\phi_2(\vec{x},t) = +\frac{\partial}{\partial x_3}\int_S \frac{[q_2]_{ret}}{r}\,dS$$

$$= -\frac{\partial}{\partial x_3}\int_S \frac{1}{r}(\rho^2+1)\,e^{i\omega(t-\frac{r}{c})}\,dS$$

$$= x_3 e^{i\omega t}\int_0^1\int_0^{2\pi}\frac{e^{-ikr}}{r^3}(\rho^2+1)(1+ikr)\,\rho\,d\varphi\,d\rho$$

$k = \frac{\omega}{c}$: WAVE NUMBER

RESULT BASED ON FORMULATION 4:

WE ASSUME THAT THE OBSERVER IS ALWAYS ON x_1x_3-PLANE

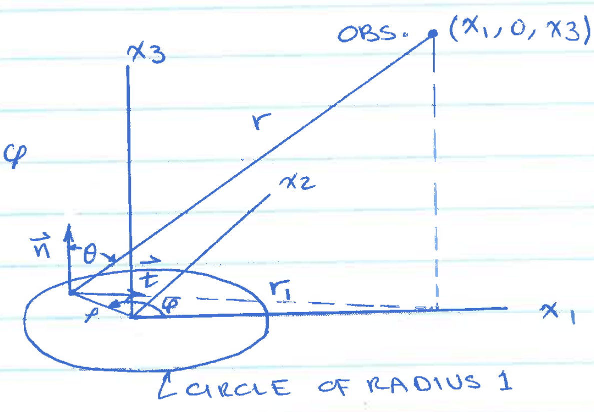
$$r^2 = \rho^2 + x_1^2 + x_3^2 - 2\rho x_1\cos\varphi$$

$$r_1^2 = \rho^2 + x_1^2 - 2\rho x_1\cos\varphi$$

$$\cos\theta = \frac{x_3}{r}$$

$$\sin\theta = \frac{r_1}{r}$$

$$\cot\theta = \frac{x_3}{r_1}$$



$$\vec{t}_1 = \left(\frac{x_1 - \rho\cos\varphi}{r_1}, -\frac{\rho\sin\varphi}{r_1}, 0\right)$$

$$e^{-i\omega t}\vec{t}_1 \cdot \nabla q_2 = -\left(\frac{x_1 - \rho\cos\varphi}{r_1}, \frac{-\rho\sin\varphi}{r_1}, 0\right)(2\rho\cos\varphi, 2\rho\sin\varphi, 0)$$

$$= -2\rho\,(x_1\cos\varphi - \rho)/r_1$$

$$K_1 = 0 \qquad k_g = \frac{1}{r_1}$$

$$\cos\tilde{\theta} = \vec{\nu}\cdot\vec{t}_1 \sin\theta$$

$$= (-\cos\varphi, -\sin\varphi, 0)\left(\frac{x_1 - \cos\varphi}{r_1}, -\frac{\sin\varphi}{r_1}, 0\right)\sin\theta$$

$$= (1 - x_1\cos\varphi)\sin\theta / r_1 \checkmark$$

$$4\pi\phi_2(\vec{x}, t) = -e^{i\omega t}\int_S \Big[\frac{2\rho(x_1\cos\varphi - \rho)x_3}{r r_1^2}$$

$$+ \frac{x_3(\cancel{1}+\rho^2)}{r\,\cancel{r_1}^2} - \frac{x_3(1\cancel{+}\rho^2)}{r\,\cancel{r_1}^2}\Big] e^{-ikr}\,dS$$

$$-e^{i\omega t}\int_0^{2\pi}\Big[\frac{x_3(1 - x_1\cos\varphi)}{r_1^2}(1+\rho^2)\,e^{-ikr}\Big]_{\rho=1} d\varphi$$

$$+ 2\pi(1 + x_1^2)\,e^{-ikx_3}\,H[1 - |x_1|^2]$$

WHERE $H(\cdot)$ IS THE HEAVISIDE FUNCTION.

$$e^{-i\omega t}\phi_2(\vec{x}, t) = -\frac{x_3}{2\pi}\int_0^1\int_0^{2\pi}\frac{\rho^2(x_1\cos\varphi - \rho)}{r r_1^2}e^{-ikr}\,d\varphi\,d\rho$$

$$+ \frac{x_3}{2\pi}\int_0^{2\pi}\Big[\frac{x_1\cos\varphi - 1}{r_1^2}e^{-ikr}\Big]_{\rho=1} d\varphi$$

$$+ \frac{1}{2}(1 + x_1^2)\,e^{-ikx_3}\,H[1 - |x_1|^2]$$

WE WRITE $e^{-i\omega t}\phi_2(\vec{x},t)\Big|_{\text{CLASSICAL}} = \text{Agen}$

$e^{-i\omega t}\phi_2(\vec{x},t)\Big|_{\text{FORM 4}} = \text{Bgen}$

I HAVE WRITTEN TWO FORMULAS USING MATHEMATICA3 AS FOLLOWS (SEE NEXT PAGE FOR Bgen):

$$\text{Agen} = \text{NIntegrate}\Big[\frac{x3}{4\pi}\left(\rho\,(\rho^2+1)\left(1+\text{I}\,k\sqrt{\rho^2-2\,\rho\,x1\,\text{Cos}[\varphi]+x1^2+x3^2}\right)\right)\Big/\left(\sqrt{\rho^2-2\,\rho\,x1\,\text{Cos}[\varphi]+x1^2+x3^2}\right)^3 \text{Exp}\Big[-\text{I}\,k\sqrt{\rho^2-2\,\rho\,x1\,\text{Cos}[\varphi]+x1^2+x3^2}\Big],\ \{\varphi,0,2\pi\},\ \{\rho,0,1\}\Big]$$

I DEFINED THE HEAVISIDE FUNCTION AS FOLLOWS:

```
H[x_] := If[x < 1, 1, 0]
```

THIS IS EXACTLY $H[1-|x_1|^2]$ ON PAGE 10.

$$\text{Bgen} = \frac{x3}{2\pi}\Bigg(-\text{NIntegrate}\Bigg[\left(\frac{\rho}{\sqrt{\rho^2 - 2\rho\, x1\, \text{Cos}[\varphi] + x1^2 + x3^2}}\;\frac{x1\,\rho\,\text{Cos}[\varphi] - \rho^2}{\rho^2 + x1^2 - 2\rho\, x1\, \text{Cos}[\varphi]}\right)\text{Exp}\Big[-\text{I}\, k\sqrt{\rho^2 - 2\rho\, x1\, \text{Cos}[\varphi] + x1^2 + x3^2}\Big], \{\varphi, 0, 2\pi\}, \{\rho, 0, 1\}\Bigg] + \text{NIntegrate}\Big[\Big((x1\,\text{Cos}[\varphi] - 1)\Big/\Big(\sqrt{1 - 2\,x1\,\text{Cos}[\varphi] + x1^2 + x3^2}\,(1 + x1^2 - 2\,x1\,\text{Cos}[\varphi])\Big)\Big)\,\text{Exp}\Big[-\text{I}\,k\sqrt{1 - 2\,x1\,\text{Cos}[\varphi] + x1^2 + x3^2}\Big], \{\varphi, 0, 2\pi\}\Big]\Bigg) + \frac{1}{2}(1 + x1^2)\,\text{Exp}[-\text{I}\,k\,x3]\,\text{H}[x1]$$

I HAVE RUN THE TWO FORMULAS AGAINST EACH OTHER AND I GET THE FOLLOWING RESULTS FOR $x_3 = 5$, $k = 10$. MATHEMATICA 3 IS EXCRUCIATINGLY SLOW FOR THESE FORMULAS ON MY G3 POWER PC WHICH IS SUPPOSED TO HAVE A FAST PROCESSOR.

x_1	$Agen = e^{-i\omega t}\phi_2\big\vert_{CLASS.}$	$Bgen = e^{-i\omega t}\phi_2\big\vert_{FORM 4}$
0	0.21098 + 0.673096 i	0.21098 + 0.673096 i
0.25	0.238068 + 0.637725 i	0.238067 + 0.637724 i
0.50	0.30169 + 0.531172 i	0.301689 + 0.531171 i
0.75	0.355626 + 0.3618 i	0.355625 + 0.3618 i
0.975	0.353059 + 0.184642 i	0.353058 + 0.184641 i
0.995	0.34963 + 0.169323 i	0.349588 + 0.169312 i
0.9995	0.348784 + 0.165908 i	0.348784 + 0.165908 i
0.99995	0.348698 + 0.165567 i	0.348698 + 0.165567 i
1.00005	0.348679 + 0.165491 i	0.348679 + 0.165491 i
1.25	0.263827 + 0.00687612 i	0.263827 + 0.00687612 i
5	0.00397175 − 0.0118504 i	0.00397175 − 0.0118504 i

AT $x_1 = 1$, THE NIntegrate ROUTINE OF MATHEMATICA CAN NOT HANDLE THE SINGULAR INTEGRAL (WHICH IS INTEGRABLE) IN Bgen. I DID NOT TRY TO FIND A WAY AROUND THIS. AS SEEN FROM THE ABOVE CALCULATIONS, THE AGREEMENT BEETWEEN THE CLASSICAL RESULT AND FORMULATION 4 IS EXCELLENT!

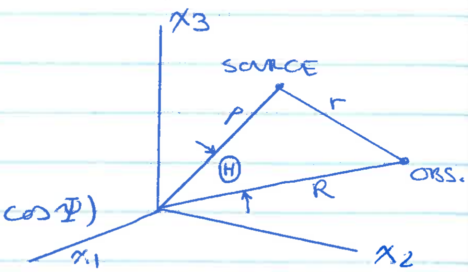
WE NEXT VALIDATE FORMULATION 4 FOR DIPOLES ON A UNIT SPHERE AND OBSERVER IN THE FAR FIELD. THE D.E. WE CONSIDER IS

$$\Box^2 \phi_2 = -\cos\Psi \, e^{i\omega t} \, \delta'(R-1)$$

WHERE (R, Ψ, Φ) ARE THE SPHERICAL POLAR COORDINATES. WE WILL USE (R, Ψ, Φ) AND (ρ, ψ, φ) AS THE OBSERVER AND THE SOURCE

SPHERICAL POLAR COORDINATES.

WE FIRST ESTABLISH SOME GEOMETRIC RESULTS.



$r^2 = R^2 + \rho^2 - 2R\rho\cos\Theta$

$\cos\Theta = \vec{e}_R \cdot \vec{e}_\rho$

$= (\sin\Psi\cos\Phi, \sin\Psi\sin\Phi, \cos\Psi)$

$\cdot (\sin\psi\cos\varphi, \sin\psi\sin\varphi, \cos\psi)$

$= \sin\Psi\sin\psi\cos(\Phi-\varphi) + \cos\Psi\cos\psi$

$$\frac{\partial r}{\partial \rho} = \frac{\rho - R\cos\Theta}{r}, \quad e^{-ikr} = e^{-ikr_0} \cdot e^{ik\cos\psi} \quad \text{(FAR FIELD)}$$

WE WILL PUT THE OBSERVER ON THE X_3-AXIS WITH $R = r_0 >> 1 \Longrightarrow \Theta = \psi$ AND $\Psi = 0$

$$\frac{\partial r}{\partial \rho} = -\cos\psi \qquad \text{(FAR FIELD)}$$

<u>CLASSICAL RESULT</u> $\quad d\vec{y} = 2\pi\rho^2\sin\psi\, d\rho\, d\psi$

$$4\pi\phi_2(\vec{x},t) = -\int \cos\psi \frac{e^{i(\omega t - kr)}}{r} \delta'(\rho-1)\, d\vec{y}$$

$$= 2\pi e^{i\omega t}\int_0^\pi \cos\psi \left[\rho^2 \frac{\partial}{\partial r}\left(\frac{e^{-ikr}}{r}\right)\frac{\partial r}{\partial \rho} + 2\rho\frac{e^{-ikr}}{r}\right]_{\rho=1}$$

$$\times \sin\psi\, d\psi$$

I USED MATHEMATICA 3 HERE $\} \rightarrow$ $$= 2\pi e^{i(\omega t - kr_0)}\int_0^\pi \frac{2 + ik\cos\psi}{r_0}\cos\psi\sin\psi\, e^{ik\cos\psi}\, d\psi$$

$$= 4\pi i \frac{\sin k}{r_0} e^{i(\omega t - kr_0)} \quad \text{(GEOMETRIC FAR FIELD)}$$

NOTE: $r = r_0 - \rho\cos\Theta = r_0 - \cos\psi$ IN THE FAR FIELD.

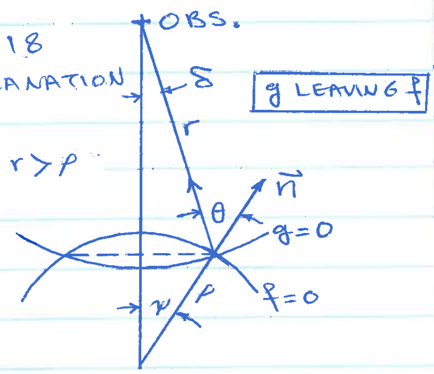
WHEN I TRIED TO APPLY THE SINGLE TERM $[2\pi q_2 \Lambda |1 - M_n|]_\tau$ TO A SPHERE, I GOT THE WRONG RESULT! I WAS SURPRISED FROM THE BEGINNING THAT THIS TERM HAS NO DEPENDENCE ON r. IT TURNS OUT THAT THIS TERM IS ONLY VALID FOR A FLAT SURFACE. I WILL FIRST DERIVE THIS TERM FOR A STATIONARY SPHERE OF RADIUS ρ AND THEN VERIFY THAT THE CLASSICAL RESULT FOR A SPHERE (PREVIOUS PAGE) CAN BE OBTAINED FROM FORMULATION 4. I THEN GENERALIZE THE LIMITING TERM

$$\lim_{d\to 0} I = \lim_{d\to 0} \int \frac{\cos\tilde{\theta}\cos\theta \, q_2}{r\Lambda_0 \sin^2\theta}\, dL \qquad (*)$$

FOR A SURFACE WITH POSITIVE GAUSSIAN CURVATURE WHICH IS STATIONARY. FINALLY, I WILL CONSIDER A MOVING SURFACE.

$\lim_{d\to 0} I$ FOR A SPHERE OF RADIUS ρ (STATIONARY)

$$I = -\int \frac{|\cos\theta|}{r\sin\theta}\, d\ell \qquad \left\{ \begin{array}{l}\text{SEE P 18}\\ \text{FOR EXPLANATION}\end{array}\right.$$



$\theta = \pm\delta + \psi \quad \left\{ \begin{array}{l} + \text{ LEAVING} \\ - \text{ ENTERING, FOR } r > \rho \end{array}\right.$

WE ASSUME $\psi \ll 1$ SINCE $\psi \to 0$ AS $\sin\theta \to 0$. WE HAVE

$r\sin\delta = \rho\sin\psi$ OR

$r\delta = \rho\psi$ FOR SMALL ψ.

$\sin\theta = \sin(\pm\delta + \psi) = \pm\delta + \psi = (1 \pm \frac{\rho}{r})\psi \quad \left\{ \begin{array}{l} + \text{ LEAVING} \\ - \text{ ENTERING} \end{array}\right.$

$d\ell = \rho\sin\psi \, d\phi$, ϕ AZIMUTHAL (POLAR) ANGLE

$|\cos\theta| = 1$ AS $\psi \to 0$. WE THUS HAVE:

(*) SEE P7, FIGURE, FOR DEFN OF d. IN ABOVE FIG., $\psi \to 0$ AS $d \to 0$.

$$\lim_{\psi\to 0} I = -\lim_{\psi\to 0} \int_0^{2\pi} \frac{1}{r(1\pm\frac{\rho}{r})\psi} q_2 \rho\, \psi\, d\phi$$

$$= -\left[\frac{2\pi\rho\, q_2}{r\pm\rho}\right]_T \qquad \begin{cases} + & \text{LEAVING} \\ - & \text{ENTERING} \end{cases}$$

$$= -\left[\frac{2\pi\rho\, q_2}{r}\right]_T \qquad \begin{cases} \text{GEOMETRIC} \\ \text{FAR FIELD: } \rho \ll r_T \end{cases}$$

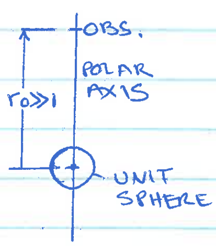
$$\left.\begin{matrix} -\text{ LEAVING} \\ +\text{ ENTERING} \end{matrix}\right\} = \mp \left[2\pi q_2\right]_T \qquad \begin{cases} \text{GEOMETRIC} \\ \text{NEAR FIELD} \end{cases} \quad \rho \gg r_T$$

THIS LAST RESULT IS VALID $\forall r$ AND $\rho = \infty$, i.e.; f FLAT AT T ($r_T = r$ AT TANGENCY OF $f=0$ AND $g=0$). THE CONDITION $r=\rho$ FOR ENTERING CONDITION MEANS THAT THE OBSERVER IS INSIDE AND ON THE CENTER OF THE SOURCE SPHERE WHICH WE RULE OUT FOR NOW.

VERIFICATION OF FORMULATION 4 FOR A SPHERE

GOING BACK TO THE PROBLEM OF RADIATION FROM DIPOLES ON A UNIT SPHERE, PAGE 13, WE HAVE FROM FORMULATION 4 FOR OBSERVER IN GEOMETRIC FAR FIELD:

$$4\pi\, \phi_2(\vec{x},t) = \frac{1}{r_0}\int \left[\cot\theta\, \vec{t}_1 \cdot \nabla_2 q_2\right]_{ret} dS$$



$$-\frac{1}{r_0}\int \left[\left(\frac{K_1}{\sin^2\theta} + K_g \cot\theta\right) q_2\right]_{ret} dS$$

$$-\left[\frac{2\pi\rho\, q_2}{r_0}\right]_{\text{ENTER.}} - \left[\frac{2\pi\rho\, q_2}{r_0}\right]_{\text{LEAV.}}$$

$\theta = \psi$, $K_1 = -1$, $\rho = 1$, $K_g = \cot\psi$

$$\vec{t}_1 \cdot \nabla_2 q_2 = -\frac{\partial}{\partial \psi}\left[-\cos\psi\, e^{i\omega\tau}\right]$$
$$= -\sin\psi\, e^{i\omega\tau}$$

$$\frac{\kappa_1}{\sin^2\theta} + \kappa_g \cot\psi = -\frac{1}{\sin^2\psi} + \cot^2\psi = -1$$

$$-\frac{2\pi}{r_0}\left\{\left[q_2\right]_{LEAV.} + \left[q_2\right]_{ENT.}\right\} = \frac{2\pi}{r_0} e^{i(\omega t - kr_0)} \times$$

$-\cos\psi\big|_{LEAV.} = -1$, $-\cos\psi\big|_{ENT.} = +1$ } FROM q_2 ONLY

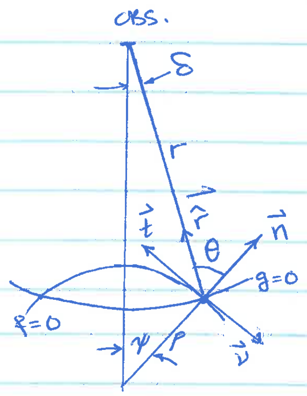
$$\left[e^{ik} - e^{-ik}\right]$$
$$= +\frac{4\pi i}{r_0} \sin k\, e^{i(\omega t - kr_0)}$$

$$4\pi\phi(\vec{x}, t) = -\frac{1}{r_0} e^{i(\omega t - kr_0)} \int_0^\pi \cos\psi \sin\psi\, e^{ik\cos\psi}\, d\psi$$
$$+ \frac{1}{r_0} e^{i(\omega t - kr_0)} \int_0^\pi \cos\psi \sin\psi\, e^{ik\cos\psi}\, d\psi$$
$$+ \frac{4\pi i}{r_0} \sin k\, e^{i(\omega t - kr_0)}$$
$$= +\frac{4\pi i}{r_0} \sin k\, e^{i(\omega t - kr_0)} \quad \text{(GEOMETRIC FAR FIELD)}$$

THIS IS EXACTLY AS THE CLASSICAL RESULT (P 14).

I WILL NOW ADDRESS THE EVALUATION OF $\lim_{d \to 0} I$ FOR THE CASE WHERE WE HAVE TWO DISTINCT VALUES OF PRINCIPAL CURVATURES κ_1 AND κ_2. NOTE κ_1 HERE IS NOT THE SAME AS IN FORMULATION 4. HERE κ_1 AND κ_2 ARE PRINCIPAL CURVATURES.

THE EXPLANATION FOR THE FORM OF I ON PAGE 15

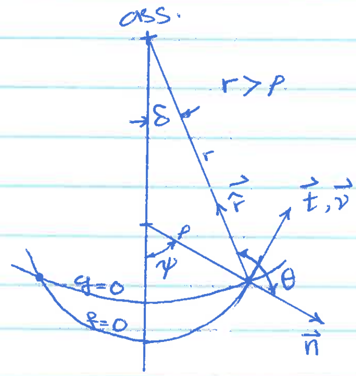


CASE 1: LEAVING

$\theta = \delta + \psi$

$\vec{t} \cdot \vec{\nu} = -1$

$\cos\theta \to 1$



CASE 2: ENTERING

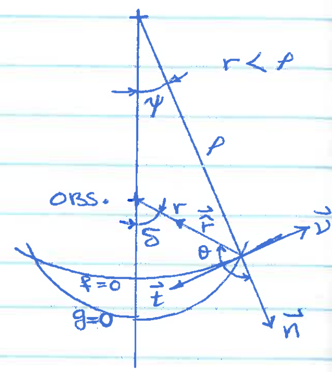
$\rho < r$

$\pi - \theta = -\delta + \psi$

$\vec{t} \cdot \vec{\nu} = 1$

$\cos\theta \to -1$

$\theta = \pi + \delta - \psi$



CASE 3: ENTERING

$\rho > r$

$\pi - \theta = \delta - \psi$

$\vec{t} \cdot \vec{\nu} = -1$

$\cos\theta \to -1$

$\theta = \pi - \delta + \psi$

$$\cos\tilde{\theta} = \vec{t} \cdot \vec{\nu} \sin\theta \quad , \quad dl = \rho \sin\psi \, d\phi$$

$$I = \int \frac{\cos\tilde{\theta} \cos\theta}{r \sin^2\theta} \, dl$$

$$= \int \frac{\vec{t} \cdot \vec{\nu} \cos\theta}{r \sin\theta} \, dl$$

IN ALL THE ABOVE CASES, WE HAVE $r\delta = \rho\psi$ AS $\psi \to 0$, OR $\delta = \frac{\rho}{r}\psi$

$$\frac{\vec{t} \cdot \vec{\nu} \cos\theta}{r \sin\theta} = \frac{-1}{(r+\rho)\psi} \qquad \text{CASE 1}$$

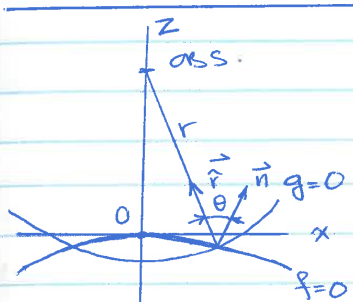
$$= \frac{-1}{(r-\rho)\psi} \qquad \text{CASE 2} \quad [\sin\theta = -\sin(\delta-\psi) = -\delta+\psi]$$

$$= \frac{-1}{(r-\rho)\psi} \qquad \text{CASE 3} \quad [\sin\theta = \sin(\delta-\psi) = \delta-\psi]$$

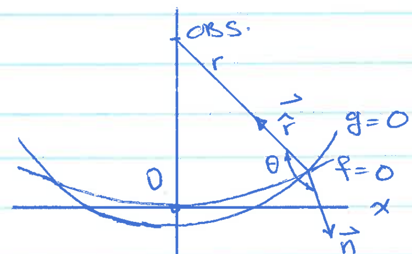
SO WE HAVE $\lim_{\psi \to 0} I = -\left[\frac{2\pi \rho q_2}{r \pm \rho}\right]_T$ $\left\{\begin{array}{l} + : \text{LEAVING} \\ - : \text{ENTERING} \end{array}\right.$

FOR ALL CASES.

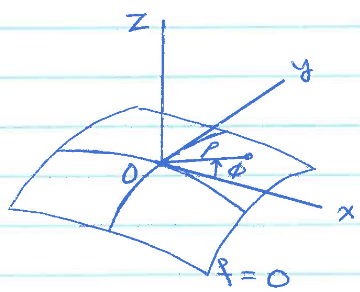
THE GENERAL CASE (STATIONARY)



g LEAVING f



g ENTERING f



x & y AXES ALONG THE DIRECTION OF LOCAL PRINCIPAL CURVATURES,

K_1 AND K_2 PRINCIPAL CURVATURES,

$K_r = \frac{1}{r}$ $\left\{\begin{array}{l}\text{NOTE: } \vec{n} = \vec{e}_z \text{ UNIT VEC. ALONG Z-AXIS} \\ \text{FOR DETERMINING SIGN OF } K_1 \& K_2\end{array}\right.$

LOCALLY f AND g ARE DESCRIBED IN THE ABOVE COORDINATE SYSTEM AS

f : $Z_f = \frac{1}{2}(K_1 x^2 + K_2 y^2)$ (SEE P 290, GOETZ, PROB. 8)

g : $Z_g = \frac{1}{2} K_r (x^2 + y^2) - \varepsilon$

FOR f, WE CAN WRITE

$$Z_g = \frac{1}{2}(K_1 \cos^2\phi + K_2 \sin^2\phi)\rho^2$$
$$= \frac{1}{2} K(\phi) \rho^2$$

FOR g: $Z = \frac{1}{2} K_r \rho^2 - \varepsilon$

THE CURVE OF INTERSECTION OF $f=0$ AND $g=0$ IN LOCAL POLAR COORDINATES IS FOUND BY EQUATING Z_f AND Z_g : $\frac{1}{2} K(\phi)\rho^2 = \frac{1}{2} K_r \rho^2 - \varepsilon$

$$\Rightarrow \quad \rho^2 = \frac{2\varepsilon}{\kappa_r - \kappa(\phi)} \qquad \kappa_r \neq \kappa(\phi) \ \forall \phi \in [0, 2\pi]$$

IN ORDER THAT $f=0$ AND $g=0$ INTERSECT, THE SIGN OF ε MUST BE SELECTED IN SUCH A WAY THAT $2\varepsilon/[\kappa_r - \kappa(\phi)] > 0$. WE WILL ADDRESS THIS PROBLEM LATER.

· ELEMENT OF LENGTH OF THE CURVE OF INTERSECTION:

$$d\ell^2 = \rho^2 d\phi^2 + d\rho^2 + dz^2$$

$$z = \frac{1}{2}\kappa(\phi)\rho^2 = \frac{\varepsilon\kappa(\phi)}{\kappa_r - \kappa(\phi)}$$

$$dz = \frac{\varepsilon\kappa_r\kappa'(\phi)}{[\kappa_r - \kappa(\phi)]^2}\, d\phi$$

$$d\rho^2 = \rho'^2 d\phi^2 = \frac{\varepsilon\kappa'^2(\phi)\, d\phi^2}{2[\kappa_r - \kappa(\phi)]^3}$$

$$d\ell^2 = \frac{\varepsilon\, d\phi^2}{\kappa_r - \kappa(\phi)}\left[2 + \frac{\kappa'^2(\phi)}{2[\kappa_r - \kappa(\phi)]^2}\right] + \frac{\varepsilon^2\kappa_r^2\kappa'^2(\phi)\, d\phi^2}{[\kappa_r - \kappa(\phi)]^4}$$

WE CAN THUS NEGLECT dz^2 BECAUSE IT IS $O(\varepsilon^2)$.

WE HAVE

$$\nabla f = \pm(-\kappa_1 x, -\kappa_2 y, 1) \qquad \begin{cases} + & \text{LEAVING} \\ - & \text{ENTERING} \end{cases}$$

$$\nabla g = (-\kappa_r x, -\kappa_r y, 1)$$

$$|\nabla f|^2 = 1 + (\kappa_1^2\cos^2\phi + \kappa_2^2\sin^2\phi)\rho^2$$

$$= 1 + \frac{2\varepsilon}{\kappa_r - \kappa(\phi)}(\kappa_1^2\cos^2\phi + \kappa_2^2\sin^2\phi)$$

$$|\nabla f| = 1 + \frac{\varepsilon}{\kappa_r - \kappa(\phi)}(\kappa_1^2\cos^2\phi + \kappa_2^2\sin^2\phi) + O(\varepsilon^2)$$

$$|\nabla g| = \sqrt{1 + \kappa_r^2\rho^2} = \left[1 + \frac{2\varepsilon\kappa_r^2}{\kappa_r - \kappa(\phi)}\right]^{1/2} = 1 + \frac{\varepsilon\kappa_r^2}{\kappa_r - \kappa(\phi)} + O(\varepsilon^2)$$

$$\vec{n} = \frac{\nabla f}{|\nabla f|}, \quad \vec{\hat{r}} = \frac{\nabla g}{|\nabla g|}$$

$$\vec{n}\cdot\vec{\hat{r}} = \pm\left[1 - \frac{\epsilon K_r^2}{K_r - K(\phi)}\right]\left[1 - \frac{\epsilon(K_1^2\cos^2\phi + K_2^2\sin^2\phi)}{K_r - K(\phi)}\right] \times$$

$$\left[1 + K_r K(\phi)\frac{2\epsilon}{K_r - K(\phi)}\right]$$

$$= \pm 1 \pm \frac{(-\epsilon)}{K_r - K(\phi)}\left[K_1^2\cos^2\phi + K_2^2\sin^2\phi - 2K_r K(\phi) + K_r^2\right]$$

$$+ O(\epsilon^2) \qquad \begin{cases} + & \text{LEAVING} \\ - & \text{ENTERING}\end{cases}$$

$$= \cos\theta$$

$$K_1^2\cos^2\phi + K_2^2\sin^2\phi - 2K_r K(\phi) + K_r^2 \equiv A(\phi)$$

$$A(\phi) = \cos^2\phi\,(K_1^2 - 2K_rK_1) + \sin^2\phi\,(K_2^2 - 2K_rK_2)$$

$$+ K_r^2(\sin^2\phi + \cos^2\phi)$$

$$= \cos^2\phi\,(K_1 - K_r)^2 + \sin^2\phi\,(K_2 - K_r)^2$$

$$\equiv K_{1-}^2\cos^2\phi + K_{2-}^2\sin^2\phi \geq 0$$

WHERE WE HAVE DEFINED $K_{1-} = K_r - K_1$, $K_{2-} = K_r - K_2$.

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \left[1 - \frac{\epsilon A(\phi)}{K_r - K(\phi)}\right]^2$$

$$= \cancel{1} - \cancel{1} + \frac{2\epsilon A(\phi)}{K_r - K(\phi)} + O(\epsilon^2)$$

$$\sin\theta = \sqrt{\frac{2\epsilon A(\phi)}{K_r - K(\phi)}}$$

DEFINE $B(\phi) = K_r - K(\phi) = (K_r - K_1)\cos^2\phi + (K_r - K_2)\sin^2\phi$

$$= K_{1-}\cos^2\phi + K_{2-}\sin^2\phi$$

ASSUMING $r > \rho(\phi)\ \forall \phi \in [0, 2\pi]$, CASES 1 AND 2, P18, APPLIES IN EACH PLANE NORMAL TO $\xi = 0$ AND $\vec{t} \cdot \vec{\nu} \cos\theta \to -1$ AS $\theta \to 0$. CASE 3 WILL BE STUDIED LATER. WE HAVE

$$I = \int \frac{\cos\tilde{\theta} \cos\theta}{r \sin^2\theta} q_2 \, dl$$

$$= \int \frac{\vec{t} \cdot \vec{\nu} \cos\theta}{r \sin\theta} q_2 \, dl$$

$$\lim_{\varepsilon \to 0} I = - \lim_{\varepsilon \to 0} \frac{q_{2T}}{r_T} \int_0^{2\pi} \sqrt{\frac{B(\phi)}{2\varepsilon A(\phi)}} \cdot \sqrt{\frac{\varepsilon}{B(\phi)}} \left[2 + \frac{K'^2(\phi)}{2B^2(\phi)}\right] d\phi$$

$$= - \frac{q_{2T}}{r_T} \int_0^{2\pi} \frac{4B^2(\phi) + K'^2(\phi)}{\sqrt{4A(\phi) B^2(\phi)}} d\phi$$

$$4B^2(\phi) + K'^2(\phi) = 4[(K_{1-}\cos^2\phi + K_{2-}\sin^2\phi)^2 + (K_{1-} - K_{2-})^2 \sin^2\phi \cos^2\phi]$$
$$= 4A(\phi)$$

$$\boxed{\lim_{\varepsilon \to 0} I = - \frac{q_{2T}}{r_T} \int_0^{2\pi} \frac{d\phi}{|B(\phi)|}} \qquad r \geq \rho(\phi)\ \forall \phi \in [0, 2\pi]$$

$$B(\phi) = K_r - K(\phi)$$

FOR A SPHERE $K(\phi) = \pm \frac{1}{\rho}$ (*) $\begin{cases} + & \text{ENTERING} \\ - & \text{LEAVING} \end{cases}$

$$B(\phi) = \frac{1}{r_T} \mp \frac{1}{\rho} = \frac{\rho \mp r_T}{\rho r_T} \qquad \begin{cases} - & \text{ENTERING} \\ + & \text{LEAVING} \end{cases}$$

$$\lim_{\varepsilon \to 0} I = - \left[\frac{2\pi \rho q_2}{|r \mp \rho|}\right]_T \qquad \text{CASES 1 \& 2 (P18)} \quad r > \rho(\phi)\ \forall \phi \in [0, 2\pi]$$

$$= - \left[\frac{2\pi \rho q_2}{r \mp \rho}\right]_T \qquad \begin{cases} - & \text{ENTERING} \\ + & \text{LEAVING} \end{cases}$$

FOR CASE 3, $\vec{t} \cdot \vec{\nu} \cos\phi \to 1$ AS $\theta \to 0$ AND WE HAVE

(*) SEE SIGN CONVENTION FOR K_1 & K_2, MIDDLE OF P19.

$$\lim_{\varepsilon \to 0} I = \lim_{\varepsilon \to 0} \frac{\varrho_{2T}}{r_T} \int_0^{2\pi} \frac{d\phi}{|B(\phi)|}$$ ENTERING $r < \rho(\phi)$ $\forall \phi \in [0, 2\pi]$

FOR A SPHERE, CASE 3, $K(\phi) = +\frac{1}{\rho}$

$$\lim_{\varepsilon \to 0} I = \left[\frac{2\pi \rho \varrho_2}{\rho - r}\right]_T = -\left[\frac{2\pi \rho \varrho_2}{r - \rho}\right]_T$$ LEAVING $r < \rho(\phi)$ $\forall \phi \in [0, 2\pi)$

SO WE HAVE VERIFIED THE GENERAL FORMULA FOR THE SURFACE $f = 0$ A SPHERE. WE CAN COMBINE THE RESULTS FOR CASES 1 & 2 AND 3 AS FOLLOWS: CONVENTION OF SIGN FOR $K(\phi)$: ASSUME $\vec{n}_T = \hat{r}_T \Longrightarrow$

$$\boxed{\lim_{\varepsilon \to 0} I = \left[\frac{\varrho_2}{r} \int_0^{2\pi} \frac{\mathrm{Sig}[K(\phi)]\, d\phi}{K_r - K(\phi)}\right]_T}$$ THIS IS THE MOST GENERAL RESULT

$K(\phi) > 0$ IF CENTER OF CURVATURE IS ON THE SIDE $\hat{r}_T$ POINTS INTO.

WE SEE THAT IF $K_r \to 0$, i.e. $r \to \infty$, WE HAVE

$$\lim_{\varepsilon \to 0} I = -\left[\frac{\varrho_2}{r} \int_0^{2\pi} \frac{d\phi}{|K(\phi)|}\right]_T$$

WE HAVE

$$\int_0^{2\pi} \frac{d\phi}{|K(\phi)|} = \frac{2\pi}{\sqrt{K}},$$

$K = \sqrt{k_1 k_2}$ GAUSSIAN CURVATURE. WE THUS ASSUME THAT $k_1 k_2 > 0$ FOR NOW. BUT WHAT HAPPENS IF $k_1 k_2 = 0$, i.e. EITHER A CYLINDRICAL OR A FLAT POINT IS LOCATED AT TANGENCY POINT T OF $f = 0$ AND $g = 0$. AT A CYLINDRICAL POINT,

LET US ASSUME $k_1 \neq 0$ AND $k_2 = 0$. AT A FLAT POINT $k_1 = k_2 = 0$. WE FIRST ASSUME $k_1 k_2 > 0$ AND THEN SPECIALIZE TO THE CASE $k_1 k_2 = 0$. THE GENERAL INTEGRAL FOR $\lim\limits_{\varepsilon \to} I$ IS INTEGRABLE AS FOLLOWS

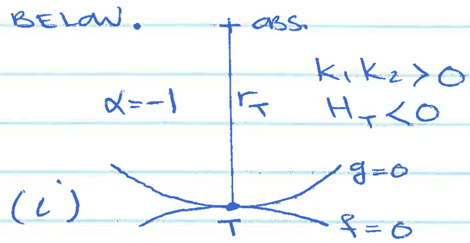
$$\boxed{\lim_{\varepsilon \to 0} I = \left[\frac{2\pi \alpha \varrho_2}{r\sqrt{(k_r - k_1)(k_r - k_2)}} \right]_T} \quad \begin{array}{l} k_1 k_2 \geqslant 0 \\ k_1 \geqslant k_2 \end{array}$$

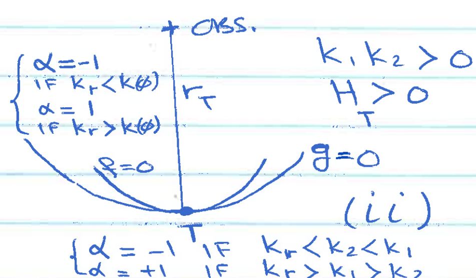
WHERE

$$\begin{aligned} \alpha &= \mathrm{Sig}(k_1) \cdot \mathrm{Sig}(k_r - k_1) \quad \text{IF } k_1 k_2 > 0 \\ &= \mathrm{Sig}(k_1) \quad\quad\quad\quad\quad\quad\quad \text{IF } k_2 = 0 \\ &= \begin{cases} +1 & \text{ENTERING } \vec{n} \cdot \vec{r} \to -1 \\ -1 & \text{LEAVING } \vec{n} \cdot \vec{r} \to +1 \end{cases} \end{aligned}$$

DISCUSSION (WE USE SIGN CONVENTION FOR k_1 & k_2 ASSUMING $\vec{n}_T = \vec{r}_T$)

CASE $k_1 k_2 > 0$ — THIS MEANS THAT THE SIGNS OF k_1 AND k_2 ARE THE SAME. THE MEAN CURVATURE $H = (k_1 + k_2)/2$ AND $\mathrm{Sig}(H) = \mathrm{Sig}(k_1) = \mathrm{Sig}(k_2) \Longrightarrow$ *i*) IF $H_T < 0 \Rightarrow [(k_r - k_1)(k_r - k_2)]_T > 0$ AND $\alpha = -1$. THE OBSERVER CAN BE ANYWHERE. *ii*) IF $\mathrm{sig}\, H_T > 0$, THEN $k_1 > 0, k_2 > 0$ AND LET $k_1 > k_2$. THEN IF r_T IS NOT IN THE INTERVAL $[\frac{1}{k_1}, \frac{1}{k_2}]$, WE HAVE $(k_r - k_1)(k_r - k_2) > 0$ AND $\alpha = \pm 1$. SITUATIONS *i*) AND *ii*) ARE SHOWN BELOW.





BY ASSUMING $k_1 k_2 > 0$, WE ARE RULING OUT SADDLE POINTS ON $\xi = 0$. RIGHT NOW, I DO NOT KNOW WHAT HAPPENS WHEN $r_T \in [1/k_1, 1/k_2]$, $k_1 > k_2 > 0$. I FEEL THAT ONE MUST RETAIN TERMS OF $O(\varepsilon^2)$ IN ANALYZING $\lim_{\varepsilon \to 0} I$ HERE AND ONE WILL THEN GET A REAL FINITE VALUE FOR THIS LIMIT. I SAY THIS BECAUSE WE KNOW FROM FORMULATION 1A THAT THERE IS NO OBSERVER LIMITATION TO THE SUBSONIC NOISE PROBLEM AND NOTHING EXTRAORDINARY HAPPENS — I THINK! (SEE NEXT PAGE!)

<u>CASE $k_1 k_2 = 0$</u> — i) CYLINDRICAL POINT $k_2 = 0$

$$\lim_{\varepsilon \to 0} I = \left[\frac{2\pi \, \mathrm{sig}(k_1) \, q_2}{\sqrt{1 - k_1 r}} \right]_T$$

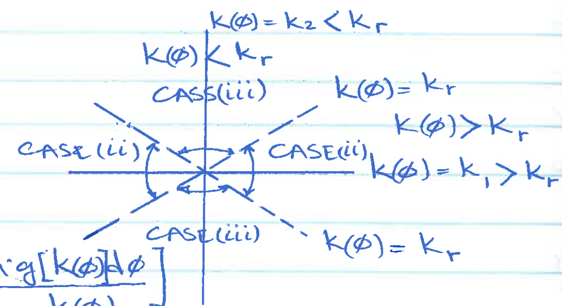
IF $k_1 < 0$ (SEE FIG (i), LAST PAGE) THERE IS NO PROBLEM. IF $k_1 > 0$, THEN WE MUST HAVE $k_1 r < 1$ OR $r < \rho_1 = 1/k_1$. THIS RESULT IS VERY STRANGE! I MUST THINK ABOUT THIS BECAUSE IN MOST APPLICATIONS, $r > \rho_1$. (SEE NEXT PAGE!)

ii) FLAT POINT $k_1 = k_2 = 0$. WE GET WHAT WE DERIVED EARLIER

$$\lim_{\varepsilon \to 0} I = \pm \left[2\pi q_2 \right]_T \qquad \begin{array}{ll} + & \vec{n} \cdot \hat{r} \to -1 \\ - & \vec{n} \cdot \hat{r} \to +1 \end{array}$$

IN THE CASE $k_1 k_2 > 0$, $k_1 > 0$, $k_2 < k_r < k_1$, WE HAVE $k_{1-} = k_r - k_1 < 0$, $k_{1+} = k_r - k_2 > 0$ SO THAT THERE IS A ϕ_0 SUCH THAT $(k_r - k_1) \cos^2 \phi_0 + (k_r - k_2) \sin^2 \phi_0 = 0$. WE GET $\tan \phi_0 = \pm \sqrt{(k_r - k_1)/(k_r - k_2)}$. SINCE

$k_r - k(\phi_0) = 0 \Rightarrow k(\phi_0) = k_r$ THIS HAPPENS ONCE IN EVERY QUADRANT. THIS MEANS THAT $\rho(\phi)$ CAN BE LARGER THAN r FOR SOME $\phi \in [0, 2\pi]$ AND SMALLER THAN r FOR SOME $\phi \in [0, 2\pi]$. THEREFORE, CASE (ii), P18, APPLIES FOR PORTION OF $\phi \in [0, 2\pi]$ AND CASE (iii) APPLIES FOR THE REST OF INTERVAL AS SHOWN IN THE FIGURE ON THE RIGHT. SO IN FACT, WE MUST WRITE



$$\lim_{\varepsilon \to 0} I = \left[\frac{q_2}{r} \, PV \int_0^{2\pi} \frac{\mathrm{Sig}[k(\phi)] \, d\phi}{k_r - k(\phi)} \right]_T$$

$$= \left[\frac{4 q_2}{r} \, PV \int_0^{\pi/2} \frac{\mathrm{Sig}[k(\phi)] \, d\phi}{k_r - k(\phi)} \right]_T$$

THE PRINCIPAL VALUE APPLIES WHEN $k(\phi) > 0$ AND $k_1 < k_r < k_1$ SO THAT, WE MUST WRITE

$$\boxed{\lim_{\varepsilon \to 0} I = \left[\frac{4 q_2}{r} \, PV \int_0^{\pi/2} \frac{d\phi}{k_r - k(\phi)} \right]_T}$$

FOR A CYLINDRICAL POINT WHEN $k_1 > 0, k_2 = 0$ AND $k_r < k_1$, WE HAVE

$$\boxed{\lim_{\varepsilon \to 0} I = \left[\frac{4 q_2}{r} \, PV \int_0^{\pi/2} \frac{d\phi}{k_r - k_1 \cos^2\phi} \right]_T}$$

NOTE THAT IN THE ABOVE CASES, WE HAVE LEFT OUT THE CASES $k_r = k_1$ OR $k_r = k_2$. THESE CORRESPOND TO THE POSITION OF CAUSTICS AND, GENERALLY, ARE NOT IMPORTANT BECAUSE THEY ARE ISOLATED

POINTS IN SPACE.

EXAMPLE: LET $K(\phi) = 4\cos^2\phi + \sin^2\phi$, i.e. $K_1 = 4$, $K_2 = 1$. LET K_r VARY AS FOLLOWS:

K_r	0.3	0.7	0.9	0.95	0.98
$\lim_{\varepsilon\to 0} I$	-3.9042	-6.3148	-11.2849	-16.0896	-25.5659

K_r	0.995	0.999	1.005	1.1	2.0
$\lim_{\varepsilon\to 0} I$	-51.2593	-114.696	-2.08×10^{-10}	-9.98×10^{-10}	5.26×10^{-14}

K_r	3.00	3.5	3.9	3.95
$\lim_{\varepsilon\to 0} I$	-1.087×10^{-13}	6.40×10^{-13}	4.10×10^{-12}	-5.05×10^{-12}

K_r	3.99	4.01	4.1	5.0	6.0
$\lim_{\varepsilon\to 0} I$	1.85×10^{-12}	36.2157	11.2849	3.14159	1.9869

IT APPEARS THAT $\lim_{\varepsilon\to 0} I = 0$ FOR $K_2 < K_r < K_1$ IN THIS CASE. THE SAME THING HAPPENS FOR A CYLINDRICAL POINT WHEN $K_r < K_1$. I MUST STUDY THIS FURTHER.

$\lim_{\varepsilon \to 0} I$ WHEN $f = 0$ IS IN MOTION

IN THIS CASE, $\Lambda_0 = \Lambda \tilde{\Lambda} \sin\theta'$, $\cos\theta' = \vec{N} \cdot \vec{\tilde{N}}$

$$\cos\theta' = \frac{1}{\Lambda\tilde{\Lambda}} (\vec{n} - M_n \vec{\hat{r}}) \cdot (\vec{\nu} - M_\nu \vec{\hat{r}})$$

$$= \frac{1}{\Lambda\tilde{\Lambda}} \left(M_n M_\nu - M_\nu \cos\theta - M_n \vec{\hat{r}} \cdot \vec{\nu} \right)$$

AS $\varepsilon \to 0$, i.e. $\theta \to 0$ OR π, WE HAVE $\vec{\hat{r}} \cdot \vec{\nu} \to 0$ AND $\cos\theta' = M_\nu (M_n \pm 1)/\Lambda\tilde{\Lambda} \Rightarrow \sin\theta' = |1 \mp M_n|/\Lambda\tilde{\Lambda}$ WHICH, IN GENERAL, IS NONZERO UNLESS $M_n|_T = \pm 1$. BUT EVEN THIS SITUATION IS HARMLESS BECAUSE Q_2 HAS THE FACTOR $M_n^2 - 1$. THE BEST WAY TO DIRECTLY RELATE WHAT WE DID IN THE STATIONARY CASE TO THE PRESENT CASE IS TO WRITE

$$\frac{dL}{\Lambda_0} = \frac{d\ell}{|1 - M_r|}$$

WHERE $d\ell$ IS THE ELEMENT OF LENGTH OF THE EDGE OF THE HOLE FOR WHICH WE CALCULATE I (SEE FIG. ON P7). WE SEE THAT $M_r \to \pm M_n$ AS $\sin\theta \to 0$ AND THE REST OF OUR ANALYSIS STAYS THE SAME SO THAT

$$\boxed{\lim_{\varepsilon \to 0} I = \left[\frac{4 Q_2}{r |1 - M_r|} \int_0^{\pi/2} \frac{\mathrm{sig}[k(\phi)]}{k_r - k(\phi)} d\phi \right]_T}$$

THIS RESULT MUST AGAIN BE REINTERPRETED IF $k(\phi) > 0$ (WITH CONVENTION FOR SIGN $\vec{n}_T = \vec{\hat{r}}_T$) AND $k_2 < k_r < k_1$ BY TAKING THE PRINCIPAL VALUE OF THE INTEGRAL ABOVE. THE ABOVE RESULT IS SINGULARITY FREE EVEN FOR $M_r = 1$!

A CHECK FOR SINGULARITY $\theta = 0$ OF THE SURFACE INTEGRALS OF FORMULATION 4

WE KNOW THAT WE CAN TOLERATE ONE POWER OF $\sin\theta$ IN THE DENOMINATOR SINCE $d\Sigma/\Lambda = \frac{c\,d\Gamma\,d\tau}{\sin\theta}$. HOWEVER, WE HAVE TWO TERMS IN FORMULATION 4 THAT HAVE $\sin^2\theta$ IN THE DENOMINATOR. WE NOW COLLECT 3 TERMS IN FORMULATION 4 WHICH INCLUDE THE HIGH ORDER SINGULARITY. WE THEN SHOW THAT FOR A SPHERE, THESE TERMS ARE INTEGRABLE. WE CONSIDER THE FOLLOWING EXPRESSION

$$E = \iint \left[\left(\frac{\cos\theta}{r^2\sin^2\theta} - \frac{k_1}{r\sin^2\theta} - \frac{k_g\cos\theta}{r\sin\theta}\right) q_2\right]_{ret} \frac{c\,d\Gamma\,d\tau}{\sin\theta}$$

WE WRITE THE NUMERATOR AS

$$E_N = \cos\theta - rk_1 - rk_g\cos\theta\sin\theta$$

USING THE SYMBOLS ON PAGE 18, WE HAVE FOR θ VERY SMALL AND

CASE (i) : $\cos\theta \to 1$, $rk_1 = -\frac{r}{\rho}$, $k_g = \frac{1}{\rho}\cot\psi$

$\theta = \psi + \delta = (1 + \frac{\rho}{r})\psi$

$$E_N \to 1 + \frac{r}{\rho} - r\left[\frac{1}{\rho}\,\frac{1}{\psi}\,(1 + \frac{\rho}{r})\psi\right] \to 0 \,!$$

CASE (ii) : $\cos\theta \to -1$, $rk_1 = -\frac{r}{\rho}$, $k_g = -\frac{\cot\psi}{\rho}$

$\theta = \pi + \delta - \psi$, $\sin\theta \to (1 - \frac{\rho}{r})\psi$

$$E_N \to -1 + \frac{r}{\rho} + r\left[\frac{1}{\rho}\,\frac{1}{\psi}\,(1 - \frac{\rho}{r})\psi\right] \to 0$$

CASE (iii) : $\cos\theta \to -1$, $rk_1 = -\frac{r}{\rho}$, $k_g = \frac{\cot\psi}{\rho}$

$\theta = \pi - \delta + \psi$, $\sin\theta \to \frac{\rho}{r} - 1$

$$E_N \to -1 + \frac{r}{\rho} - r\left[\frac{1}{\rho}\,\frac{1}{\psi}\,(\frac{\rho}{r} - 1)\psi\right] \to 0$$

SINCE THE NUMERATOR GOES TO ZERO TO THE ORDER ψ AND THE DENOMINATOR IS $O(\psi^2)$ AS $\psi \to 0 \Rightarrow$ THE INTEGRAL IS CONVERGENT AS $\theta \to 0$. THE ABOVE ARGUMENT APPLIES TO THE GENERAL CASE SINCE WE HAVE PROVED $E_N \to 0$ AT ANY POINT ON THE Γ-CURVE. THE MOTION OF $\xi = 0$ DOES NOT CHANGE THIS ARGUMENT ALSO.

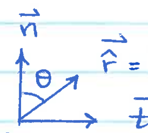
A NEW DEVELOPMENT (9/8/98)

THERE IS A SIMPLIFICATION OF SURFACE TERMS OF FORMULATION 4. WE CONSIDER THE CURVATURE TERMS FIRST

$$\frac{k_1}{r\sin^2\theta} + \frac{k_g \cos\theta}{r\sin\theta} = \frac{1}{r\sin^2\theta}\left(k_1 + k_g \sin\theta\cos\theta\right)$$

$$= \frac{\vec{k}}{r\sin^2\theta}\cdot\left(\vec{n} + \vec{t}_1 \sin\theta\cos\theta\right)$$

WHERE $\vec{k}$ IS THE CURVATURE OF THE Γ-CURVE IN 3D SPACE. WE HAVE



$\vec{r} = \vec{t}_1 \sin\theta + \vec{n}\cos\theta$

$$\vec{n} + \vec{t}_1 \sin\theta\cos\theta$$
$$= \vec{n} + (\vec{t}_1 \sin\theta + \vec{n}\cos\theta)\cos\theta - \vec{n}\cos^2\theta$$
$$= \vec{n}\sin^2\theta + \vec{r}\cos\theta$$

WE NOTE THAT $\vec{k}\cdot\vec{r} = k_r = \frac{1}{r}$

$$\therefore \quad (\vec{n} + \vec{t}_1 \sin\theta\cos\theta)\cdot\vec{k} = (\vec{n}\sin^2\theta + \vec{r}\cos\theta)\cdot\vec{k}$$
$$= k_1 \sin^2\theta + \frac{\cos\theta}{r}$$

$$\frac{k_1}{r\sin^2\theta} + \frac{k_g\cos\theta}{r\sin\theta} = \frac{k_1}{r} + \frac{\cot\theta}{r^2\sin\theta}$$

$\Rightarrow$ THE SOLUTION OF $\Box^2\phi_2 = \underset{\sim}{q}_2\, \delta'(\xi)$ IS

$$\boxed{\begin{aligned} 4\pi\,\phi_2(\vec{x},t) = &\int \frac{1}{r}\left[\cot\theta\,\vec{E}_1\cdot\nabla q_2 - K_1 q_2\right]_{ret}\frac{d\Sigma}{\Lambda} \\ &+ \int\left[\frac{\cos\tilde{\theta}\,\cot\theta}{r\Lambda_0\sin\theta}\,q_2\right]_{ret} dL \\ &+ \sum\left[\frac{4q_2}{r|1-M_r|}\int_0^{\pi/2}\frac{\mathrm{sig}[k(\phi)]}{k_r-k(\phi)}\,d\phi\right]_T \end{aligned}}$$

THIS RESULT GIVES THE CLASSICAL SOLUTION FOR DIPOLES ON THE UNIT CIRCLE AND UNIT SPHERE DISCUSSED EARLIER. NOTE THAT $\cot\theta$ IN THE FIRST TERM OF SURFACE INTEGRAL HAS A TERM OF $O(1/r^2)$ IN IT. THEREFORE, WE DO HAVE A NEAR FIELD TERM IN $\phi_2(\vec{x},t)$.

[signature] 9/27/98

WHAT HAPPENED TO THE FAR FIELD TERM IN FORMULATION 4?

IT SEEMS THAT WE HAVE LOST ALL TERMS OF $O(1/r^2)$ IN FORMULATION 4. THIS BAFFLED ME A LOT! HERE IS THE ANSWER. LET $\frac{\partial}{\partial 6} = \vec{t}_1 \cdot \nabla_2 \Rightarrow$ FROM $\cos\theta = \vec{n} \cdot \hat{\vec{r}}$ THAT

$$-\sin\theta \frac{\partial\theta}{\partial 6} = \frac{\partial \vec{n}}{\partial 6} \cdot \hat{\vec{r}} + \vec{n} \cdot \frac{\partial \hat{\vec{r}}}{\partial 6}$$

$$= -K_1 \vec{t}_1 \cdot \hat{\vec{r}} + \vec{n} \cdot \left(-\frac{\vec{t}_1}{r} + \frac{\hat{\vec{r}}}{r}\sin\theta\right)$$

$$= \sin\theta\left(-K_1 + \frac{\cos\theta}{r}\right)$$

$$-\frac{\partial\theta}{\partial 6} = -K_1 + \frac{\cos\theta}{r}$$

$$E = \cot\theta\, \vec{t}_1 \cdot \nabla q_2 - K_1 q_2 = \cot\theta \frac{\partial q_2}{\partial 6} - \left(\frac{\partial\theta}{\partial 6} + \frac{\cos\theta}{r}\right) q_2$$

$$= \frac{1}{\sin\theta}\left(\cos\theta \frac{\partial q_2}{\partial 6} - q_2 \sin\theta \frac{\partial\theta}{\partial 6}\right) - \frac{\cos\theta}{r} q_2$$

$$= \frac{1}{\sin\theta} \frac{\partial}{\partial 6}(q_2 \cos\theta) - \frac{\cos\theta}{r} q_2$$

SINCE WE ALSO HAVE

$$\frac{\partial}{\partial 6}\left(\frac{1}{r}\right) = -\frac{\sin\theta}{r}$$

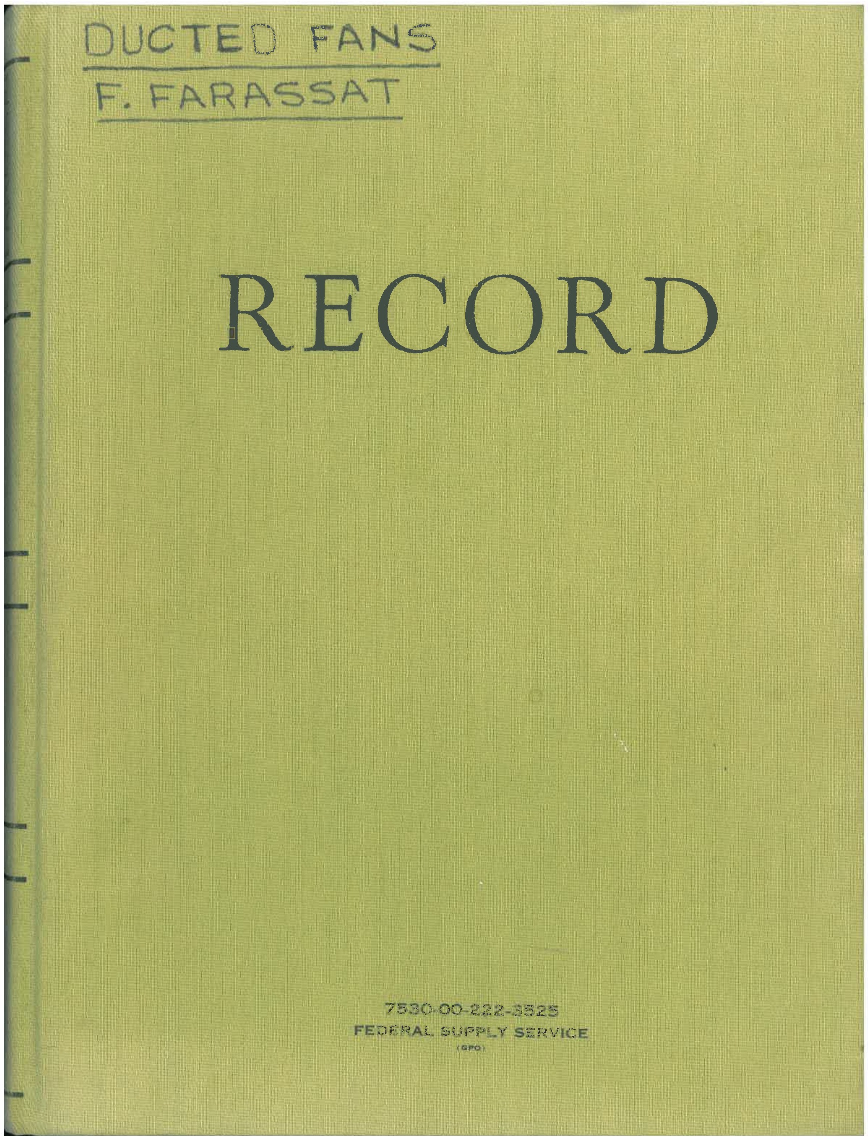
WE GET

$$\frac{E}{r} = \frac{1}{r\sin\theta} \frac{\partial}{\partial 6}(q_2 \cos\theta) - \boxed{\frac{\cos\theta}{r^2} q_2}$$ NEAR FIELD TERM

$$= \frac{1}{\sin\theta} \frac{\partial}{\partial 6}\left(\frac{q_2 \cos\theta}{r}\right)$$

$\therefore$ THE NEAR FIELD TERM IS HIDDEN IN OUR FORMULATION!

13 Ducted Fans



SMALL PERTURBATION EQUATIONS FOR ACOUSTIC WAVES IN FLUID WITH MEAN FLOW

NOTATION: MEAN (BACKGROUND) QUANTITIES ARE DENOTED BY SUBSCRIPT 0, PERTURBATION VALUES HAVE A PRIME

$$\rho = \rho_0 + \rho'$$
$$\vec{V} = \vec{V}_0 + \vec{V}' \qquad \text{ALL QUANTITIES DIMENSIONAL}$$
$$p = p_0 + p'$$

MASS CONTINUITY & MOMENTUM EQS.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \\ \dfrac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\dfrac{\nabla p}{\rho} \end{cases}$$

MASS CONTINUITY EQUATIONS

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho'}{\partial t} + \nabla \cdot \left[(\rho_0 + \rho')(\vec{V}_0 + \vec{V}')\right] = 0$$

$$\begin{cases} \dfrac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \vec{V}_0) = 0 & \text{MEAN} \\ \dfrac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' \vec{V}_0 + \rho_0 \vec{V}') = 0 & \text{ACOUSTIC} \end{cases}$$

MOMENTUM EQUATION

$$\frac{\partial}{\partial t}(\vec{V}_0 + \vec{V}') + (\vec{V}_0 + \vec{V}') \cdot \nabla (\vec{V}_0 + \vec{V}') = -\frac{\nabla (p_0 + p')}{\rho_0 + \rho'}$$
$$= -\frac{\nabla (p_0 + p')}{\rho_0}\left(1 - \frac{\rho'}{\rho_0}\right)$$

MOMENTUM EQUATIONS

$$\begin{cases} \dfrac{\partial \vec{V}_0}{\partial t} + \vec{V}_0 \cdot \nabla \vec{V}_0 = -\dfrac{\nabla p_0}{\rho_0} & \text{MEAN} \\ \dfrac{\partial \vec{V}'}{\partial t} + \vec{V}_0 \cdot \nabla \vec{V}' + \vec{V}' \cdot \nabla \vec{V}_0 = -\dfrac{\nabla p'}{\rho_0} + \dfrac{\rho' \nabla p_0}{\rho_0^2} & \text{ACOUSTIC} \end{cases}$$

EQUATION OF STATE

$p = A\rho^{\gamma}$ WHERE A IS FOUND FROM REFERENCE CONDITIONS

$$\frac{p}{p_r} = \left(\frac{\rho}{\rho_r}\right)^{\gamma} \Rightarrow p = (p_r \rho_r^{-\gamma})\rho^{\gamma}$$

$$\therefore A = p_r \rho_r^{-\gamma}$$

$$p_0 + p' = A(\rho_0 + \rho')^{\gamma}$$

$$= A\rho_0^{\gamma}\left(1 + \frac{\rho'}{\rho_0}\right)^{\gamma}$$

$$= A\rho_0^{\gamma}\left(1 + \gamma\frac{\rho'}{\rho_0}\right)$$

$$p_0 = A\rho_0^{\gamma}$$

$$p' = \gamma A\rho_0^{\gamma}\frac{\rho'}{\rho_0} = \frac{\gamma p_0}{\rho_0}\rho' = c_0^2\rho'$$

EQUATIONS OF STATE

$$\begin{cases} p_0 = A\rho_0^{\gamma} & \underline{\text{MEAN}} \\ p' = c_0^2\rho' & \underline{\text{ACOUSTIC}} \end{cases}$$

ENERGY EQUATION (ISENTROPIC)

$T\,ds = du + p\,d\left(\frac{1}{\rho}\right) = 0 \Rightarrow$ ISENTROPIC EQS. OF STATE

FROM $p = A\rho^{\gamma} \Rightarrow$

$$dp - \gamma\frac{p}{\rho}d\rho = 0 \qquad (dp - c^2 d\rho = 0)$$

$$\frac{Dp}{Dt} - \gamma\frac{p}{\rho}\frac{D\rho}{Dt} = 0$$

$$\frac{Dp}{Dt} - \gamma p\,\nabla\cdot\vec{V} = 0$$

$$\frac{\partial}{\partial t}(p_0 + p') + (\vec{V}_0 + \vec{V}')\cdot\nabla(p_0 + p') - \gamma(p_0 + p')\,\nabla\cdot(\vec{V}_0 + \vec{V}') = 0$$

ENERGY EQUATIONS

$$\begin{cases} \dfrac{\partial p_0}{\partial t} + \vec{V}_0 \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \vec{V}_0 = 0 & \text{MEAN} \\ \dfrac{\partial p'}{\partial t} + \vec{V}_0 \cdot \nabla p' + \vec{V}' \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \vec{V}' - \gamma p' \nabla \cdot \vec{V}_0 = 0 & \text{ACOUSTIC} \end{cases}$$

IRROTATIONAL FLOW

$$\nabla \times \vec{V} = \nabla \times \vec{V}_0 + \nabla \times \vec{V}_0' = 0 \Rightarrow \nabla \times \vec{V}_0 = 0, \nabla \times \vec{V}' = 0$$

$$\phi = \phi_0 + \phi' \quad \text{VELOCITY POTENTIAL}$$

$$\vec{V} \cdot \nabla \vec{V} = \frac{1}{2} \nabla V^2$$

$$\frac{\partial \vec{V}}{\partial t} + \frac{1}{2} \nabla V^2 = -\frac{\nabla p}{\rho}$$

FROM $\quad c_p dT - \dfrac{dp}{\rho} = 0$, WE HAVE

$$\frac{dp}{\rho} = c_p dT = \frac{\gamma R\, dT}{\gamma - 1}$$

$$\frac{\nabla p}{\rho} = \frac{1}{\gamma - 1} \nabla(\gamma R T) = \frac{1}{\gamma - 1} \nabla c^2$$

$$\Rightarrow \frac{\partial \vec{V}}{\partial t} + \frac{1}{2} \nabla V^2 = -\frac{1}{\gamma - 1} \nabla c^2$$

$$\nabla \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{1}{\gamma - 1} c^2 \right] = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + \frac{1}{\gamma - 1} c^2 = \frac{1}{2} V_\infty^2 + \frac{1}{\gamma - 1} c_\infty^2$$

ASSUMING THAT V_∞ AND C_∞ ARE TIME INDEPENDANT. NOW WE USE $\phi = \phi_0 + \phi'$

$$\begin{cases} \dfrac{\partial \phi_0}{\partial t} + \dfrac{1}{2} |\nabla \phi_0|^2 + \dfrac{1}{\gamma - 1} c_0^2 = \dfrac{1}{2} V_\infty^2 + \dfrac{1}{\gamma - 1} c_\infty^2 & \text{MEAN} \\ \dfrac{\partial \phi'}{\partial t} + \nabla \phi_0 \cdot \nabla \phi' + c_0^2 \dfrac{p'}{p_0} = 0 & \text{ACOUSTIC} \end{cases}$$

$$\frac{1}{\gamma-1}c^2 = \frac{1}{\gamma-1}\gamma R(T_0+T')$$

$$= \frac{\gamma R T_0}{\gamma-1}\left(1+\frac{T'}{T_0}\right)$$

$$\frac{T}{T_0} = \frac{T_0+T'}{T_0} = \left(\frac{\rho_0+\rho'}{\rho_0}\right)^{\gamma-1}$$

$$1+\frac{T'}{T_0} = 1+(\gamma-1)\frac{\rho'}{\rho_0}$$

$$\frac{1}{\gamma-1}c^2 = \frac{c_0^2}{\gamma-1}\left[1+(\gamma-1)\frac{\rho'}{\rho_0}\right]$$

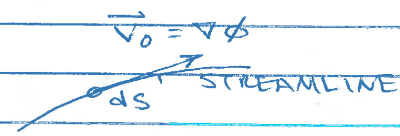
$$= \frac{1}{\gamma-1}c_0^2 + \frac{\rho'}{\rho_0}c_0^2$$

FROM ACOUSTIC EQUATION

$$\rho' c_0^2 = p'$$

$$= -\rho_0\left[\frac{\partial\phi'}{\partial t} + \nabla\phi_0 \cdot \nabla\phi'\right]$$

$$\nabla\phi_0 \cdot \nabla\phi' = V_0\frac{\partial\phi'}{\partial s}$$



$$\rho' c_0^2 = p' = -\rho_0\left[\frac{\partial\phi'}{\partial t} + V_0\frac{\partial\phi'}{\partial s}\right]$$

IF THE MEAN FLOW IS TIME INDEPENDENT, $\partial\phi_0/\partial t = 0$ AND WE HAVE

$$c_0^2 = \frac{\gamma-1}{2}\left[V_\infty^2 - |\nabla\phi_0|^2\right] + c_\infty^2$$

THE GOVERNING EQUATION FOR ϕ'

FROM PREVIOUS PAGE

$$\rho' = -\frac{\rho_0}{c_0^2}\left[\frac{\partial \phi'}{\partial t} + \nabla\phi_0 \cdot \nabla\phi'\right]$$

$$\equiv -\frac{\rho_0}{c_0^2} L\,\phi' \qquad \left\{\begin{array}{l}\text{NOTE THIS COMES FROM}\\ \text{THE MOMENTUM EQ.}\end{array}\right.$$

THE EQUATION FOR ϕ' IS OBTAINED FROM MASS CONTINUITY EQUATION BY SUBSTITUTING FOR ρ' FROM ABOVE

$$-\frac{\partial}{\partial t}\left[\frac{\rho_0}{c_0^2} L\,\phi'\right] - \nabla\cdot\left[\frac{\rho_0}{c_0^2} L\,\phi'\,\nabla\phi_0\right]$$

$$+ \rho_0 \nabla^2\phi' + \nabla\rho_0 \cdot \nabla\phi' = 0$$

IF THE MEAN FLOW IS TIME INDEPENDENT, WE GET

$$-\frac{\rho_0}{c_0^2} L\,\phi'_t - \nabla\cdot\left[\frac{\rho_0}{c_0^2} L\,\phi'\,\nabla\phi_0\right] + \rho_0\nabla^2\phi' + \nabla\rho_0\cdot\nabla\phi' = 0$$

IF THE MEAN FLOW IS UNIFORM AND ALONG x_1-AXIS, THEN $\nabla\phi_0 = V_1\vec{e}_1$, c_0 AND ρ_0 ARE CONSTANT. WE HAVE

$$\rho' = -\frac{\rho_0}{c_0^2}\left[\frac{\partial\phi'}{\partial t} + V_1\frac{\partial\phi'}{\partial x_1}\right],$$

AND THE EQUATION FOR ϕ' IS

$$-\frac{\rho_0}{c_0^2}\left[\frac{\partial^2\phi'}{\partial t^2} + V_1\frac{\partial^2\phi'}{\partial x_1\partial t}\right] - \frac{\rho_0}{c_0^2}V_1\left[\frac{\partial^2\phi'}{\partial x_1\partial t} + V_1\frac{\partial^2\phi'}{\partial x_1^2}\right]$$

$$+ \rho_0\nabla^2\phi' = 0$$

OR

$$\frac{1}{c_0^2}\left[\frac{\partial}{\partial t} + V_1\frac{\partial}{\partial x_1}\right]^2\phi' - \nabla^2\phi' = 0$$

FROM THIS EQUATION, USING

$$p' = -\rho_0 \left[\frac{\partial \phi'}{\partial t} + V_1 \frac{\partial \phi'}{\partial x_1} \right]$$

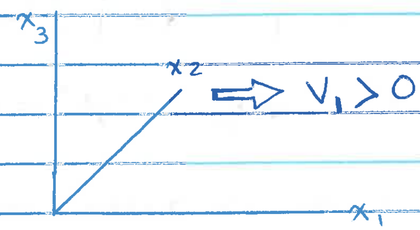
$$= -\rho_0 L \phi'$$

WE GET, AFTER NOTING THAT L IS LINEAR,

$$\frac{1}{c_0^2} \left[\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x_1} \right]^2 [-\rho_0 L \phi'] - \nabla^2 [-\rho_0 L \phi'] = 0$$

OR

$$\frac{1}{c_0^2} \left[\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x_1} \right]^2 p' - \nabla^2 p' = 0$$



WE, OF COURSE, EXPECT THIS SINCE THIS IS JUST THE WAVE EQUATION IN A MOVING FRAME. WE DEFINE $M = V_1 / c_0$ AND WRITE THE ABOVE EQUATION AS

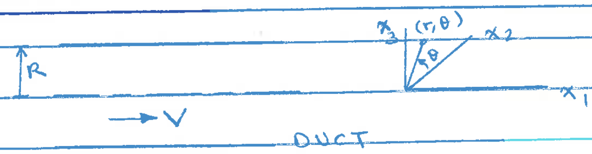
$$\left(\frac{1}{c_0} \frac{\partial}{\partial t} + M \frac{\partial}{\partial x_1} \right)^2 p' - \nabla^2 p' = 0$$

THE HARDWALL BC IS

$$\left. \frac{\partial p'}{\partial n} \right|_{\text{WALL}} = 0$$

ACOUSTICS OF A CYLINDRICAL DUCT WITH UNIFORM FLOW

ALL QUANTITIES
DIMENSIONAL



$$\begin{cases} \left(\frac{1}{c_0}\frac{\partial}{\partial t} + M\frac{\partial}{\partial x_1}\right)^2 p' - \nabla^2 p' = 0 & \text{(1-a)} \\ \left.\frac{\partial p'}{\partial r}\right|_{r=R} = 0 \quad \text{B.C.} & \text{(1-b)} \end{cases}$$

ASSUME A SOLUTION OF THE FORM

$$p' = P(x_1, r, \theta)\, e^{i\omega t} \qquad (2)$$

WE WILL SAY MORE ABOUT ω LATER. WE DEFINE

$$k = \frac{\omega}{c_0} \qquad \text{WAVE NUMBER}$$

THE EQUATION FOR P IS

$$(1-M^2)\frac{\partial^2 P}{\partial x_1^2} + \nabla_c^2 P - 2ikM\frac{\partial P}{\partial x_1} + k^2 P = 0 \qquad (3)$$

WHERE ∇_c^2 IS LAPLACIAN IN POLAR COORDINATES (r,θ).

$$\nabla_c^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad (4)$$

AND THE BOUNDARY CONDITION ON P IS

$$\left.\frac{\partial P}{\partial r}\right|_{r=R} = 0 \qquad (5)$$

WE DEFINE $k_{a,mn}$ AND $k_{r,mn}$ AS AXIAL AND RADIAL WAVE NUMBERS FOR MTH CIRCUMFERENTIAL AND NTH RADIAL MODES, RESPECTIVELY.

THE SOLUTION FOR (m,n) MODE IS

$$p'_{mn} = A_{mn} J_m(k_{r,mn} r) \exp i(\omega t - m\theta - k_{a,mn} x_1) \qquad (6)$$

WHERE $k_{r,mn}$ IS OBTAINED FROM

$$J'_m(k_{r,mn} R) = 0 \qquad (7)$$

WHERE $J_m(\cdot)$ IS THE BESSEL FUNCTION OF FIRST KIND OF ORDER m. LET $\beta^2 = 1 - M^2$, THEN $k_{a,mn}$ IS GIVEN BY

$$k_{a,mn} = \frac{k}{\beta^2}\left[-M \pm \sqrt{1-\beta^2 (k_{r,mn}/k)^2}\right] \tag{8}$$

DEPENDING ON THE SIGN OF THE QUANTITY UNDER RADICAL SIGN, WE HAVE TWO REAL OR COMPLEX ROOTS.

CASE 1 : TWO REAL ROOTS. WE HAVE

$$\left(\frac{k_{r,mn}}{k}\right)^2 < \frac{1}{\beta^2} \tag{9}$$

WE HAVE TWO ROOTS WITH OPPOSITE SIGNS OR TWO NEGATIVE ROOTS.

i) IF $\left(\frac{k_{r,mn}}{k}\right)^2 < 1$ ($< \frac{1}{\beta^2}$ AUTOMATICALLY, SINCE $M<1$), THEN WE HAVE TWO ROOTS OF OPPOSITE SIGNS. THE POSITIVE ROOT FOR $M>0$ CORRESPONDS TO ENERGY TRANSFER DOWNSTREAM AND IS OBTAINED USING THE + SIGN IN $k_{a,mn}$ ABOVE. SIMILARLY THE NEGATIVE ROOT CORRESPONDS TO ENERGY TRANSFER UPSTREAM. NOTE THAT

$$|k_{a,mn+}| < |k_{a,mn-}| \tag{10-a}$$

i.e.

$$\lambda_{a,mn+} > \lambda_{a,mn-} \tag{10-b}$$

AS EXPECTED FROM PHYSICS.

ii) IF $1 < \left(\frac{k_{r,mn}}{k}\right)^2 < \frac{1}{\beta^2}$, WE HAVE TWO NEGATIVE ROOTS. AGAIN RELATIONS (10-a,b) HOLD AND ENERGY TRANSFER TO DOWNSTREAM AND UPSTREAM REGIONS CORRESPOND TO + AND − SIGNS IN EQ.(8), RESPECTIVELY.

TO SEE HOW THE CONDITION IN (i) IS OBTAINED, LET US DENOTE $k_{r,mn}/k$ BY α. THEN TO GET A POSITIVE ROOT FROM EQ. (8), WE MUST HAVE

$$\sqrt{1-\alpha^2\beta^2} > M$$

$$1 - \alpha^2\beta^2 > M^2$$

$$\alpha^2\beta^2 < 1 - M^2 \qquad (= \beta^2)$$

$$\alpha^2 < 1$$

CASE 2 TWO COMPLEX ROOTS. WE HAVE $\left(\frac{k_{r,mn}}{k}\right)^2 > \frac{1}{\beta^2}$ AND

$$k_{a,mn\pm} = \frac{k}{\beta^2}\left[-M \pm i\sqrt{\beta^2(k_{r,mn}/k)^2 - 1}\right]$$

$$= -\frac{kM}{\beta^2} \pm i\frac{k}{\beta^2}\sqrt{\beta^2(k_{r,mn}/k)^2 - 1} \qquad (11)$$

$$= \mathrm{Re}\, k_{a,mn\pm} + i\, \mathrm{Im}\, k_{a,mn\pm}$$

FOR PROPAGATION IN +VE AND -VE DIRECTIONS, WE HAVE

$$p'_{mn} = A_{mn} J_m(k_{r,mn} r)\, e^{\mp x_1 \mathrm{Im}\, k_{a,mn\pm}}$$

$$\times \exp i\,(\omega t - m\theta \pm x_1 \mathrm{Re}\, k_{a,mn\pm}) \qquad (12)$$

WHICH IS AN EXPONENTIALLY DECAYING WAVE.

THE CUT-OFF RATIO: LET US DEFINE THE FOLLOWING RATIO

$$\beta_{mn} = \frac{k}{\beta k_{r,mn}} \qquad (13)$$

THIS IS CALLED THE CUT-OFF RATIO FOR THE FOLLOWING REASON. CASES 1 AND 2, ABOVE INDICATE THAT

IF $\beta_{mn} > 1$ MODE (m,n) PROPAGATES

IF $\beta_{mn} < 1$ MODE (m,n) DECAYS

WE NOTE THAT SINCE $k_{r,mn}$ IS DETERMINED BY DUCT GEOMETRY, β_{mn} IS DETERMINED BY THE MODE, GEOMETRY AND FLOW MACH NUMBER. WE CAN, THEREFORE, CALCULATE IF A MODE IS PROPAGATING BY SPECIFYING k, M AND $k_{r,mn}$. ALSO NOTE THAT $k_{r,mn}$ IS INDEPENDENT OF M.

IF $M \uparrow \Rightarrow \beta^2 \downarrow \therefore$ IF MODE (m,n) IS CUT-ON FOR M_1, IT IS ALSO CUT-ON FOR $1 > M_2 > M_1$ SINCE $\beta_2^2 < \beta_1^2$ AND THUS $\beta_{mn2} > \beta_{mn1} > 1$.

WHAT IS k IN THE ABOVE EQUATIONS?

IF WE HAVE A FREE ROTOR WITH B BLADES, THEN THE PRESSURE FIELD ROTATES WITH ROTOR AND HAS A FOURIER SERIES WITH PERIODICITY $\frac{2\pi}{B}$ IN CIRCUMFERENTIAL DIRECTION. IF THE ROTOR ANGULAR VELOCITY IS Ω, THEN THE PRESSURE PATTERN WILL HAVE $\Omega t - \theta$ DEPENDENCE IN STATIONARY FRAME. THE CIRCUMFERENTIAL PERIODICITY GIVES THE RESULT THAT HARMONICS OF ORDER mB CAN ONLY APPEAR. THEREFORE

$$k = \frac{mB\Omega}{c_0} \qquad (14)$$

i.e. k DOES DEPEND ON CIRCUMFERENTIAL MODE.

NOW, LET US HAVE V EXIT GUIDE VANES (E.G.V.). THEN IN ROTATING FRAME, THE PRESSURE FIELD

HAS THE FORM

$$p' = \sum_{-\infty}^{\infty} A_m(\theta)\, e^{-imB\theta} \qquad (15.)$$

AN OBSERVER IN ROTATING FRAME SEES A WAVE OF PERIODICITY V. THEREFORE,

$$A_m(\theta) = \sum_{-\infty}^{\infty} B_{mq}\, e^{-iqV\theta} \qquad (16)$$

SUBSTITUTING (16) IN (15), WE GET

$$p' = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} B_{mq}\, e^{-i(mB+qV)\theta} \qquad (17)$$

THIS MEANS THAT WE HAVE CIRCUMFERENTIAL MODES OF ORDER

$$mB+qV \qquad m, q \text{ INTEGERS} \qquad (18)$$

WITH CORRESPONDING k

$$k = \frac{mB\Omega}{c_0} \qquad (19)$$

NOTE THAT ONLY θ IN $mB\theta$ IN EQ.(15) IS REPLACED BY $\Omega t - \theta$ TO GET TIME DEPENDENCE IN NONROTATING FRAME, i.e. $p'(\theta, t)$ IS

$$p'(\theta, t) = \sum_{q}\sum_{m\geq 1} B_{mq}\, e^{-imB\Omega t}\, e^{-i(mB+qV)\theta} \qquad (20)$$

THE RIGOROUS PROOF OF THIS RESULT IS AS FOLLOWS. SUPPOSE AGAIN WE HAVE V VANES EQUALLY SPACED AT $\frac{2\pi}{V}$ RADIANS AND B BLADES ROTATING AT ANGULAR VELOCITY Ω PASS OVER EACH VANE. THE BLADES ARE SPACED $\frac{2\pi}{B}$ RADIANS APART. LET US NUMBER THE VANES BY INDEX $m = 0$ TO $V-1$ AND THE BLADES BY INDEX $k = 0$ TO $B-1$.

LET US ASSUME THAT AT TIME $t=0$, BLADE 0 CROSSED OVER VANE 0. THEN A PULSE OF THE TYPE $\delta(t)\delta(\theta)$ IS GENERATED. WE HAVE ASSUMED THAT VANE 0 IS AT $\theta=0$ ANGLE. OTHER BLADES PASSING OVER VANE 0 PRODUCE A PULSE TRAIN OF THE FOLLOWING TYPE:

$$p' = \sum_{k=0}^{B-1} \delta\left(t - \frac{2\pi k}{B\Omega}\right)\delta(\theta) \qquad (21)$$

THE KTH BLADE PASSING OVER MTH VANE PRODUCES A PULSE OF THE TYPE

$$\delta\left(t - \frac{2\pi k}{B\Omega} - \frac{2\pi m}{V\Omega}\right)\delta\left(\theta - \frac{2\pi m}{V}\right) \qquad (22)$$

THEREFORE, B BLADES PASSING OVER V VANES PRODUCE A PULSE TRAIN OF THE FORM

$$p' = \sum_{m=0}^{V-1}\sum_{k=0}^{B-1} \delta\left(t - \frac{2\pi k}{B\Omega} - \frac{2\pi m}{V\Omega}\right)\delta\left(\theta - \frac{2\pi m}{V}\right) \qquad (23)$$

NOW WE FIND FOURIER TRANSFORM OF THIS PULSE:

$$\left\langle p', e^{i(p\Omega t + q\theta)}\right\rangle = \sum_{m=0}^{V-1}\sum_{k=0}^{B-1} \exp 2\pi i\left[p\left(\frac{k}{B}+\frac{m}{V}\right) + \frac{qm}{V}\right]$$

$$= \sum_{m=0}^{V-1} \exp \frac{2\pi i m}{V}(p+q) \times \sum_{k=0}^{B-1} \exp \frac{2\pi i p k}{B} \qquad (24)$$

USING THE RESULT

$$1 + r + r^2 + \dots + r^{n-1} = \frac{1-r^n}{1-r}$$

WE GET BOTH SUMS ARE ZERO UNLESS p IS A MULTIPLE OF B AND $p+q$ IS A MULTIPLE OF V. THAT IS $p = \alpha B$ AND $q = -\alpha B + \beta V$ FOR INTEGER VALUES OF α AND β.

THE SOLUTION FOR P' IS, THEREFORE, OF THE TYPE OF EQ. (20).

AN ANALYSIS OF CIRCUMFERENTIAL MODES

FROM ABSTRACT ALGEBRA, THE SMALLEST POSITIVE INTEGER $mB + nV$ CAN TAKE IS THE GREATEST COMMON DIVISOR OF B AND V. THEREFORE, IF B AND V ARE RELATIVELY PRIME, THEN THE SMALLEST VALUE OF $mB + nV$ IS 1. THIS MEANS THAT LOW CIRCUMFERENTIAL MODES CAN BE GENERATED BY ROTOR-STATOR INTERACTION. WE CAN CONFINE OUR STUDY OF THE MODES TO POSITIVE VALUES SINCE CORRESPONDING TO EACH $mB + nV > 0$, WE HAVE $(-m)B + (-n)V < 0$ WHILE THEY ARE EQUAL IN ABSOLUTE VALUE. LET $p = (B, V)$, THE GREATEST COMMON DIVISOR OF B AND V $\Rightarrow \exists\, m', n' \ni m'B + n'V = p$. ALL MULTIPLES OF p ARE ALSO AMONG NUMBERS GENERATED BY $mB + nV$ SINCE $\alpha m'B + \alpha n'V = \alpha p$. WHAT ARE THE OTHER NUMBERS? OBVIOUSLY $B+V$ AND $|B-V|$ ARE AMONG THESE NUMBERS.

LET US ASSUME $V > B$ AND V AND B ARE RELATIVELY PRIME. THE EQUATION $mB + nV = k$ IS A DIOPHANTINE EQUATION. WE NOTE THAT IF THIS EQUATION HAS A SOLUTION (m', n') FOR A GIVEN POSITIVE INTEGER k, THEN $k + \alpha B$ AND $k + \beta V$ ALSO HAS SOLUTIONS $(m'+\alpha, n')$ AND $(m', \beta + n')$. SINCE $k = B$ AND $k = V$ HAS SOLUTIONS $(1, 0)$ AND $(0, 1)$, RESPECTIVELY, ALL MULTIPLES OF B AND V ARE GENERATED

TRIVIALLY BY $mB+nV$. TO FIND ALL NUMBERS GENERATED BY $mB+nV$, WE MUST SEE FOR WHICH $1<k<V$, THE EQUATION $mB+nV=k$ HAS A SOLUTION. THIS HAS BEEN ANSWERED BY THE SOLUTION OF THE DIOPHANTINE EQUATION (GREEK MATHEMATICIAN DIOPHANTOS, ~250 AD).

ONE SOLUTION OF $mB+nV=1$ CAN BE FOUND BY EUCLID'S ALGORITHM. SINCE WE HAVE ASSUMED $V>B$, WE HAVE

$$
\begin{array}{ll}
V = q_1 B + r_1 & B > r_1 \\
B = q_2 r_1 + r_2 & r_1 > r_2 \\
r_1 = q_3 r_2 + r_3 & r_2 > r_3 \\
\vdots & \\
r_{k-2} = q_k r_{k-1} + r_k & r_{k-1} > r_k \ , \ r_k = 1 \\
r_{k-1} = q_{k+1} r_k = q_{k+1} & r_{k+1} = 0
\end{array}
\qquad (25)
$$

$r_k = 1$ IS THEN GCD. NOW WE WORK FORWARD AS FOLLOWS

$$
\begin{aligned}
r_1 &= V - q_1 B \\
r_2 &= B - q_2 r_1 = B - q_2 (V - q_1 B) \\
&= (1 - q_1) B - q_2 V \\
&\ \vdots \\
1 = r_k &= r_{k-2} + q_k r_{k-1} \\
&= \text{LIN. COMB. OF } B \text{ AND } V
\end{aligned}
\qquad (26)
$$

AT EVERY STEP, WE KNOW r_i AND r_{i+1} ON EACH SIDE OF $r_i = q_{i+2} r_{i+1} + r_{i+2}$ ARE LINEAR COMBINATION OF B AND V.

EXAMPLE: $B=15$, $V=41$

$$41 = 2 \times 15 + 11$$
$$15 = 1 \times 11 + 4$$
$$11 = 2 \times 4 + 3$$
$$4 = 1 \times 3 + 1 \qquad GCD\ 41 \ \& \ 15 = 1$$

NOW WE HAVE

$$11 = 41 - 2 \times 15$$
$$4 = 15 - 11$$
$$= 15 - (41 - 2 \times 15)$$
$$= 3 \times 15 - 41$$
$$3 = 11 - 2 \times 4$$
$$= 41 - 2 \times 15 - 2(3 \times 15 - 41)$$
$$= -8 \times 15 + 3 \times 41$$
$$1 = 4 - 3$$
$$= 3 \times 15 - 41 - (-8 \times 15 + 3 \times 41)$$
$$= 11 \times 15 - 4 \times 41$$
$$= 11 \times B - 4 \times V$$
$$\therefore (m, n) = (11, -4)$$

FROM EUCLID'S ALGORITHM ONE (m, n) IS FOUND FOR THE SOLUTION OF $mB + nV = 1$. NOW, IF (m', n') IS ANOTHER SOLUTION, THEN

$$m'B + n'V = 1 \tag{27}$$

SUBTRACTING THIS FROM $mB + nV = 1$, WE GET

$$(m - m')B + (n - n')V = 0$$

OR

$$(m - m')B = (n' - n)V \tag{28}$$

SINCE B AND V ARE RELATIVELY PRIME, WE MUST HAVE

$$\begin{cases} m - m' = k'V \\ n' - n = k'B \end{cases} \qquad k' \text{ ANY INTEGER}$$

$$\therefore \begin{cases} m' = m - k'V & k' \text{ ANY INTEGER } (+VE, 0, -VE) \\ n' = n + k'B & (29.) \end{cases}$$

THIS CAN BE CHECKED AS FOLLOWS

$$(m - k'V)B + (n + k'B)V = mB + nV = 1$$

REFERENCE : L. A. KALUZHNIN "THE FUNDAMENTAL THEOREM OF ARITHMETIC", MIR PUB., MOSCOW, 1979.
I LIKE THE PROOF IN HUNGERFORD "ABSTRACT ALGEBRA – AN INTRODUCTION", 1990, SAUNDERS, OF THE THEOREM THAT THE SMALLEST POSITIVE INTEGER $mB + nV$, B & V INTEGERS (+VE), IS GCD OF B AND V.

NOW THE NEXT QUESTION IS DOES $mB + nB = k$ HAVE A SOLUTION FOR ANY INTEGER $k > 1$? THE ANSWER IS YES! THE PROOF IS EASY. IF (m', n') IS A SOLUTION OF $m'B + n'V = 1 \Rightarrow m'kB + n'kV = k$, i.e. (km', kn') IS A SOLUTION OF $mB + nV = 1$. THIS MEANS THAT IF GCD OF B AND V IS 1 $\Rightarrow$ ALL CIRCUMFERENTIAL MODES ARE EXCITED.

IF GCD OF B AND V IS NOT 1, SAY IT IS C' $\Rightarrow$ THAT $mB + nV = k$ HAS A SOLUTION IF AND ONLY IF k IS A MULTIPLE OF C'. THE GCD CAN BE AGAIN FOUND BY EUCLID'S ALGORITHM AND EQ. (29) MUST BE MODIFIED. ONLY CIRCUMFERENTIAL MODES WHICH ARE MULTIPLES OF C' ARE EXCITED. EQUATION (29) BECOMES $m' = m + k'\frac{V}{C'}$, $n' = n - k'\frac{B}{C'}$ — (30)
WE SEE THAT THE TIME DEPENDENCE OF THE MODES IS ALWAYS $e^{imB\Omega t}$ BUT CIRCUMFERENTIAL MODES

ARE OF THE TYPE $\exp -i(mB+nV)\theta$. THIS MEANS THAT THE ROTATIONAL SPEED OF THE MODE IS

$$\dot{\theta} = \frac{mB\Omega}{mB+nV} \qquad (31)$$

THEREFORE, THE MACH NUMBER $R\dot{\theta}/C_0$ CAN BE GREATER THAN 1 AND THE MODE CAN BE PROPAGATING (BECAUSE OF SUPERSONIC TRACE VELOCITY). NOTE THAT (m,n) HERE ARE NOT THE SAME (m,n) IN $K_{r,mn}$ AND $K_{a,mn}$.

THE CIRCUMFERENTIAL MODES WHICH ARE MULTIPLE OF C ARE EXCITED. HOWEVER, ALTHOUGH MATHEMATICALLY THE SOLUTION OF THE DIOPHANTINE EQUATION IS EASY, THERE ARE SOME SUBTLETIES IN APPLICATION. CONSIDER THE EQUATION

$$mB+nV = C', \quad C' = GCD(B,V)$$

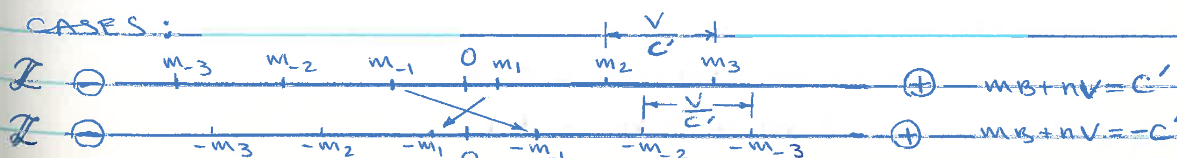
LET US FIND THE SMALLEST POSITIVE SOLUTION FOR m FROM THIS EQUATION. THEN

$$\begin{cases} m' = m + k'\dfrac{V}{C'} \\ n' = n - k'\dfrac{B}{C'} \end{cases} \qquad (32)$$

FOR ALL INTEGERS k' GIVE OTHER SOLUTIONS. THE NEGATIVES OF ALL THESE SOLUTIONS ARE THE SOLUTION OF

$$mB+nV = -C'$$

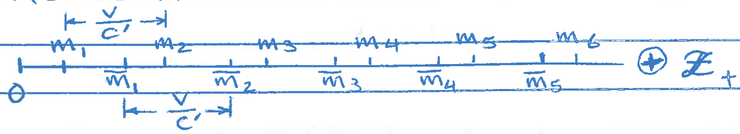
WHICH IS AGAIN FOR THE CIRCUMFERENTIAL MODE C'. LET US DRAW THE SOLUTIONS FOR m IN THESE TWO CASES:



FROM THIS FIGURE, WE CONCLUDE THAT WE CAN ALSO FIND THE POSITIVE SOLUTIONS OF

$$mB + nV = \pm c'$$

IN GENERAL, ONE OF THE EQUATIONS GIVES THE SMALLEST POSITIVE SOLUTION FOR m, SAY m_1. THE OTHER EQUATION GIVES SMALLEST POSITIVE SOLUTION $\overline{m}_1$. THE DISTRIBUTION OF THE SOLUTIONS IS AS FOLLOWS:



WE NOTE THAT NOW

$$\begin{cases} m_i = m_1 + \dfrac{(i-1)V}{c'} & i \in \mathbb{Z}_+ \\ \overline{m}_i = \overline{m}_1 + \dfrac{(i-1)}{c'}V & i \in \mathbb{Z}_+ \end{cases} \quad (33)$$

THESE GIVE THE FREQUENCIES $m_i B \Omega$ AND $\overline{m}_i B \Omega$ THAT ARE EXCITED. NOTE THAT $\overline{m}_1 = \frac{V}{c'} - m_1$.

HIGHER CIRCUMFERENTIAL MODES ARE SIMPLY THE SOLUTIONS OF

$$mB + nV = \pm q c' \qquad m \in \mathbb{Z}_+ \quad (34)$$

THE SOLUTIONS ARE SEEN TO BE $m_i q$ AND $\overline{m}_i q$ WHERE m_i AND $\overline{m}_i$ ARE GIVEN BY EQ.(33).(*)

EXAMPLE: LANGLEY DUCTED FAN RIG HAS $B=16$, AND $V = 40$. WE FIRST FIND THE SMALLEST SOLUTIONS OF

$$16m + 40n = \pm 8$$

OR $$2m + 5n = \pm 1$$

(*) NOT COMPLETELY CORRECT! SEE P26

$m_1 = 2$, FOR $n = -5$, $\bar{m}_1 = 3$, $\frac{V}{c'} = -5$

$$\begin{cases} m_i = 2 + 5(i-1) & i = 1, 2, 3, \ldots \\ \bar{m}_i = 3 + 5(i-1) & i = 1, 2, 3, \ldots \end{cases}$$

$m_i B = 32, 112, 192, \ldots$

$\bar{m}_i B = 48, 128, 208, \ldots$

OR BASED ON BLADE PASSAGE FREQUENCY

2 BPF, 7 BPF, 12 BPF, 17 BPF, ...

3 BPF, 8 BPF, 13 BPF, 18 BPF, ...

THESE ARE ALL FOR FIRST(*) CIRCUMFERENTIAL MODE.

FOR $B = 16$ AND $V = 20$(*), WE HAVE $c' = 4$

$mB + nV = \pm 4$

$4m + 5n = \pm 1$

$\Rightarrow$ $m_1 = 1$ $\quad m_i = 1 + 5(i-1) \quad i = 1, 2, 3 \ldots$

$\bar{m}_1 = 4$ $\quad \bar{m}_i = 4 + 5(i-1) \quad i = 1, 2, 3$

THE MULTIPLES OF BPF FOR FIRST CIRCUMFERENTIAL MODE ARE

1 BPF 6 BPF 11 BPF 16 BPF ....

4 BPF 9 BPF 14 BPF 19 BPF ....

(END OF EXAMPLE)

NOT ALL THE ABOVE FREQUENCIES MAY PROPAGATE. WE NEED TO USE THE OPERATING CONDITIONS OF THE TUNNEL TEST FOR PROPAGATING FREQUENCIES.

(*) THE FIRST CIRCUMFERENTIAL MODE FOR $B = 16$, $V = 40$ IS ± 8, AND FOR $B = 16$, $V = 20$, IT IS ± 4.

FURTHER ANALYSIS OF INTERACTION MODES

TO PREDICT PROPAGATING MODES, WE MUST CALCULATE THE CUTOFF RATIO. WHAT $k_{r,mn}$ DO WE TAKE IN

$$\beta_{mn} = \frac{k}{\beta\, k_{r,mn}} \ ?$$

WHEN WE SOLVE FOR INTERACTION MODES, WE FIRST FIND p AND q FROM

$$pB + qV = \pm c' \qquad c' = GCD(B,V)$$

THEREFORE, k AND $k_{r,mn}$ ARE OBTAINED AS FOLLOWS

$$k = pB\Omega \qquad \Omega \text{ SHAFT RAD/SEC}$$

$$m = c',$$

AND $k_{r,mn}$ ARE THE ZEROS OF $J_c'(k_r r_o)$. AS AN EXAMPLE, FOR LANGLEY'S DUCTED FAN MODEL WITH $B=16$, $V=40$, WE FOUND THAT

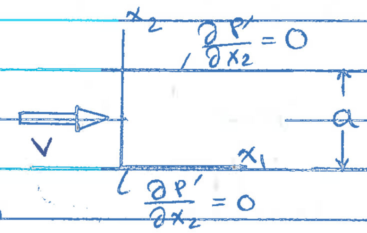
$$m = c' = 8$$

$$p = m_1 = 2 \ \therefore \ k = 2B\Omega,$$

THUS, $k_{r,mn}$ ARE THE ZEROS OF $J_8'(k_r r_o)$, WHERE r_o IS THE DUCT RADIUS. FOR $\overline{m}_1 = 3 = p$, $k = 3B\Omega$ AND AGAIN $k_{r,mn}$ ARE THE ZEROS OF $J_8'(k_r r_o)$. ACTUALLY, THE LANGLEY DUCTED FAN HAS A LARGE HUB SO THAT THE ZEROS OF $J_8' + QY_8'$ HAVE TO BE FOUND, BUT THESE ARE DETAILS.

A GRAPHICAL CONSTRUCTION FOR PROPAGATING MODES IN A DUCT WITH UNIFORMLY MOVING FLUID.

2D CASE



LET $p' = P\, e^{i\omega t}$, AND SUBSTITUTE IN

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x_1}\right)^2 p' \quad \nabla^2 p' = 0$$

$$\overbrace{(1-M^2)}^{\beta^2} P_{11} + P_{22} - ikMP_1 + k^2 P = 0$$

USING SEPARATION OF VARIABLES AND $P = X_1(x_1)X_2(x_2)$, WE GET

$$\beta^2 \frac{X_1''}{X_1} - 2ikM\frac{X_1'}{X_1} + k^2 = -\frac{X_2''}{X_2} = k_2^2$$

$$X_2'' + k_2^2 X_2 = 0$$

$$X_2 = \cos k_2 x_2$$

$$\frac{\partial X_2}{\partial x_2} = -k_2 \sin k_2 x_2 = 0 \quad \text{FOR } x_2 = 0 \text{ AND } x_2 = a$$

$$\Longrightarrow \quad k_2 a = n\pi, \quad k_2 = \frac{n\pi}{a} \quad n \in \mathbb{Z}$$

THE EQUATION FOR X_1 IS THEN

$$\beta^2 X_1'' - 2ikMX_1' - k^2 + k_2^2 = 0$$

WE ASSUME A SOLUTION OF THE TYPE $e^{-ik_1x_1}$

THIS GIVES

$$\beta^2 k_1^2 + 2kMk_1 + k^2 - k_2^2 = 0$$

DIVIDE BY k^2

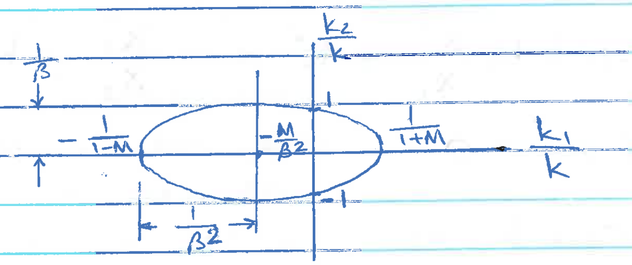
$$\beta^2\left(\frac{k_1^2}{k^2} + 2\frac{M}{\beta^2}\frac{k_1}{k}\right) + \frac{k_2^2}{k^2} = 1$$

NOW COMPLETE SQUARES

$$\beta^2\left(\frac{k_1}{k}+\frac{M}{\beta^2}\right)^2+\frac{k_2^2}{k^2}=1+\frac{M^2}{\beta^2}=\frac{1}{\beta^2}$$

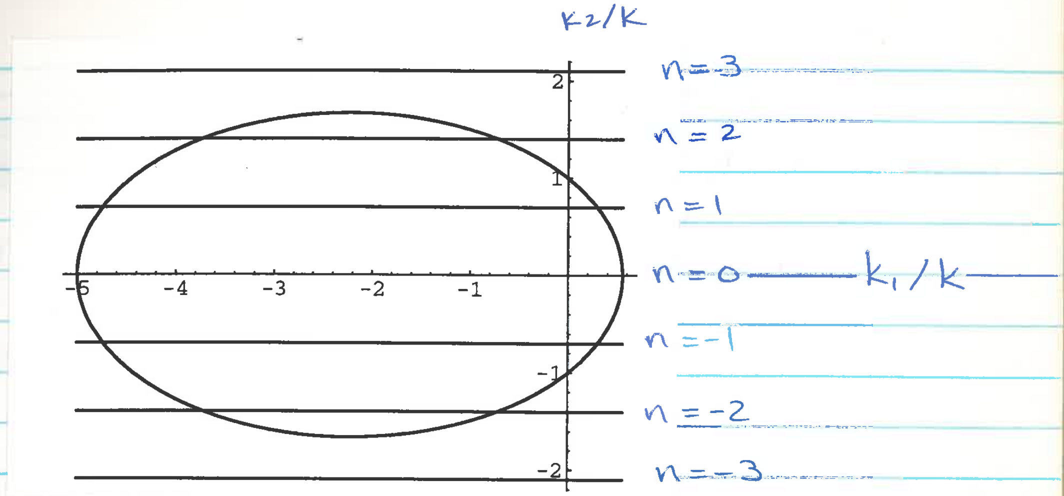
$$\frac{\left(\frac{k_1}{k}+\frac{M}{\beta^2}\right)^2}{1/\beta^4}+\frac{\left(\frac{k_2}{k}\right)^2}{1/\beta^2}=1 \qquad \text{AN ELLIPSE}$$

THE CENTER OF THIS ELLIPSE IS AT $\left(-\frac{M}{\beta^2}, 0\right)$ WITH MAJOR AND MINOR AXES OF $\frac{1}{\beta^2}$ AND $\frac{1}{\beta}$, RESPECTIVELY. THE ELLIPSE CROSSES THE k_1/k AXIS AT $-\frac{M}{\beta^2} \pm \frac{1}{\beta^2} = \frac{-M \pm 1}{1-M^2} = \frac{\pm 1}{1 \pm M}$

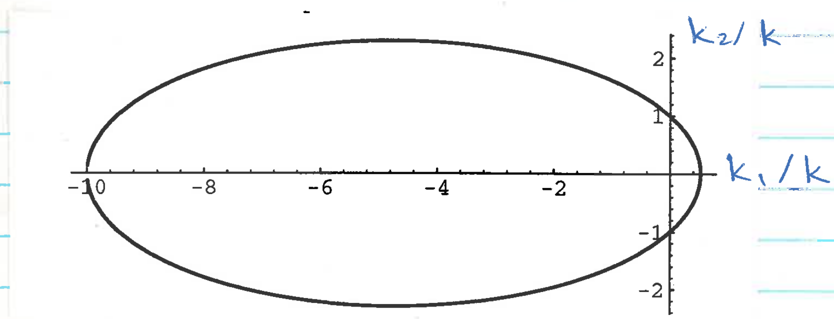


THE ASPECT RATIO (AR) OF THIS ELLIPSE IS $\frac{1}{\beta}$. THUS THE ELLIPSE STRETCHES AS M INCREASES. HOW DO WE FIND THE PROPAGATING MODES GRAPHICALLY? WE PLOT THE ABOVE ELLIPSE AND THEN PLOT $k_2/k = \frac{n\pi}{ka}$. IF THESE LINES INTERSECT THE ELLIPSE, THE CORRESPONDING MODE PROPAGATES. WE NOTE THE WE CAN ONLY TAKE POSITIVE n FOR THIS STUDY. ALSO IF WE WERE WORKING WITH SINES AND CO-SINES, WE WOULD ONLY HAVE POSITIVE n.

WE GIVE AN EXAMPLE FOR $M = 0.8$, $\frac{\pi}{ka} = 0.7$. WE SEE FROM THE FIGURE THAT PLANE WAVE $(n=0)$, $n=1$ AND $n=2$ MODES PROPAGATE

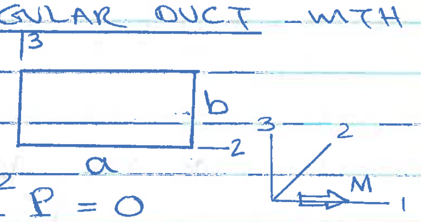


THE ELLIPSE FOR $M=0.9$ IS SHOWN BELOW



IT IS INTERESTING THAT THE SAME ELLIPSE CAN BE USED FOR BOTH 3D RECTANGULAR AND CIRCULAR, AS WELL AS ANNULAR DUCTS.

WE FIRST DISCUSS 3D RECTANGULAR DUCT WITH DIMENSIONS a AND b.



THE D.E. FOR P IS

$$\beta^2 P_{11} + P_{22} + P_{33} - ikMP + k^2 P = 0$$

THE EQUATION FOR FINDING k_1 IS AGAIN

$$\frac{\left(\frac{k_1}{k} + \frac{M}{\beta^2}\right)^2}{1/\beta^4} + \frac{(k'/k)^2}{1/\beta^2} = 1, \quad k'^2 = k_2^2 + k_3^2$$

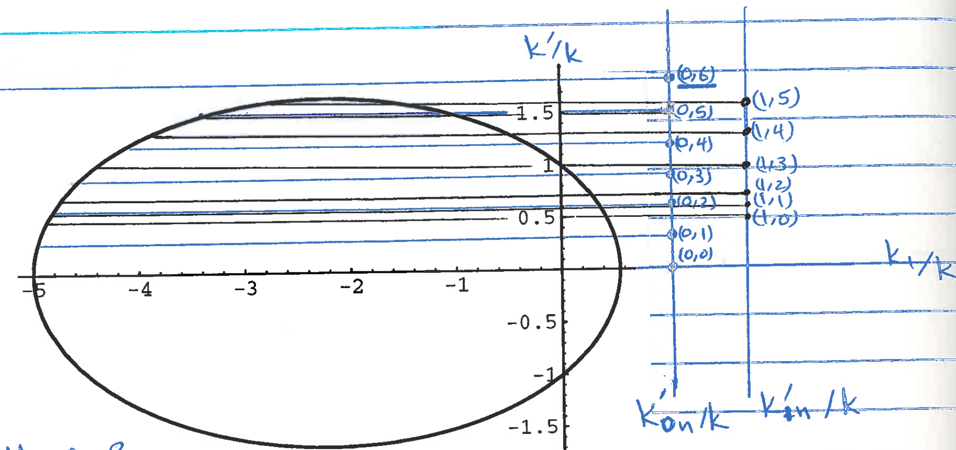
THE EQUATIONS FOR k_2 AND k_3 ARE

$$k_2 = \frac{m\pi}{a}, \quad k_2 = \frac{n\pi}{b}$$

ASSUME $a > b$ AS SHOWN ON PREVIOUS PAGE, AND

$$k'_{mn} = \sqrt{\left(\frac{m\pi}{ak}\right)^2 + \left(\frac{n\pi}{bk}\right)^2}$$

NOW WE USE THE FOLLOWING GRAPHICAL METHOD



M=0.8

— NONPROPAGATING

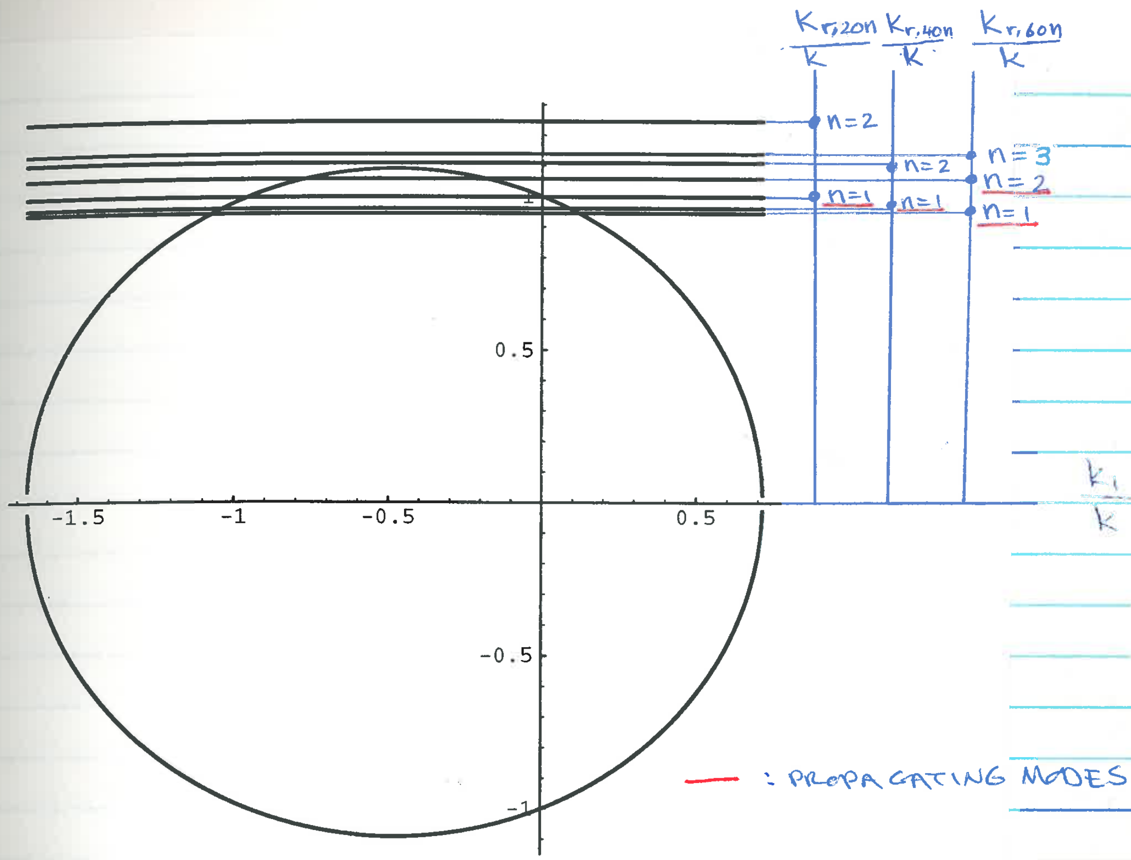
WE HAVE TAKEN $\frac{\pi}{ak} = 0.5$, $\frac{\pi}{bk} = 0.3$ ABOVE. WE CAN DRAW k'_{mn}/k FOR ALL $m \in \mathbb{Z}_+$.

CIRCULAR DUCT

WE HAVE THE SAME ELLIPSE:

$$\frac{(k_1/k + M/\beta^2)^2}{1/\beta^4} + \frac{(k_{r,mn}/k)^2}{1/\beta^2} = 1$$

WE PLOT THE ELLIPSE, THEN DRAW VERTICAL LINES FOR $k_{r,0n}/k$, $k_{r,01}/k$, $k_{r,02}/k$, WHERE $a k_{r,mn}$ ARE THE ZEROS OF $J'_m(k_{r,mn} a) = 0$, a RADIUS OF THE DUCT. WE GIVE THE EXAMPLE OF BOEING DUCTED FAN RIG FOR $M = 0.40$ NEAR FAN FACE, $a = 0.229$ m, $m = 20, 40, 60$.



M = 0.40

NOTE THAT k IS CHANGING WHEN m CHANGES SINCE $k = \frac{m\Omega}{c}$, WHERE Ω (rad/s) IS BASED ON SHAFT SPEED: $k = 97.4$ FOR $m = 20$, AND FOR $m = 40$ AND $m = 60$, k IS 2×97.4 AND 3×97.4, RESPECTIVELY.

TENTATIVE DATA ON LARGE DUCTED FAN ENGINES

THE FOLLOWING DATA WERE SUPPLIED BY P&W FOR LARGE DUCTED FAN ENGINES. MORE DATA WILL BE SENT TO ME LATER.

	P&W 4084	GE 90	R.R. TRENT
FAN DIA.	112" (2.84 m)	123." (3.12 m)	110" (2.79 m)
BLADE NO.	22	22	26
VANE NO.	82	54	58
FAN TIP SPEED (TAKE OFF)	1340 fps (408 m/s)	1200–1250 fps (366–381 m/s)	1420 fps (433 m/s)

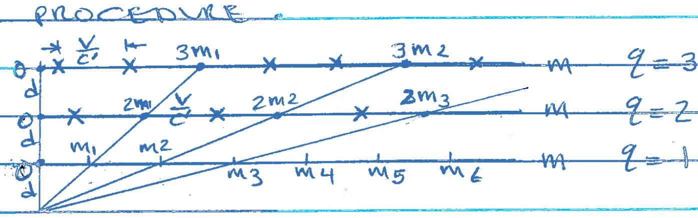
CORRECTION TO SOLUTION OF $mB + nV = qc'$, P18

FROM THE RELATION

$$B\left(m + \frac{V}{c'}\right) + V\left(n - \frac{V}{c'}\right) = mB + nV = qc',$$

IT IS CLEAR THAT THE SOLUTIONS FOR m ARE $\frac{V}{c'}$ APART. BUT $m_i \cdot q$ AND $\bar{m}_i \cdot q$ ARE qV/c' APART. WE HAVE THEREFORE MISSED SOME SOLUTIONS. WE CAN SIMPLY USE EUCLID'S ALGORITHM TO FIND SMALLEST +VE SOLUTION m OF $mB + nV = \pm qc'$ FOR n ZERO, +VE OR −VE ($n \in \mathbb{Z}$); OR FILL IN THE MISSING SOLUTIONS (SHOWN × BELOW) BY A GRAPHICAL PROCEDURE.

d: ARBITRARY



IN PRACTICE, THE MULTIPLE OF BPF IS USUALLY ASSUMED KNOWN SO THAT mB IS GIVEN, WE ARE

THEN INTERESTED IN THE MODE NUMBERS qc' THAT ARE EXCITED. WE SUBSTITUTE VALUES FOR n IN $mB + nV = qc'$ WHERE n IS +, − OR ZERO, i.e. $n \in \mathbb{Z}$. WE USUALLY START WITH $n = 0, \pm 1, \pm 2 \ldots$.

EXAMPLE: THE LANGLEY RIG, $B = 16$, $V = 40$

BPF: $m = 1$, $c' = 8$

$$qc' = 16 + nV$$

$n =$	0	1	−1	2	−2	3	−3
$qc' =$	16	56	−24	96	−64	136	−104

NOT ALL THESE MODES PROPAGATE. WE NOTE THAT FOR PROPAGATION STUDY, WE MUST FIND THE SOLUTIONS OF $J'_{qc'}(\alpha) = 0$.

HOW A ROTATING MICROPHONE CAN SEPARATE MODES

LET US LOOK AT THE EXPONENTIAL TERM OF SOLUTION FOR PROPAGATING MODES

$$\exp i\,[\omega t - m'\theta - k_a x).$$

IN THE CASE OF ROTOR-STATOR INTERACTION, $\omega = BmΩ$ AND $m' = qc' = mB + nV$ AND LET US ASSUME WE HAVE A ROTATING MICROPHONE AT FIXED x SO THAT $e^{-ik_a x}$ ONLY INTRODUCES A PHASE CORRECTION. LET US ASSUME THAT THE MICROPHONE ROTATES WITH ANGULAR VELOCITY $\alpha Ω$ WHERE $Ω$ IS THE SHAFT ANG. VELOCITY (IN LEWIS' RIG $\alpha = \frac{1}{250}$). THEN THE MICROPHONE SENSES A TIME HARMONIC

SIGNAL OF THE TYPE WHICH IS MULTIFREQUENCY IN TIME:

$$\exp -i(-mB\Omega + \alpha q c'\Omega)t$$

$$= \exp\, i(mB + \alpha q c')\Omega t, \quad q = \pm 1, \pm 2, \ldots$$

THIS MEANS THAT IF WE USE A NARROW BAND FILTER, WE CAN SEPARATE ALL CIRCUMFERENTIAL MODE NOW SINCE THEY CORRESPOND TO $q = \pm 1, \pm 2, \ldots$. INFACT, THE DIRECTION OF ROTATION OF THESE MODES ARE ALSO DETERMINED. THE MODES CORRESPONDING TO -VE q ARE ROTATING IN THE DIRECTION OPPOSITE TO THE SHAFT ROTATION. USING AN ARRAY OF MICROPHONE IN RADIAL DIRECTION AND ROTATING IN THE DIRECTION OF SHAFT ROTATION, WE CAN SEPARATE THE RADIAL MODE BY A HANKEL-LIKE TRANSFORM. SPINNING MODES HAVE BAD RADIAL PROPERTY. ALSO EXPANSIONS INVOLVING $Y_n(x)$ ARE VERY TROUBLESOME. THE ENGINE INDUSTRY HAS GROWN USED TO SPINNING MODE CONCEPT. ONE SHOULD LOOK FOR FUNCTIONS WITH BETTER PROPERTIES IN A DUCT.

NOTE THAT, FOR A GIVEN CIRCUMFERENTIAL MODE, MANY MULTIPLES OF BPF CAN CONTRIBUTE. HOWEVER, THESE MODES ROTATE AT DIFFERENT SPEEDS AND CAN BE SEPARATED BY A ROTATING MICROPHONE.

MORE DATA ON LARGE DUCTED FAN ENGINES

P & W (BOB MANZI) HAS VERIFIED THE DATA ON ALL THREE LARGE DUCTED FAN ENGINES, PARTICULARLY THE NUMBER OF EGV FOR GE-90. HERE ARE MORE USEFUL DATA:

	P&W-4084	GE-90	RR TRENT
% SPACING TO EGV @ TIP	300 %	310 %	240 %
V_{TIP} FAN (CORR.)	1345 ft/s (410. m/s)	1234 ft/s (376. m/s)	1415 ft/s (431. m/s)
FAN PR	1.67	1.50	1.63
PRIMARY JET EXHAUST VEL. V_{JE}	1356 ft/s (413. m/s)	1355 ft/s (413. m/s)	1557 ft/s (475. m/s)
SEC. EXHAUST JET VEL. V_{JD}	1047 ft/s (319. m/s)	945 ft/s (288 m/s)	1027 ft/s (313. m/s)

RPM FOR TAKE-OFF AND LANDING, PW-4084

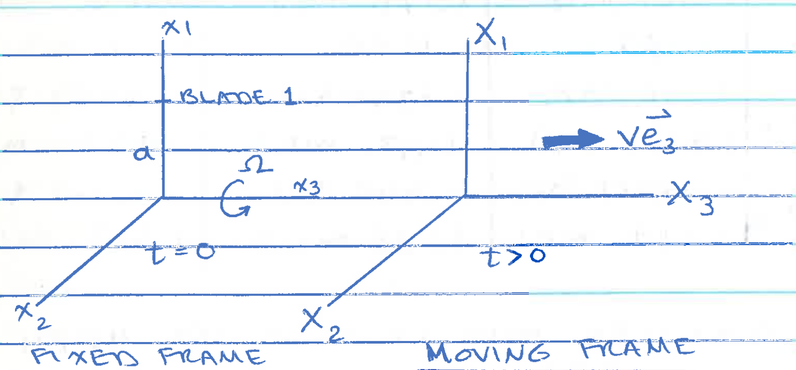
TAKE-OFF	2750 RPM
LANDING	1700 RPM

ROTATING SOURCES IN A MOVING FRAME

TO MODEL ROTATING BLADES IN THE DUCT SCATTERING PROBLEM, WE NEED THE INCIDENT PRESSURE FROM ROTATING POINT FORCES. THIS MODEL IS USED ONLY IN VALIDATING THE DUCT SCATTERING CODE. WE WANT TO SOLVE

$$\Box^2 p' = -\nabla \cdot \vec{f}$$

WHERE $\vec{f} = \sum_{\alpha=1}^{B} \vec{F}(\theta)\,\delta[\vec{x} - \vec{x}_{s_\alpha}(t)]$



$$\vec{x}_{s_\alpha}(t) = [a\cos(\Omega t + \phi_\alpha),\ a\sin(\Omega t + \phi_\alpha),\ Vt]$$

$$\phi_\alpha = \frac{(\alpha-1)2\pi}{B}, \quad B: \text{BLADE NUMBER}$$

$$\delta[\vec{x} - \vec{x}_s(t)] = \frac{\delta(\rho - a)}{\rho}\,\delta(\theta - \Omega t - \phi_\alpha)\,\delta(x_3 - Vt)$$

WE USE (ρ, θ) AS POLAR VARIABLES. WE NOW FIND THE FOURIER EXPANSION OF $\vec{f}$ IN CIRCUMFERENTIAL DIRECTION

$$\langle \vec{f}, e^{in\theta} \rangle = \frac{\delta(\rho - a)}{\rho}\,\delta(x_3 - Vt) \sum_{\alpha=1}^{B} \langle \vec{F}(\theta)\,\delta(\theta - \Omega t - \phi_\alpha), e^{in\theta} \rangle$$

WE ASSUME $F(\theta)$ IS PERIODIC, WITH PERIOD 2π

$$\vec{F}(\theta) = \sum_{-\infty}^{\infty} \vec{F}_j\, e^{ij\theta}$$

$$\Rightarrow \frac{1}{2\pi}\langle \vec{f}, e^{in\theta}\rangle = \frac{\delta(\rho-a)}{2\pi\rho}\delta(x_3-vt)\sum_{j=-\infty}^{\infty}\vec{F}_j e^{i(n+j)\Omega t} \sum_{\alpha=1}^{B} e^{-i(n+j)\phi_\alpha}$$

$$\sum_{\alpha=1}^{B} e^{i(n+j)\phi_\alpha} = \begin{cases} 0 & n+j \neq mB \\ B & n+j = mB \end{cases}$$

$n+j = mB$, B NO. OF BLADES.

IT IS CLEAR THAT, FOR A GIVEN n, ALL $m \in (-\infty,\infty)$ ARE POSSIBLE. THE FUNCTION $\vec{F}(\theta)$ ASSUMED ABOVE HAS A ONE PER REV DISTURBANCE AND THIS IS EQUIVALENT TO TAKING V (THE NUMBER OF VANES) AS 1 SO THAT $mB + jV = mB + j = n$. NOTE THAT SINCE j IS AN INTEGER, THE SIGN OF j IN THIS RELATION DOES NOT MATTER.

$$\frac{1}{2\pi}\langle \vec{f}, e^{in\theta}\rangle = \sum_{m=-\infty}^{\infty} B\frac{\delta(\rho-a)}{2\pi\rho}\delta(x_3-vt)\vec{F}_{mB-n}\, e^{imB\Omega t}$$

$$\vec{f} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} B\frac{\delta(\rho-a)}{2\pi\rho}\delta(x_3-vt)\vec{F}_{mB-n}\, e^{i(mB\Omega t-n\theta)}$$

WE CONSIDER ONLY ONE COMPONENT OF THE FORCE:

$$\Box^2 p' = -\nabla\cdot\left[\vec{F}\frac{\delta(\rho-a)}{2\pi\rho}\delta(x_3-vt)e^{i(mB\Omega t-n\theta)}\right]$$

WE HAVE DROPPED THE SUBSCRIPT $mB-n$ FROM $\vec{F}_{mB-n}$.

WE HAVE A RING OF SOURCES MOVING IN x_3 DIRECTION.

THE SOLUTION IS

$$4\pi p'(\vec{x},t) = -\nabla\cdot\int_\Gamma \vec{Q}\,\delta(g)\,dy\,d\tau$$

[ABANDONED! MARK NO LONGER NEEDS THIS]

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PIPES, CAVITIES, AND WAVEGUIDES

9.1 INTRODUCTION. Thus far, we have considered the behavior of acoustic waves under relatively simple geometric conditions. There are, however, many more complicated situations in acoustics for which the geometry of the boundaries confines the wave to a limited region of space. These situations will occupy our attention for most of the remainder of this book. In this and the next chapter we will discuss the basic properties of sound in pipes, cavities, and waveguides.

When sound propagates in a rigid-walled pipe with wavelength larger than the radius, the acoustic motion is essentially planar, much as longitudinal waves in a bar. The resonance properties of pipes driven at one end and terminated at the other have important application in the laboratory for measurement of acoustical impedances and of the absorptive properties of materials. This study of pipes will also reveal many properties of brasses, woodwinds, and organ pipes. Pipes also serve as models for ventilation ducts.

For larger spaces in which the dimensions are not smaller than a wavelength, two- and three-dimensional standing waves can be stimulated. The basic properties of the normal modes describing these standing waves in rigid-walled volumes offer some simple explanations for the behavior of lower-frequency sound in rooms, auditoriums, concert halls, and so forth.

Finally, we will study the simple waveguide with uniform cross section and develop the concepts of the group speed and the phase speed associated with a sound wave propagating in a waveguide. Application of acoustic waveguides is found in surface-wave delay lines, in high-frequency electronic systems, and in the propagation of sound in the oceans and the atmosphere.

9.2 RESONANCE IN PIPES. Assume that the fluid in a pipe of cross-sectional area S and length L is driven by a piston at $x = 0$ and that the pipe is terminated at $x = L$ in a mechanical impedance $\mathbf{Z}_{mL}$. If the piston vibrates harmonically at a frequency sufficiently low that only plane waves propagate (see Sect. 9.8), the wave in the pipe will be of the form

$$\mathbf{p} = \mathbf{A}e^{j[\omega t + k(L-x)]} + \mathbf{B}e^{j[\omega t - k(L-x)]} \tag{9.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are determined by the boundary conditions at $x = 0$ and $x = L$.

At $x = L$, the continuities of force and particle speed require that the mechanical impedance of the wave at $x = L$ equals the mechanical impedance of the termination, $\mathbf{Z}_{mL}$. Since the force of the fluid on the termination is $\mathbf{p}(L, t)S$ and the particle

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speed is $\mathbf{u}(L, t) = -(1/\rho_0)\int(\partial p/\partial x)\, dt$,

$$\mathbf{Z}_{mL} = \rho_0 c S \frac{\mathbf{A} + \mathbf{B}}{\mathbf{A} - \mathbf{B}} \tag{9.2}$$

The input mechanical impedance $\mathbf{Z}_{m0}$ at $x = 0$ is correspondingly given by

$$\mathbf{Z}_{m0} = \rho_0 c S \frac{\mathbf{A}e^{jkL} + \mathbf{B}e^{-jkL}}{\mathbf{A}e^{jkL} - \mathbf{B}e^{-jkL}} \tag{9.3}$$

Combining these equations to eliminate $\mathbf{A}$ and $\mathbf{B}$, we obtain

$$\frac{\mathbf{Z}_{m0}}{\rho_0 c S} = \frac{\dfrac{\mathbf{Z}_{mL}}{\rho_0 c S} + j \tan kL}{1 + j \dfrac{\mathbf{Z}_{mL}}{\rho_0 c S} \tan kL} \tag{9.4}$$

This equation is identical with (3.33) with the replacement of $\rho_L c$ with $\rho_0 c S$, the *characteristic mechanical impedance* of the fluid, and the substitution

$$\frac{\mathbf{Z}_{mL}}{\rho_0 c S} = r + jx \tag{9.5}$$

leads directly to (3.34). Recalling the discussion following that equation, we see that the frequencies of resonance and antiresonance are determined by the vanishing of the reactance,

$$-j \frac{x \tan^2 kL + (r^2 + x^2 - 1)\tan kL - x}{(r^2 + x^2)\tan^2 kL - 2x \tan kL + 1} = 0 \tag{9.6}$$

The solution identified with *small* input resistance denotes *resonance*, and that identified with *large* input resistance denotes *antiresonance*. (In the limiting case $r = 0$, there is only one solution, corresponding to resonance.)

Let the pipe be driven at $x = 0$ and *closed* at $x = L$ by a rigid cap. To obtain the condition of resonance most simply, let $|\mathbf{Z}_{mL}/(\rho_0 c S)| \rightarrow \infty$ in (9.4). This yields

$$\frac{\mathbf{Z}_{m0}}{\rho_0 c S} = -j \cot kL \tag{9.7}$$

The reactance is zero when $\cot kL = 0$,

$$k_n L = (2n - 1)\pi/2 \qquad n = 1, 2, 3, \ldots \tag{9.8a}$$

or

$$f_n = \frac{2n - 1}{4} \frac{c}{L} \tag{9.8b}$$

This formula is identical with (2.22) for the forced-fixed string. The resonance frequencies are the odd harmonics of the fundamental. The driven, closed pipe has a pressure antinode at $x = L$ and a pressure node at $x = 0$. Notice that this requires that the driver presents a vanishing mechanical impedance to the tube. The impli-

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cation of this, and the effects of the mechanical properties of the driver on the behavior of the driver-pipe system, will be discussed in Sect. 9.6.

Now, consider a pipe driven at $x = 0$ and *open-ended* at $x = L$. On first examination, it might be thought that this will lead to $\mathbf{Z}_{mL} = 0$ for which $\mathbf{Z}_{m0}/(\rho_0 cS) = j \tan kL$ with resonance occurring at $f_n = (n/2)c/L$, $n = 1, 2, 3, \ldots$. However, this is not the case, most elementary physics textbooks notwithstanding. The condition at $x = L$ is not $\mathbf{Z}_{mL} = 0$ since the open end of the tube radiates sound into the surrounding medium. The appropriate value for $\mathbf{Z}_{mL}$ is therefore

$$\mathbf{Z}_{mL} = \mathbf{Z}_r \tag{9.9}$$

where $\mathbf{Z}_r$ is the radiation impedance of the open end of the pipe.

For example, assume that the open end of a circular pipe of radius a is surrounded by a *flange* large with respect to the wavelength of the sound. Consistent with the assumption that the wavelength is large compared to the transverse dimensions of the tube ($\lambda \gg a$), the opening resembles a baffled piston in the low-frequency limit. We have, therefore, from (8.68) *et seq.*

$$\frac{\mathbf{Z}_{mL}}{\rho_0 cS} = \frac{1}{2}(ka)^2 + j\frac{8}{3\pi}ka \qquad \text{(flanged)} \tag{9.10}$$

where both $r = (ka)^2/2$ and $x = 8ka/(3\pi)$ are much less than unity and $r \ll x$. Solution of (9.6) under these conditions gives $\tan kL = -x$ for the resonance frequencies. Since $x \ll 1$, this yields

$$\tan(n\pi - k_n L) = \frac{8}{3\pi}ka \doteq \tan\left(\frac{8}{3\pi}ka\right)$$

where $n = 1, 2, 3, \ldots$. Therefore,

$\frac{8}{3\pi} = 0.849$

$$n\pi = k_n L + \frac{8}{3\pi}k_n a \tag{9.11a}$$

and the resonance frequencies are

$$f_n = \frac{n}{2}\frac{c}{L + \frac{8}{3\pi}a} \tag{9.11b}$$

These resonance frequencies are all harmonics of the fundamental, and it is apparent that the *effective length* L_{eff} of such a pipe is not L but rather $L + 8a/(3\pi)$. This predicted *end correction* for a flanged pipe is in reasonable agreement with experimentally measured values of around $0.82a$.

For an *unflanged*, open pipe, both experiments and theory indicate that the radiation impedance is approximately

$$\frac{\mathbf{Z}_{mL}}{\rho_0 cS} = \tfrac{1}{4}(ka)^2 + j0.6ka \qquad \text{(unflanged)} \tag{9.12}$$

The end correction for an unflanged, open pipe is therefore $0.6a$, so that $L_{eff} = L + 0.6a$.

In both cases, the end corrections are independent of frequency, so that the resonance frequencies of flanged and unflanged open pipes are harmonics of the fundamental (so long as $\lambda_n \gg a$).

These considerations reveal that the resonances of a suitably driven, open-ended organ pipe correspond to the harmonics of the driving frequency. It should be noted that this result has been obtained only for pipes of constant cross section. The presence of any *flare* in the pipe, as found in many wind instruments and some organ pipes, modifies these results. In particular, the resonance frequencies may no longer be harmonics of the fundamental. Indeed, the design of the flare is very important in emphasizing or reducing certain of the harmonics present in the forcing function and therefore in controlling the quality or *timbre* of the sound radiated by the pipe.

9.3 POWER RADIATION FROM OPEN-ENDED PIPES. Solution of (9.2) for $\mathbf{B}/\mathbf{A}$ yields

$$\frac{\mathbf{B}}{\mathbf{A}} = \frac{\mathbf{Z}_{mL}/(\rho_0 cS) - 1}{\mathbf{Z}_{mL}/(\rho_0 cS) + 1} \tag{9.13}$$

and the power transmission coefficient can be found from

$$T_\pi = 1 - |\mathbf{B}/\mathbf{A}|^2 \tag{9.14}$$

once the termination impedance $\mathbf{Z}_{mL}$ is known.

For an open-ended pipe terminated in a flange, $\mathbf{Z}_{mL}$ is given by (9.10), and (9.13) becomes

$$\frac{\mathbf{B}}{\mathbf{A}} = -\frac{[1 - \frac{1}{2}(ka)^2] - j\dfrac{8}{3\pi}ka}{[1 + \frac{1}{2}(ka)^2] + j\dfrac{8}{3\pi}ka} \tag{9.15}$$

This, in turn, yields

$$T_\pi = \frac{2(ka)^2}{[1 + \frac{1}{2}(ka)^2]^2 + \left(\dfrac{8}{3\pi}\right)^2 (ka)^2} \tag{9.16a}$$

Since $ka \ll 1$, the power transmission coefficient is extremely small and can be further simplified,

$$T_\pi \doteq 2(ka)^2 \qquad \text{(flanged)} \tag{9.16b}$$

and (9.15) shows that $\mathbf{B}/\mathbf{A}$ is very nearly -1. The pressure amplitude of the reflected wave is only slightly less than that of the incident wave, and at $x = L$ its pressure differs in phase by nearly 180°; a condensation is reflected as a rarefaction. In contrast, the incident and reflected particle speeds are nearly in phase at the orifice of the pipe, so that this position is approximately an antinode of particle speed. Thus, in spite of the fact that the amplitude of the particle speed at the orifice is almost twice that of the incident wave alone, only a small percentage of the incident power is transmitted out of a flanged pipe. This is another statement of the fact that

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sources whose dimensions are small compared with the wavelength of the sound are very inefficient as radiators of acoustic energy.

For an unflanged pipe, $\mathbf{Z}_{mL}$ is given by (9.12), and the transmission coefficient becomes

$$T_\pi = \frac{(ka)^2}{[1 + \frac{1}{4}(ka)^2]^2 + (0.6ka)^2} \tag{9.17a}$$

or LOW FREQUENCY !

$$T_\pi \doteq (ka)^2 \qquad \text{(unflanged)} \tag{9.17b}$$

so that the presence of a wide flange at the end of a pipe approximately doubles the radiation of sound at low frequencies.

Note that when a pipe is terminated in a gradual flare the low-frequency power transmission is still further increased.

In the vicinity of resonance we can write $\omega = \omega_n + \Delta\omega$, and the input impedance of the unflanged pipe (9.7) is then well-approximated by

$$\frac{\mathbf{Z}_{m0}}{\rho_0 cS} \approx \tfrac{1}{4}(k_n a)^2 + j\,\Delta\omega\,\frac{L}{c}$$

The half-power points are

$$\omega_{u,l} = \omega_n \pm \tfrac{1}{4}(k_n a)^2\,\frac{c}{L}$$

and the Q for the nth resonance is

$$Q_n = \frac{\omega_n}{\omega_u - \omega_l} = \frac{2}{n\pi}\,\frac{L}{a}\,\frac{L + 0.6a}{a}$$

The radiated power, $\Pi = F^2 R_{m0}/(2Z_{m0}^2)$ where $R_{m0} = \mathrm{Re}\{\mathbf{Z}_{m0}\}$ and F is the force amplitude, has the value

$$\Pi_n = \frac{F^2}{\rho_0 cS}\,\frac{2}{(k_n a)^2} = \frac{2}{(n\pi)^2}\,\frac{F^2}{\rho_0 cS}\left(\frac{L + 0.6a}{a}\right)^2$$

Thus, we see that the Q's of the resonances decrease as $1/n$ and that the power radiated at resonance decreases as $1/n^2$ in the low-frequency region for constant applied force amplitude.

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ACOUSTIC ENERGY AND INTENSITY RELATIONS IN MOVING MEDIA

HOMENTROPIC FLOW $S = \text{CONST.}$ — THE BEST SOURCE FOR THIS ANALYSIS IS MYERS, AN EXACT ENERGY COROLLARY FOR HOMENTROPIC FLOW, JSV, VOL 109(2), 1986, 277–284. WE ASSUME THAT THE BACKGROUND (ZEROTH ORDER) FLOW IS TIME INDEPENDENT AND EXPAND EACH VARIABLE $q(\vec{x},t)$ AS FOLLOWS

$$q(\vec{x},t) = q_0(\vec{x}) + \sum_{n=1}^{\infty} \varepsilon^n q_n(\vec{x},t)$$

THE CONTINUITY AND MOMENTUM EQS ARE

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{m} = 0, \quad \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{\nabla p}{\rho} = 0$$

WHERE $\vec{m} = \rho \vec{u}$. LET e BE THE SPECIFIC INTERNAL ENERGY, THEN

$$de = \frac{p}{\rho^2} d\rho$$

ALSO LET h BE THE SPECIFIC ENTHALPY, THEN

$\nabla h = \frac{\nabla p}{\rho}$ OR $dh = \frac{dp}{\rho}$ (FROM $Tds = dh - \frac{dp}{\rho} = 0$)

FROM CONTINUITY AND MOMENTUM EQS, WE CAN DERIVE CONSERVATION LAWS THAT ARE BASED ON VARIOUS ORDERS OF CONTINUITY AND MOMENTUM EQS. MYERS HAS SHOWN THAT THERE IS A BETTER WAY BASED ON THE FULL FLUID ENERGY RELATION

$$\frac{\partial E}{\partial t} + \nabla \cdot \vec{W} = \frac{\partial}{\partial t}\left(\rho e + \frac{\rho u^2}{2}\right) + \nabla \cdot \left[\vec{m}\left(h + \frac{u^2}{2}\right)\right] = 0$$

LET US FIRST RIGHT THE VARIOUS ORDERS OF CONT. & MOM EQS. :

$$\begin{cases} \nabla \cdot \vec{m}_0 = 0 \\ \vec{u}_0 \cdot \nabla \vec{u}_0 + \nabla h_0 = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial \rho_1}{\partial t} + \nabla \cdot \vec{m}_1 = 0 \\ \dfrac{\partial \vec{u}_1}{\partial t} + \vec{u}_0 \cdot \nabla \vec{u}_1 + u_1 \cdot \nabla u_0 + \nabla h_1 = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial \rho_2}{\partial t} + \nabla \cdot \vec{m}_2 = 0 \\ \dfrac{\partial \vec{u}_2}{\partial t} + \vec{u}_0 \cdot \nabla \vec{u}_2 + \vec{u}_1 \cdot \nabla \vec{u}_1 + \vec{u}_2 \cdot \nabla \vec{u}_0 + \nabla h_2 = 0 \end{cases}$$

ALSO WE HAVE

$$\begin{cases} \rho e = \rho_0 e_0 + \varepsilon h_0 \rho_1 + \varepsilon^2 \underbrace{\left(\dfrac{c_0^2 \rho_1^2}{2\rho_0} + h_0 \rho_2 \right)}_{(\rho e)_2} + \cdots \\ h = h_0 + \varepsilon \dfrac{c_0^2 \rho_1}{\rho_0} + \cdots \end{cases}$$

NOTE THAT WE OBTAINED THESE RESULTS AS FOLLOWS:

$$\rho e = \rho_0 e_0 + \frac{d}{d\rho}(\rho e)\Big|_0 (\rho - \rho_0) + \frac{d^2}{d\rho^2}(\rho e)\Big|_0 \frac{(\rho-\rho_0)^2}{2} + \cdots$$

$$= \rho_0 e_0 + \frac{d(\rho e)}{d\rho}\Big|_0 (\varepsilon \rho_1 + \varepsilon^2 \rho_2 + \cdots) + \frac{1}{2} \frac{d^2(\rho e)}{d\rho^2}\Big|_0 (\varepsilon \rho_1 + \varepsilon^2 \rho_2 + \cdots)^2 + \cdots$$

$$= \rho_0 e_0 + \varepsilon \frac{d(\rho e)}{d\rho}\Big|_0 \rho_1 + \varepsilon^2 \left(\rho_2 \frac{d(\rho e)}{d\rho}\Big|_0 + \frac{\rho_1^2}{2} \frac{d^2(\rho e)}{d\rho^2}\Big|_0 \right) + \cdots$$

$$\frac{d(\rho e)}{d\rho}\Big|_0 = e_0 + \rho_0 \frac{de}{d\rho}\Big|_0 = e_0 + \frac{p_0}{\rho_0} = h_0$$

$$\frac{d^2(\rho e)}{d\rho^2}\Big|_0 = \left[\frac{de}{d\rho} + \frac{d}{d\rho}\left(\frac{p}{\rho}\right) \right]_0 = \cancel{\frac{p_0}{\rho_0^2}} - \cancel{\frac{p_0}{\rho_0^2}} + \frac{c_0^2}{\rho_0}$$

SIMILARLY $h = h_0 + \dfrac{dh}{d\rho}\Big|_0 (\rho - \rho_0) + \cdots$

$$= h_0 + \varepsilon \frac{dh}{d\rho}\Big|_0 \rho_1 + \cdots$$

$$\frac{dh}{d\rho} = \left[\frac{de}{d\rho} + \frac{d}{d\rho}\left(\frac{p}{\rho}\right) \right]_0 = \frac{c_0^2 \rho_1}{\rho_0}$$

THE ZEROTH ORDER ENERGY RELATION IS

$$\nabla \cdot [\vec{m}_0 (h_0 + \frac{u_0^2}{2})] = 0$$

THE FIRST ORDER ENERGY RELATION IS IDENTICALLY ZERO (SEE MYERS, JSV, VOL 109(2), 1986, 277-284). THE SECOND ORDER EQ. IS:

$$\frac{\partial}{\partial t}\left[(\rho e)_2 + \frac{\rho_2 u_0^2}{2} + \rho_1 \vec{u}_0 \cdot \vec{u}_1 + \rho_0 \frac{u_1^2}{2} + \rho_0 \vec{u}_0 \cdot \vec{u}_2\right]$$
$$+ \nabla \cdot [\vec{m}_0 (h_2 + \frac{u_1^2}{2} + \vec{u}_0 \cdot \vec{u}_2) + \vec{m}_1 (h_1 + \vec{u}_0 \cdot \vec{u}_1)$$
$$+ \vec{m}_2 (h_0 + u_0^2/2)] = 0$$

WE HAVE, FROM PREVIOUS PAGE $(\rho e)_2 = \frac{c_0^2 \rho_1^2}{2\rho_0} + h_0 \rho_2$

$$E_1 = (\rho e)_2 + \frac{1}{2} \rho_2 u_0^2 + \rho_1 \vec{u}_0 \cdot \vec{u}_1 + \frac{1}{2} \rho_0 u_1^2 + \rho_0 \vec{u}_0 \cdot \vec{u}_2$$
$$= \underbrace{\frac{c_0^2 \rho_1^2}{2\rho_0} + \frac{1}{2} \rho_0 u_1^2 + \rho_1 \vec{u}_0 \cdot \vec{u}_1}_{\tilde{E}} + (h_0 + \frac{u_0^2}{2}) \rho_2$$

$$\frac{\partial E_1}{\partial t} = \frac{\partial}{\partial t}\left[\frac{c_0^2 \rho_1^2}{2\rho_0} + \frac{1}{2} \rho_0 u_1^2 + \rho_1 \vec{u}_0 \cdot \vec{u}_1\right]$$
$$+ (h_0 + \frac{u_0^2}{2}) \frac{\partial \rho_2}{\partial t} + \rho_0 \vec{u}_0 \cdot \frac{\partial \vec{u}_2}{\partial t}$$

$$\vec{E}_2 = \vec{m}_0 (h_2 + \frac{u_1^2}{2} + \vec{u}_0 \cdot \vec{u}_2) + \vec{m}_1 (h_1 + \vec{u}_0 \cdot \vec{u}_1)$$
$$+ \vec{m}_2 (h_0 + \frac{u_0^2}{2})$$

$$\nabla \cdot \vec{E}_2 = \vec{m}_0 \cdot \nabla (h_2 + \frac{u_1^2}{2} + \vec{u}_0 \cdot \vec{u}_2) + \nabla \cdot [\vec{m}_1 (h_1 + \vec{u}_0 \cdot \vec{u}_1)]$$
$$+ (h_0 + \frac{u_0^2}{2}) \nabla \cdot \vec{m}_2 + \vec{m}_2 \cdot \nabla (h_0 + \frac{u_0^2}{2})$$

$$\frac{\partial E_1}{\partial t} + \nabla \cdot \vec{E}_2 = \frac{\partial \tilde{E}}{\partial t} + (h_0 + \frac{u_0^2}{2}) \overbrace{\left(\frac{\partial \rho_2}{\partial t} + \nabla \cdot \vec{m}_2\right)}^{= 0 \text{ CONT. EQ.}}$$
$$+ \rho_0 \vec{u}_0 \cdot \left[\frac{\partial \vec{u}_2}{\partial t} + \nabla h_2 + \nabla (\frac{u_1^2}{2}) + \nabla (\vec{u}_0 \cdot \vec{u}_2)\right]$$
$$+ \nabla \cdot [\vec{m}_1 (h_1 + \vec{u}_0 \cdot \vec{u}_1)] + \vec{m}_2 \cdot \nabla (h_0 + \frac{u_0^2}{2}) = 0$$

NOW WE SIMPLIFY SOME TERMS AS FOLLOWS. WE HAVE

$$\nabla(\vec{u}_m\cdot\vec{u}_n) = \vec{u}_m\cdot\nabla\vec{u}_n + \vec{u}_n\cdot\nabla\vec{u}_m - \xi_n\times\vec{u}_m - \vec{\xi}_m\times\vec{u}_n \quad , \quad \vec{\xi}_i = \nabla\times\vec{u}_i$$

$$\frac{\partial \vec{u}_2}{\partial t} + \nabla\frac{u_1^2}{2} + \nabla(\vec{u}_0\cdot u_2) + \nabla h_2 = \frac{\partial\vec{u}_2}{\partial t} + \vec{u}_1\cdot\nabla\vec{u}_1 + \vec{u}_0\cdot\nabla u_2 + u_2\cdot\nabla u_0 + \nabla h_2 - (\vec{\xi}_0\times\vec{u}_2 + \vec{\xi}_1\times\vec{u}_1 + \vec{\xi}_2\times\vec{u}_0)$$

$$= -(\vec{\xi}_0\times\vec{u}_2 + \vec{\xi}_1\times\vec{u}_1 + \vec{\xi}_2\times\vec{u}_0)$$

WE USED THE 2ND ORDER MOM. EQ HERE.

SIMILARLY

$$\nabla\left(h_0 + \frac{u_0^2}{2}\right) = \nabla h_0 + \nabla(\vec{u}_0\cdot\vec{u}_0/2) = \underbrace{\nabla h_0 + \vec{u}_0\cdot\nabla u_0}_{=0 \text{ BY 0TH ORDER MOM. EQ.}} - \vec{\xi}_0\times\vec{u}_0$$

THE FINAL CONSERVATION LAW IS:

$$\boxed{\frac{\partial E_a}{\partial t} + \nabla\cdot\vec{W}_a = \rho_0\vec{u}_0\cdot(\vec{\xi}_1\times\vec{u}_1) + \rho_1\vec{u}_1\cdot(\vec{\xi}_0\times\vec{u}_0)}$$

$$\boxed{E_a = \frac{c_0^2\rho_1^2}{2\rho_0} + \frac{\rho_0 u_1^2}{2} + \rho_1\vec{u}_0\cdot\vec{u}_1}$$

$$\boxed{\vec{W}_a = \left(\frac{c_0^2\rho_1}{\rho_0} + \vec{u}_0\cdot\vec{u}_1\right)(\rho_0\vec{u}_1 + \rho_1\vec{u}_0)}$$

MYERS WRITE $\vec{W}_a$ IS FOLLOWS:

$$\vec{W}_a = (c_0^2\rho_1 + \rho_0\vec{u}_0\cdot\vec{u}_1)\left(\vec{u}_1 + \frac{\rho_1}{\rho_0}\vec{u}_0\right)$$

NOTE THAT THIS HAS THE DIMENSION OF $p\vec{u}$.

THE EXACT ENERGY COROLLARY OF MYERS

LET $H = h + \frac{u^2}{2} = e + \frac{p}{\rho} + \frac{u^2}{2}$ BE THE STAGNATION ENTHALPY. THE ENERGY EQ. CAN BE WRITTEN AS

$$\frac{\partial}{\partial t}(\rho H - p) + \nabla\cdot(\vec{m} H) = 0 \qquad (1)$$

THE MOM. EQ. CAN BE WRITTEN AS

$$\frac{\partial \vec{u}}{\partial t} + \nabla H + \vec{\xi} \times \vec{u} = 0 \qquad \vec{\xi}: \text{VORTICITY}$$

NOW LET US USE SUBSCRIPT O FOR THE STEADY BACKGROUND FLOW. WE ADD TWO VANISHING TERMS TO THE ENERGY EQ. (1) :

$$\frac{\partial}{\partial t}(\rho H - p) + \nabla\cdot(\vec{m} H) - H_o\left(\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{m}\right)$$
$$- \vec{m}_o \cdot \left(\frac{\partial \vec{u}}{\partial t} + \nabla H + \vec{\xi} \times \vec{u}\right) = 0$$

$$\frac{\partial}{\partial t}\left[\rho(H - H_o) - p - \vec{m}_o\cdot\vec{u}\right] + \nabla\cdot(\vec{m} H) - \nabla\cdot(\vec{m} H_o)$$
$$+ \vec{m}\cdot\nabla H_o - \nabla\cdot(\vec{m}_o H) - \vec{m}_o\cdot\vec{\xi}\times\vec{u} = 0$$

NOTE THAT $\nabla\cdot\vec{m}_o = 0$ BY MASS CONT. EQ.

$$\frac{\partial}{\partial t}[\cdot] + \nabla\cdot\left[\vec{m}(H - H_o) - \vec{m}_o H\right]$$
$$= \vec{m}_o\cdot\vec{\xi}\times\vec{u} - \vec{m}\cdot\nabla H_o$$

BUT $\vec{u}_o \cdot \nabla \vec{u}_o + \nabla h_o = \nabla\left(h_o + \frac{u_o^2}{2}\right) + \vec{\xi}_o \times \vec{u}_o$
$$= \nabla H_o + \vec{\xi}_o \times \vec{u}_o = 0$$

$$\frac{\partial}{\partial t}[\cdot] + \nabla\cdot\left[\vec{m}(H - H_o) - \vec{m}_o H\right] =$$
$$= \vec{m}_o\cdot\vec{\xi}\times\vec{u} + \vec{m}\cdot\vec{\xi}_o\times\vec{u}_o \qquad (2)$$

NOW FOR STEADY FLOW, THIS EQ. BECOMES

$$\underbrace{\frac{\partial}{\partial t}(P_0 + \rho_0 u^2)}_{=0,\ \text{STEADY FLOW}} + \underbrace{\nabla\cdot(\vec{m}_0 H_0)}_{=0\ \text{ENERGY EQ. FOR BACKGROUND FLOW}} = -\underbrace{2\vec{m}_0\cdot\vec{\xi}_0\times\vec{u}_0}_{=0,\ \vec{m}_0 \parallel \vec{u}_0}$$

SUBTRACT THIS FROM EQ. (2), TO GET

EXACT ENERGY COROLLARY

$$\boxed{\frac{\partial E^*}{\partial t} + \nabla\cdot\vec{W}^* = -(\vec{m}-\vec{m}_0)\cdot(\vec{\xi}\times\vec{u} - \vec{\xi}_0\times\vec{u}_0)}$$

$$\boxed{E^* = \rho(H-H_0) - (P-P_0) - \vec{m}_0\cdot(\vec{u}-\vec{u}_0)}$$

$$\boxed{\vec{W}^* = (\vec{m}-\vec{m}_0)(H-H_0)}$$

MYERS SHOWS THAT $E^*_1 = 0$ AND $E^*_2 = E_a$.
ALSO $\vec{W}^*_0 = \vec{W}^*_1 = 0$ AND $W^*_2 = \vec{m}_1 H_1 = W_a$.
THE RIGHT SIDE OF THE EXACT ENERGY COROLLARY IS ZERO UPTO THE FIRST ORDER AND TO 2ND ORDER IT IS $\rho_0\vec{u}_0\cdot\vec{\xi}_1\times\vec{u}_1 + \rho_1\vec{u}_1\cdot\vec{\xi}_0\times\vec{u}_0$.

W. MÖHRING'S RESULTS

I FOUND THREE OF MÖHRING'S PAPERS AS FOLLOWS:

i) ENERGY FLUX IN DUCT FLOW, JSV, VOL. 18(1), 1971, 101-109

ii) ON ENERGY, GROUP VELOCITY AND SMALL DAMPING OF SOUND WAVES IN DUCTS WITH SHEAR FLOW, JSV, VOL. 29(1), 1973, 93-101

iii) ACOUSTIC ENERGY FLUX IN NONHOMOGENEOUS DUCTS, AIAA-77-1280, 4 PAGES

SOME OF MÖHRING'S WORKS ARE IN GERMAN.

ANALYSIS IN i)

A CONSERVATION EQ. IS DEFINED BY THE RELATION

$\vec{n}$, V, $S = \partial V$

$$\frac{d}{dt}\int_V w\, d\vec{y} + \int_{\partial V} \vec{U}\cdot d\vec{S} = 0 \qquad (1)$$

$$\frac{\partial w}{\partial t} + \nabla\cdot\vec{U} = 0 \qquad (2)$$

SINCE $\langle \partial f/\partial t\rangle = \frac{1}{T}\int_0^T \frac{\partial f}{\partial t}\,dt = \frac{f(T)-f(0)}{T} \to 0$ AS $T\to\infty$, WE HAVE

$$\int_{\partial V} \vec{\bar{U}}\cdot d\vec{S} = 0 \quad , \quad \nabla\cdot\vec{\bar{U}} = 0 \quad , \quad \vec{\bar{U}} = \langle\vec{U}\rangle$$

I AM NOT FOLLOWING ALL OF MÖHRING'S NOTATION HERE.

MÖHRING INTRODUCES CLEBSCH POTENTIALS ϕ, η, α AND β BY THE RELATIONS

$$\begin{cases} \frac{\partial \eta}{\partial t} + \vec{u}\cdot\nabla\eta = -T \quad , \quad T \text{ ABS. TEMP} & (3) \\ \vec{u}\cdot d\vec{x} - (h + \frac{u^2}{2})\,dt = d\phi + s\,d\eta + \alpha\, d\beta & (4) \end{cases}$$

s : ENTROPY, $\vec{u}$: FLUID VELOCITY ($\vec{v}$ IN MÖHRING!)

WE HAVE

$$\begin{cases} \partial\phi/\partial t + \vec{u}\cdot\nabla\phi = \frac{u^2}{2} - g \quad , \quad g = h - Ts \text{ FREE ENTHALPY} \\ \partial\alpha/\partial t + \vec{u}\cdot\nabla\alpha = 0 \\ \partial\beta/\partial t + \vec{u}\cdot\nabla\beta = 0 \end{cases} \qquad \text{(EQS. 5-7)}$$

THE FUNCTION $\mathcal{H} = h + \frac{u^2}{2} = -\phi_t - s\eta_t - \alpha(\beta_t + 1)$

SATISFIES

$$\frac{\partial \mathcal{H}}{\partial t} + \vec{u}\cdot\nabla\mathcal{H} = \frac{1}{\rho}\frac{\partial p}{\partial t} \qquad (8)$$

FOR STEADY BACKGROUND FLOW, THE CLEBSCH POTENTIALS ϕ_0, η_0, α_0 AND β_0 ARE INTRODUCED BY EQS (3) AND (4). IF $h + u^2/2 \neq 0$, EQ. (4) SHOWS THAT AT LEAST ONE OF THE CLEBSCH POTENTIAL WILL BE TIME DEPENDENT. WE ASSUME THAT $\beta_0 = B(\vec{x}) - t$ IS TIME DEPENDENT AND ϕ_0, η_0 AND α_0 ARE FUNCTIONS OF TIME. FOR FLUCTUATING QUANTITIES, THE CLEBSCH POTENTIALS SATISFY

$$\begin{cases} \dfrac{\partial \eta'}{\partial t} + \vec{u}_0\cdot\nabla\eta' + \vec{u}_1\cdot\nabla\eta_0 = -T' & (9)\\ \vec{u}'\cdot d\vec{x} - (h' + \vec{u}_0\cdot\vec{u}_1)\,dt = d\phi' + s_0\,d\eta' + s'\,d\eta_0 + \alpha_0\,d\beta' + \alpha'\,d\beta_0 & (10)\end{cases}$$

THESE ARE SOLUTIONS OF

$$\begin{cases} \partial\phi'/\partial t + \vec{u}_0\cdot\nabla\phi' + \vec{u}'\cdot\nabla\phi_0 = u_0 u' - g' & (11)\\ \partial\alpha'/\partial t + \vec{u}_0\cdot\nabla\alpha' + \vec{u}'\cdot\nabla\alpha_0 = 0 & (12)\\ \partial\beta'/\partial t + \vec{u}_0\cdot\nabla\beta' + \vec{u}'\cdot\nabla\beta_0 = 0 & (13)\end{cases}$$

AGAIN WE DEFINE A FUNCTION $\mathcal{H}'$ AS FOLLOWS

$$\mathcal{H}' = h' + \vec{u}_0\cdot\vec{u}' - \alpha'$$
$$= -\phi'_t - s_0\eta'_t - \alpha_0\beta'_t$$

$$\Rightarrow \quad \frac{\partial\mathcal{H}'}{\partial t} + \vec{u}_0\cdot\nabla\mathcal{H}' = \frac{1}{\rho_0}\frac{\partial p'}{\partial t} \qquad (14)$$

IT IS OBVIOUS TO ME THAT THIS IS AN AREA THAT SYMPLECTIC GEOMETRY COULD BE APPLIED FRUITFULLY. THE CONVENTIONAL APPROACH IS ALGEBRAICALLY VERY COMPLICATED AS EVIDENCED FROM MÖHRING'S WORK. MÖHRING THEN PROPOSES (BASED ON HIS GERMAN PAPER ZAMM, VOL 50, 1970, 196-198) THE FOLLOWING ENERGY

FLUX

$$\vec{U} = \mathcal{H}'(\rho_0 \vec{u}' + \rho' \vec{u}_0) - \rho_0 (s' \eta'_t + \alpha' \beta'_t) \vec{u}_0 \quad (15)$$

WHICH RESULTS IN EQ. (2): $w_t + \nabla \cdot \vec{U} = 0$ WITH w DEFINED BY

$$w = \rho' \mathcal{H}' + \rho_0 \vec{u}_0 \cdot (s' \nabla \eta' + \alpha' \nabla \beta') + \frac{1}{2} \rho_0 u'^2$$
$$- \frac{\rho'^2}{2\rho_0} \left(\frac{\partial p}{\partial \rho_0} \right)_{s_0} + \frac{1}{2} \rho_0 s'^2 \left(\frac{\partial T}{\partial s_0} \right)_{\rho_0} \quad (16)$$

MÖHRING THEN MAKES A COMMENT ABOUT $\vec{U}$ THAT IN THE ABSENCE OF ENTROPY FLUCTUATIONS, $\vec{U}$ BECOMES

$$\vec{U}^B = \left(\frac{p'}{\rho_0} + \vec{u}_0 \cdot \vec{u}' \right) \left(\rho_0 \vec{u}' + \frac{p' \vec{u}_0}{c_0^2} \right)$$
$$= \left(p' + \rho_0 \vec{u}_0 \cdot \vec{u}' \right) \left(\vec{u}' + \frac{p'}{\rho_0} \vec{u}_0 \right) \quad (17)$$

WHERE THE SUPERSCRIPT B STANDS FOR BLOKHINTSEV. MÖHRING THEN APPLIES THE ABOVE RESULTS TO A TWO-DIMENSIONAL DUCT.

ANALYSIS IN ii) JSV, 29(1), 1973, 93-101

HERE AGAIN HAMILTON'S PRINCIPLE USING CLEBSCH POTENTIALS η, ϕ, α AND β BY EQS. (3) AND (4). THE CHOICE OF CLEBSCH POTENTIAL IS NOT UNIQUE. HAMILTON'S PRINCIPLE IS

$$\delta \int \mathcal{L} \, d\vec{y} \, dt = 0 \quad (18)$$

WHERE

$$\mathcal{L} = -\rho \left[E(\rho, S) + \frac{\partial \phi}{\partial t} + S \frac{\partial \eta}{\partial t} + \alpha \frac{\partial \beta}{\partial t} + \frac{1}{2} \left| \nabla \phi + S \nabla \eta + \alpha \nabla \beta \right|^2 \right] \quad (19)$$

"LAGRANGIAN DENSITY"

FROM EQ. (3) $\Rightarrow \mathcal{L} = p$, i.e. $\mathcal{L}$ IS INDEPENDENT OF THE CHOICE OF CLEBSCH POTENTIALS. FOR A STEADY FLOW, WE TAKE $\beta_0 = B(\vec{x}) - t$, ϕ, η AND α

FUNCTIONS OF $\vec{x}$ ONLY. THIS IS CALLED NORMALIZATION OF THE POTENTIAL. NOW LET THE PRIMED QUANTITIES BE THE FLUCTUATIONS TO THE STEADY VALUES. LET $\vec{\psi} = (\rho', s', \phi', \alpha', \beta', \eta')$, THEN THE EULER EQUATION OF THE VARIATIONAL PRINCIPLE

$$\delta^2 \int \mathcal{L}\, d\vec{y}\, dt = \delta \int \mathcal{L}'\, d\vec{y}\, dt = 0 \qquad (20)$$

IS

$$\frac{\partial \mathcal{L}'}{\partial \psi^i} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}'}{\partial \psi^i_t} - \frac{\partial}{\partial x^j}\frac{\partial \mathcal{L}'}{\partial \psi^i_j} = 0 \qquad (21)$$

$$i = 1-6$$

WHERE $\psi^i_j = \partial \psi^i / \partial x^j$. HERE THE LAGRANGIAN DENSITY $\mathcal{L}'$ IS DEFINED BY

$$\mathcal{L}' = -\Big[\frac{1}{2}\frac{\partial^2(\rho E)}{\partial s_0^2} s'^2 + \frac{\partial^2(\rho E)}{\partial \rho_0 \partial s_0}\rho' s' + \frac{1}{2}\frac{\partial^2(\rho E)}{\partial \rho_0^2}\rho'^2$$

$$+ \rho'\phi'_t + \rho_0 s'\eta'_t + \rho' s_0 \eta'_t + \rho_0 \alpha' \beta'_t$$

$$+ \rho' \alpha_0 \beta'_t - \rho'\alpha' + \rho_0\big[\vec{u}_0 \cdot (s'\nabla\eta' + \alpha'\nabla\beta') + \frac{u'^2}{2}\big]$$

$$+ \rho' \vec{u}_0 \cdot \vec{u}'\Big] \qquad (22)$$

IN THIS EQ., USE HAS BEEN MADE OF NORMALIZATION OF ϕ, η, α AND β. NOW, THE EULER EQ. CAN BE INTERPRETED AS

$$\frac{\partial w'}{\partial t} + \nabla \cdot \vec{U}' = 0 \qquad (23)$$

WHERE

$$\begin{cases} w' = \dfrac{\partial \mathcal{L}'}{\partial \psi^i_t}\psi^i_t - \mathcal{L}' & (25) \\ U'^j = \dfrac{\partial \mathcal{L}'}{\partial \psi^i_j}\psi^i_t & (26) \end{cases}$$

THIS IS BECAUSE $\mathcal{L}'$ DOES NOT EXPLICITLY DEPEND ON t. WE HAVE

$$w' = \frac{1}{2}\frac{\partial^2(\rho E)}{\partial s_0^2} s'^2 + \frac{\partial^2(\rho E)}{\partial s_0 \partial \rho_0} \rho' s' + \frac{1}{2}\frac{\partial^2(\rho E)}{\partial \rho_0^2} \rho'^2$$
$$+ \frac{1}{2}\rho_0 u'^2 + \rho' \vec{u}_0 \cdot \vec{u}' - \rho' \alpha'$$
$$+ \rho_0 \vec{u}_0 \cdot (s' \nabla \eta' + \alpha' \nabla \beta') \qquad (27)$$

$$\vec{U}' = (h' + \vec{u}_0 \cdot \vec{u}' - \alpha')(\rho_0 \vec{u}' + \rho' \vec{u}_0)$$
$$- \rho_0 (s' \eta'_t + \alpha' \beta'_t) \vec{u}_0 \qquad (28)$$

MÖHRING MENTIONS THAT, IN CONTRAST TO $\mathcal{L}$, NOW w' AND $\vec{U}'$ ARE DEPENDENT ON THE CHOICE OF CLEBSCH POTENTIAL. THIS IS ALSO TRUE FOR $\mathcal{L}'$. THESE QUANTITIES, i.e. w', $\vec{U}'$, AND $\mathcal{L}'$ ARE NOT GAUGE INVARIANT. DIFFERENT CLEBSCH POTENTIALS LEAD TO DIFFERENT $\mathcal{L}'$, w' AND $\vec{U}'$. HAMILTONIAN FORMALISM CAN BE USED TO FIND THE RELATION BETWEEN THESE QUANTITIES. MÖHRING ACTUALLY DOES THIS.

<u>NOTE ADDED ON APRIL 12, 2000</u>

I WILL CONTINUE THIS STUDY LATER. IN MY TALK IN BERLIN, WILLIE MÖHRING MADE SOME IMPORTANT REMARKS ABOUT ENERGY IDENTITIES. LATER, WE EXCHANGED A FEW E-MAIL MESSAGES ABOUT THE SUBJECT THAT I HAVE SAVED ON MY COMPUTER. MARK FARRIS WILL BE WORKING ON THIS PROBLEM SUMMER OF 2000 AS AN ASEE FELLOW.

CIRCULAR MICROPHONE ARRAY – CHECK OF THEORETICAL RESULTS

LORENZO AND CARL HAVE REPEATED THE CIRCULAR MICROPHONE ARRAY EXPERIMENT PAYING MUCH ATTENTION TO PHASE MEASUREMENTS. LARRY BECKER AND I HAVE CHECKED PHASE VALUES FROM 3 REFERENCE MICRO-PHONES FOR SEVERAL MICROPHONES ON THE HOOP. FOR BPF AND 2BPF, THE AGREEMENT WAS GOOD TO VERY GOOD. FOR 3BPF, THERE WERE SOME LARGE DISAGREEMENTS (5°–9°). THIS GIVES US SOME REASON TO SUSPECT THE MODE DETECTION FROM 3BPF MODES. WE WILL KNOW MORE AFTER RUNNING THE MODE DETECTION CODE.

DERIVATION OF THE THEORETICAL RESULTS

THE ACOUSTIC PRESSURE DESCRIPTION IN THE DUCT IS GIVEN AS

$$p'(r,\theta,x,t) = e^{i[\alpha B(\Omega t-\theta)]} \sum_n \sum_q A_{qn} J_m[K_r(m,n)r] \times e^{-i[qV\theta + K_a(m,n)x]} \checkmark$$

$$u_{(m,n)} = \frac{p' K_a(m,n)}{\alpha B \Omega \rho_0} \checkmark$$

$$u(r,\theta,0,t) = \frac{1}{\alpha B \Omega \rho_0} e^{i[\alpha B(\Omega t-\theta)]} \times \sum_n \sum_q A_{qn} K_a(m,n) J_m[K_r(m,n)r] e^{-iq\ldots}$$

$m = \alpha B + qV$, $q = 0, \pm 1, \pm 2, \ldots$

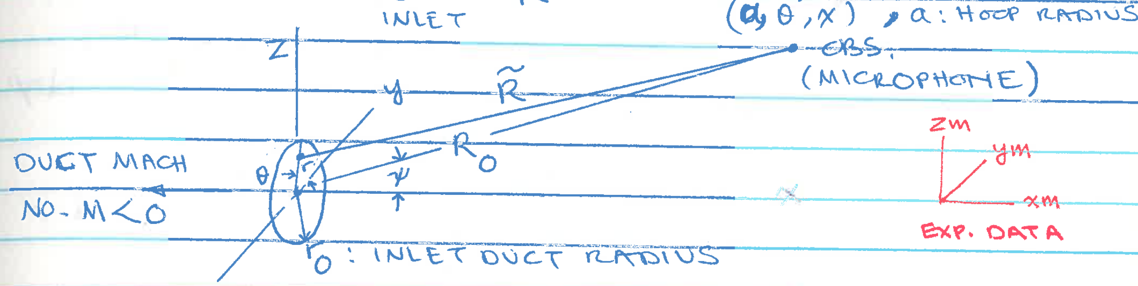
V NO. OF PERIODIC CIRCUMFERENTIAL DISTURBANCES

NOTE THAT GIVEN α, q SPECIFIES m, i.e. $m = m(q)$.

RAYLEIGH'S PISTON IN THE WALL FORMULA:

$$4\pi p'(\vec{x},t) = \int_{\text{INLET}} \frac{\rho_0[\dot{u}]_{ret}}{\tilde{R}}\, dS$$

(a, θ', x), a : HOOP RADIUS



$$\tilde{R} = R_0 - r\sin\psi\cos(\theta-\theta')$$

$$\sin\psi = \frac{a}{R_0}, \quad k = \alpha B\Omega/c$$

$$\dot{u} = \frac{i}{\rho_0} e^{i[\alpha B(\Omega t-\theta)]} \sum_n \sum_q A_{qn} k_a J_m(k_r r)\, e^{-iqv\theta}$$

$$\frac{1}{\tilde{R}} = \frac{1}{R_0}\left[1 + \frac{r}{R_0}\sin\psi\cos(\theta-\theta') + \dots\right]$$

$$= \frac{1}{R_0} + O\left(\frac{1}{R_0^2}\right)$$

$$[\dot{u}]_{ret} = \frac{i}{\rho_0} e^{i\alpha B\left[\Omega\left(t-\frac{R_0}{c}\right)-\theta'\right]}$$

$$\times \sum_n \sum_q e^{-iqv\theta'} A_{qn} k_a J_m(k_r r)\, e^{i[kr\sin\psi\cos(\theta-\theta') - \overbrace{(\alpha B+qv)}^{m}(\theta-\theta')]}$$

$$= \frac{i}{\rho_0} e^{i\alpha B\left[\Omega\left(t-\frac{R_0}{c}\right)-\theta'\right]}$$

$$\times \sum_n \sum_q e^{-iqv\theta'} A_{qn} k_a J_m(k_r r)\, e^{i[kr\sin\psi\cos(\theta-\theta') - m(\theta-\theta')]}$$

$$\int_0^{2\pi} e^{i[kr\sin\psi\cos(\theta-\theta') - m(\theta-\theta')]}\, d\theta = 2\pi e^{\frac{im\pi}{2}} J_m(kr\sin\psi)$$

(SEE NEXT PAGE)

THIS IS OBTAINED FROM THE FOLLOWING RELATION

$$J_m(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\varphi - m\varphi)}\, d\varphi$$

LET $\varphi = \psi + \frac{\pi}{2} \Rightarrow \sin\varphi = \cos\psi$

$$J_m(x) = \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} e^{i(x\cos\psi - m\psi - \frac{m\pi}{2})}\, d\psi$$

$$= \frac{1}{2\pi}\, e^{i\frac{m\pi}{2}} \int_0^{2\pi} e^{i(x\cos\psi - m\psi)}\, d\psi$$

$$\therefore \int_0^{2\pi} e^{i(x\cos\psi - m\psi)}\, d\psi = 2\pi\, e^{\frac{i m\pi}{2}} J_m(x) \checkmark$$

I CHECKED THIS NUMERICALLY USING MATHEMATICA!
(THIS ψ IS NOT THE SAME AS ψ IN PREVIOUS PAGE!)
WE HAVE

$$4\pi p'(\vec{x},t) = \frac{2\pi i \cancel{p_0}}{\cancel{p_0} R_0}\, e^{i\alpha B[\Omega(t-\frac{R_0}{c}) - \theta']} \checkmark$$

$$\times \sum_n \sum_q A_{qn} k_a\, e^{-iqV\theta' + i\frac{m\pi}{2}} \int_0^{r_0} r J_m(kr\sin\psi)\, J_m(k_r r)\, dr$$

$$p'(\vec{x},t) = \frac{i}{2R_0}\, e^{i\alpha B[\Omega(t-\frac{R_0}{c}) - (\theta' - \pi/2)]} \checkmark$$

$$\times \sum_n \sum_q A_{qn} k_a^{(m,n)} e^{-iqV(\theta' - \pi/2)}\, C(m,n,\psi)$$

$$C(m,n,\psi) = \int_0^{r_0} r J_m\!\left(\frac{\alpha B\Omega r}{c}\sin\psi\right) J_m[k_r(m,n) r]\, dr$$

$$m = \alpha B + qV \Rightarrow m = m(q)$$

LET $p'(\vec{x}, f)$ BE F.T. IN TIME OF $p'(\vec{x},t)$, WHERE $f = \alpha B \Omega$, Ω: ANG. VEL. OF THE SHAFT

$$\rightarrow \quad \varphi'(\vec{x}, \alpha B \Omega) = \frac{i}{2R_0} e^{-i\alpha B\left[\frac{\Omega R_0}{c} + \theta' - \pi/2\right]}$$

$$\sum_n \sum_q A_{qn} K_a(m,n) e^{-iqV(\theta' - \pi/2)} C(m,n,\psi)$$

LET US NOW DEFINE

$$P(\vec{x}, \alpha B \Omega, \theta') = -2iR_0 \, e^{i\alpha B\left(\frac{\Omega R_0}{c} + \theta' - \pi/2\right)} \varphi'(\vec{x}, \alpha B \Omega)$$

$$= \sum_n \sum_q A_{qn} K_a(m,n) e^{-iqV(\theta' - \pi/2)} C(m,n,\psi)$$

IF WE NOW TAKE F.T. OF P IN θ' AS FOLLOWS

$$\hat{P}_q = \frac{1}{\pi} \int_0^{\pi} P \, e^{iqV\theta'} d\theta' \qquad \left\{ \begin{array}{l} \text{NOTE SIGN OF} \\ \text{EXPONENTIAL HERE!} \end{array} \right.$$

$$= \sum_n C[m(q), n, \psi] K_a[m(q), n] e^{iqV\pi/2} A_{qn}$$

$$= D_q(R_0, \psi) e^{iqV\pi/2}$$

NOTE THAT WE HAVE MICROPHONES TRAVERSING π RADIANS!

BY TAKING MEASUREMENTS AT SEVERAL AXIAL STATIONS, WE CAN FIND A_{qn}. WE SOLVE THE SET OF LINEAR EQUATIONS

$$D_q(R_0, \psi) = e^{-iqV\pi/2} \hat{P}_q$$

FOR A_{qn}. HERE α, i.e. MULTIPLE OF BPF, IS FIXED. NOTE THAT n GIVES THE RADIAL MODES. WE USUALLY HAVE AN OVERDETERMINED SYSTEM OF LINEAR EQUATIONS. WE USE TIKHONOV REGULARIZATION TO SOLVE

FOR PROPAGATING MODES.

NOTE OF F.T. IN TIME OF MICROPHONE SIGNALS

WE DEFINE $p'(\vec{x}, f)$ AS FOLLOWS:

$$p'(\vec{x}, f) = \int_{-\infty}^{\infty} p'(\vec{x}, t)\, e^{-2\pi i f t}\, dt,$$

IN FACT, THE F.T. SOFTWARE WE USED DOES HAVE THE NEGATIVE SIGN OF THE EXPONENT OF THE EXPONENTIAL TERM. I CONSULTED LARRY BECKER ON THIS.
[LOCKHEED MARTIN INTERNAL DOC. # 98-DFOS-14, JUNE 98, J.J. KELLY & L.E. BECKER]

SUMMARY OF THE RESULTS

$$D_q(R_0, \psi) = \sum_n C[m(q), n, \psi]\, k_a[m(q), n]\, A_{qn}$$

$$= e^{-iq\nu\pi/2}\, \hat{P}_q$$

$$\hat{P}_q = \frac{1}{\pi}\int_0^{\pi} P\, e^{iq\nu\theta'}\, d\theta'$$

θ' AZIMUTHAL POSITION OF MICROPHONE

$$P(\vec{x}, \alpha B\Omega, \theta') = -2iR_0\, e^{i\alpha B\left(\frac{\Omega R_0}{c} + \theta' - \pi/2\right)}\, p'(\vec{x}, \alpha B\Omega$$

HERE $p'(\vec{x}, \alpha B\Omega)$ IS THE COMPLEX F.T. IN TIME OF $p'(\vec{x}, t)$. READ THE NOTE ABOVE ABOUT THE SIGN CONVENTION OF F.T. IN TIME.

ANALYSIS OF THE RESULTS

IN THIS METHOD, WE REQUIRE THE KNOWLEDGE OF:

- V : THE PERIODIC CIRCUMFERENTIAL DISTURBANCE
- M : THE FLOW MACH NUMBER IN THE INLET
- r_0 : RADIUS OF THE DUCT (m)
- $k = \alpha B \Omega / c$
  - B : NO. OF BLADES IN ROTOR
  - Ω : SHAFT ANGULAR FREQUENCY
  - c : SPEED OF SOUND m/s
  - α : MULTIPLE OF BPF
- $m = \alpha B + q V$, $q = 0, \pm 1, \pm 2, \ldots$
- $k_r(m,n)$ IS FOUND FROM THE SOLUTION OF

$$J'_m [k_r(m,n) r_0] = 0 \quad \left\{ \begin{array}{l} \text{INDEPENDENT OF} \\ \text{MACH NO. } M \end{array} \right.$$

THE PROPAGATING WAVES SATISFY

$$\underline{\text{CUT-OFF RATIO}} \; \beta(m,n) = \frac{k}{\beta k_r(m,n)} > 1 \quad \left\{ \begin{array}{l} \text{DEPENDENT ON} \\ M \text{ THRU } \beta \end{array} \right.$$

$$\beta = \sqrt{1 - M^2}$$

- THE AXIAL WAVE NUMBER $k_a(m,n)$ IS FOUND FROM

$$k_a(m,n) = \frac{k}{\beta^2} \left[-M \pm \sqrt{1 - 1/\beta^2(m,n)} \right]$$

FOR UPSTREAM RADIATION $M < 0$ AND

$$k_a(m,n) = \frac{k}{\beta^2} \left[|M| + \sqrt{1 - 1/\beta^2(m,n)} \right] > 0$$

NOTE: SINCE OUR x-AXIS POINTS OUT OF THE INLET, $\Rightarrow M < 0$ AND WE SELECT THE SIGN OF THE SQ. ROOT TO GIVE US THE WAVE NUMBER WITH THE LARGER

ABSOLUTE VALUE. IT IS POSITIVE BECAUSE OF THE DIRECTION OF PROPAGATION IS IN POSITIVE x DIREC- TION. IN EQ. FOR p', WE HAVE THE COMBINATION OF $(\alpha B \Omega t' - k_\alpha x) \Rightarrow \dot{x} = \alpha B \Omega / k_\alpha > 0$, AS IT SHOULD BE.

DERIVATION OF ENERGY EQUATION FOR UNIFORM FLOW

UNIFORM BACKGROUND FLOW u_0, PERTURBATION VELOCITY u'

MOMENTUM EQ.:

$$\rho_0 \frac{\partial u'}{\partial t} + \rho_0 \vec{u}_0 \cdot \nabla \vec{u}' + \nabla p' = 0$$

$$\rho_0 \vec{u}' \cdot \frac{\partial \vec{u}'}{\partial t} + \rho_0 (\vec{u}_0 \cdot \nabla \vec{u}') \cdot \vec{u}' + \vec{u}' \cdot \nabla p' = 0$$

$$\frac{\partial}{\partial t}\left[\frac{\rho_0 u'^2}{2}\right] + \vec{u}_0 \cdot \nabla \left(\frac{\rho u'^2}{2}\right) + \nabla \cdot (p'\vec{u}') - p' \nabla \cdot u' = 0$$

CONTINUITY EQ.:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{u}' + \vec{u}_0 \cdot \nabla \rho' = 0$$

$$\frac{\partial p'}{\partial t} + \rho_0 c^2 \nabla \cdot \vec{u}' + \vec{u}_0 \cdot \nabla p' = 0$$

$$-p' \nabla \cdot \vec{u}' = \frac{1}{\rho_0 c^2} p' \frac{\partial p'}{\partial t} + \frac{1}{\rho_0 c^2} p' \nabla \cdot (p' \vec{u}_0)$$

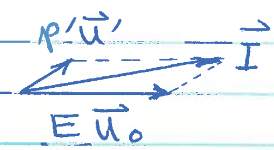
$$= \frac{\partial}{\partial t}\left(\frac{p'^2}{2\rho_0 c^2}\right) + \nabla \cdot \left(\frac{p'^2}{2\rho_0 c^2} \vec{u}_0\right)$$

SUBSTITUTE THIS RESULT IN MOM. EQ:

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{\rho_0 u'^2}{2} + \frac{p'^2}{2\rho_0 c^2}\right)}_{E} + \nabla \cdot \Big[\underbrace{p'\vec{u}' + \left(\frac{\rho u'^2}{2} + \frac{p'^2}{2\rho_0 c^2}\right)\vec{u}_0}_{\vec{I} = p'\vec{u}' + E\vec{u}_0}\Big]$$

ACOUSTIC ENERGY DENSITY — ACOUSTIC ENERGY INTENSITY VECTOR

$$\frac{\partial E}{\partial t} + \nabla \cdot \vec{I} = 0$$



IT IS INTERESTING TO NOTE THAT WE HAVE USED ONLY THE CONDITION $\nabla \cdot \vec{u}_0 = 0$ AND NOT THE UNIFORMITY OF BACKGROUND VELOCITY $\vec{u}_0$. THIS MEANS THAT THIS EQUATION IS VALID FOR NONUNIFORM, LOW SUBSONIC SPEED FLOWS.

14 Sound Propagation in a Duct and Interaction Tones

SOUND PROPAGATION IN A DUCT
AND INTERACTION TONES

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1

SMALL PERTURBATION EQUATIONS FOR ACOUSTIC WAVES IN FLUIDS WITH MEAN FLOW

NOTATION : MEAN (BACKGROUND) QUANTITIES DENOTED BY SUBSCRIPT O., PERTURBATION VALUES BY A PRIME. ALL QUANTITIES DIMENSIONAL

$\rho = \rho_0 + \rho', \quad p = p_0 + p', \quad \vec{V} = \vec{V}_0 + \vec{V}'$

NOTE : $\rho_0 = \rho_0(\vec{x}, t), \; p_0 = p_0(\vec{x}, t), \; \vec{V}_0 = \vec{V}_0(\vec{x}, t)$

MASS CONTINUITY EQUATIONS

$$\begin{cases} \dfrac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \vec{V}_0) = 0 & \text{MEAN} \quad (1) \\ \dfrac{\partial \rho'}{\partial t} + \nabla \cdot (\rho' \vec{V}_0 + \rho_0 \vec{V}') = 0 & \text{ACOUSTIC} \quad (2) \end{cases}$$

MOMENTUM EQUATIONS

$$\begin{cases} \dfrac{\partial \vec{V}_0}{\partial t} + \vec{V}_0 \cdot \nabla \vec{V}_0 = - \dfrac{\nabla p_0}{\rho_0} & \text{MEAN} \quad (3) \\ \dfrac{\partial \vec{V}'}{\partial t} + \vec{V}_0 \cdot \nabla \vec{V}' + \vec{V}' \cdot \nabla \vec{V}_0 = - \dfrac{\nabla p'}{\rho_0} + \dfrac{\rho' \nabla p_0}{\rho_0^2} & \text{ACOUSTIC} \quad (4) \end{cases}$$

2

EQUATION OF STATE $\quad p = A \rho^{\gamma}, \quad A = p_r \rho_r^{-\gamma}$

WHERE p_r AND ρ_r ARE REFERENCE CONDITIONS

$$\begin{cases} p_0 = A \rho_0^{\gamma} & \text{MEAN} \quad (5) \\ p' = c_0^2 \rho' & \text{ACOUSTIC} \quad \text{WHERE } c_0^2 = \gamma R T_0 = \dfrac{\gamma p_0}{\rho_0} \quad (6) \end{cases}$$

ENERGY EQUATIONS (ISENTROPIC) $\quad dp - c^2 d\rho = 0$

$$\frac{Dp}{Dt} - \frac{\gamma p}{\rho} \frac{D\rho}{Dt} = 0 \Rightarrow \frac{Dp}{Dt} + \gamma p \, \nabla \cdot \vec{V} = 0 \qquad (7)$$

$$\begin{cases} \dfrac{\partial p_0}{\partial t} + \vec{V}_0 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \vec{V}_0 = 0 & \text{MEAN} \quad (8) \\ \dfrac{\partial p'}{\partial t} + \vec{V}_0 \cdot \nabla p' + \vec{V}' \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \vec{V}' + \gamma p' \nabla \cdot \vec{V}_0 = 0 & (9) \; \text{ACOUSTIC} \end{cases}$$

THESE EQUATIONS SIMPLIFY IF WE HAVE IRROTATIONAL FLOW. THERE ARE SITUATIONS OF INTEREST IN DUCTED FANS WHERE WE DO NOT HAVE IRROTIONAL FLOW.

<u>IRROTATIONAL FLOW</u> $\quad \nabla \times \vec{V} = 0$

$$\begin{cases} \nabla \times \vec{V}_0 = 0 & \text{MEAN} \qquad (10) \\ \nabla \times \vec{V}' = 0 & \text{ACOUSTIC} \qquad (11) \end{cases}$$

$\phi = \phi_0 + \phi'$ VELOCITY POTENTIAL

IN TERMS OF ϕ, THE MOMENTUM EQUATION BECOMES

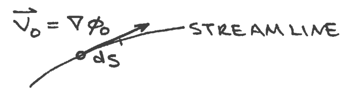
$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + \frac{1}{\gamma - 1} c^2 = \frac{1}{2} V_\infty^2 + \frac{1}{\gamma-1} c_\infty^2 \qquad (12)$$

ASSUMING THAT V_∞ AND C_∞ ARE TIME INDEPENDENT

$$\begin{cases} \dfrac{\partial \phi_0}{\partial t} + \dfrac{1}{2}|\nabla \phi_0|^2 + \dfrac{1}{\gamma-1} c_0^2 = \dfrac{1}{2} V_\infty^2 + \dfrac{1}{\gamma-1} c_\infty^2 & \text{MEAN} \quad (13) \\ \dfrac{\partial \phi'}{\partial t} + \nabla \phi_0 \cdot \nabla \phi' + c_0^2 \dfrac{\rho'}{\rho_0} = 0 & \text{ACOUSTIC} \quad (14) \end{cases}$$

— FROM $p' = c_0^2 \rho'$ AND THE ABOVE EQUATION

$$p' = -\rho_0 \left[\frac{\partial \phi'}{\partial t} + \nabla \phi_0 \cdot \nabla \phi' \right] = -\rho_0 \left[\frac{\partial \phi'}{\partial t} + V_0 \frac{\partial \phi'}{\partial s} \right] \qquad (15)$$



IF THE MEAN FLOW IS TIME INDEPENDENT, $\dfrac{\partial \phi_0}{\partial t} = 0$

$$c_0^2 = \frac{\gamma - 1}{2} \left[V_\infty^2 - |\nabla \phi_0|^2 \right] + c_\infty^2 \qquad (16)$$

NOTE THAT THE MOMENTUM EQUATION IS WRITTEN IN AN INERTIAL FRAME. A ROTATING FRAME IS NOT AN INERTIAL FRAME.

<u>THE GOVERNING EQUATION FOR ϕ'</u>

FROM THE ACOUSTIC MOMENTUM EQUATION, WE GET

$$\rho' = -\frac{\rho_0}{c_0^2} \left[\frac{\partial \phi'}{\partial t} + \nabla \phi_0 \cdot \nabla \phi' \right] \equiv -\frac{\rho_0}{c_0^2} L \phi' \qquad (17)$$

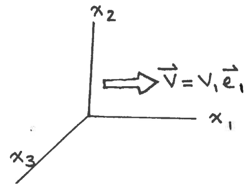
NOW SUBSTITUTE FOR ρ' IN THE ACOUSTIC MASS CONTINUITY EQUATION (2), TO GET THE DESIRED EQUATION FOR ϕ':

$$-\frac{\partial}{\partial t} \left[\frac{\rho_0}{c_0^2} L \phi' \right] - \nabla \cdot \left[\left(\frac{\rho_0}{c_0^2} L \phi' \right) \nabla \phi_0 \right] + \rho_0 \nabla^2 \phi'$$
$$+ \nabla \rho_0 \cdot \nabla \phi' = 0 \qquad (18)$$

A SECOND ORDER LINEAR EQUATION, BUT A COMPLICATED EQUATION!

5

TIME INDEPENDENT UNIFORM MEAN FLOW : $\nabla \phi_o = V_1 \vec{e}_1$



$$p' = -\frac{\rho_o}{C_o^2}\left[\frac{\partial \phi'}{\partial t} + V_1 \frac{\partial \phi'}{\partial x_1}\right] \qquad (19)$$

ρ_o AND C_o CONSTANT

THE EQUATION FOR ϕ' IS

$$\frac{1}{C_o^2}\left(\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x_1}\right)^2 \phi' - \nabla^2 \phi' = 0 \qquad (20)$$

NOTE THAT THE LINEAR OPERATOR L IN EQ.(17) IS

$$L = \frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x_1} \qquad (21)$$

THE ACOUSTIC PRESSURE p' ALSO SATISFIES AN EQUATION SIMILAR TO EQ. (20) :

$$\frac{1}{C_o^2}\left(\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial x_1}\right)^2 p' - \nabla^2 p' = 0 \qquad (22)$$

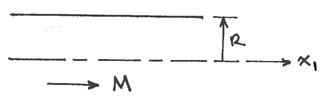
OR

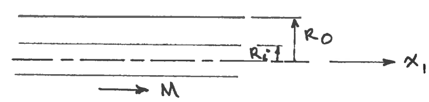
$$\left(\frac{1}{C_o}\frac{\partial}{\partial t} + M \frac{\partial}{\partial x_1}\right)^2 p' - \nabla^2 p' = 0 \qquad (23)$$

WHERE $M = V_1 / C_o$ IS THE MACH NUMBER OF THE MEAN FLOW.

6

ACOUSTICS OF A CYLINDRICAL DUCT WITH UNIFORM FLOW





CIRCULAR DUCT

BC : $\left.\frac{\partial p'}{\partial r}\right|_{r=R} = 0$

ANNULAR DUCT

BC : $\left.\frac{\partial p'}{\partial r}\right|_{r=R_i} = \left.\frac{\partial p'}{\partial r}\right|_{r=R_o} = 0$

GOVERNING EQ. : $\left(\frac{1}{C_o}\frac{\partial}{\partial t} + M \frac{\partial}{\partial x_1}\right)^2 p' - \nabla^2 p' = 0 \qquad (23)$

ASSUME A SOLUTION OF THE FORM

$$p' = P(x_1, r, \theta)\, e^{i\omega t} \qquad (24)$$

THE EQUATION FOR P IS

$$(1 - M^2)\frac{\partial^2 P}{\partial x_1^2} + \nabla_c^2 P - 2iMk \frac{\partial P}{\partial x_1} + k^2 P = 0 \qquad (25)$$

$$\nabla_c^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad (26)$$

$k = \frac{\omega}{C_o}$ WAVE NUMBER

7

TO SOLVE EQ. (25), WE USE SEPARATION OF VARIABLE TECHNIQUE. WE CONFINE OURSELVES TO CIRCULAR DUCT. THE EIGENFUNCTIONS, INCLUDING THE TIME DEPENDENCE ARE OF THE FOLLOWING FORM:

$$\begin{cases} p'_{mn} = A_{mn} J_m(k_{r,mn} r) \exp i(\omega t - m\theta - k_{a,mn} x_1) \\ J_m(\cdot) \text{ BESSEL FN OF FIRST KIND \& ORDER } m \end{cases} \tag{27}$$

WHERE $k_{r,mn}$ AND $k_{a,mn}$ ARE THE RADIAL AND AXIAL WAVE NUMBERS. WHEN m IS A GIVEN POSITIVE INTEGER, $k_{r,mn}$ FOR $n = 1, 2, \ldots$ ARE OBTAINED FROM

$$J'_m(k_{r,mn} R) = 0, \tag{28}$$

i.e., THEY ARE $\frac{1}{R} \times$ ZEROS OF $J'_m(x)$. THIS RESULT FOLLOWS FROM THE APPLICATION OF THE BC: $\left.\frac{\partial p'}{\partial r}\right|_{r=R} = 0$ TO p'_{mn}, EQ. (27).

THE RELATION BETWEEN $k_{a,mn}$ AND $k_{r,mn}$ IS GIVEN BY

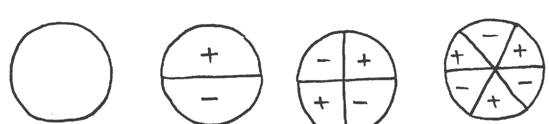
$$k_{a,mn} = \frac{k}{\beta^2}\left[-M \pm \sqrt{1 - (\beta k_{r,mn}/k)^2}\,\right] \tag{29}$$

8

WHERE $\beta^2 = 1 - M^2$. WE WILL GIVE AN INTERESTING GRAPHICAL INTERPRETATION OF THIS RELATION LATER. FIRST LET US NOTICE A FEW THINGS FROM EQ. (27):

– i) AT A FIXED x_1, THE AXIAL DISTANCE, $p'_{mn} \propto e^{i(\omega t - m\theta)}$, i.e. THE PATTERN IS ROTATING WITH ANGULAR VELOCITY $\dot{\theta} = \frac{\omega}{m}$. $\therefore$ IF $\omega > 0$ (AS WE ALWAYS ASSUME), SIGN $(\dot{\theta})$ = SIGN(m).

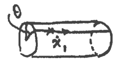
– ii) FOR A FIXED (x_1, t), $p'_m \propto A(r)\, e^{-im\theta}$, i.e. m IS THE <u>CIRCUMFERENTIAL MODE</u>. WE DRAW THE ANTINODES (WRT θ ONLY!) BELOW FOR SOME m



, ETC!

NOTE CAREFULLY THAT THE ORDER OF THE BESSEL FUNCTION IN p'_{mn} IS PRECISELY m.

– iii) ASSUMING $k_{a,mn}$ REAL AND NONZERO, FOR A FIXED r AND θ, $p'_{mn} \propto e^{i(\omega t - k_{a,mn} x_1)}$. THIS MEANS THAT THE POINTS OF CONSTANT PHASE TRAVEL AXIALLY WITH SPEED $\dot{x}_1 = \frac{\omega}{k_{a,mn}}$ AND $\text{SIGN}(\dot{x}_1) = \text{SIGN}(k_{a,mn})$.

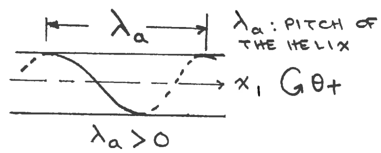


THUS IF $k_{a,mn} > 0$, THE POINTS OF CONSTANT PHASE MOVE WITH THE SPEED $\dot{x}_1$ ALONG POSITIVE x_1-AXIS AND NEGATIVE x_1-AXIS IF $k_{a,mn} < 0$. WE WILL DISCUSS THE CONDITION FOR WHICH k_{mn} IS REAL. IN THIS CASE WE SAY THAT THE MODE (m,n) IS <u>PROPAGATING</u>.

– iv) FOR (r,t) FIXED AND $k_{a,mn}$ REAL, THE CURVES OF CONSTANT PHASE ON THE CYLINDER $r = a$ ARE HELICES $m\theta + k_{a,mn} x_1 = \text{CONST}.$

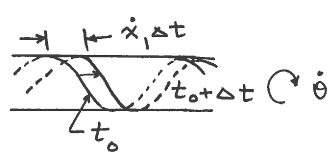
$$m\underbrace{\Delta\theta}_{2\pi} + k_{a,mn}\underbrace{\Delta x}_{\lambda_a} = 0$$

$$\lambda_a = -\frac{2\pi m}{k_{a,mn}}$$



WHAT IS THE MEANING OF THE NEGATIVE SIGN? ASSUMING $m > 0$ AND $k_{a,mn} > 0$, IF WE TRACE THE HELIX OF CONSTANT PHASE, FOR INCREASING θ, WE MOVE TOWARD THE NEGATIVE AXIS. IN THE FIGURE ON P9, WE HAVE SHOWN THE CASE OF $\lambda_a > 0$ SO THAT $\text{SIGN}(m) \cdot \text{SIGN}(k_{a,mn}) < 0$.

V) COMBINING THE ABOVE PICTURE OF THE CURVE ON CONSTANT PHASE WITH THE RESULT OF (i), P8, WE SEE THAT FOR $k_{a,mn}$ REAL, i.e. <u>PROPAGATING WAVE</u>, THE HELICES OF CONSTANT PHASE ROTATE AT ANGULAR VELOCITY $\dot{\theta} = \frac{\omega}{m}$ THUS APPEAR AS BARBER POLE AS SHOWN BELOW:



– NOTE THAT UPTO HERE ω, m AND n ARE ARBITRARY WE WILL LATER USE PERIODICITY OF BLADES AND VANES TO RESTRICT EIGENFUNCTIONS.

VI) NOW LET $k_{a,mn}$ BE COMPLEX, i.e. LET

$$k_{a,mn} = \kappa_{a,mn} + i\,\mu_{a,mn}$$

THEN $\quad p_{mn} \propto e^{-\mu_{a,mn} x_1}$, i.e. DECAYING IF $\mu_{a,mn} > 0$ AND x_1 IS INCREASING. WE HAVE A DECAYING MODE.

THE CONCEPT OF MODE CUT-OFF

TO HAVE A PROPAGATING MODE, WE NEED REAL $k_{a,mn}$. LET US REITERATE WHAT THE PROCEDURE FOR FINDING $k_{a,mn}$ IS:

$k = \frac{\omega}{c_0} \Rightarrow$ DECIDE ABOUT CIRCUMFERENTIAL MODE $m (=0,1,2,\dots)$

$\Rightarrow$ SOLVE $J_m'(x) = 0$, GET $\alpha_1, \alpha_2, \alpha_3 \dots$ IN INCREASING ORDER,

THEN $k_{r,mn} = \frac{\alpha_n}{R} \Rightarrow$ PLUG IN EQ.(29) TO SEE IF

$k_{a,mn}$ IS REAL OR COMPLEX, i.e. MODE (m,n) PROPAGATING OR DECAYING

THIS PROCEDURE WILL EVENTUALLY COME TO AN END SINCE $k_{r,mn}$ IS AN INCREASING SEQUENCE OF POSITIVE NUMBERS

THIS ALREADY SHOWS THAT FOR ANY GIVEN CIRCUMFERENTIAL MODE m, THERE IS A RADIAL MODE n SUCH THAT FOR ALL RADIAL MODES GREATER THAN n, $k_{a,mn}$ IS COMPLEX AND THEREFORE DECAYING, i.e. NONPROPAGATING.

LET US DEFINE THE CUT-OFF RATIO β_{mn}

$$\beta_{mn} = \frac{k}{\beta\, k_{r,mn}} \qquad (30) \qquad (*)$$

FROM EQ.(29)

IF $\beta_{mn} < 1$, MODE (m,n) DECAYING

IF $\beta_{mn} > 1$, MODE (m,n) PROPAGATING

LET US NOTICE A FEW IMPORTANT FACTS ABOUT THE CUT-OFF RATIO:

– i) SINCE k AND $k_{r,mn}$ ARE INDEPENDENT OF MACH NUMBER OF THE FLOW, THE DEPENDENCE OF β_{mn} ON

(*). EQ.(29) FOR $k_{a,mn}$ CAN BE WRITTEN IN TERMS OF β_{mn}, SEE EQ.(33), P 19

M IS THROUGH $\beta = \sqrt{1-M^2}$.

– ii) SINCE $|M| < 1$ ALWAYS IN OUR ANALYSIS, $\beta \downarrow$ AS $M \uparrow$. THEREFORE $\beta_{mn} \uparrow$ AS $M \uparrow$, i.e. IF THE MODE (m,n) IS PROPAGATING FOR FLOW MACH NUMBER M_1, IT WILL SURELY PROPAGATE FOR $M > M_1$. SIMILARLY, A NONPROPAGATING MODE MAY PROPAGATE IF M IS INCREASED. <u>ALL</u> MODES PROPAGATE AT $M = 1$!

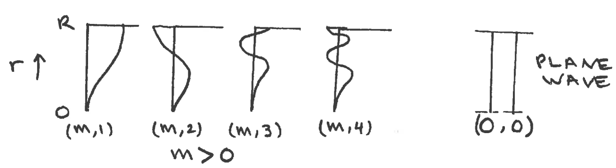
– iii) $k_{r,mn}$ IS DEPENDENT ON DUCT GEOMETRY ONLY.

BEFORE WE GIVE THE GRAPHICAL CONSTRUCTION OF EQ.(29), LET US SAY SOMETHING ABOUT THE ZEROES OF BESSEL FUNCTIONS. THERE IS NO SIMPLE FORMULA FOR SOLUTIONS OF $J'_m(x) = 0$. THE FOLLOWING ARE SOME USEFUL FACTS TO REMEMBER:

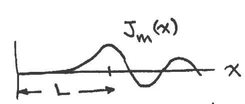
i) FOR $m = 0, 1, 2, \ldots$, $J'_m(x) = 0$ HAS AN INFINITE NUMBER OF SOLUTIONS ON THE POSITIVE REAL AXIS.

ii) FOR LARGE x, THE ZEROES ARE SPACED π APART.

iii) ALL p'_{mn}, $m > 0$ ARE ZERO ON THE AXIS. THE RADIAL DISTRIBUTION OF p'_{mn} IS EASY TO CONSTRUCT.



iv) I FIND THAT MATHEMATICA IS THE EASIEST SOFTWARE FOR GETTING THE ZEROES OF $J'_m(x)$. THERE IS ALSO A PACKAGE FOR GETTING THE ZEROES OF BESSEL FUNCTIONS IN MATHEMATICA.



v) HERE IS A GOOD RULE OF THUMB. FOR $m > 1$, THE DISTANCE L WHERE THE FIRST PEAK OF $J_m(x)$ APPEARS, OR THE FIRST ZERO $J'_m(x)$ APPEARS, IS APPROXIMATELY m, i.e. THE THE ORDER OF THE BESSEL FUNCTION.

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MODE CUT-ON AND CUT-OFF

WE KNOW THAT $k_{r,mn}$'S ARE GEOMETRIC QUANTITIES. WE CAN THINK ABOUT THE CUT-OFF PHENOMENON IN TWO WAYS:

i) GIVEN k AND A CIRCUMFERENTIAL MODE m, WE KNOW THAT $k_{r,m1} < k_{r,m2} < k_{r,m3} \cdots$. THEREFORE, THERE IS AN n SUCH THAT $\beta_{mn} = \dfrac{k}{\beta k_{r,mn}} > 1$ BUT $\beta_{m,n+1} = \dfrac{k}{\beta k_{r,m,n+1}} < 1$

WE SAY MODES $(m,1)$, $(m,2)$ .... , (m,n) ARE <u>CUT ON</u> WHILE MODE $(m,n+1)$, $(m,n+2)$ .... ARE ALL <u>CUT OFF</u>.

ii) GIVEN A MODE (m,n), WE ASK FOR THE FREQUENCY f_{mn} SUCH THAT THIS MODE IS CUT ON <u>ABOVE</u> f_{mn}, i.e.

$$\frac{2\pi(f_{mn}/c_0)}{\beta\, k_{r,mn}} = \beta_{mn} = 1 \quad \therefore \quad f_{mn} = \frac{\beta c_0 k_{r,mn}}{2\pi} \qquad (31)$$

THIS CALLED THE <u>CUT-OFF FREQUENCY</u> OF MODE (m,n).

— NOTE FOR A PLANE WAVE, i.e. MODE $(0,0)$, $\beta_{00} = \infty > 1$, i.e. THIS MODE ALWAYS PROPAGATES.

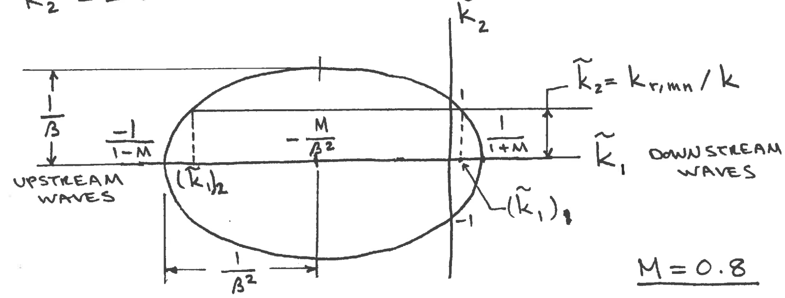
16

A GRAPHICAL CONSTRUCTION

LET US WRITE $\tilde{k}_1 = k_{a,mn}/k$ AND $\tilde{k}_2 = k_{r,mn}/k$. THEN EQ. (29) CAN BE WRITTEN AS

$$\frac{(\tilde{k}_1 + M/\beta^2)^2}{1/\beta^4} + \frac{\tilde{k}_2^{\,2}}{1/\beta^2} = 1 \qquad (32)$$

THIS IS THE EQUATION OF AN ELLIPSE WITH CENTER AT $\tilde{k}_1 = -\dfrac{M}{\beta^2}$, SEMIMAJOR AXIS $= \dfrac{1}{\beta^2}$, SEMIMINOR AXIS $= \dfrac{1}{\beta}$, CROSSING $\tilde{k}_1$-AXIS AT $\tilde{k}_1 = \dfrac{-1}{1-M}$ AND $\tilde{k}_1 = \dfrac{1}{1+M}$ AND $\tilde{k}_2$-AXIS AT $\tilde{k}_2 = \pm 1$ AS SHOWN BELOW.



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HERE ARE A FEW FACTS ABOUT THIS ELLIPSE:

i) FOR EACH MACH NUMBER, THERE IS A UNIQUE ELLIPSE IN $\tilde{k}_1\tilde{k}_2$-PLANE. EXACTLY THE SAME ELLIPSE CAN BE USED FOR STUDYING WAVE PROPAGATION IN 2D, 3D RECTANGULAR AND ANNULAR DUCT.

ii) THE ASPECT RATIO OF THE ELLIPSE IS $\frac{1}{\beta}$, i.e. THE ELLIPSE LOOKS MORE ELONGATED AS M INCREASES. ALSO BOTH SEMIMAJOR AND SEMIMINOR AXES INCREASE AS M INCREASES AND THE CENTER MOVES FARTHER FROM THE ORIGIN ALONG NEGATIVE $\tilde{k}_1$-AXIS.

iii) THE <u>REAL</u> SOLUTIONS FOR $k_{a,mn}$ FROM EQ.(29) ARE OBTAINED BY DRAWING A HORIZONTAL LINE AT $\tilde{k}_2 = k_{r,mn}/k$ AS SHOWN IN THE FIGURE ON P16. IF THIS LINE INTERSECTS THE ELLIPSE AT $(\tilde{k}_1)_1$ AND $(\tilde{k}_1)_2$ THEN THE CORRESPONDING $k_{a,mn}$ ARE $k(\tilde{k}_1)_1$ AND $k(\tilde{k}_1)_2$. IN THIS CASE MODE (m,n) IS PROPAGATING. NOTE THAT THE ELLIPSE ALWAYS INTERSECTS $\tilde{k}_2$-AXIS AT -1 AND 1. IF $\tilde{k}_2 > 1$, WE HAVE $(\tilde{k}_1)_1 < 0$ AND $(\tilde{k}_1)_2 < 0$.

iv) FROM THIS FIGURE, IF $\tilde{k}_2 = \frac{k_{r,mn}}{k} > \frac{1}{\beta}$ (SEMIMINOR AXIS),

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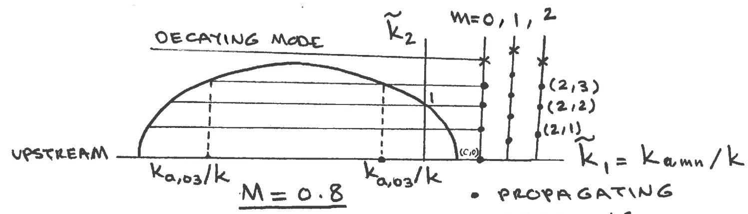
THEN THE HORIZONTAL LINE DOES NOT INTERSECT THE ELLIPSE, i.e. WE HAVE NO REAL SOLUTIONS OF EQ.(29) AND THEREFORE THE MODE IS DECAYING. BUT THE CONDITION $\frac{k_{r,mn}}{k} > \frac{1}{\beta}$ IS EXACTLY $\beta_{mn} = \frac{k}{\beta k_{r,mn}} < 1$, i.e. IN THIS CASE THE CUT-OFF RATIO IS LESS THAN ONE.

v) FROM THE FACT THAT THE SEMIMINOR AXIS SIZE INCREASES AS M INCREASES, AND THE ABOVE GEOMETRIC CONSTRUCTION, WE FIND THAT IF THE MODE (m,n) PROPAGATE FOR M_1, IT WILL ALSO PROPAGATE FOR $M > M_1$. REMEMBER $k_{r,mn}$ DOES NOT DEPEND ON MACH NUMBER.

vi) HERE IS THE WAY WE GRAPHICALLY CAN GET ALL PROPAGATING MODES IN A DUCT. FOR CIRCUMFERENTIAL MODE $m = 0, 1, 2, \ldots$, MARK OFF $\tilde{k}_2 = k_{r,mn}/k$ ON A VERTICAL AXIS, DRAW HORIZONTAL LINES THAT CROSS THE ELLIPSE. IT IS EASIER IF WE DRAW A SEPARATE VERTICAL LINE FOR EACH CIRCUMFERENTIAL MODE. THIS IS SHOWN ON THE NEXT PAGE.

19

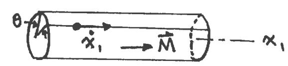


NOTE : AT M=0, THE ELLIPSE BECOMES A CIRCLE OF RADIUS 1 WITH CENTER AT THE ORIGIN.

PHASE AND GROUP VELOCITY

FROM THE RELATION PHASE $= \omega t - m\theta - k_{a,mn} x_1$

WE HAVE SHOWN THAT



FOR θ FIXED, THE AXIAL PHASE SPEED IS $\dot{x}_1 = \dfrac{\omega}{k_{a,mn}}$ $\therefore$ SIGN$(\dot{x}_1) =$ SIGN$(k_{a,mn})$.

LET US WRITE EQ. (29) AS FOLLOWS

$$k_{a,mn\pm} = \frac{k}{\beta^2}\left[-M \pm \sqrt{1 - 1/\beta_{mn}^2}\,\right] \qquad (33)$$

WE ALWAYS HAVE $|k_{a,mn+}| < |k_{a,mn-}|$ FOR $M \geqslant 0$. BUT

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AS SEEN FROM THE FIGURE ON THE TOP OF P19, ALTHOUGH $k_{a,mn-} < 0$, $k_{a,mn+}$ CAN BE BOTH POSITIVE OR NEGATIVE.

IF i) $k_{a,mn+} > 0 \Rightarrow \dot{x}_{1+} > 0$ DOWNSTREAM MOVING PHASE,

ii) $k_{a,mn+} < 0 \Rightarrow \dot{x}_{1+} < 0$ UPSTREAM MOVING PHASE,

iii) $k_{a,mn-} < 0$ (ALWAYS) $\Rightarrow \dot{x}_{1-} < 0$ UPSTREAM MOVING PHASE.

HOWEVER, IN CASE ii) SINCE $\dot{x}_{1-} < \dot{x}_{1+} < 0$, THE WAVE ASSOCIATED WITH $k_{a,mn-}$ MOVES UPSTREAM <u>FASTER</u> THAN THE WAVE ASSOCIATED WITH $k_{a,mn+}$.

ENERGY TRANSFER IS ASSOCIATED WITH THE GROUP VELOCITY. EVERSMAN HAS SHOWN THAT REGARDLESS OF THE SIGN OF $k_{a,mn+}$, THE WAVES ASSOCIATED WITH $k_{a,mn+}$ AND $k_{a,mn-}$ TRANSFER ENERGY TO DOWNSTREAM AND UPSTREAM REGIONS OF THE DUCT, RESPECTIVELY. WE WILL EXPLAIN THIS FURTHER BELOW.

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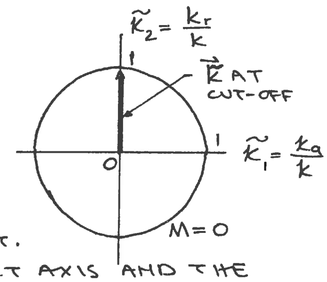
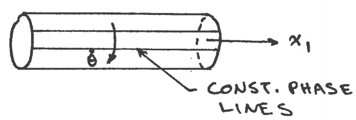
WHAT IS THE CUT-OFF PHENOMENON ?



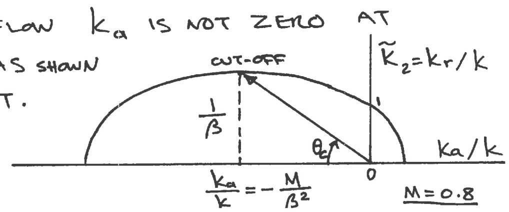
FOR A DUCT WITH NO FLOW, $\beta_{mn} = \frac{k}{k_r}$ AND AT CUT-OFF $\beta_{mn} = 1$, i.e. $k_r = k$ AND $k_a = 0$, i.e. WE HAVE NO AXIAL PROPAGATION AND HAVE A STANDING WAVE IN RADIAL DIRECTION.
THE CONSTANT PHASE LINES ON $r =$ CONST. SURFACES ARE PARALLEL TO THE DUCT AXIS AND THE WHOLE PRESSURE PATTERN ROTATES AT $\dot{\theta} = \frac{\omega}{m}$.



$$p'_{mn} = A_{mn} J_m(k_{r,mn} r)\, e^{i(\omega t - m\theta)} \qquad (34)$$

(NO x_1 DEPENDENCE !)

THE ENERGY FLUX IN x_1 DIRECTION IS ZERO.
IN THE CASE OF DUCT WITH FLOW k_a IS NOT ZERO AT CUT-OFF, IN FACT, $k_a < 0$ AS SHOWN IN THE FIGURE ON THE RIGHT.
WHAT IS SO PARTICULAR ABOUT $k_a/k = -M/\beta^2$?



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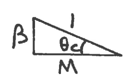
THE ACOUSTIC ENERGY TRAVELS AT GROUP VELOCITY $\vec{V}_G$

$$\vec{V}_G = \vec{V} + c\,\vec{e}_{k'} = c\,(\vec{M} + \vec{e}_k) \qquad (35)$$

WHERE $\vec{e}_k$ IS THE UNIT VECTOR ALONG WAVE NUMBER VECTOR $\vec{k}'$. HERE $\vec{V}$ AND $\vec{M}$ ARE THE FLUID VELOCITY AND MACH NUMBER VECTORS, RESPECTIVELY. ON PREVIOUS PAGE, FROM THE BOTTOM FIGURE, WE HAVE

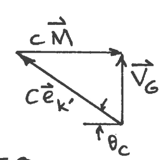
$$\tan\theta_c = \frac{1/\beta}{M/\beta^2} = \frac{\beta}{M} \quad (36)$$



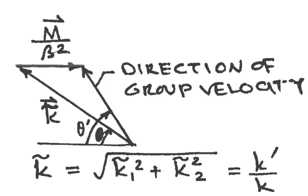
$$\boxed{\cos\theta_c = M}$$

THE COMPONENT OF $\vec{V}_G$ ALONG x_1-AXIS, i.e. THE DUCT AXIS, IS

$$\vec{V}_G \cdot \vec{e}_1 = c\,(M - \cos\theta_c) = c\,(M - M) = 0\,!$$



THIS MEANS FOR THE ENERGY FLUX VECTOR IS NORMAL TO THE WALL AND THUS CUT-OFF TO THE OUTSIDE.
TO FIND THE DIRECTION OF GROUP VELOCITY IN OUR GRAPHICAL CONSTRUCTION, FOLLOW THE DIAGRAM ON THE RIGHT. NOTE THAT $\frac{-M}{\beta^2}$ IS THE $(k_a/k)_{\text{CUT-OFF}}$.



WHY DO WE TREAT $(k_a, k_r) = \vec{k}'$ AS PROPAGATION VECTOR?

CONSIDER $p' = A \, J_m(k_r r) \exp i[\omega t - m\theta - k_a x_1]$

LET US WRITE $k_r = \frac{\alpha_n}{r_0}$ AND ASSUME α_n IS LARGE. REMEMBER α_n IS THE nTH ZERO OF $J_m'(x) = 0$. THEN

$$J_m(k_r r) = J_m\left(\frac{r}{r_0}\alpha_n\right) \approx \sqrt{\frac{2r_0}{\pi r \alpha_n}} \cos\left(\frac{\alpha_n r}{r_0} - \frac{\pi}{4} - \frac{m\pi}{2}\right) \quad (37)$$

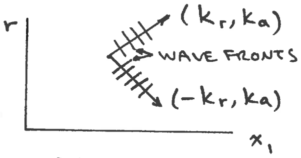
FOR r NEAR r_0 (DUCT WALL)

LET $\frac{\pi}{4} + \frac{m\pi}{2} = \psi$ (PHASE), THEN USING $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$,

WE HAVE (AFTER SUBSTITUTING $k_r = \frac{\alpha_n}{r_0}$):

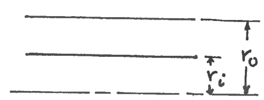
$$p' \approx \frac{A}{2}\sqrt{\frac{2}{\pi k_r r}} \left\{ \exp i[\omega t + k_r r - m\theta - k_a x - \psi] + \exp i[\omega t - k_r r - m\theta - k_a x + \psi] \right\} \quad (38)$$

NOW IF WE TAKE θ = FIXED, i.e. IN (r, x_1)-PLANE, WE HAVE TWO WAVES AS SHOWN WHICH LOOK ESSENTIALLY THE SAME AS THE PICTURE OF WAVE NUMBER VECTORS IN A 2D DUCT. BUT REMEMBER THAT OUR PICTURE IN A CIRCULAR DUCT IS <u>APPROXIMATE</u>.



<u>ANNULAR DUCT</u>

THE DUCT EIGENFUNCTIONS NOW INCLUDE BESSEL FUNCTIONS OF 2ND KIND:



$$p'_{mn} = A_{mn}[J_m(k_{r,mn} r) + Q\, Y_m(k_{r,mn})] \exp i[\omega t - m\theta - k_{a,mn} x_1]$$

k_r'S AND Q ARE FOUND FROM

$$\begin{cases} J_m'(r_0 z) + Q\, Y_m'(r_0 z) = 0 \\ J_m'(r_i z) + Q\, Y_m'(r_i z) = 0 \end{cases} \Rightarrow \boxed{J_m'(r_0 z) Y_m'(r_i z) - Y_m'(r_0 z) J_m'(r_i z) = 0} \quad (39)$$

$$Q = -\frac{J_m'(r_0 z)}{Y_m'(r_0 z)} = -\frac{J_m'(r_i z)}{Y_m'(r_i z)} \quad (40)$$

AS THE CASE OF CIRCULAR DUCT, THE BOXED EQ. ABOVE HAS AN INFINITE NUMBER OF ROOTS $z_1 < z_2 < z_3 < \cdots < z_n < \cdots$

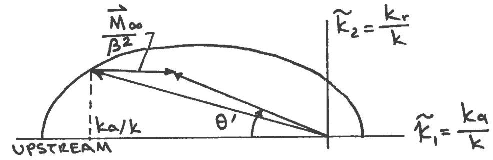
AND $k_{r,mn} = z_n$, $n = 1, 2, \ldots$. <u>NOTE</u>, USE z_n IN Q, i.e. $Q = Q(z_n)$

BECAUSE Y_m' HAS A NASTY SINGULARITY NEAR THE AXIS, THE SOLUTION OF THE ABOVE EQUATION IS SUBJECT TO ERRORS FOR SMALL r_i/r_0, i.e. SMALL HUB-TO-TIP RATIOS EVEN ON A MAIN FRAME COMPUTER. TO USE MATHEMATICA, PLOT $f(z) = J_m'(r_0 z) Y_m'(r_i z) - Y_m'(r_0 z) J_m'(r_i z)$ AND USE FindRoot[$f(z)$, {z, z_1, z_2}] TO GET THE ROOT IN THE INTERVAL (z_1, z_2).

DIRECTION OF RADIATION PEAK FOR A GIVEN MODE

IF THE TUNNEL OR FLIGHT MACH NUMBER IS M_∞, RICE, HEIDMANN AND SOFRIN (AIAA-79-0183) SUGGEST THAT THE DIRECTION OF PEAK RADIATION FOR A MODE IS THE DIRECTION OF GROUP VELOCITY BASED ON M_∞. HERE IS THE CONSTRUCTION BASED ON OUR GRAPHICAL METHOD. NOTE THAT THE ELLIPSE IN $\tilde{k}_1\tilde{k}_2$-PLANE IS BASED ON THE DUCT MACH NUMBER.

θ' IS THE ANGLE OF PEAK RADIATION. WE CAN CALCULATE THIS ANGLE FROM



$$\tan\theta' = \frac{\tilde{k}_2}{|\tilde{k}_1 + M_\infty/\beta^2|} = \frac{k_r/k}{|M_\infty - M - \sqrt{1-1/\beta_{mn}^2}|/\beta^2} = \frac{\beta}{|(M_\infty - M)\beta_{mn} - \sqrt{\beta_{mn}^2 - 1}|} \quad (41)$$

THIS IS EQUIVALENT TO WHAT RICE ET AL HAVE GIVEN. THEIR EXPRESSION LOOKS MORE COMPLICATED SINCE THEY CALCULATED $\cos\theta'$. OUR GRAPHICAL METHOD HAS SIMPLIFIED THE ANALYSIS. NOTE THAT β_{mn} IS THE CUT-OFF RATIO OF THE MODE.

MODES IN A DUCT WITH ROTOR AND EGV

ASSUME THAT WE HAVE B BLADES ON THE ROTOR AND V VANES ON EGV AND THE ROTOR IS TURNING WITH ANGULAR VELOCITY Ω. WE ASSUME THE BLADES AND VANES ARE CIRCUMFERENTIALLY SPACED $2\pi/B$ AND $2\pi/V$, RESPECTIVELY.

ROTOR ALONE

IN THE PHASE RELATION $\omega t - m\theta - k_{a,mn}x_1$, WE WANT TO FIND HOW ω AND m ARE RELATED TO Ω AND B. FOR A MICROPHONE IN THE DUCT, WE DETECT A PERIODIC SIGNAL IN TIME WITH FUNDAMENTAL FREQUENCY $B\Omega$. USING FOURIER TRANSFORMATION, THIS PERIODIC SIGNAL CAN BE DECOMPOSED INTO COMPONENTS WITH FREQUENCIES WHICH ARE MULTIPLES OF $B\Omega$.

$\therefore$ ω IS A MULTIPLE OF $B\Omega$: $B\Omega, 2B\Omega, 3B\Omega \ldots$

THE ACOUSTIC DISTURBANCE MUST ROTATE WITH ANGULAR VELOCITY Ω, i.e. WITH THE ROTOR. THIS MEANS THAT $\dot{\theta} = \Omega$ AND $\theta = \Omega t + \text{CONST}$. THEREFORE, $\Omega t - \theta$ MUST APPEAR IN THE PHASE RELATION. WE HAVE DISCOVERED THAT FOR THE ROTOR ALONE CASE $\boxed{\text{PHASE} = mB(\Omega t - \theta) - k_{a,mB,n}x_1}$ (42)

THIS MEANS THAT CIRCUMFERENTIAL MODES OF ORDER B, 2B,

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3B, ... , mB, ... CAN BE GENERATED. NOTE THAT CORRESPONDING TO EACH FREQUENCY $mB\Omega$, WE HAVE ONLY ONE CIRCUMFERENTIAL MODE mB. ALSO m CANNOT BE ZERO SINCE TIME DEPENDENCE DISAPPEARS. THE EIGENFUNCTIONS HAVE THE FOLLOWING FORM:

$$p' = A\, J_{mB}(k_{r,mB,n} r) \exp i\left[mB(\Omega t - \theta) - k_{a,mB,n} x_1\right] \qquad (43)$$

WE CAN WRITE $k_{r,mn} \equiv k_{r,mB,n}$, $k_{a,mn} = k_{a,mB,n}$.

NOTE THAT FOR mBTH CIRCUMFERENTIAL MODE, k IN EQ. (29) IS $k_m = \frac{mB\Omega}{c_0}$. WE CAN HAVE SEVERAL RADIAL MODES CORRESPONDING TO A CIRCUMFERENTIAL MODE mB. NOW THIS IS THE PROCEDURE WE USE TO FIND PROPAGATING AND DECAYING MODES:

FOR CIRCUMFERENTIAL MODE mB, $m = 1, 2, 3, \ldots$, FIND THE ZEROES OF $\quad J'_{mB}(x) = 0 \;:\; \alpha_1, \alpha_2, \alpha_3, \alpha_4 \ldots$

$$\Longrightarrow \quad k_{r,mn} = \frac{\alpha_n}{R} \quad \Longrightarrow \quad \beta_{mn} = \frac{k_m}{\beta\, k_{r,mn}}$$

WHERE $k_m = \frac{mB\Omega}{c_0}$ $\Longrightarrow$ IF $\beta_{mn} > 1$, PROPAGATING; IF $\beta_{mn} < 1$,

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DECAYING $\Longrightarrow$ FIND $k_{a,mn} = \frac{k_m}{\beta^2}\left[-M \pm \sqrt{1 - 1/\beta_{mn}^2}\right]$

OUR GRAPHICAL ANALYSIS STAYS THE SAME. REMEMBER TO CHECK IF YOU HAVE USED THE RIGHT $k = k_m$ FOR EACH CIRCUMFERENTIAL MODE mB.

A USEFUL RULE OF THUMB. $\quad \beta_{mn} = \frac{k_m}{\beta\, k_{r,mn}} = \frac{\frac{mB\Omega}{c_0}}{\beta\, k_{r,mn}}$

LET R BE THE DUCT RADIUS, THEN $R\Omega/c_0$ IS THE BLADE TIP MACH NUMBER M_T, ALSO $k_{r,mn} R = \alpha_n$, THE NTH ZERO OF $J'_{mB}(x)$. USING THESE TWO PARAMETERS, WE CAN WRITE β_{mn} AS

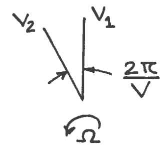
$$\beta_{mn} = \frac{mB\, M_T}{\beta\, \alpha_n} \qquad (44)$$

$$\beta_{m1} = \frac{mB\, M_T}{\beta\, \alpha_1} < \frac{M_T}{\beta} \quad \text{SINCE } \alpha_1 > mB \text{ (USUALLY } \alpha_1 >> mB)$$

$\therefore$ FOR SMALL M, i.e. SMALL DUCT MACH NUMBER WHEN $\beta \approx 1$, $\beta_{m1} < M_T$. THIS MEANS THAT FOR ROTORS WITH SUBSONIC TIP SPEED AND SMALL DUCT MACH NUMBERS, NO MODE CAN PROPAGATE OUT. THIS IS A VERY USEFUL RESULT.

INTERACTION MODES

THE STUDY OF INTERACTION MODES CAN BE BASED MAINLY ON KINEMATICS. WE HAVE V VANES SPACED $\frac{2\pi}{V}$ RADIANS. LET US HAVE B BLADES ROTATING WITH ANGULAR VELOCITY Ω. CONCENTRATING ON VANE NUMBER 1, V_1, FIRST, THE WAKES FROM B BLADES HIT THIS VANE AT TIME INTERVALS $\frac{2\pi}{B\Omega}$ SECONDS, i.e. WITH FREQUENCY $B\Omega$ WHICH IS THE FUNDAMENTAL FREQUENCY OF OSCILLATIONS IN TIME. ASSUME V_1 IS AT $\theta = 0$. A LINE PERTURBATION (RADIAL) AT VANE 1 HAS THE REPRESENTATION



$$\delta(\theta) \sum_{1}^{\infty} A_m e^{imB\Omega t} \qquad (45)$$

NOW THE nTH VANE AT $\theta = \frac{2n\pi}{V}$ SEES A PERTURBATION AT V_1, EXACTLY $\frac{2\pi n}{V\Omega}$ SECONDS LATER. THIS MEANS THAT THE PERTURBATION AT THE nTH VANE HAS THE REPRESENTATION

$$\delta\left(\theta - \frac{2\pi n}{V}\right) \sum_{1}^{\infty} A_m \exp i\left[mB\Omega\left(t - \frac{2\pi n}{V\Omega}\right)\right] \qquad (46)$$

THEREFORE, V VANES PRODUCE A PERTURBATION OF THE FORM

$$\sum_{m=1}^{\infty} \sum_{n=0}^{V-1} A_m \delta\left(\theta - \frac{2\pi n}{V}\right) \exp i\left[mB\Omega\left(t - \frac{2\pi n}{V\Omega}\right)\right] \qquad (47)$$

A CIRCUMFERENTIAL FOURIER TRANSFORM, SUMMATION ON n AND NOTING THAT UNLESS SOME SPECIAL COMBINATIONS OF B AND V ARE TAKEN, THE RESULTING FOURIER COMPONENTS ADD UP TO ZERO, WE FIND ONLY THE FOLLOWING CIRCUMFERENTIAL MODES CAN EXIST

$$\boxed{mB + nV = q} \qquad \begin{array}{l} m = 1, 2, 3, \dots \\ n = 0, \pm 1, \pm 2 \\ q \text{ CIRCUMFERENTIAL MODE} \end{array} \qquad (48)$$

WE NOTE THAT q CAN BE A POSITIVE OR A NEGATIVE INTEGER. IF $q > 0$, THE MODE IS ROTATING IN THE DIRECTION OF THE ROTOR. OTHERWISE, IT IS ROTATING IN OPPOSITE DIRECTION TO THE ROTOR.

WE MENTION SOME IMPORTANT FACTS HERE.

i) m IN EQ.(48) INDICATES THE MULTIPLE OF BLADE PASSAGE FREQUENCY. THEREFORE, THE CIRCUMFERENTIAL MODES

CORRESPONDING TO BPF, i.e. $m = 1$, ARE

..... $B-2V$ $B-V$ B $B+V$ $B+2V$ ......

AND FOR 2BPF, i.e. $m=2$, ARE

..... $2B-2V$ $2B-V$ $2B$ $2B+V$ $2B+2V$ ....

ii) LET $\tilde{C}$ BE GREATEST COMMON DIVISOR OF B AND V, THEN $q = mB + nV$ CAN BE ONLY A MULTIPLE OF $\tilde{C}$. THEREFORE TO PRODUCE ALL CIRCUMFERENTIAL MODES, B AND V MUST BE RELATIVELY PRIME.

iii) THE EIGENFUNCTIONS FOR INTERACTION MODES ARE OF THE FORM:

$$p'_{mq} = A_{mq} J_q(k_{r,qn} r) \exp i \left[mB\Omega t - q\theta - k_{a,qn} x_1 \right] \quad (49)$$

NOTE $k = k_m = \frac{mB\Omega}{C_0}$, THE ORDER OF BESSEL FUNCTION IS q, i.e. THE CIRCUMFERENTIAL MODE NUMBER.

WE HAVE COME UP WITH THE FOLLOWING PROCEDURE

DECIDE MULTIPLE OF BPF, i.e. mPF $\Rightarrow k_m = \frac{mB\Omega}{C_0}$

$\Rightarrow$ FIND ALL CIRCUMFERENTIAL MODES $mB + nV$, $n = 0, \pm 1, \dots$

$\Rightarrow$ FOR EACH CIRCUMFERENTIAL MODE q, FIND ROOTS OF $J'_{|q|}(x) = 0$ GETTING $\alpha_1, \alpha_2, \dots \alpha_n, \dots$

$\Rightarrow k_{r,qn} = \frac{\alpha_n}{R}$ $\Rightarrow$ FIND CUT-OFF RATIO $\beta_{qn} = \frac{k_m}{\beta k_{r,qn}}$

$\Rightarrow$ FIND $k_{a,qn} = \frac{k_m}{\beta^2} \left[-M \pm \sqrt{1 - \frac{1}{\beta^2_{qn}}} \right]$

NOTE $k_{r,qn}$ DEPENDS ON q AND DOES NOT DEPEND ON M AND m.

iv) THE SAME CIRCUMFERENTIAL MODE CAN BE PRODUCED BY MANY MULTIPLES OF BPF'S. THIS CAN BE SEEN EASILY FROM $q = mB + nV = (m + p\frac{V}{\tilde{C}})B + (n - p\frac{B}{\tilde{C}})V$ WHERE $\tilde{C}$ IS GCD OF B AND m AND p IS AN INTEGER WHICH MAKES $m' = m + p\frac{V}{\tilde{C}} > 0$.

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V) WHY WORRY ABOUT INTERACTION MODES? WE KNOW THAT k_r'S COME FROM SOLUTION OF $J'_q(x) = 0$. THE SMALLER q IS, THE LOWER IS THE FIRST ROOT OF THIS EQUATION AND THE CUT-OFF RATIO IS

$$\beta_{qn} = \frac{k_m}{\beta k_{r,qn}}$$

NOW β_{qn} CAN BE ABOVE ONE AND THUS MODE (q,n) WILL BE PROPAGATING. NOTE THAT WE SHOULD SAY MODE (q,n) FOR mBPF, i.e. THE MULTIPLE OF BLADE PASSAGE FREQUENCY FOR THE MODE (q,n) MUST BE MENTIONED.

<u>EXAMPLE</u> $B = 16$, $V = 20$

$$q = -4 = 16 - 20 = \left(1 + \tfrac{20}{4}\right)16 + \left(-1 - \tfrac{16}{4}\right)20$$
$$= 6 \times 16 - 5 \times 20$$

$\therefore$ 4TH CIRCUMFERENTIAL MODE IS PRODUCED, AMONGST OTHERS, BY BPF AND 6BPF.

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<u>HOW DOES A ROTATING MICROPHONE SEPARATE MODES?</u>

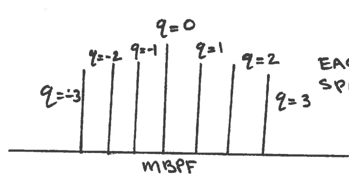
WE CONSIDER THE PHASE OF A PROPAGATING MODE (q,n) FOR mBPF, DISREGARDING $k_{a,qn}\, x_1$

$$\text{PHASE} = mB\Omega t - q\theta$$

FOR A ROTATING MICROPHONE WITH ANGULAR VELOCITY $\alpha\Omega$, WE HAVE $\theta = \alpha\Omega t$ SO THAT

$$\text{PHASE} = (mB + \alpha q)\Omega t \qquad (50)$$

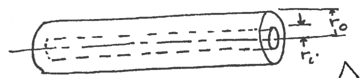
THIS MEANS THAT A NARROWBAND FILTER WILL GIVE A SPECTRUM AS SHOWN BELOW



EACH FREQUENCY SPACED $\frac{\alpha}{B} \times$ BPF APART

IN LEWIS ROTATING MICROPHONE DESIGN $\alpha = \frac{1}{250}$.

SPINNING MODES IN DUCTS WITH RIGID WALLS
(CIRCULAR DUCTS WITH NO FLOW)



$\Lambda = \frac{r_i}{r_o}$

$\boxed{k = \frac{\omega}{c}}$

$$\Box^2 p' = 0$$

$$p' = \mathrm{Re}\, p$$

$$p = e^{i(k_a x - \omega t)} G(r)\, H(\theta)$$

$$\begin{cases} H'' + m^2 H = 0 \\ G'' + \frac{1}{r} G' + \Big[\underbrace{(k^2 - k_a^2)}_{k_r^2} - \frac{m^2}{r^2}\Big] G = 0 \end{cases}$$

$$\begin{cases} H(\theta) = e^{im\theta} \qquad m \text{ AN INTEGER} : 0, \pm 1, \pm 2, \ldots \\ G(r) = A\, J_m(k_r r) + B\, Y_m(k_r r) \end{cases}$$

$$\left.\frac{\partial p}{\partial n}\right|_{\substack{r = r_i \\ r = r_o}} = 0 \implies G'(r_i) = G'(r_o) = 0$$

THIS WILL RESULT IN DISCRETE VALUES OF k_r :

(OVER)

$$k_{m0}\,,\ k_{m1}\,,\ k_{m2}\,, \ldots\ldots,\ k_{mn}\,, \ldots$$

m CIRCUMFERENCIAL MODE NO.

n RADIAL MODE NO.

— THE AXIAL WAVE NO. k_a IS NOW DEPENDENT ON m AND n

$$k_a \equiv k'_{mn} = \sqrt{k^2 - k_{mn}^2}$$

— IF k'_{mn} REAL, THE (m,n) MODE PROPAGATES WITH AXIAL SPEED $V_a = \frac{\omega}{k'_{mn}}$

— IF k'_{mn} IMAGINARY, THE (m,n) MODE DECAYS IN AXIAL DIRECTION

— NOTE k_{mn}'S DEPEND ON DUCT GEOMETRY AND CAN BE CALCULATED WHEN HUB-TO-TIP RATIO Λ IS KNOWN. THESE WAVE NUMBERS ARE THEREFORE ASSUMED GIVEN. THEY CORRESPOND TO STANDING WAVES IN RADIAL DIRECTION.

– NOW LET $A J_m(k_{mn} r) + B Y_m(k_{mn} r) = P_{mn} R_m(k_{mn} r)$

THE GENERAL SOLUTION OF $\Box^2 p' = 0$, CORRESPONDING TO A <u>SINGLE</u> FREQUENCY ω IS THUS

$$p(r,\theta,t) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} P_{mn} e^{i(m\theta + k'_{mn}x - \omega t)} R_m(k_{mn} r)$$

– AXIAL WAVE SPEED IS FOUND FROM $k'_{mn}x - \omega t = \text{CONST.}$,

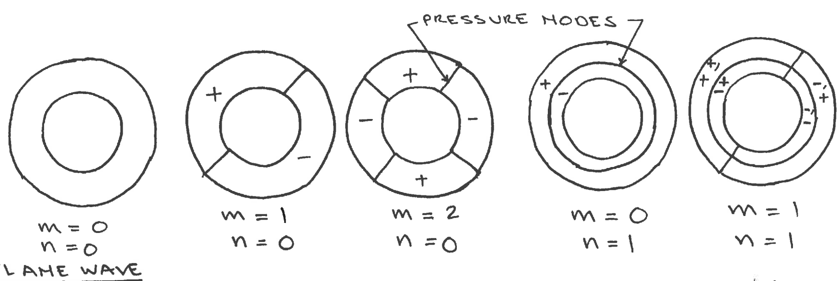
i.e. $V_a = \dot{x} = \dfrac{\omega}{k'_{mn}}$, k'_{mn} REAL

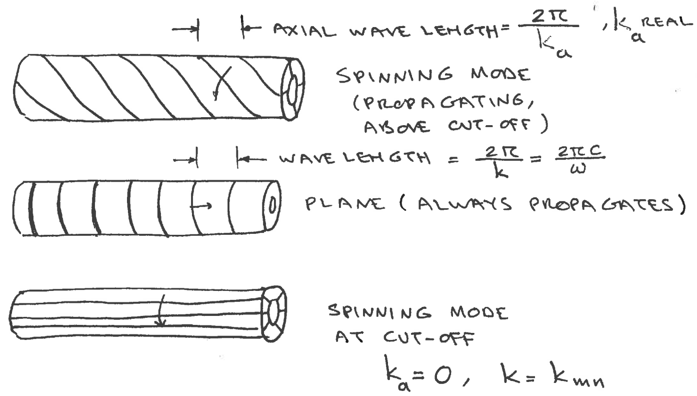
– CIRCUMFERENCIAL ANGULAR VELOCITY IS FOUND FROM

$m\theta - \omega t = \text{CONST.}$, i.e. $\dot{\theta} = \dfrac{\omega}{m}$ $\therefore$ $m > 0$ (+) , $m < 0$ (−)

CIRCUMFERENCIAL TIP SPEED $= \dfrac{r_0 \omega}{m}$

– CUT-OFF FREQUENCY FOR (m,n) MODE $= \dfrac{c\,k_{mn}}{2\pi}$





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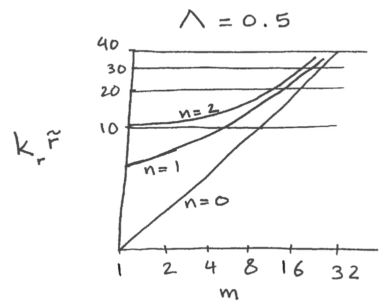

$$\tilde{r} = \sqrt{r_i^2 + r_o^2}$$

NOTE $k_{m,n+1} > k_{mn}$

$k_{m+1,n} > k_{mn}$ (AT LEAST FOR LARGE m

SOUND TRANSMITTED ALONG A DUCT, α REAL $= \dfrac{k'_{mn}}{k}$

$$I = \frac{1}{2}\frac{\alpha}{\rho c}|p|^2 \quad , \ |p| \text{ PRES. AMP.}$$

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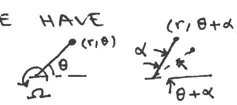
— SUPPOSE WE HAVE B BLADES. FIX A MICROPHONE IN THE DUCT. WE GET A PERIODIC TIME HISTORY WITH PERIOD $T = \dfrac{1}{BPF}$, $BPF = B \times REV/SEC = \dfrac{1}{2\pi} B\Omega$

$$\therefore \quad p = \sum_{n'=-\infty}^{\infty} p_{n'} \, e^{i n' B \Omega t} \qquad (*)$$

i.e. $\omega = n' B \Omega$, $n' = \pm 1, \pm 2, \ldots$

— THE θ AND t DEPENDENCE OF A ROTATING PRESSURE PATTERN IS OF A SPECIAL FORM. WE HAVE

$$p\left(\theta + \alpha, t + \frac{\alpha}{\Omega}\right) = p(\theta, t)$$



TAKE $\left.\dfrac{\partial}{\partial \alpha}\right|_{\alpha=0} \Rightarrow \quad \dfrac{\partial p}{\partial \theta} + \dfrac{1}{\Omega}\dfrac{\partial p}{\partial t} = 0$, $\therefore \quad p(\theta, t) = f(\theta - \Omega t)$

— FROM TIME DEPENDENCE (*), ABOVE, WE HAVE

$$p(r,\theta,t) = \sum_{-\infty}^{\infty} P_{n'}(r) \, e^{i n' B (\theta - \Omega t)}$$

ROTOR ALONE, ROTATING PRESSURE PATTERN

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— NOW LET US ADD V VANES (STATIC) TO OUR B ROTATING BLADES. WE WILL HAVE A PRESSURE PATTERN WHICH IS ROTATING WITH ANGULAR VELOCITY Ω BUT ITS AMPLITUDE IS PERIODIC WRT θ WITH PERIOD $\frac{2\pi}{V}$, i.e.

$$p(r,\theta,t) = \sum_{n'=-\infty}^{\infty} P_{n'}(r,\theta)\, e^{in'B(\theta-\Omega t)}$$

$$P_n(r,\theta) = \sum_{m'=-\infty}^{\infty} \tilde{P}_{n'm'}(r)\, e^{im'V\theta}$$

$$\Rightarrow \quad p(r,\theta) = \sum_{n'=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty} \tilde{P}_{n'm'}(r)\, e^{i[(n'B+m'V)\theta - n'B\Omega t]}$$

$\therefore$ CIRCUMFERENTIAL MODES OF ORDERS

$$\boxed{n'B + m'V}$$

n' AND m' INTEGERS ARE GENERATED.

— THE ROTATIONAL SPEED OF THE $nB+mV$ MODE IS $n'B\Omega / (n'B+m'V) = \dot{\theta}$.

$\dot{\theta} > 0 \quad n'B+m'V > 0$

$\dot{\theta} < 0 \quad n'B+m'V < 0$

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— WE NOW GO BACK TO OUR MODE SOLUTION ON P3. WE NOTE THAT

$$m = n'B + m'V, \quad n', m' \text{ INTEGERS}$$

$$\omega = n'B\Omega$$

$$\Omega = 2\pi \times \text{REV/SEC}$$

m CIRCUMFERENTIAL MODE

n RADIAL MODE

$$k = \frac{\omega}{c} = \frac{n'B\Omega}{c} = \frac{m}{r_0}\,\frac{n'Br_0\Omega}{mc} = \frac{m}{r_0} M_m$$

M_m CIRCUMFERENTIAL MACH NO. OF mTH CIR. MODE

$$k'_{mn} = \sqrt{k^2 - k_{mn}^2} = \sqrt{\frac{m^2}{r_0^2}M_m^2 - k_{mn}^2}$$

$$= \frac{m}{b}\sqrt{M_m^2 - \left(\frac{r_0 k_{mn}}{m}\right)^2} = \frac{m}{r_0}\sqrt{M_m^2 - M_{mn}^{*2}}$$

k'_{mn} AXIAL MACH NO.

$M^*_{mn} = \frac{r_0 k_{mn}}{m}$ CRITICAL MACH NO. FOR (m,n) MODE

— WE SEE THAT WE HAVE A <u>PROPAGATING</u> MODE

IF $M_m > M^*_{mn}$

AND A <u>DECAYING</u> MODE

IF $M_m < M^*_{mn}$

— WE HAVE $M^*_{mn} > M^*_{m0} \equiv M^*_m$ FOR $n > 0$.

$\therefore$ IF $M_m < M^*_m$, <u>ALL</u> (m,n) MODES ARE DECAYING. WE CAN SHOW THAT $M^*_m > 1$.

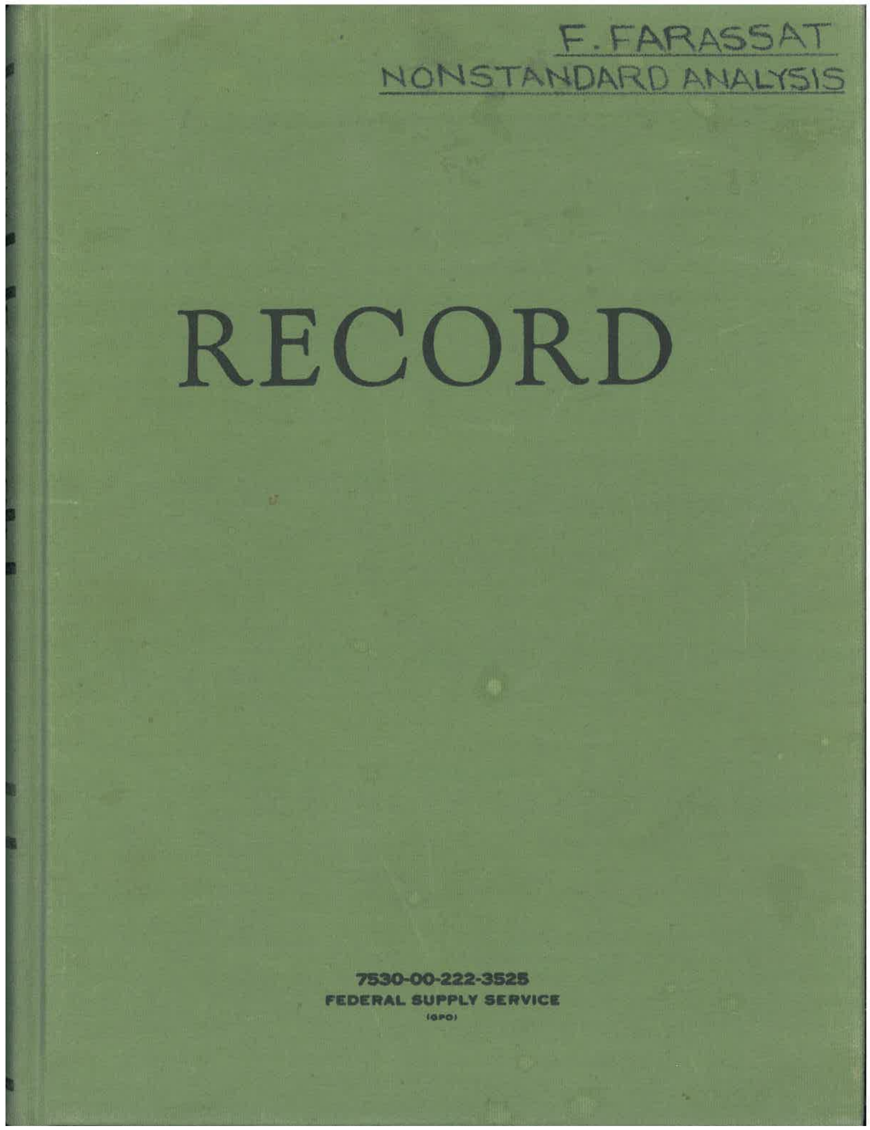
$\Rightarrow$ WE MUST ALWAYS HAVE $M_m > 1$ TO HAVE ANY PROPAGATING MODE!

— WE NOTE THAT ROTOR / STATOR INTERACTION MAY MAKE MANY MODES PROPAGATING BY MAKING

$$M_m = \frac{r_0 \dot{\theta}}{c} = \frac{r_0 n' B \Omega}{c(n'B + m'V)}$$

LARGE ENOUGH.

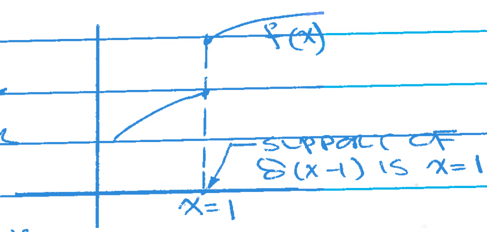
15 Nonstandard Analysis



JAN. 20, 1998 1

APPLICATION OF SOME CONCEPTS OF NONSTANDARD ANALYSIS TO FINDING SHOCK JUMP CONDITIONS FROM CONSERVATION LAWS IN NONCONSERVATIVE FORM

THE USE OF CONSERVATION LAWS IN NONCONSERVATIVE FORM TO FIND SHOCK JUMP CONDITIONS LEADS TO PRODUCTS OF GENERALIZED FUNCTIONS OF SINGULAR AND DISCONTINUOUS TYPES. THIS PRODUCES AN AMBIGUITY OF INTERPRETATION. FOR EXAMPLE, THE PRODUCT OF $f(x)$, WITH JUMP DISCONTINUITY AT $x=1$ AND $\delta(x-1)$ WITH SUPPORT AT $x=1$, IS NOT DEFINED. HERE $\delta(x-1)$ IS A SINGULAR GF WHILE $f(x)$ IS A REGULAR AND DISCONTINUOUS FUNCTION.



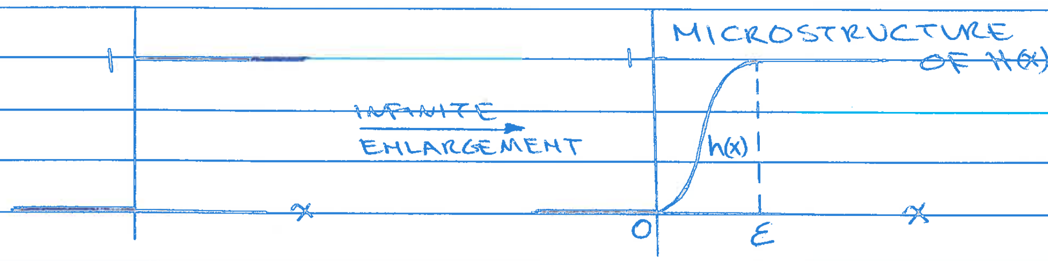
WE PROPOSE THAT THIS PROBLEM CAN BE ALEVIATED IF WE USE SOME CONCEPTS FROM NONSTANDARD ANALYSIS. IN NONSTANDARD ANALYSIS (NSA), WE HAVE TRUE INFINITESIMALS WHICH ARE SMALLER THAN ANY FINITE NON-ZERO REALS. WE USE ε FOR A GIVEN NONZERO INFINITESIMAL HERE. A TYPICAL INFINITESIMAL IS $\varepsilon = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} = \{\frac{1}{n}\}$. OUR WORK DEPENDS ON A NEW KIND OF HEAVISIDE FUNCTION WHERE THE JUMP FROM 0 TO 1 OCCURS ON AN INFINITESIMAL INTERVAL ε. WE THUS DEFINE A HEAVISIDE FUNCTION $H(X)$ AS

2

FOLLOWS:

$$H(x) = \begin{cases} 0 & x < 0 \\ h(x) & 0 \leq x \leq \varepsilon \\ 1 & x > \varepsilon \end{cases}$$

WHERE $h(0)=0$, $h(\varepsilon)=1$ AND $h(x)$ IS A SMOOTH FUNCTION OF x. NOTE THAT $H(x)$ IS A FUNCTION OF ε AND $h(x)$ SO THAT WE ARE DEALING WITH A WHOLE CLASS OF FUNCTIONS WE CALL HEAVISIDE FUNCTION. AN INFINITE ENLARGEMENT THE HEAVISIDE FUNCTION IN THE x DIRECTION WILL GIVE THE FOLLOWING PICTURE:



WE NOTE THAT $H'(x) = h'(x)$, $x \in [0,\varepsilon]$ AND $H'(x)=0$ IF $x \notin [0,\varepsilon]$. WE CAN SHOW THAT $h'(x)$ THE PROPERTIES OF THE DIRAC DELTA FUNCTION WITH SUPPORT $[0,\varepsilon]$. THIS FACT IS BEHIND THE SUCCESS OF THE METHOD USED HERE. THE MOST IMPORTANT ADVANTAGE IS THAT THERE IS NO MORE ANY AMBIGUITIES ABOUT THE INTERPRETATION OF $H'K$ AND HK', WHERE H AND K ARE TWO HEAVISIDE FUNCTIONS WHICH DIFFER IN MICROSTRUCTURES.

WE WILL NOT SPECIFY MICROSTRUCTURES OF THE

HEAVISIDE FUNCTIONS AND WE WILL NOT NEED THEM HERE

SHOCK JUMP CONDITIONS

REF.: M. D. SALAS AND A. IOLLO "ENTROPY JUMP ACROSS AN INVISCID SHOCK WAVE", THEO. & COMP. FL. MECH., 8, 365-375, 1996

SYMBOLS: SAME AS THE ABOVE REFERENCE WITH THE EXCEPTION OF USING $\Delta(\cdot)$ FOR JUMP INSTEAD OF $[\cdot]$ IN THE REF. ABOVE.

1D CONSERVATION LAWS

$$\begin{cases} v_t - v u_x + u v_x = 0 & (1) \\ u_t + u u_x + v p_x = 0 & (2) \\ p v (S_t + u S_x) = C_v [v (p_t + u p_x) + \gamma p (v_t + u v_x)] & (3) \end{cases}$$

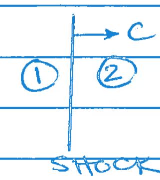
HERE v IS SPECIFIC VOLUME, u VELOCITY, p PRESSURE, S ENTROPY/UNIT MASS AND γ IS THE RATIO OF SPECIFIC HEATS.

THE ENTROPY LAW IN CONSERVATIVE FORM IS

$$\{\rho [S - C_v \ln(p v^\gamma)]\}_t + \nabla \cdot \{\rho \vec{u} [S - C_v \ln(p v^\gamma)]\} = 0 \quad (4)$$

WE WILL USE THIS EQUATION IN NONCONSERVATIVE FORM IN 1D, EQ. (3), TO FIND THE ENTROPY JUMP CONDITION ACROSS THE SHOCK LATER.

IT IS CONJECTURED HERE THAT v, u, p AND s HAVE DIFFERENT MICROSTRUCTURE AT $x = 0$. WE, THEREFORE, USE DIFFERENT HEAVISIDE FUNCTIONS. WE ASSUME A 1D SHOCK MOVING WITH THE SPEED c WITH SIDES 1 AND 2. WE DEFINE THE FOLLOWING



$$\begin{cases} v = v_1 + \Delta v \, H(\xi) & (5) \\ u = u_1 + \Delta u \, K(\xi) & (6) \\ p = p_1 + \Delta p \, L(\xi) & (7) \\ s = s_1 + \Delta s \, T(\xi) & (8) \end{cases}$$

$$\Delta(\cdot) = (\cdot)_2 - (\cdot)_1$$

WHERE H, K, L AND T ARE FOUR HEAVISIDE FUNCTIONS POSSIBLY DIFFERING IN MICROSTRUCTURE NEAR $\xi = x - ct = 0$. WE CLAIM THAT THE CONSERVATION LAWS IN NONCONSERVATIVE FORM CAN GIVE INFORMATION ABOUT THE RELATION BETWEEN THESE HEAVISIDE FUNCTIONS WITHOUT ANY KNOWLEDGE OF THE MICROSTRUCTURE.

USING EQS. (5) AND (6) IN EQ. (1), WE GET

$$-c \Delta v H' - (v_1 + \Delta v H) \Delta u K' + (u_1 + \Delta u K) \Delta v H' = 0$$

$$(u_1 - c) \Delta v H' - v_1 \Delta u K' - \Delta v \Delta u H K' + \Delta v \Delta u K H' = 0$$

LET $a = (u_1 - c)/\Delta u \equiv \tilde{u}_1/\Delta u$, WE HAVE

$$(a + K) \frac{dH}{dK} - H = \frac{v_1}{\Delta v} \qquad (9)$$

THE SOLUTION OF THIS EQUATION IS

$$H = -\frac{v_1}{\Delta v} + b(a + K) \qquad (10)$$

WHERE b IS THE CONSTANT OF INTEGRATION. FROM $H(0) = K(0) = 0$, WE GET $b = \frac{v_1}{a \Delta v}$, AND FROM $H(\varepsilon) \equiv H(0+) = K(0+) = 1$, WE GET $b = 1 - \frac{v_1}{a \Delta v}$.

$\therefore$ FROM EQ. (1), WE FIND $K = H$, i.e. K AND H HAVE THE SAME MICROSTRUCTURE!

USING EQS. (2), (6) AND (7), WE GET

$$-c\,\Delta u\,K' + (u_1 + \Delta u\,K)\Delta u\,K' + (v_1 + \Delta v\,H)\Delta p\,L' = 0$$

USING THE RESULT $H = K$ HERE,

$$\Big[\underbrace{\frac{u_1 - c}{\Delta u}}_{\alpha} + K\Big](\Delta u)^2 K' + \Big[\underbrace{\frac{v_1}{\Delta v}}_{=\alpha \text{ (BOTTOM OF P4!)}} + K\Big]\Delta v\,\Delta p\,L' = 0$$

$$\therefore \quad \frac{dL}{dK} + \frac{(\Delta u)^2}{\Delta v\,\Delta p} = 0 \qquad (11)$$

$$L = -\frac{(\Delta u)^2}{\Delta v\,\Delta p}K + d \text{ , } d \text{ CONST.}$$

$$L(0) = K(0) = 0 \implies d = 0$$

$$L(0_+) = K(0_+) = 1 \implies \frac{(\Delta u)^2}{\Delta v\,\Delta p} = -1$$

THIS GIVES PRANDTL'S RELATION

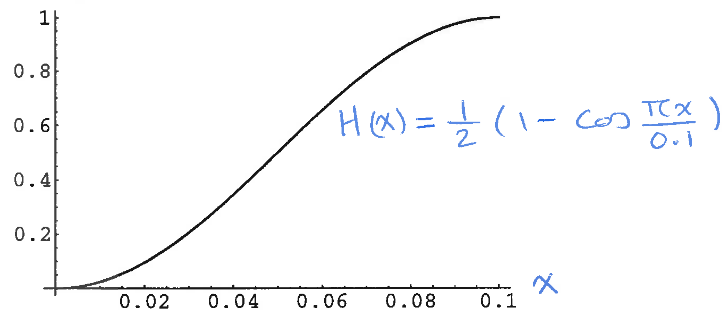
$$\tilde{u}_1\tilde{u}_2 = \frac{\Delta p}{\Delta \rho} = \frac{p_2 - p_1}{\rho_2 - \rho_1}$$

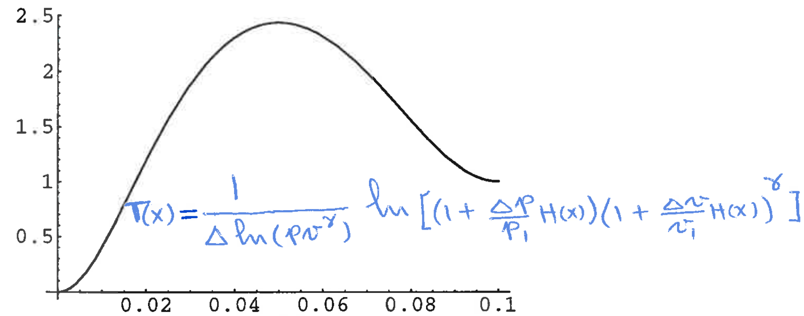
WHERE $\tilde{u} = u - c$.

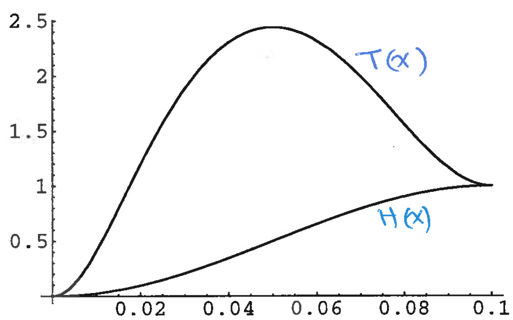
ALSO, WE HAVE $L = K$ $\therefore$ $L = K = H$, i.e. WE HAVE THE SAME MICROSTRUCTURE FOR L, K AND H! THE SITUATION FOR T IS DIFFERENT.

GO TO PAGE 7

6







$M_1 = 3$

$$\frac{\Delta p}{p_1} = \frac{7}{6}(M_1^2 - 1), \quad \frac{\Delta v}{v_1} = -\frac{5(M_1^2 - 1)}{6 M_1^2}.$$

$$\Delta \ln(p v^{\gamma}) = \ln\left(\frac{7M_1^2 - 1}{6}\right) - 1.4 \ln\left(\frac{6 M_1^2}{M_1^2 + 5}\right)$$

I GOT THE ABOVE FIGURES USING MATHEMATICA.

THE HEAVISIDE FUNCTION T

THE ENTROPY LAW IN CONSERVATIVE FORM IS

$$[\rho(S - C_v \ln p + C_v \ln \rho^\gamma)]_t + \nabla\cdot[\rho\vec{u}(S - C_v \ln p + C_v \ln \rho^\gamma)] = 0 \quad (12)$$

THIS GIVES THE CORRECT ENTROPY JUMP

$$\Delta S = C_v \,\Delta \ln\left(\frac{p}{\rho^\gamma}\right)$$

$$= C_v \,\Delta \ln(p v^\gamma) \quad (13)$$

BY USING SCHWARTZ DISTRIBUTION THEORY.

IN NONCONSERVATIVE FORM, EQ.(15) IS

$$[S - C_v \ln(p v^\gamma)]_t + \vec{u}\cdot\nabla[S - C_v \ln(p v^\gamma)] = 0$$

IN 1D, WE HAVE

$$S_t + u S_x = \frac{C_v}{p v^\gamma}\left[(p v^\gamma)_t + u(p v^\gamma)_x\right]$$

$$= \frac{C_v}{p v^\gamma}\left[v^\gamma(p_t + u p_x) + \gamma p(v_t + u v_x) v^{\gamma-1}\right]$$

$$p v(S_t + u S_x) = C_v\left[v(p_t + u p_x) + \gamma p(v_t + u v_x)\right] \quad (14)$$

USING K = L = H AND EQS.(5-8) IN THE ABOVE EQ., WE GET

$$(p_1 + \Delta p H)(v_1 + \Delta v H)(\tilde{u}_1 + \Delta u H)\Delta S T' =$$

$$C_v\{(v_1 + \Delta v H)(\tilde{u}_1 + \Delta u H)\Delta p H' + \gamma(p_1 + \Delta p H)(\tilde{u}_1 + \Delta u H)\Delta v H'\}$$

$$\Delta S\, T' = C_v \Delta p \frac{H'}{p_1 + \Delta p H} + \gamma C_v \Delta v \frac{H'}{v_1 + \Delta v H}$$

$$\Delta S\, T = C_v \ln[(p_1 + \Delta p H)(v_1 + \Delta v H)^\gamma] + e$$

WHERE e IS A CONSTANT. LETTING $T(0-)=H(0-)=0$, WE GET

$$\Delta S T = C_v \ln[(p_1+\Delta p H)(v_1+\Delta v H)^\gamma] - C_v \ln(p_1 v_1^\gamma) \quad (15)$$

FROM $T(0+)=H(0+)=1$, WE GET

$$\Delta S = C_v \Delta \ln(p v^\gamma)$$

WE CAN WRITE T AS FOLLOWS

$$T = \frac{C_v}{\Delta S} \ln\left[(1+\frac{\Delta p}{p_1}H)(1+\frac{\Delta v}{v_1}H)^\gamma\right] \quad (17)$$

SEE PAGE 6 FOR PLOTS OF H AND T FOR $M_1 = 3$. THE MOST REMARKABLE THING ABOUT EQ. (17) IS THAT $T(x)$ BEHAVES AS THE ENTROPY PROFILE THROUGH VISCOUS SHOCK LAYER (FIG. 1, SALAS & IOLLO).

A MUCH MORE DIRECT METHOD FOR FINDING T

$$S = S_1 + \Delta S T = C_v \ln(p v^\gamma)$$

$$= C_v \ln[(p_1+\Delta p H)(v_1+\Delta v H)^\gamma]$$

$$= \underbrace{C_v \ln(p_1 v_1^\gamma)}_{S_1} + C_v \ln\left[(1+\frac{\Delta p}{p_1}H)(1+\frac{\Delta v}{v_1}H)^\gamma\right]$$

$$\therefore \Delta S T = C_v \ln\left[(1+\frac{\Delta p}{p_1}H)(1+\frac{\Delta v}{v_1}H)^\gamma\right]$$

THIS IS THE SAME AS EQ. (17). SO IT IS THE EQUATION OF THE ENTROPY AS A FUNCTION OF (p, v) THAT DETERMINES THE SHAPE OF $T(x)$.

COMMENTS ON ENTROPY LAW

EQUATION (4) FOR ENTROPY IS EQUIVALENT TO

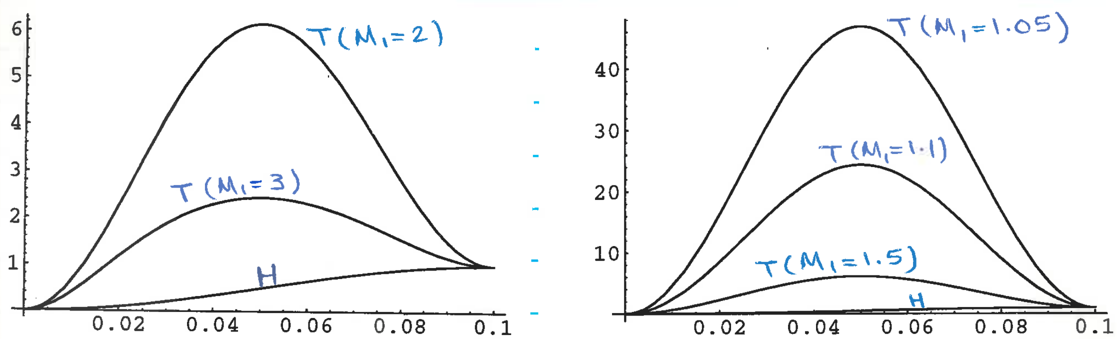
$$\frac{DS}{Dt} = C_v \frac{D}{Dt} \ln(p v^\gamma)$$

$$= C_v \left[\frac{D}{Dt} \ln p + \gamma \frac{D}{Dt} \ln v \right] \quad (18)$$

THIS RELATES THE RATE OF CHANGE OF S FOR A PARTICLE OF FLUID AS A FUNCTION OF p AND v. SINCE ENTROPY IS NOT CONSERVED ACROSS THE SHOCK, EQ. (18) IS THE APPROPRIATE CONSERVATION LAW FOR FINDING SHOCK JUMP CONDITION AND NOT $\frac{DS}{Dt} = 0$!

T AND H AS A FUNCTION OF M_1

BELOW WE PLOT T AND H AS A FUNCTION OF x WHEN WE VARY M_1 (H IS INDEPENDENT OF M_1)



NOTE THAT THE MAXIMUM OF T INCRESES AS M_1 APPROACHES 1 FROM ABOVE. THIS CAN BE PROVEN AS FOLLOWS. AS $M_1 \downarrow 1 \Rightarrow \frac{\Delta p}{p_1} \ll 1$ AND $\frac{\Delta v}{v_1} \ll 1$ AND

$$T \approx \frac{C_v}{\Delta S} \ln\left[1 + \left(\frac{\Delta p}{p_1} + \gamma \frac{\Delta v}{v_1}\right) H\right] \approx \frac{C_v}{\Delta S}\left(\frac{\Delta p}{p} + \gamma \frac{\Delta v}{v_1}\right) H \quad (19)$$

WE HAVE $\frac{\Delta p}{p_1} = \frac{7}{6}(M_1^2-1)$, $\frac{\Delta v}{v_1} = -\frac{5(M_1^2-1)}{6M_1^2}$

$$\frac{\Delta S}{C_v} = \frac{7}{108}(M_1^2-1)^3 + O(M_1^2-1)^4$$

$$\frac{\Delta p}{p_1} + \gamma\frac{\Delta v}{v_1} = \frac{M_1^2-1}{6}\left[7 - \frac{7}{M_1^2}\right] = \frac{7(M_1^2-1)^2}{6M_1^2}$$

$$\therefore\ T \approx \frac{108}{7}\,\frac{1}{(M_1^2-1)^3}\,\frac{7}{6M_1^2}(M_1^2-1)^2 H$$

$$= \frac{18}{M_1^2(M_1^2-1)} H \qquad \text{NOT ACCEPTABLE !}$$

SINCE $T(0+) \neq 1$. WE HAVE NOT USED ENOUGH EXPANSION OF $\ln\left[(1+\frac{\Delta p}{p_1}H)(1+\frac{\Delta v}{v_1}H)^\gamma\right]$.

$$\ln\left[(1+\frac{\Delta p}{p_1}H)\left(1 + \frac{\gamma\Delta v}{v_1}H + \frac{\gamma(\gamma-1)}{2}\left(\frac{\Delta v}{v_1}\right)^2 H^2 + \cdots\right]\right.$$

$$\approx \underbrace{\left(\frac{\Delta p}{p_1} + \gamma\frac{\Delta v}{v_1}\right)}_{\frac{7(M_1^2-1)^2}{6M_1^2}} H + \underbrace{\left[\gamma\frac{\Delta p}{p_1}\frac{\Delta v}{v_1} + \frac{\gamma(\gamma-1)}{2}\left(\frac{\Delta v}{v_1}\right)^2\right]}_{\frac{7(M_1^2-1)^2(-7M_1^2+1)}{36M_1^4}} H^2$$

$+ O(M_1^2-1)^3$. USING MATHEMATICA, WE GET

$$= \frac{7}{6}(H-H^2)(M_1^2-1)^2 + \frac{7}{108}H(-18+15H+4H^2)(M_1^2-1)^3 + O(M_1^2-1)^4$$

$$T = \frac{18(H-H^2)}{M_1^2-1} + H(-18+15H+4H^2) + O(M_1^2-1)\cdot H$$

$$T(0) = 0, \quad T(0+) = 1$$

THE FUNCTION $\frac{18(H-H^2)}{M_1^2-1}$ HAS A MAXIMUM AT $H=\frac{1}{2}$ EQUAL TO $\frac{9}{2(M_1^2-1)}$. AT $M_1 = 1.1$, THIS MAXIMUM IS

2.14. FOR $M_1 = 1.05$, THE MAXIMUM IS 43.9. THESE VALUES AGREE WELL WITH THOSE IN THE FIGURES ON PAGE 9. THESE FIGURES ALSO CONFIRM THAT THE MAXIMUM OF T APPEARS AT $H = \frac{1}{2}$.

QUESTION: WHY MAXIMUM OF $T \to \infty$ AS $M_1 \downarrow 1$?

A: $T(\alpha)$ DOES NOT HAVE PHYSICAL MEANING. IT IS $\Delta S(M_1)\,T(\alpha)$ THAT HAS PHYSICAL MEANING.

$$\Delta S(M_1)\,T(\alpha) = \frac{7}{6} C_v (H - H^2)(M_1^2 - 1)^2 + \frac{7}{108} C_v H(-18 + 15H + 4H^2)(M_1^2 - 1)^3$$

THE PEAK OF $\Delta S(M_1)\,T(\alpha)$ STILL OCCURS NEAR $H = \frac{1}{2}$ BUT $\Delta S(M_1)\,T(\alpha) \to 0$ AS $M_1 \downarrow 1$

DEC. 13, 1999

I AM AGAIN READY TO WORK ON APPLICATION OF NSA TO THE SHOCK PROBLEM. ROY AND I SELECTED THE VISCOUS PROBLEM WITH CONSTANT VISCOSITY. WE WERE HOPING TO COME UP WITH A METHOD WHERE THE MICROSTRUCTURE OF ONE OF THE HEAVISIDE FUNCTIONS COULD BE FOUND. YESTERDAY, I WORKED ON THIS PROBLEM A FEW HOURS AND THE FOLLOWING RESULTS WERE OBTAINED.

THE GOVERNING EQUATIONS – 1-D CASE

v : SP. VOLUME, $\alpha = \dfrac{1}{\gamma - 1}$

u : VELOCITY.

p : PRESSURE $\beta = \dfrac{4\mu}{3}$, μ : VISCOSITY

$$\begin{cases} v_t + u v_x - v u_x = 0 & \text{(MASS)} \\ u_t + u u_x + v p_x - \beta v u_{xx} = 0 & \text{(MOM.)} \\ \alpha (pv)_t + \alpha u (pv)_x + pv\, u_x - \beta v u_x^2 = 0 & \text{(ENER.)} \end{cases}$$

I HAVE CHECKED THE LAST EQ. AND IT IS CORRECT.

ASSUME

$$\begin{cases} v = v_1 + \Delta v\, H(\xi) , & \xi = u - ct \\ u = u_1 + \Delta u\, K(\xi) \\ p = p_1 + \Delta p\, L(\xi) \end{cases}$$

WE HAVE

$$\begin{cases} v_t = -c \Delta v H' , & v_x = \Delta v H' \\ u_t = -c \Delta u K' , & u_x = \Delta u K' \\ p_t = -c \Delta p L' , & p_x = \Delta p L' \end{cases}$$

$$pv = p_1 v_1 + v_1 \Delta p L + p_1 \Delta v H + \Delta v \Delta p\, HL$$

$$(pv)_t = -c\,[v_1 \Delta p L' + p_1 \Delta v H' + \Delta v \Delta p (HL)']$$

$$= -c\,(pv)_x$$

CONS. OF MASS

$$-c\,\Delta v\,H' + (u_1 + \Delta u\,K)(\Delta v\,H') - (v_1 + \Delta v\,H)(\Delta u\,K') = 0$$

$$(u_1 - c)\,\Delta v\,H' - v_1\,\Delta u\,K' + \Delta v\,\Delta u\,(KH' - HK') = 0$$

$$[(u_1 - c)\,\Delta v + \Delta v\,\Delta u\,K]\,dH - [v_1\,\Delta u + \Delta v\,\Delta u\,H]\,dK = 0$$

$$\frac{dH}{\Delta u\,[v_1 + \Delta v\,H)} - \frac{dK}{\Delta v\,[(u_1 - c) + \Delta u\,K]} = 0$$

$$\frac{u_1 - c}{\Delta u} + K = A\left(\frac{v_1}{\Delta v} + H\right), \quad A = \text{CONST.}$$

$H, K \to 0$ AS $\xi \to -\infty$, $\Longrightarrow$ $A = \dfrac{u_1 - c}{v_1}\dfrac{\Delta v}{\Delta u}$

$H, K \to 1$ AS $\xi \to \infty$ $\Longrightarrow$

$$\frac{u_1 - c}{\Delta u} + 1 = \frac{u_1 - c}{\Delta u} + \frac{u_1 - c}{v_1}\frac{\Delta v}{\Delta u}$$

$$\Longrightarrow \frac{u_1 - c}{v_1}\frac{\Delta v}{\Delta u} = 1$$

$$u_1 - c = \frac{\Delta u}{\Delta v}\,v_1$$

$$c = u_1 - \frac{\Delta u}{\Delta v}\,v_1$$

$$\Longrightarrow \boxed{H = K}$$

FROM NOW ON, WE REPLACE K BY H EVERYWHERE.

CONS. OF MOMENTUM

$$-c\,\Delta u\,H' + (u_1 + \Delta u\,H)(\Delta u\,H') + (v_1 + \Delta v\,H)\,\Delta p\,L'$$
$$- \beta\,(v_1 + \Delta v\,H)\,\Delta u\,H'' = 0$$

$$\frac{(u_1 - c)\Delta u + (\Delta u)^2 H}{v_1 + \Delta v H} H' + \Delta p L' - \beta \Delta u H'' = 0$$

WE SUBSTITUTE FOR $u_1 - c$ FROM P13, SIMPLIFY TO GET

$$\frac{(\Delta u)^2}{\Delta v} H' + \Delta p L' - \beta \Delta u H'' = 0 \checkmark$$

$$H'' - \frac{\Delta u}{\beta \Delta v} H' = \frac{\Delta p}{\beta \Delta u} L'$$

$$\Rightarrow H' - \frac{\Delta u}{\beta \Delta v} H = \frac{\Delta p}{\beta \Delta u} L + \text{CONST.}$$

AT $\xi \to -\infty$, $H = H' = L = 0 \Rightarrow$ CONST. $= 0$!

AS $\xi \to \infty$, $L = H = 1$, $H' = 0 \Rightarrow$

$$(\Delta u)^2 = -\Delta p \Delta v$$

$$H' - \frac{\Delta u}{\beta \Delta v} H = H' + \frac{\Delta p}{\beta \Delta u} H = \frac{\Delta p}{\beta \Delta u} L$$

LET $A_1 = \dfrac{\Delta p}{\beta \Delta u}$ (A VERY LARGE NUMBER!) (SEE P24)

$$H' + A_1 H = A_1 L$$

$$\frac{d}{d\xi}\left(e^{A_1 \xi} H\right) = A_1 e^{A_1 \xi} L$$

$$H(\xi) = A_1 e^{-A_1 \xi} \int_0^{\xi} e^{A_1 \xi'} L(\xi') d\xi'$$

WE HAVE USED $H(0) = L(0) = 0$ HERE !

WE CAN WRITE H AS FOLLOWS

$$H(\xi) = A_1 \int_0^{\xi} e^{A_1(\xi' - \xi)} L(\xi') d\xi'$$

THIS ESTABLISHES THE FUNCTIONAL RELATION BETWEEN H AND L, i.e. $H = H(L)$, $H' = A_1(L - H)$ ✓

ENERGY EQ.

$$E = \alpha(\rho v)_t + \alpha u (\rho v)_x + \rho v u_x - \beta v u_x^2 = 0$$

$$(\rho v)_x = v_1 \Delta \rho L' + \rho_1 \Delta v H' + \Delta v \Delta \rho H' L$$
$$+ \Delta v \Delta \rho H L'$$
$$= \Delta \rho (v_1 + H \Delta v) L' + \Delta v (\rho_1 + L \Delta \rho) H'$$
$$= v \Delta \rho L' + \rho \Delta v H'$$
$$= v \Delta \rho L' + A_1 \rho \Delta v (L - H)$$

$$E = \alpha (u - c)[v \Delta \rho L' + A_1 \rho \Delta v (L - H)]$$
$$+ \rho v \Delta u H' - \beta v (\Delta u)^2 H'^2 = 0$$

$$u - c = u_1 - c + \Delta u H$$
$$= \frac{\Delta u}{\Delta v} v_1 + \Delta u H$$
$$= \frac{\Delta u}{\Delta v}(v_1 + \Delta v H) = \frac{\Delta u}{\Delta v} v$$

v IS ∴ CANCELED FROM E, ABOVE. SO IS Δu

$$E' = \alpha v \frac{\Delta \rho}{\Delta v} L' + \alpha A_1 \rho (L - H)$$
$$+ A_1 \rho (L - H) - \beta A_1^2 \Delta u (L - H)^2$$
$$= \alpha v \frac{\Delta \rho}{\Delta v} L' + (1 + \alpha) A_1 \rho (L - H)$$
$$- \beta A_1^2 \Delta u (L - H)^2$$
$$= 0$$

$$A_1 = \frac{\Delta \rho}{\beta \Delta u}$$

$$E' = \alpha v \frac{\Delta \rho}{\Delta v} L' + (1 + \alpha) \frac{\Delta \rho}{\beta \Delta u} \rho (L - H)$$
$$- \beta \frac{(\Delta \rho)^2}{\beta^2 (\Delta u)^2} \Delta u (L - H)^2 = 0$$

$$\alpha \beta \nu \Delta u L' + (1+\alpha) p \Delta \nu (L-H) - \Delta p \Delta \nu (L-H)^2 = 0 \qquad (*)$$

CHECK: DIMENSIONS OKAY!

ALSO AT $\xi = 0$ $L = H = L' = 0 \Rightarrow$ OKAY!

AT $\xi \to \infty^{(*)}$ $L = H = 1$, $L' = 0 \Rightarrow$ OKAY!

(*) ACTUALLY ξ = INFINITESIMAL $\neq 0$!

THE ABOVE EQUATION CAN BE SOLVED NUMERICALLY USING $L(0) = 0$. NOTE THAT β CAN BE ASSUMED INFINITESIMAL AND WE CAN USE $\beta\xi$ AS NEW VARIABLE OF DIFFERENTIATION. NUMERICAL INTEGRATION OF THE ABOVE O.D.E. WILL NOT BE EASY BECAUSE STARTING OF THE SCHEME FROM $\xi = 0$ WILL CAUSE PROBLEMS (SINCE $L = H = 0$ AT $\xi = 0$). WE WILL DO A QUALITATIVE ANALYSIS OF THE D.E. FIRST.

QUALITATIVE ANALYSIS OF D.E. FOR L

1) TAKING DERIVATIVES OF D.E. (*), ABOVE AND LETTING $\xi = 0$, WE GET

$L''(0) = 0$ SIMILARLY $L''(1) = 0$

FROM $H'' = A_1(L' - H') \Rightarrow$

$H''(0) = H''(1) = 0$

CONTINUING THIS PROCESS, WE FIND

$$L^{(n)}(0) = L^{(n)}(1) = H^{(n)}(0) = H^{(n)}(1) = 0.$$

$\Rightarrow$ i) NEAR $\xi = 0$, BOTH L AND H BEHAVE AS

$A e^{-\frac{B}{\xi^2}}$, $B > 0$

ii) NEAR $\xi = 1$, BOTH L AND H BEHAVE AS

$$A e^{-\frac{B}{(1-\xi)^2}} , \quad B > 0$$

THIS IS AN IMPORTANT RESULT.

2) AT THE POINT ξ_0 WHERE H' IS MAXIMUM, WE HAVE $H'' = A_1(L' - H') = 0$ OR $L' = H'$ $= A_1(L-H)$. FROM THE D.E. (*), WE GET

$$\alpha \beta A_1 v \Delta u (L-H) + (1+\alpha) p \Delta v (L-H) - \Delta p \Delta v (L-H)^2 = 0$$

$$(L-H)_{H'=MAX} = \alpha \beta A_1 \frac{v \Delta u}{\Delta p \Delta v} + (1+\alpha) \frac{p \Delta v}{\Delta p \Delta v}$$

$$= \alpha \beta \frac{\cancel{\Delta p}}{\cancel{\beta \Delta u}} \frac{v \cancel{\Delta u}}{\cancel{\Delta p} \Delta v} + (1+\alpha) \frac{p \cancel{\Delta v}}{\Delta p \cancel{\Delta v}}$$

$$= \alpha \frac{v}{\Delta v} + (1+\alpha) \frac{p}{\Delta p} \qquad (**)$$

NO! SEE BELOW!

$$H'_{MAX} = L'_{H_{MAX}} = A_1 \left[\alpha \frac{v}{\Delta v} + (1+\alpha) \frac{p}{\Delta p} \right]$$

THE RIGHT SIDE IS A KNOWN QUANTITY. NO!

3) H IS MAX OR MIN IF $H' = A_1(L-H) = 0$, i.e. $L - H = 0$, $L = H \Rightarrow$ FROM THE D.E. (*) THAT $L' = 0$ i.e. H IS ALSO MAX OR MIN. THIS SITUATION ALREADY HAPPENS AT $\xi = 0$ AND $\xi = \infty$ (OR SHOCK WIDTH, AN INFINITESIMAL $\neq 0$).

2 (CONT'D)

$$\alpha \frac{v}{\Delta v} + (1+\alpha) \frac{p}{\Delta p} = \alpha \frac{v_1}{\Delta v} + (1+\alpha) \frac{p_1}{\Delta p} + \underbrace{\alpha H + (1+\alpha) L}_{(1+\alpha)(L-H) + \cancel{(1+2\alpha) H}}$$

SUBSTITUTE THIS IN EQ. (**), P17, TO GET

$$\alpha(L - H) = -(1+2\alpha) H_{H'=MAX} - \alpha\frac{v_1}{\Delta v} - (1+\alpha)\frac{p_1}{\Delta p}$$

$$\text{OR } (L - H)_{H'=MAX} = -(\gamma+1) H_{H'=MAX} - \frac{v_1}{\Delta v} - \gamma\frac{p_1}{\Delta p}$$

$$H'_{MAX} = A_1\left[-(\gamma+1) H_{H'=MAX} - \frac{v}{\Delta v} - \gamma\frac{p_1}{\Delta p}\right]$$

$$L_{H'=MAX} = -\gamma H_{H'=MAX} - \frac{v_1}{\Delta v} - \gamma\frac{p_1}{\Delta p} > H_{H'=MAX}$$

WE ALSO KNOW THAT $L'_{H'=MAX} = H'_{MAX}$

WE NOTE THAT FOR ISENTROPIC SHOCK $\frac{v_1}{\Delta v} + \gamma\frac{p_1}{\Delta p} = 0$.

4. WE NOTE THAT WHEN THE VISCOSITY $\mu = 0$, $\Rightarrow \beta = 0$ AND EQ. (*), P16, BECOMES AN ALGEBRAIC EQUATION:

$$(1+\alpha)\, p\, \Delta v\,(L-H) = \Delta p\, \Delta v\,(L-H)^2$$

ONE SOLUTION IS $L = H$. ANOTHER IS

$$L - H = (1+\alpha)\frac{p}{\Delta p}$$

THIS SOLUTION IS NOT ACCEPTABLE BECAUSE

$$L(0) - H(0) = 0 \neq (1+\alpha)\frac{p_1}{\Delta p} \neq 0$$

$$L(1) - H(1) = 0 \neq (1+\alpha)(p_1 + \Delta p)/\Delta p \neq 0$$

THEREFORE, ENERGY EQUATION WILL GIVE $H = L$ WHICH IS CONSISTENT WITH MOMENTUM EQ. RESULT.

5. WE WILL SHOW THAT THE INITIAL STAGE OF THE JUMP IS ISENTROPIC! SINCE $H(0)=L(0)=0$, WE ASSUME THAT $|L-H| \ll 1$ AND THUS $(L-H)^2 \ll L-H$. THEN EQ. (*) CAN BE APPROXIMATED AS

$$\alpha \beta v \Delta u L' + \frac{1+\alpha}{A_1} p \Delta v H' = 0$$

$$\alpha \beta \frac{\Delta p}{\beta \Delta u} v \Delta u L' + (1+\alpha) p \Delta v H' = 0$$

$$\alpha \frac{\Delta p L'}{p_1 + \Delta p L} + (1+\alpha) \frac{\Delta v H'}{v_1 + \Delta v H} = 0$$

THE SOLUTION IS

$$(p_1 + \Delta p L)^{\alpha} (v_1 + \Delta v H)^{1+\alpha} = \text{CONST.}$$

$$p^{\alpha} v^{1+\alpha} = \text{CONST.}$$

$$p^{\frac{1}{\gamma-1}} v^{\frac{\gamma}{\gamma-1}} = \text{CONST.}$$

$$p v^{\gamma} = \text{CONST.} = p_1 v_1^{\gamma}$$

THIS IS THE ISENTROPIC LAW!

$\therefore$ INITIALLY, THE JUMP FUNCTIONS SATISFY

$$(p_1 + \Delta p L)(v_1 + \Delta v H)^{\gamma} = p_1 v_1^{\gamma}$$

$$\left(1 + \frac{\Delta p}{p_1} L\right)\left(1 + \frac{\Delta v}{v_1} H\right)^{\gamma} = 1$$

$$H = -\frac{v_1}{\Delta v}\left[1 - \left(1 + \frac{\Delta p}{p_1} L\right)^{-\frac{1}{\gamma}}\right]$$

LET US CHECK HOW H VARIES INITIALLY AS L IS VARIED. WE HAVE FROM NACA REP. # 1135,

$$\frac{\Delta p}{p_1} = \frac{7(M_1^2-1)}{6}, \quad \text{(PAGE 7 OF NACA REP.)}$$

$$\frac{\Delta v}{v_1} = \frac{1}{v_1}\left(\frac{1}{p_2} - \frac{1}{p_1}\right) = -v_2 \Delta p = -\frac{\Delta p}{p_2}$$

$$\frac{p_2}{p_1} = \frac{6M_1^2}{M_1^2+5} \Rightarrow \frac{p_2}{\Delta p} = \frac{6M_1^2}{5(M_1^2-1)}$$

$$\frac{\Delta p}{p_2} = \frac{5(M_1^2-1)}{6M_1^2}$$

FOR $M_1 = 2$, $\frac{\Delta p}{p_1} = \frac{7\times 3}{6} = 3.5$

$$\frac{\Delta v}{v_1} = -\frac{5\times \cancel{3}}{\cancel{6}_2\times 4} = -0.625$$

FROM MATHEMATICA:

$L = 0.01 \qquad H = 0.03884$

$L = 0.05 \qquad H = 0.17409$

$L = 0.1 \qquad H = 0.30871$

$\therefore$ INITIALLY $L < H$. (NOTE: ACTUALLY $A_1 < 0$, SEE P24, AND THEREFORE $L < H$ IS CORRECT!)

WE CAN NOW INTEGRATE (*) IN STEPS AS FOLLOWS. WE WILL ASSUME v AND p ARE CONSTANTS EVALUATED AT LEFT POINT OF AN INTERVAL $\Delta\xi$. WE WRITE EQ.(*) AS FOLLOWS FIRST:

$$\alpha\beta v \Delta u (L'-H') + \overbrace{(1+\alpha)}^{\gamma\alpha} p \Delta v (L-H) + \alpha\beta A_1 v \Delta u (L-H) - \Delta p \Delta v (L-H)^2 = 0$$

$$C_2 = (1+\alpha) p\Delta v + \alpha\beta A_1 v \Delta u = (1+\alpha) p \Delta v + \alpha \cancel{\beta} \frac{\Delta p}{\cancel{\beta}\cancel{\Delta u}} v \cancel{\Delta u}$$

$$= (1+\alpha) p \Delta v + \alpha v \Delta p$$

LET $C_1 = \alpha\beta v \Delta u$, $C_3 = \Delta p \Delta v$

$$\frac{C_1\, d(L-H)}{C_3(L-H)^2 - C_2(L-H)} = d\xi$$

THIS IS INTEGRABLE. LET $y = L - H$

$$\frac{\Delta\xi}{C_1} = \int_{y_1}^{y_2} \frac{dy}{C_3 y^2 - C_2 y}$$

$$= \frac{1}{C_2} \log\left[\frac{y_1}{C_2 - C_3 y_1} \cdot \frac{C_2 - C_3 y_2}{y_2}\right]$$

$$\frac{y_2}{C_2 - C_3 y_2} = e^{-\frac{C_2\Delta\xi}{C_1}} \frac{y_1}{C_2 - C_3 y_1}$$

$$y_2 = C_2 \frac{y_1 e^{-\frac{C_2\Delta\xi}{C_1}}}{C_2 - C_3\left(1 - e^{-\frac{C_2\Delta\xi}{C_1}}\right) y_1} \qquad (\star)$$

ALGORITHM:

GIVEN L, H AT START, $\Delta\xi$ (INFINITESIMAL)

1. CALCULATE $p = p_1 + \Delta p L$, $v = v_1 + \Delta v H$

CALCULATE $y_1 = (L-H)_{OLD}$

CALCULATE C_1, C_2, C_3

CALCULATE $y_2 = (L-H)_{NEW}$ FROM $(\star)$ ABOVE

$$\Delta H = \frac{A_1}{2}\left[(L-H)_{NEW} + (L-H)_{OLD}\right]\Delta\xi$$

$$H_{NEW} = H_{OLD} + \Delta H$$

$$L_{NEW} = (L-H)_{NEW} + H_{NEW}$$

THIS ALGORITHM WORKS BEAUTIFULLY. REMEMBER THAT BASED ON LENGTH SCALE $-\frac{1}{A_1}$, THE SHOCK WIDTH CAN BE VERY LARGE. SEE PAGES 27–31. I USED MATHEMATICA 4 TO PROGRAM THIS ALGORITHM. 12/30/99

GO TO 1, REPEAT THE ALGORITHM.

WE NOTE THAT WE CAN MAKE $A_1\Delta\xi$ & $\frac{\Delta\xi}{C_1}$ A FINITE, BUT NOT AN INFINITESIMAL, NUMBER.

22

$$\frac{C_2}{C_1} = \frac{\alpha v \Delta p + \overbrace{(1+\alpha)}^{\gamma\alpha} p \Delta v}{\alpha \underbrace{\frac{\beta \Delta u}{\Delta p}}_{A_1^{-1}} \Delta p v}$$

$$= \left(1 + \gamma \frac{\Delta v}{\Delta p} \frac{p}{v}\right) A_1 = B_1 A_1$$

$$\frac{C_3}{C_2} = \frac{\Delta p \Delta v}{\alpha v \Delta p + \gamma \alpha p \Delta v} = \frac{\gamma - 1}{\frac{v}{\Delta v} + \gamma \frac{p}{\Delta p}}$$

$$= \frac{\Delta v}{v} \frac{\gamma - 1}{1 + \gamma \frac{\Delta v}{\Delta p} \frac{p}{v}} = \frac{\gamma - 1}{B_1} \frac{\Delta v}{v} < 0$$

$$\therefore \quad y_2 = \frac{y_1 e^{-B_1 A_1 \Delta \xi}}{1 - \frac{\gamma-1}{B_1} \frac{\Delta v}{v} (1 - e^{-B_1 A_1 \Delta \xi}) y_1}$$

$$B_1 = 1 + \gamma \frac{\Delta v}{\Delta p} \frac{p}{v}$$

$$= 1 + \gamma \frac{\Delta v / v_1}{\Delta p / p_1} \frac{(p_1 + \Delta p L)/p_1}{(v_1 + \Delta v H)/v_1}$$

$$= 1 - \gamma \frac{5(\cancel{M_1^2-1})/\cancel{6} M_1^2}{7(\cancel{M_1^2-1})/\cancel{6}} \frac{1 + \frac{7(M_1^2-1)}{6} L}{1 - \frac{5(M_1^2-1)}{6 M_1^2} H}$$

$$= 1 - \frac{6 + 7(M_1^2-1)L}{6 M_1^2 - 5(M_1^2-1)H} = -\frac{(M_1^2-1)[6 + 7L - 5H]}{6 M_1^2 - 5(M_1^2-1)H}$$

$$\frac{\Delta v}{v} = \frac{\Delta v}{v_1} \frac{v_1}{v_1 + \Delta v H}$$

$$= \frac{5(M_1^2-1)}{6 M_1^2} \frac{1}{1 - \frac{5(M_1^2-1)}{6 M_1^2} H}$$

$$= -\frac{5(M_1^2-1)}{6 M_1^2 - 5(M_1^2-1)H}$$

$$\frac{C_3}{C_2} = -\frac{2}{\cancel{5}}\,\frac{\cancel{5}(\cancel{M_1^2-1})}{(\cancel{M_1^2-1})(6-7L-5H)}$$

$$= -\frac{2}{6-7L-5H}$$

$$y_2 = \frac{y_1 e^{-B_1 A_1 \Delta\xi}}{1+\dfrac{2}{6-7L-5H}\left(1-e^{-B_1 A_1 \Delta\xi}\right) y_1}$$

WE NOTE THAT IN B_1

$$0 < \frac{M_1^2-1}{6M_1^2-5(M_1^2-1)H} < \frac{M_1^2-1}{6M_1^2-5(M_1^2-1)} = \frac{M_1^2-1}{M_1^2+5}$$

(WE ALWAYS HAVE $M_1^2-1>0$) $\therefore$ B_1 CHANGES SIGN DEPENDING ON SIGN OF $\mathcal{E}=6-7L-5H$. IN PARTICULAR WHEN $L<0.5$, $H<0.5$, WE HAVE $\mathcal{E}>0$. AND IF $L>0.5$, $H>0.5$, THEN $\mathcal{E}<0$. THIS MEANS THAT THE NATURE OF THE SOLUTION CHANGES FOR $L<0.5$, $H<0.5$ AND $L>0.5$, $H>0.5$.

WE WILL CALCULATE L AND H FOR $M_1=2$

$$B_1 = \frac{3(6-7L-5H)}{24-15H} = \frac{6-7L-5H}{8-5H}$$

$$A_1 = \frac{\Delta p}{\beta \Delta u} = \frac{-\Delta p}{\beta\sqrt{-\Delta p \Delta v}} = \frac{-1}{\beta}\sqrt{-\frac{\Delta p}{\Delta v}} \qquad \boxed{\Delta u<0}$$

$$-\frac{\Delta p/p_1}{\Delta v/v_1}\,\frac{p_1}{v_1} = +\frac{7(\cancel{M_1^2-1})/\cancel{6}}{5(\cancel{M_1^2-1})/\cancel{6}M_1^2}\,\frac{p_1}{v_1} = \frac{7M_1^2}{5}\,\frac{p_1}{v_1}$$

$$\beta = \frac{4}{3}\mu = \frac{4}{3}\frac{\mu}{\rho_1}\rho_1 = \frac{4}{3}\nu_1 \rho_1, \quad \boxed{A_1 = \frac{-3M_1}{4\nu_1}\sqrt{\gamma R T_1} = \frac{-3u_1}{4\nu_1}}$$

KIN. VISC.

WE TAKE $\nu_1 = 1.461 \times 10^{-5}$ Kg s^2/m^4

$u_1 = 680$ m/S

$$A_1 = \frac{-4 \times 680}{3 \times 1.461} \times 10^5$$

$= -6.2058 \times 10^7$ VERY LARGE INDEED!

WE TAKE THE THICKNESS OF THE SHOCK $\varepsilon = \frac{1}{|A_1|}$

I PROGRAMED THE ABOVE RESULTS GETTING APPROX. $H \approx L$.
BUT I FOUND A BETTER
RESULT! SEE BELOW.

12/28/99

NEW ANALYSIS (12/29/99)

I AM GETTING SOME VERY INTERESTING RESULTS WHEN I TRY TO GET L OR H USING APPROXIMATIONS. IT IS CLEAR THAT WE HAVE A STIFF ODE HERE. TODAY I MANAGED TO GET A NONLINEAR ODE, 2ND ORDER, FOR H THAT HAS A CLOSED FORM SOLUTION. I WILL WRITE THE DETAILS BELOW. NOTE: $1+\alpha = \gamma\alpha$

$$\alpha\beta\Delta u(\nu_1 + \Delta\nu H)L' + \gamma\alpha\Delta\nu(p_1 + \Delta p L)\frac{H'}{A_1} - \frac{\Delta p \Delta\nu}{A_1^2}H'^2 = 0 \quad (1)$$

ALSO $\frac{1}{A_1}(H' + A_1 H) = \frac{H'}{A_1} + H = L \quad (2)$

$$\alpha\beta\Delta u(\nu_1 + \Delta\nu H)\left(\frac{H''}{A_1} + H'\right) + \gamma\alpha\Delta\nu\left[p_1 + \Delta p\left(\frac{H'}{A_1} + H\right)\right]\frac{H'}{A_1}$$

$$- \frac{\Delta p \Delta\nu}{A_1^2}H'^2 = 0 \quad (3$$

$$\frac{\alpha\beta\nu_1\Delta u}{A_1}H'' + \alpha(\beta\nu_1\Delta u + \gamma p_1\Delta\nu)\frac{H'}{A_1} + \frac{1}{A_1}\Big[\alpha\beta\Delta\nu\Delta$$

$$HH'' + \frac{\Delta p\Delta\nu}{A_1}(\underbrace{\gamma\alpha - 1}_{\alpha})H'^2\Big] + \alpha\Delta\nu\left(\beta\Delta u + \frac{\gamma\Delta p}{A_1}\right)HH$$

$= 0 \quad (4)$

MULTIPLY BY A_1 AND SIMPLIFY

$$A_1 \beta v_1 \Delta u + \gamma p_1 \Delta v = \frac{\Delta p}{\beta \Delta u} \beta v_1 \Delta u + \gamma p_1 \Delta v$$

$$= v_1 \Delta p + \gamma p_1 \Delta v$$

$$\alpha \beta \Delta v \Delta u \, H H'' + (\gamma\alpha - 1) \frac{\Delta p \Delta v}{A_1} H'^2 =$$

$$\alpha \beta \Delta v \Delta u \, H H'' + \Big(\underbrace{\frac{\gamma}{\gamma - 1}}_{\alpha} - 1\Big) \frac{\Delta p \Delta v}{\Delta p / \beta \Delta u} H'^2 =$$

$$\alpha \beta \Delta v \Delta u \, H H'' + \alpha \beta \Delta v \Delta u \, H'^2 =$$

$$\alpha \beta \Delta v \Delta u (H H')'$$

$$A_1 \beta \Delta u + \gamma \Delta p = \frac{\Delta p}{\beta \Delta u} \beta \Delta u + \gamma \Delta p =$$

$$= (1 + \gamma) \Delta p$$

EQ. (4) BECOMES, AFTER DIVIDING BY α

$$\beta v_1 \Delta u \, H'' + \beta \Delta v \Delta u (H H')' + (v_1 \Delta p + \gamma p_1 \Delta v) H'$$
$$+ (1+\gamma) \Delta p \, \Delta v \, H H' = 0 \quad (5)$$

WE INTEGRATE TO GET

$$\beta v_1 \Delta u \, H' + \beta \Delta v \Delta u \, H H' + (v_1 \Delta p + \gamma p_1 \Delta v) H$$
$$+ \frac{\gamma + 1}{2} \Delta p \Delta v \, H^2 = \text{CONST.} \quad (6)$$

LETTING $\xi = 0$, WE GET CONST. $= 0$.

WHEN WE LET $\xi \to \infty$, WE GET

$$v_1 \Delta p + \gamma p_1 \Delta v + \frac{\gamma+1}{2} \Delta p \Delta v \overset{?}{=} 0 \quad (7)$$

USING SHOCK JUMP CONDITIONS, AFTER WRITING THIS AS $\frac{\Delta p}{p_1} + \frac{\Delta v}{v_1} + \frac{\gamma + 1}{2} \frac{\Delta p}{p_1} \cdot \frac{\Delta v}{v_1}$, WE GET 0!

OUR FIRST INTEGRAL IS, THEREFORE,

$$\beta v_1 \Delta u H' + \beta \Delta v \Delta u HH' + (v_1 \Delta p + \gamma p_1 \Delta v) H + \frac{\gamma+1}{2} \Delta p \Delta v H^2 = 0 \qquad (8)$$

$$\beta \Delta u (v_1 + \Delta v H) H' + (v_1 \Delta p + \gamma p_1 \Delta v) H + \frac{\gamma+1}{2} \Delta p \Delta v H^2 = 0 \qquad (9)$$

WE WRITE THIS AS

$$\frac{1}{A_1} \frac{\Delta p}{p_1} \left(1 + \frac{\Delta v}{v_1} H\right) H' + \left(\frac{\Delta p}{p_1} + \gamma \frac{\Delta v}{v_1}\right) H + \frac{\gamma+1}{2} \frac{\Delta p}{p_1} \frac{\Delta v}{v_1} H^2 = 0 \qquad (10)$$

$$\frac{\Delta p}{p_1} \int_0^H \frac{\left(1 + \frac{\Delta v}{v_1} y\right) dy}{\frac{\gamma+1}{2} \frac{\Delta p}{p_1} \frac{\Delta v}{v_1} y^2 + \left(\frac{\Delta p}{p_1} + \gamma \frac{\Delta v}{v_1}\right) y} = -A_1 x \qquad (11)$$

WE WRITE $PR = \frac{\Delta p}{p_1}$, $VR = \frac{\Delta v}{v_1}$

$$PR \int_0^H \frac{(1 + VR\, y)\, dy}{\frac{\gamma+1}{2} PR \cdot VR\, y^2 + (PR + \gamma VR) y} = -A_1 x \qquad (12)$$

$$\frac{H'}{A_1} = -\frac{\frac{\gamma+1}{2} PR \cdot VR\, H^2 + (PR + \gamma VR) H}{PR (1 + VR\, H)}$$

$$L = \frac{H'}{A_1} + H = \left[1 - \frac{\frac{\gamma+1}{2} PR \cdot VR\, H + PR + \gamma VR}{PR(1 + VR\, H)}\right] H$$

$$= \frac{-VR\left(\gamma + \frac{\gamma-1}{2} PR\, H\right)}{PR(1 + VR\, H)} H \qquad (13)$$

DEC. 99 27

HERE IS MY MATHEMATICA 4. OUTPUT :

Shock Jump Microstructures

F. Farassat

December 30, 1999

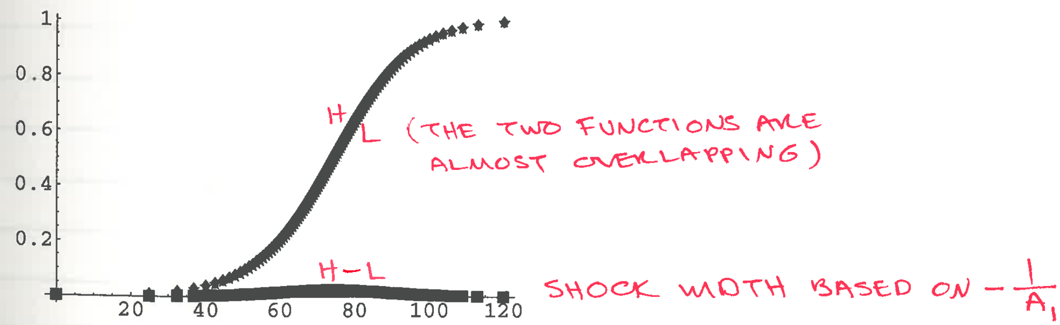
K = 0
μ = CONST.

These calculations are performed for the shock jump microstructures of u, v and p. We have used H and L for the microstructures of the Heaviside functions of u (also of v) and p, respectively. We note that shock width decreases as the Mach number increases. Also maximum of H - L increases within the shock as Mach number increases. Always H > L within the shock.

```
<< Graphics`MultipleListPlot`
```

```
M1 = 1.05; pr = 7 (M1^2 - 1)/6; vr = -5 (M1^2 - 1)/(6 M1^2); Hlist = {{0., 0.}};
Llist = {{0., 0.}}; H = 0.01; HminusL = {{0., 0.}}; ΔH = 0.01; Label[repeat];
newξ = Re[∫_{0.001}^{H} pr (1 + vr h)/((pr + 7/5 vr) h + 6/5 pr vr h^2) dh]; L = -(vr (7/5 + pr H/5) H)/(pr (1 + vr H));
Hlist = Append[Hlist, {newξ, H}]; Llist = Append[Llist, {newξ, L}];
HminusL = Append[HminusL, {newξ, H - L}]; H = H + ΔH; If[H < 1., Goto[repeat]]
```

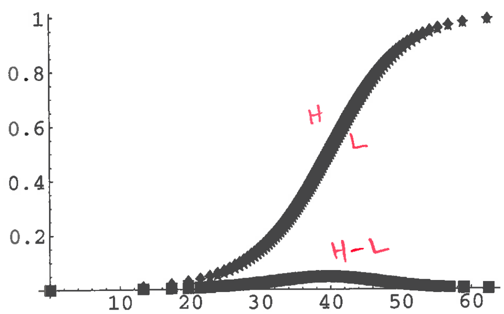
```
MultipleListPlot[Hlist, Llist, HminusL]
```



- Graphics -

```
M1 = 1.1; pr = 7 (M1^2 - 1)/6; vr = -5 (M1^2 - 1)/(6 M1^2); Hlist = {{0., 0.}};
Llist = {{0., 0.}}; H = 0.01; HminusL = {{0., 0.}}; ΔH = 0.01; Label[repeat];
newξ = Re[∫_{0.001}^{H} pr (1 + vr h)/((pr + 7/5 vr) h + 6/5 pr vr h^2) dh]; L = -(vr (7/5 + pr H/5) H)/(pr (1 + vr H));
Hlist = Append[Hlist, {newξ, H}]; Llist = Append[Llist, {newξ, L}];
HminusL = Append[HminusL, {newξ, H - L}]; H = H + ΔH; If[H < 1., Goto[repeat]]
```

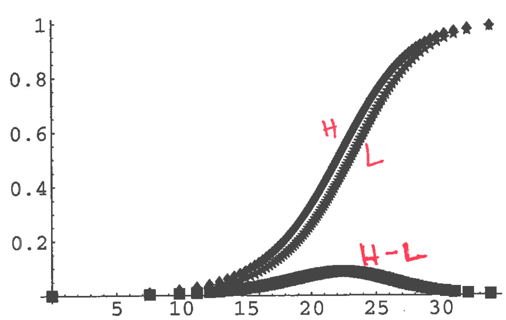
```
MultipleListPlot[Hlist, Llist, HminusL]
```



- Graphics -

```
M1 = 1.2; pr = 7 (M1^2 - 1)/6; vr = -5 (M1^2 - 1)/(6 M1^2); Hlist = {{0., 0.}};
Llist = {{0., 0.}}; H = 0.01; HminusL = {{0., 0.}}; ΔH = 0.01; Label[repeat];
newξ = Re[∫_{0.001}^{H} pr (1 + vr h)/((pr + 7/5 vr) h + 6/5 pr vr h^2) dh]; L = -(vr (7/5 + pr H/5) H)/(pr (1 + vr H));
Hlist = Append[Hlist, {newξ, H}]; Llist = Append[Llist, {newξ, L}];
HminusL = Append[HminusL, {newξ, H - L}]; H = H + ΔH; If[H < 1., Goto[repeat]]
```

```
MultipleListPlot[Hlist, Llist, HminusL]
```



- Graphics -

DEC. 99 29

$$M1 = 1.5;\ pr = \frac{7\,(M1^2 - 1)}{6};\ vr = \frac{-5\,(M1^2 - 1)}{6\,M1^2};\ Hlist = \{\{0.,\ 0.\}\};$$

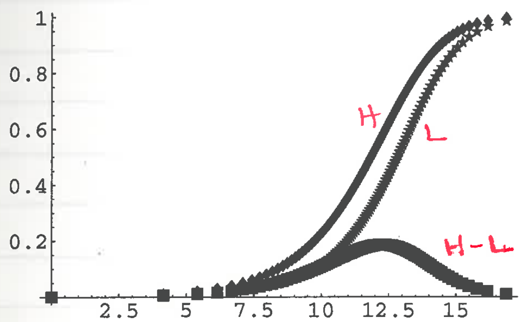
$$Llist = \{\{0.,\ 0.\}\};\ H = 0.01;\ HminusL = \{\{0.,\ 0.\}\};\ \Delta H = 0.01;\ Label[repeat];$$

$$new\xi = Re\left[\int_{0.001}^{H} \frac{pr\,(1 + vr\,h)}{(pr + \frac{7}{5}\,vr)\,h + \frac{6}{5}\,pr\,vr\,h^2}\,dh\right];\ L = -\frac{vr\,(\frac{7}{5} + \frac{pr\,H}{5})\,H}{pr\,(1 + vr\,H)};$$

$$Hlist = Append[Hlist,\ \{new\xi,\ H\}];\ Llist = Append[Llist,\ \{new\xi,\ L\}];$$

$$HminusL = Append[HminusL,\ \{new\xi,\ H - L\}];\ H = H + \Delta H;\ If[H < 1.,\ Goto[repeat]]$$

MultipleListPlot[Hlist, Llist, HminusL]



- Graphics -

$$M1 = 2;\ pr = \frac{7\,(M1^2 - 1)}{6};\ vr = \frac{-5\,(M1^2 - 1)}{6\,M1^2};\ Hlist = \{\{0.,\ 0.\}\};$$

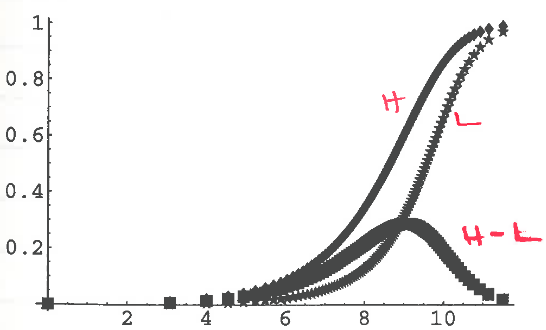
$$Llist = \{\{0.,\ 0.\}\};\ H = 0.01;\ HminusL = \{\{0.,\ 0.\}\};\ \Delta H = 0.01;\ Label[repeat];$$

$$new\xi = Re\left[\int_{0.001}^{H} \frac{pr\,(1 + vr\,h)}{(pr + \frac{7}{5}\,vr)\,h + \frac{6}{5}\,pr\,vr\,h^2}\,dh\right];\ L = -\frac{vr\,(\frac{7}{5} + \frac{pr\,H}{5})\,H}{pr\,(1 + vr\,H)};$$

$$Hlist = Append[Hlist,\ \{new\xi,\ H\}];\ Llist = Append[Llist,\ \{new\xi,\ L\}];$$

$$HminusL = Append[HminusL,\ \{new\xi,\ H - L\}];\ H = H + \Delta H;\ If[H < 1.,\ Goto[repeat]]$$

MultipleListPlot[Hlist, Llist, HminusL]



- Graphics -

30 DEC. 99

$$\text{M1} = 3;\ \text{pr} = \frac{7\,(\text{M1}^2 - 1)}{6};\ \text{vr} = \frac{-5\,(\text{M1}^2 - 1)}{6\,\text{M1}^2};\ \text{Hlist} = \{\{0.,\ 0.\}\};$$

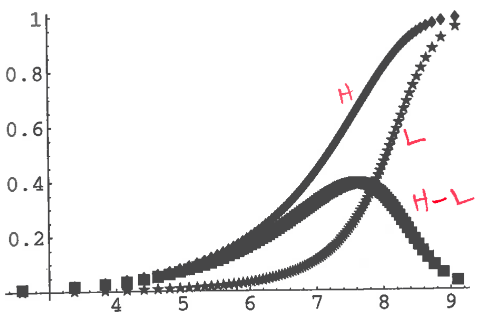
$$\text{Llist} = \{\{0.,\ 0.\}\};\ \text{H} = 0.01;\ \text{HminusL} = \{\{0.,\ 0.\}\};\ \Delta\text{H} = 0.01;\ \text{Label[repeat]};$$

$$\text{new}\xi = \text{Re}\left[\int_{0.001}^{\text{H}} \frac{\text{pr}\ (1 + \text{vr}\ \text{h})}{(\text{pr} + \frac{7}{5}\ \text{vr})\ \text{h} + \frac{6}{5}\ \text{pr}\ \text{vr}\ \text{h}^2}\, d\text{h}\right];\ \text{L} = -\frac{\text{vr}\ (\frac{7}{5} + \frac{\text{pr}\ \text{H}}{5})\ \text{H}}{\text{pr}\ (1 + \text{vr}\ \text{H})};$$

$$\text{Hlist} = \text{Append[Hlist},\ \{\text{new}\xi,\ \text{H}\}];\ \text{Llist} = \text{Append[Llist},\ \{\text{new}\xi,\ \text{L}\}];$$

$$\text{HminusL} = \text{Append[HminusL},\ \{\text{new}\xi,\ \text{H} - \text{L}\}];\ \text{H} = \text{H} + \Delta\text{H};\ \text{If}[\text{H} < 1.,\ \text{Goto[repeat]}]$$

MultipleListPlot[Hlist, Llist, HminusL]



- Graphics -

$$\text{M1} = 4;\ \text{pr} = \frac{7\,(\text{M1}^2 - 1)}{6};\ \text{vr} = \frac{-5\,(\text{M1}^2 - 1)}{6\,\text{M1}^2};\ \text{Hlist} = \{\{0.,\ 0.\}\};$$

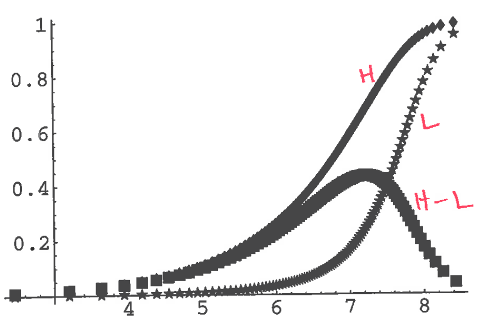
$$\text{Llist} = \{\{0.,\ 0.\}\};\ \text{H} = 0.01;\ \text{HminusL} = \{\{0.,\ 0.\}\};\ \Delta\text{H} = 0.01;\ \text{Label[repeat]};$$

$$\text{new}\xi = \text{Re}\left[\int_{0.001}^{\text{H}} \frac{\text{pr}\ (1 + \text{vr}\ \text{h})}{(\text{pr} + \frac{7}{5}\ \text{vr})\ \text{h} + \frac{6}{5}\ \text{pr}\ \text{vr}\ \text{h}^2}\, d\text{h}\right];\ \text{L} = -\frac{\text{vr}\ (\frac{7}{5} + \frac{\text{pr}\ \text{H}}{5})\ \text{H}}{\text{pr}\ (1 + \text{vr}\ \text{H})};$$

$$\text{Hlist} = \text{Append[Hlist},\ \{\text{new}\xi,\ \text{H}\}];\ \text{Llist} = \text{Append[Llist},\ \{\text{new}\xi,\ \text{L}\}];$$

$$\text{HminusL} = \text{Append[HminusL},\ \{\text{new}\xi,\ \text{H} - \text{L}\}];\ \text{H} = \text{H} + \Delta\text{H};\ \text{If}[\text{H} < 1.,\ \text{Goto[repeat]}]$$

MultipleListPlot[Hlist, Llist, HminusL]



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ONE VERY INTERESTING RESULT IS THAT THE MICRO-STRUCTURE OF THE HEAVISIDE FUNCTION T FOR ENTROPY NO LONGER EXHIBITS A MAXIMUM WITHIN THE SHOCK! HERE IS ONE CALCULATION:

```
<< Graphics`MultipleListPlot`
```

$$\text{M1} = 2;\ \text{pr} = \frac{7\,(\text{M1}^2 - 1)}{6};\ \text{vr} = \frac{-5\,(\text{M1}^2 - 1)}{6\,\text{M1}^2};$$

$$\Delta\text{sOverCv} = \text{Log}\left[\frac{7\,\text{M1}^2 - 1}{6}\right] - \frac{7}{5}\,\text{Log}\left[\frac{6\,\text{M1}^2}{\text{M1}^2 + 5}\right];\ \text{Hlist} = \{\{0.,\ 0.\}\};\ \text{Llist} = \{\{0.,\ 0.\}\};$$

$$\text{Tlist} = \{\{0.,\ 0.\}\};\ \text{HminusL} = \{\{0.,\ 0.\}\};\ \text{H} = 0.01;\ \Delta\text{H} = 0.01;\ \text{Label[repeat]};$$

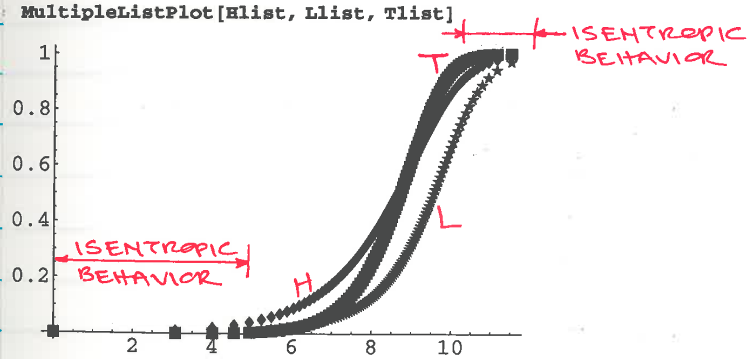
$$\text{new}\xi = \text{Re}\left[\int_{0.001}^{\text{H}} \frac{\text{pr}\,(1 + \text{vr}\,h)}{(\text{pr} + \frac{7}{5}\,\text{vr})\,h + \frac{6}{5}\,\text{pr}\,\text{vr}\,h^2}\,dh\right];\ \text{L} = -\frac{\text{vr}\,(\frac{7}{5} + \frac{\text{pr}\,\text{H}}{5})\,\text{H}}{\text{pr}\,(1 + \text{vr}\,\text{H})};$$

$$\text{T} = \frac{1}{\Delta\text{sOverCv}}\,\text{Log}\left[(1 + \text{pr}\,\text{L})\,(1 + \text{vr}\,\text{H})^{\frac{7}{5}}\right];\ \text{Hlist} = \text{Append[Hlist, \{new}\xi\text{, H\}]};$$

$$\text{Llist} = \text{Append[Llist, \{new}\xi\text{, L\}]};\ \text{Tlist} = \text{Append[Tlist, \{new}\xi\text{, T\}]};$$

$$\text{HminusL} = \text{Append[HminusL, \{new}\xi\text{, H} - \text{L\}]};\ \text{H} = \text{H} + \Delta\text{H};\ \text{If[H} < 1.\text{, Goto[repeat]]}$$

```
MultipleListPlot[Hlist, Llist, Tlist]
```



- Graphics -

DEC. 31, 1999

I HAVE SHOWN THAT AS $M_1 \to \infty$, THE SHOCK WIDTH CONVERGES TO $-\frac{8}{A_1}$. I DID THIS NUMERICALLY ALTHOUGH I CAN DO THIS ANALYTICALLY. THE NUMBER 8 IN $-8/A_1$ IS APPROXIMATE!

 DEC. 31, 1999

LITERATURE REVIEW

BESIDE THE REFERENCE ON P3 BY SALAS & IOLLO, THE FOLLOWING ARTICLES ARE ALSO RELEVANT TO OUR WORK:

1) M. MORDUCHOW AND PAUL A. LIBBY: ON A COMPLETE SOLUTION OF THE ONE-DIMENSIONAL FLOW EQUATIONS OF A VISCOUS, HEAT-CONDUCTING, COMPRESSIBLE GAS, J. AERO. SCIENCES, VOL. 16, 1949, 674-704

2) M. MORDUCHOW AND P.A. LIBBY: ON THE DISTRIBUTION OF ENTROPY THROUGH A SHOCK WAVE, J. DE MÉCHANIQUE, VOL. 4(2), 1965, 191-213

I HAVE READ BOTH THESE PAPERS AND SINCE K IS NOT ASSUMED TO BE ZERO, I WILL FIRST SOLVE THE PROBLEM FOR $\mu = \text{CONST} \neq 0$, $k = \text{CONST} \neq 0$ AND THEN DISCUSS THE RESULTS.

JAN. 8, 2000

THE CASE $k = \text{CONST.} \neq 0$, $\mu = \text{CONST.} \neq 0$

THE MASS CONT. & MOM. EQ. REMAIN THE SAME WITH THE CONCLUSIONS $K = H$ AND

$$H(\xi) = A_1 \int_0^{\xi} e^{A_1(\xi' - \xi)} L(\xi')\, d\xi'$$

OR $\quad H' = A_1(L - H)$

THE ENERGY EQ. IS:

$$E_1 = \alpha(p v)_t + \alpha u (p v)_x + v p u_x - \beta v u_x^2 - \tilde{k} v (p v)_{xx}$$

WHERE $\alpha = \frac{1}{\gamma - 1}$, $\beta = \frac{4}{3}\mu$, $\tilde{k} = \frac{k}{R}$

FROM THE ANALYSIS ON PAGE 15, WE HAVE

$$E_1 = \frac{\Delta u \cancel{v}}{2\Delta v}\Big[\alpha\beta A_1 v \Delta u L' + \gamma\alpha A_1 p \Delta v (L-H) - \beta A_1^2 \Delta u \Delta v (L-H)^2\Big] - \tilde{k}\cancel{v}(pv)_{xx} = 0 \quad (1)$$

WE WORK ON $(pv)_{xx}$ TERM NOW.

$$E_2 = [v\Delta p L' + A_1 p \Delta v (L-H)]_x \quad \text{SEE TOP OF P15}$$

$$= [v \Delta p (L' - H') + A_1 v \Delta p (L-H) + A_1 p \Delta v (L-H)]_x$$

$$= [v\Delta p (L-H)' + A_1 (v\Delta p + p \Delta v)(L-H)]_x$$

$$= A_1 \Delta v \Delta p (L-H)(L-H)' + v \Delta p (L-H)''$$

$$+ A_1 [A_1 \Delta v \Delta p (L-H) + \Delta p \Delta v (L-H)'$$

$$+ A_1 \Delta p \Delta v (L-H)](L-H)$$

$$+ A_1 (v\Delta p + p \Delta v)(L-H)'$$

$$= v\Delta p (L-H)'' + 2A_1 \Delta v \Delta p (L-H)(L-H)'$$

$$+ A_1 (v\Delta p + p\Delta v)(L-H)'$$

$$+ 2A_1^2 \Delta p \Delta v (L-H)^2 \quad (2)$$

$$A_1 = \frac{\Delta p}{\beta \Delta u} \qquad P_r = \frac{C_p \mu}{k} = \overbrace{\gamma\alpha R}^{C_p}\,\frac{\overbrace{\tfrac{4}{3}\mu}^{\beta}}{\tfrac{4}{3}k}$$

$$= \frac{3\gamma\alpha\beta}{4\tilde{k}} \quad \text{PRANDTL NO.}$$

SINCE $P_r = O(1)$, β INFINITESIMAL $\Rightarrow$ $\tilde{k}$ INFINITESIMAL

WE HAVE $\tilde{K} = \frac{3\gamma\alpha}{4P_r}\beta \equiv \delta\beta$ WITH δ DEFINED BY THE RELATION

$$\delta = \frac{3\gamma\alpha}{4P_r} = O(1)\,!$$

WE WRITE EQ.(1) AS FOLLOWS

$$E_1 \equiv \frac{\Delta u}{\Delta v} E_3 - \delta\beta E_2 = 0 \quad (*)$$

$$\Delta u\, E_1 = \frac{(\Delta u)^2}{\Delta v} E_3 - \delta\beta\Delta u\, E_2 = 0$$

$$= -\frac{\Delta p \cancel{\Delta v}}{\cancel{\Delta v}} E_3 - \delta\frac{\beta\Delta u}{\Delta p}\Delta p E_2 = 0$$

$$= -\Delta p E_3 - \frac{\delta}{A_1}\Delta p E_2 = 0$$

$$A_1 E_3 + \delta E_2 = 0 \quad (3)$$

WE NOW WORK ON E_3 AS FOLLOWS

$$E_3 = \alpha\beta A_1 v \Delta u L' + \gamma\alpha A_1 p\Delta v (L-H) - \beta A_1^2 \Delta u \Delta v (L-H)^2 \quad \text{(TOP OF P16)}$$

$$= \alpha\Delta p v (L-H)' + \alpha A_1 \Delta p v (L-H) + \alpha\gamma A_1 p\Delta v (L-H) - A_1\Delta p\Delta v (L-H)^2$$

$$= \alpha\Delta p v (L-H)' + \alpha A_1 (v\Delta p + \gamma p\Delta v) \times (L-H) - A_1 \Delta p \Delta v (L-H)^2 \quad (4)$$

EQ.(3) BECOMES:

(*) NOTE: E ON P15 IS $\frac{v\Delta u}{\Delta v}E_3$. v IS CANCELED FROM $\tilde{K}v$ (P... IN THIS EQ.

$$\delta v \Delta p (L-H)'' + 2 A_1 \delta \Delta v \Delta p (L-H)(L-H)'$$
$$+ A_1 [\alpha v \Delta p + \delta (v \Delta p + p \Delta v)] (L-H)'$$
$$+ \alpha A_1 (v \Delta p + \gamma p \Delta v)(L-H)$$
$$+ (-1 + 2\delta) A_1^2 \Delta p \Delta v (L-H)^2 = 0$$

$$(L-H)'' + 2A_1 \frac{\Delta v}{v} \delta (L-H)(L-H)'$$
$$+ A_1 \left[\frac{\alpha+\delta}{\delta} + \frac{p}{\Delta p} \frac{\Delta v}{v} \right) (L-H)'$$
$$+ \frac{\alpha}{\delta} A_1 \left(1 + \gamma \frac{p}{\Delta p} \frac{\Delta v}{v} \right) (L-H)$$
$$+ \left(2 - \frac{1}{\delta} \right) A_1^2 \frac{\Delta v}{v} (L-H)^2 = 0 \qquad (5)$$

LET $y = L-H, \quad Z = \frac{1}{y'} = \frac{dx}{dy}$

$$Z y' = 1 \Rightarrow Z' y' + Z y'' = 0$$

$$y'' = - \frac{Z'}{Z^2}, \quad y' = \frac{1}{Z}$$

WE WRITE EQ. (5) AS

$$y'' + C_1 y y' + C_2 y' + C_3 y + C_4 y^2 = 0 \qquad (6)$$

$$- \frac{Z'}{Z^2} + C_1 \frac{y}{Z} + \frac{C_2}{Z} + C_3 y + C_4 y^2 = 0$$

$$- Z' + (C_1 y + C_2) Z + (C_3 y + C_4 y^2) Z^2 = 0 \qquad (7)$$

THIS HAS A VERY COMPLICATED SOLUTION AND APPLYING BC IS DIFFICULT. FOR NOW WE ABANDON THIS METHOD. THERE IS ALWAYS A WAY!

WE NOTE THAT, IN TERMS OF H, E_3 IS INTEGRABLE! SEE PAGES 24 & 25.

$$E_3 = \alpha[\beta v_1 \Delta u H'' + \beta \Delta v \Delta u (HH')' + (v_1 \Delta p + \gamma p_1 \Delta v) H' + (1+\gamma) \Delta p \Delta v HH'] \quad (8)$$

($\frac{A_1}{\alpha} E_3$ IS EQ. (5), P25!)

$$E_2 = (pv)_{xx}$$

$\therefore$ $A_1 E_3 + \delta E_2 = 0$ IS INTEGRABLE!

THE FIRST INTEGRAL IS

$$\alpha A_1 [\beta v_1 \Delta u H' + \beta \Delta v \Delta u HH' + (v_1 \Delta p + \gamma p_1 \Delta v) H + \frac{(\gamma+1)}{2} \Delta p \Delta v H'^2] + \delta (pv)_x = \text{CONST.} \quad (9)$$

SINCE $p_x(0) = 0$, $v_x(0) = 0$, CONST. $= 0$!

$$\begin{aligned}(pv)_x &= v \Delta p L' + A_1 p \Delta v (L-H) \\ &= (v_1 + \Delta v H) \Delta p (L' - H') \\ &\quad + (v_1 + \Delta v H) \Delta p H' \\ &\quad + A_1 \Delta v [p_1 + \Delta p (1-H) + \Delta p H](L-H) \\ &= \frac{1}{A_1} v_1 \Delta p H'' + \frac{1}{A_1} \Delta v \Delta p HH'' \\ &\quad + v_1 \Delta p H' + \Delta v \Delta p HH' \\ &\quad + p_1 \Delta v H' + \frac{1}{A_1} \Delta v \Delta p H'^2 \\ &\quad + \Delta v \Delta p HH' \\ &= \frac{1}{A_1} v_1 \Delta p H'' + \frac{1}{A_1} \Delta v \Delta p (HH'' + H'^2) \\ &\quad + 2 \Delta v \Delta p HH' + (v_1 \Delta p + p_1 \Delta v) H' \quad (10)\end{aligned}$$

THIS IS INTEGRABLE BUT WE KNOW THIS BECAUSE THIS EXPRESSION IS $(pv)_x$!

WE GET

$$A_1 E_3 + \delta E_2 = C_1 H'' + C_2 (H^2)'' + C_3 H' + C_4 (H^2)' + C_5 H + C_6 H^2 = 0 \quad (11)$$

WHERE

$$\begin{cases} C_1 = \dfrac{\delta}{A_1} v_1 \Delta p \checkmark \\ C_3 = (\alpha + \delta) v_1 \Delta p + \delta p_1 \Delta v \checkmark \\ C_5 = \alpha A_1 (v_1 \Delta p + \gamma p_1 \Delta v) = -\dfrac{\alpha(\gamma+1)}{2} A_1 \Delta v \Delta p \end{cases} \quad (12)$$

$$\begin{cases} C_2 = \dfrac{\delta}{2A_1} \Delta v \Delta p \checkmark \\ C_4 = \dfrac{1}{2}(2\delta + \alpha) \Delta v \Delta p \checkmark \\ C_6 = \dfrac{\alpha(\gamma+1)}{2} A_1 \Delta v \Delta p = -C_5 \end{cases} \quad (13)$$

(SEE P25, EQ. (7))

AT LEAST WE KNOW THAT WHEN $H^2 \ll H$, i.e. NEAR THE START OF THE RISE, THE ODE FOR H IS

$$C_1 H'' + C_3 H' + C_5 H = 0 \quad (14)$$

WHICH HAS A CLOSED FORM SOLUTION.

THE GENERAL EQ. IS

$$C_1 H'' + C_3 H' + C_5 H + C_2 (H^2)'' + C_4 (H^2)' + C_6 H^2 = 0 \quad (15)$$

THIS DOES NOT HAVE A CLOSED FORM SOLUTION WITHOUT SOME KIND OF APPROXIMATION. WE CAN GET INITIAL CONDITIONS H AND H', BOTH NON-ZERO FROM $p v^\gamma = p_1 v_1^\gamma$ BECAUSE INITIALLY THE JUMP IS ISENTROPIC.

DIVIDE EQ. (15) BY $\dfrac{1}{p_1 v_1}$, WE GET

$$\tilde{C}_1 H'' + \tilde{C}_3 H' + \tilde{C}_5 H + \tilde{C}_2 (H^2)'' + \tilde{C}_4 (H^2)' + \tilde{C}_6 H^2 = 0 \quad (16)$$

$$\begin{cases} \tilde{C}_1 = \dfrac{\delta}{A_1} \dfrac{\Delta p}{p_1} \\ \tilde{C}_3 = (\alpha + \delta) \dfrac{\Delta p}{p_1} + \delta \dfrac{\Delta v}{v_1} \\ \tilde{C}_5 = -\dfrac{\alpha(\gamma+1)}{2} \dfrac{\Delta p}{p_1} \dfrac{\Delta v}{v_1} A_1 \end{cases} \qquad (17)$$

$$\begin{cases} \tilde{C}_2 = \dfrac{\delta}{2A_1} \dfrac{\Delta p}{p_1} \dfrac{\Delta v}{v_1} \\ \tilde{C}_4 = \dfrac{1}{2}(2\delta + \alpha) \dfrac{\Delta p}{p_1} \dfrac{\Delta v}{v_1} \\ \tilde{C}_6 = \dfrac{\alpha(\gamma+1)}{2} \dfrac{\Delta p}{p_1} \dfrac{\Delta v}{v_1} A_1 \end{cases} \qquad (18)$$

WE WILL NOW MAKE AN IMPORTANT SCALE CHANGE AS FOLLOWS: $\xi = -A_1 x$ (NOTE $A_1 < 0$). DIVIDE EQ (16) BY A_1:

$$\frac{\tilde{C}_1}{A_1} H'' + \frac{\tilde{C}_3}{A_1} H' + \frac{\tilde{C}_5}{A_1} H + \cdots\cdots = 0 \qquad (19)$$

LET US WRITE $\tilde{H}(\xi) = H(-A_1 x)$. EQ. (19) IS

$$D_1 \tilde{H}'' + D_3 \tilde{H}' - D_5 \tilde{H} + D_2 (\tilde{H}^2)'' + D_4 (\tilde{H}^2)' - D_6 \tilde{H}^2 = 0 \qquad (20)$$

WHERE $(\ )' = \dfrac{d}{d\xi}$. WE WILL DROP TYLDE ON $\tilde{H}$.

$$\begin{cases} D_1 = \delta \dfrac{\Delta p}{p_1} \\ D_3 = (\alpha + \delta) \dfrac{\Delta p}{p_1} + \delta \dfrac{\Delta v}{v_1} \\ D_5 = -\dfrac{\alpha(\gamma+1)}{2} \dfrac{\Delta p}{p_1} \dfrac{\Delta v}{v_1} \end{cases} \qquad (21)$$

$$\begin{cases} D_2 = \dfrac{\delta}{2} \dfrac{\Delta p}{p_1} \dfrac{\Delta v}{v_1} \\ D_4 = \dfrac{1}{2}(2\delta + \alpha) \dfrac{\Delta p}{p_1} \dfrac{\Delta v}{v_1} \\ D_6 = \dfrac{\alpha(\gamma+1)}{2} \end{cases} \qquad (22)$$

Shock Microstructure of Viscous and Heat Conducting Fluid

μ = const., k = const., Pr = $\frac{3}{4}$

F. Farassat

January 15, 2000

We are going to plot the shock microstructure of the Heaviside functions of the parameters of a viscous, heat conducting fluid. We use the following functions for the microstructure of the Heaviside functions of the jumps:

H for velocity and specific volume,
L for the pressure
N for the temperature
E for the entropy

Subscript 1 denotes the conditions upstream of the shock.

See the notes in my Research Notebook on Nonstandard Analysis, pages 32 to 38.

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Variable Definitions (M1 = 1.5)

M1 = 1.5
(* Mach number upstream of the shock *);

$$pr = \frac{7\,(M1^2 - 1)}{6} \quad (\text{* This is } \frac{\Delta p}{p1}. \text{ *});$$

$$vr = -\frac{5\,(M1^2 - 1)}{6\,M1^2} \quad (\text{* This is } \frac{\Delta v}{v1}. \text{ *});$$

$$tr = \frac{(M1^2 - 1)\,(7\,M1^2 + 5)}{36\,M1^2}$$

(* This is $\frac{\Delta T}{T1}$, T is the temperature. *);

$\gamma = 1.4$
(* This is ratio of specific heats. *);

$$\alpha = \frac{1}{\gamma - 1};\ \delta = \alpha\,\gamma$$

(* This is $\frac{3\,\alpha\,\gamma}{4\,\text{PrandtlNo}}$,
PrandtlNo is taken equal to $\frac{3}{4}$. *);

$d1 = \delta\,pr;\ d3 = (\delta + \alpha)\,pr + \delta\,vr;$

$$d5 = -\frac{\alpha\,(\gamma + 1)}{2}\,pr\,vr;\ d2 = \frac{\delta}{2}\,pr\,vr;$$

$$d4 = \frac{(2\,\delta + \alpha)}{2}\,pr\,vr;\ d6 = \frac{\alpha\,(\gamma + 1)}{2}\,pr\,vr$$

(* d1 to d6 are the coefficients of
the nonlinear ordinary differential
equation d1 H''+ d3 H' + d5 H +
d2 $(H^2)''$+ d4 $(H^2)'$+ d6 H^2 = 0 *);

Solution of the Nonlinear ODE for H (M1 = 1.5)

```
sol =
  NDSolve[
   {d1 H''[x] + d3 H'[x] - d5 H[x] +
      2 d2 (H[x] H''[x] + H'[x]^2) +
      d4 H[x] H'[x] - d6 H[x]^2 == 0,
    H[0.001] == 0.001,
    H'[0.001] == (M1^2 - 1)/M1^2 0.001}, H,   (*)
   {x, 0.001, 40}] ;
Plot[Evaluate[H[x] /. sol], {x, 0.001, 40},
 PlotRange → All]
```

(*) WE HAVE USED ISENTROPIC CONDITION HERE

$p v^{\gamma} = (p_1 + \Delta p L)(v_1 + \Delta v H)^{\gamma} = \text{CONST.}$

$\Delta p L' v_1 + \gamma p_1 \Delta v H' = 0$ (APPROXIMATION WHEN $H \ll 1, L \ll 1$)

$\Delta p v_1 L + \gamma \Delta v p_1 H = 0$

$\Delta p v_1 (H - H') + \gamma p_1 \Delta v H = 0$

$H' = \left(1 + \frac{\gamma p_1}{\Delta p} \frac{\Delta v}{v_1}\right) H$ $\left\{ \begin{array}{l} H' \text{ IS REALLY} \\ \frac{d}{d\xi} \tilde{H}(\xi) \end{array} \right.$

$= \frac{M_1^2 - 1}{M_1^2}$

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Determination of L, N and E (M1 = 1.5)

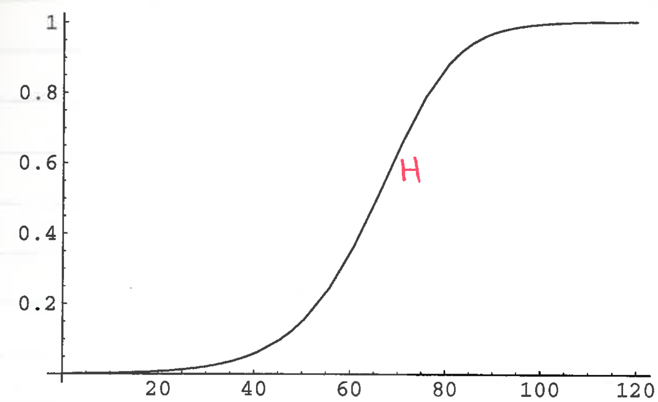
```
M1 = 1.5;
ΔsOverCv = Log[(7 M1^2 - 1)/6] - 7/5 Log[(6 M1^2)/(M1^2 + 5)]
  (* This is ΔS/Cv *);
Plot[
 Evaluate[
  {H[x], H[x] - H'[x]
    (* This is L,
     the Heaviside function of the
      pressure. *),
    1/ΔsOverCv
     Log[(1 + pr (H[x] - H'[x]))
       (1 + vr H[x]) ^ (7/5)]
     (* This is E,
      the Heaviside function of the
       entropy. *),
     1/tr
      (-1 + (1 + pr (H[x] - H'[x])) (1 + vr H[x]))}
    (* This is N,
     the Heaviside function of the
      temperature. *) /. sol], {x, 0.01, 40},
  PlotRange → All]
```

RESULTS

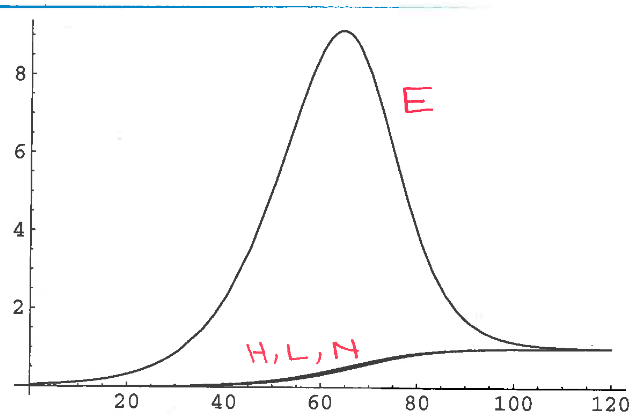
$M = 1.1$

{d1, d2, d3, d4, d5, d6, δ, pr, vr, tr}

{0.8575, -0.0620093, 0.963802, -0.168311,
0.106302, -0.106302, 3.5, 0.245, -0.144628, 0.064938}



THICKNESS (DISTANCE) BASED ON $-\frac{1}{A_1}$)

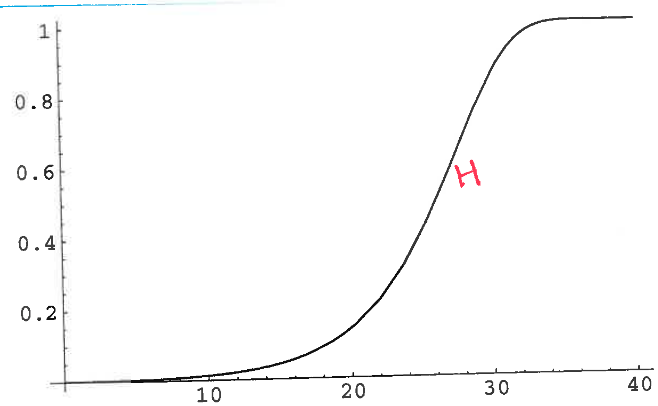


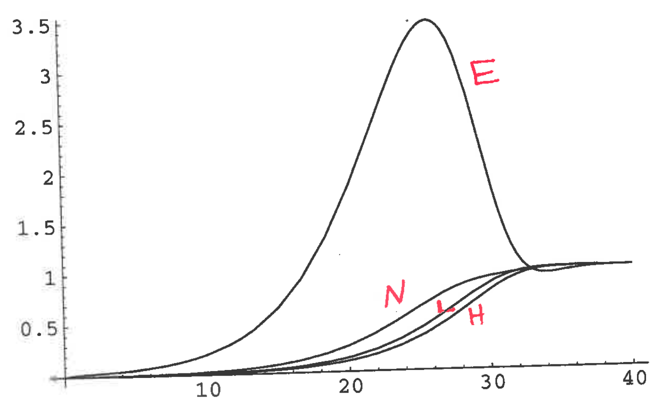
44

M = 1.5

{d1, d2, d3, d4, d5, d6, δ, pr, vr, tr}

{5.10417, -1.18152, 7.12963,
-3.20698, 2.02546, -2.02546, 3.5,
1.45833, -0.462963, 0.320216}



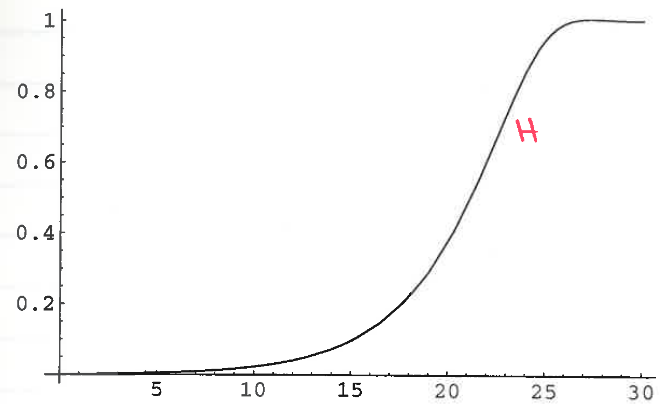


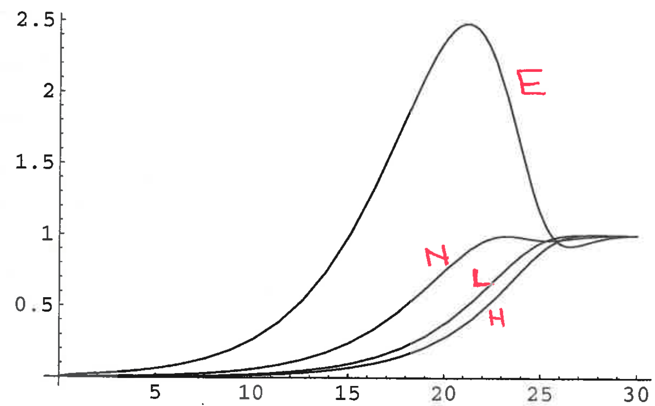
45

M = 2.0

{d1, d2, d3, d4, d5, d6, δ, pr, vr, tr}

{12.25, -3.82813, 18.8125, -10.3906, 6.5625, -6.5625, 3.5, 3.5, -0.625, 0.6875}



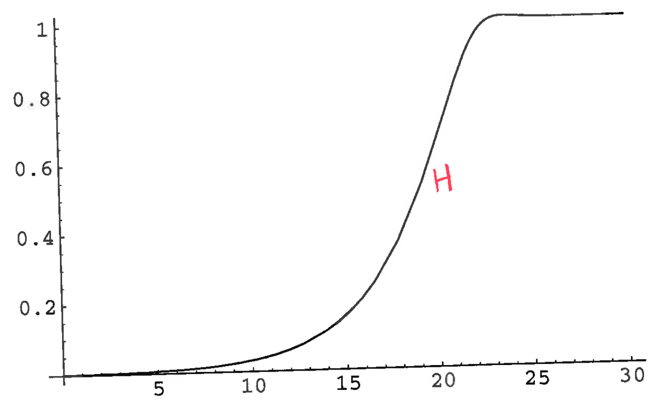


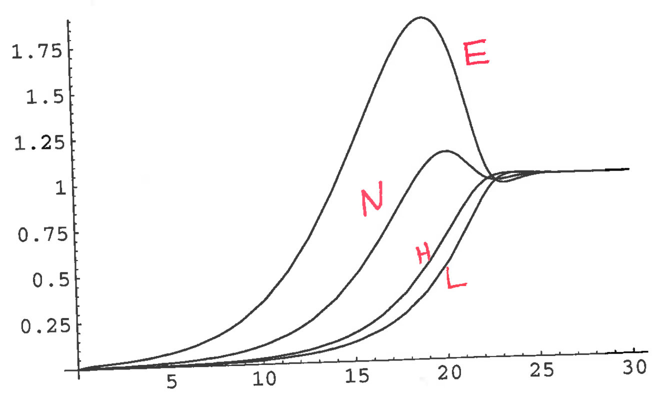
46

M = 3.0

{d1, d2, d3, d4, d5, d6, δ, pr, vr, tr}

{32.6667, -12.0988, 53.4074,
-32.8395, 20.7407, -20.7407,
3.5, 9.33333, -0.740741, 1.67901}

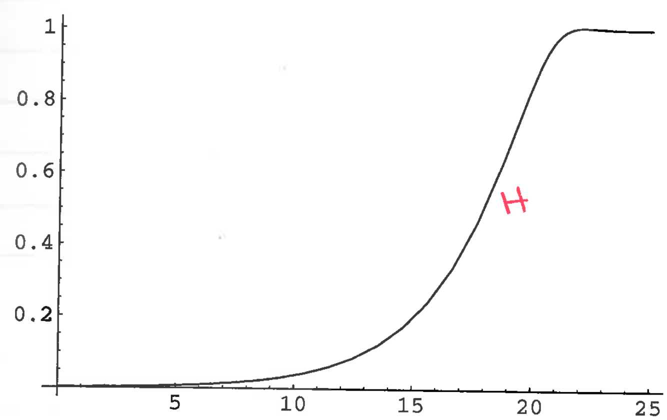


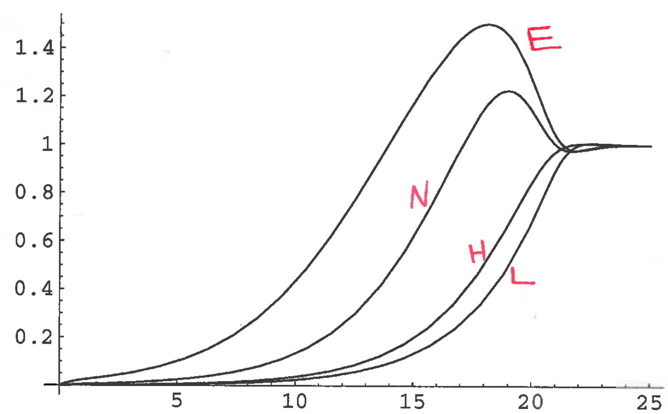


M = 5.

{d1, d2, d3, d4, d5, d6, δ, pr, vr, tr}

{98., -39.2, 165.2, -106.4,
67.2, -67.2, 3.5, 28., -0.8, 4.8}

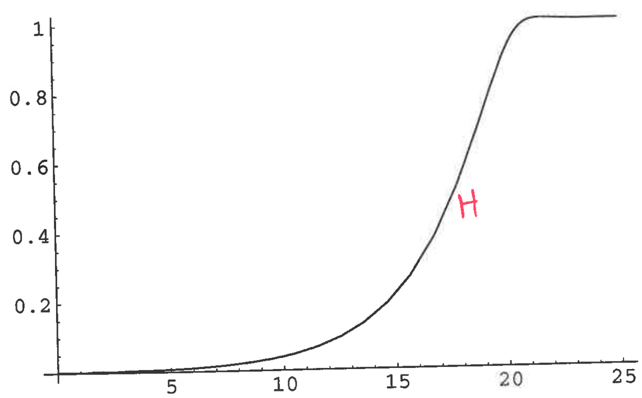


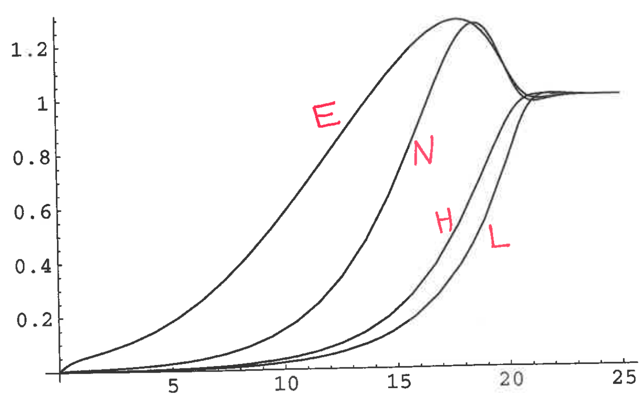


~~M = 10. (NOT REALISTIC BECAUSE WE DO NOT INCLUDE DISSOCIATION AND IONIZATION OF THE GAS)~~

{d1, d2, d3, d4, d5, d6, δ, pr, vr, tr}

{404.25, -166.753, 690.113, -452.616, 285.863, -285.863, 3.5, 115.5, -0.825, 19.3875}





WE NOTE THAT i) THE SHOCK THICKNESS DECREASES AS M_1 INCREASES, ii) THE HEAVISIDE FUNCTIONS E AND N, FOR ENTROPY AND TEMPERATURE HAVE A PEAK ABOVE THE VALUE 1 WITH A STRANGE BEHAVIOR — Max E ↓ AS M_1 ↑ WHILE MAX N ↑ AS M_1 ↑ !

$\boxed{\text{[signature] 1-18-00}}$

16 Notes on Differential Geometry

NOTES ON

DIFFERENTIAL GEOMETRY

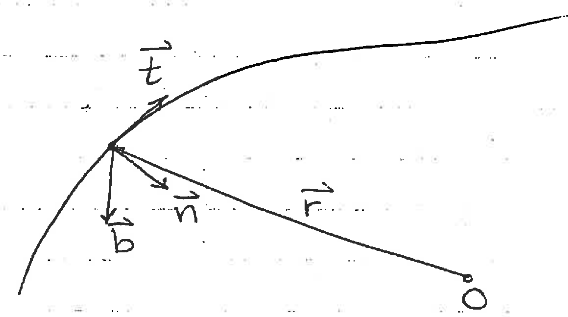
BY F. FARASSAT

SUMMER 1987

CURVE THEORY

$$\vec{t} = \frac{d\vec{r}}{ds} \equiv \vec{r}\,'$$

$\vec{n}$ = NORMAL



$$\frac{d\vec{t}}{ds} = |\vec{r}\,''|\,\vec{n}$$
$$= \kappa\,\vec{n} \qquad (1)$$

$|\vec{r}\,''| = \kappa$ CURVATURE

— $\kappa \geqslant 0$ ALWAYS

— $\vec{n}$ ALWAYS POINTS TOWARD THE CENTER OF CURVATURE

$\vec{b} = \vec{t} \times \vec{n}$ THE BINORMAL

$$(2) \qquad \frac{d\vec{b}}{ds} = -\tau\,\vec{n}$$ (SOME BOOKS DEFINE WITH +VE SIGN)

$$(3) \qquad \frac{d\vec{n}}{ds} = +\tau\,\vec{b} - \kappa\,\vec{t}$$

$$\boxed{\frac{d}{ds}\begin{bmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{bmatrix}}$$

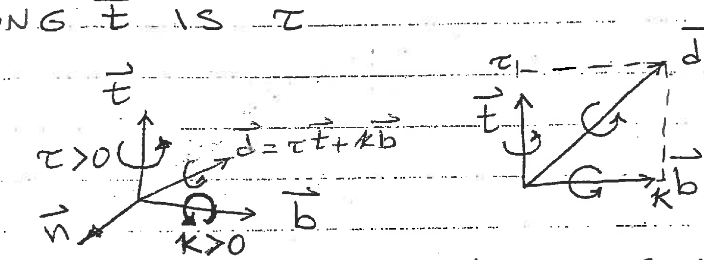
τ = TORSION CAN BE POSITIVE OR NEGATIVE

$[\vec{t}, \vec{n}, \vec{b}]$ FRENET TRIHEDRON

(1), (2) & (3) TOGETHER ARE CALLED THE FRENET - SERRET FORMULAS

WHAT IS THE MEANING OF +VE TORSION ?

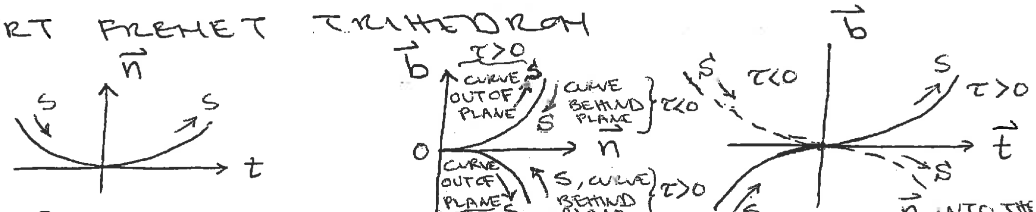
— IF WE TAKE s = TIME AND THINK OF THE FRENET TRIHEDRON AS A SOLID CUBE THEN THE ANGULAR VELOCITY OF THE CUBE ALONG $\vec{t}$ IS τ



— THE NET ANGULAR VELOCITY VECTOR IS

$$\vec{d} = \tau\vec{t} + \kappa\vec{b} \quad \text{THE DARBOUX VECTOR}$$

— THE SHAPE OF PROJECTION OF A CURVE WRT FRENET TRIHEDRON



PARABOLA
$\vec{b}$ OUT OF THE PAGE.

CUBIC PARABOLA
$\vec{t}$ OUT OF THE PAGE

CUBIC
BOTH CURVES BEHIND PLANE

$\vec{r} \approx s\vec{t} + \kappa s\vec{n} + \kappa\tau s\vec{b}$
IN THE VICINITY OF $s=0$

— FUNDAMENTAL THEOREM OF CURVE THEORY:
GIVEN TWO CONTINUOUS FNS $\kappa(s) > 0$ AND $\tau(s)$, $s \in [a,b] \Rightarrow \exists$ A CURVE WITH s AS LENGTH PARAMETER $\ni$ $\kappa(s)$ AND $\tau(s)$ ARE CURVATURE AND TORSION OF THE CURVE. TWO SUCH CURVES COINCIDE EXCEPT FOR THE POSITION IN SPACE.

— IF WE CHANGE THE DIRECTION OF INCREASING s, $\vec{t}$ AND $\vec{b}$ CHANGE SIGN BUT τ DOES NOT CHANGE SIGN. WE ALWAYS HAVE $\kappa > 0$. THEREFORE, τ IS AN INTRINSIC PROPERTY OF A CURVE. SEE THE FIG. SHOWING THE BEHAVIOR OF THE CURVE IN $\vec{t}\vec{b}$-PLANE.

THEORY OF SURFACES

SURFACES ARE SPECIFIED BY

i) $z = f(x, y)$

ii) $f(x, y, z) = 0$

iii) $x = \phi_1(u^1, u^2)$, $y = \phi_2(u^1, u^2)$, $z = \phi_3(u^1, u^2)$

OR $\vec{r} = \vec{\phi}(u^1, u^2)$ (GAUSS)

WE ASSUME THAT THE 3RD SPECIFICATION IS USED. LET $\vec{r} = \phi(u^1, u^2)$

TANGENTS TO THE SURFACE

(NATURAL) **BASIS VECTORS**
$$\begin{cases} \vec{r}_1 = \dfrac{\partial \vec{r}}{\partial u^1} \\ \vec{r}_2 = \dfrac{\partial \vec{r}}{\partial u^2} \end{cases}$$
(NOTE: THESE ARE NOT NECESSARILY UNIT VECTORS)

$$d\vec{r} = \vec{r}_1\, du^1 + \vec{r}_2\, du^2$$

$$d\vec{r} \cdot d\vec{r} = ds^2$$

$$= (\vec{r}_1 \cdot \vec{r}_1)(du^1)^2 + 2(\vec{r}_1 \cdot \vec{r}_2)du^1 du^2$$

$$+ (\vec{r}_2 \cdot \vec{r}_2)(du^2)^2$$

$$\boxed{ds^2 \equiv g_{ij}\, du^i du^j}$$
FIRST FUNDAMENTAL FORM

NOTE: $g_{ij} = g_{ij}(u^1, u^2)$

$$g_{ij} = \vec{r}_i \cdot \vec{r}_j \Rightarrow g_{12} = g_{21}$$
← COEFFICIENTS OF FIRST FUNDAMENTAL FORM

LET
$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11} g_{22} - (g_{12})^2$$

ALSO
$$g^{-1} \equiv \begin{vmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{vmatrix} \Rightarrow g^{11} = \frac{g_{22}}{g}\,;\quad g^{22} = \frac{g_{11}}{g}\,;$$
$$g^{12} = g^{21} = -\frac{g_{12}}{g}$$

(5) (4)

WE WILL SEE THE USES OF g_{ij} AND g^{ij}.

i) LENTH OF A VECTOR $\vec{a} = a^1\vec{r}_1 + a^2\vec{r}_2 \equiv (a^1, a^2)$

$$|\vec{a}| = (g_{ij}\, a^i a^j)^{1/2}$$

— LENGTH OF THE TANGENT VECTORS $\vec{r}_1 = (1,0)$, $\vec{r}_2 = (0,1)$

$$|\vec{r}_1| = \sqrt{g_{11}} \quad , \quad |\vec{r}_2| = \sqrt{g_{22}}$$

ii) COSINE OF ANGLE BETWEEN TO VECTORS $\vec{a} = (a^1, a^2)$ AND $\vec{b} = (b^1, b^2)$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{g_{ij}\, a^i b^j}{\sqrt{g_{ij}a^i a^j}\cdot\sqrt{g_{ij}b^i b^j}}$$

— ANGLE BETWEEN BASIS VECTORS

$$\cos\omega = \frac{g_{12}}{\sqrt{g_{11}\, g_{22}}}$$

<u>NOTE</u>: THIS IS A FUNCTION OF (u^1, u^2). ALSO $\omega = 90°$ IF $g_{12} = 0$

<u>NOTE</u>: LET $\vec{a} = (a^1, a^2)$, SINCE $g_{ij} = g_{ij}(u^1, u^2)$ EVEN IF $a^1 = \text{CONST.}$, $a^2 = \text{CONST.}$, THE LENGTH OF $\vec{a}$ <u>VARIES</u> OVER THE SURFACE.

— LET $G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \Rightarrow G^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}$

$$G G^{-1} = G^{-1} G = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow g_{ij}\, g^{jk} = \delta_i^k$$

ALSO NOTE THAT

$$g = \det G = g(u^1, u^2)$$

THE $\det G = g$ IS CALLED THE DISCRIMINANT OF THE FIRST FUNDAMENTAL FORM.

A USEFUL RESULT: $|\vec{r_1} \times \vec{r_2}|^2 = g$

PROOF

$$(\vec{r_1} \times \vec{r_2}) \cdot (\vec{r_1} \times \vec{r_2}) = \vec{r_1} \cdot (\vec{r_2} \times (\vec{r_1} \times \vec{r_2}))$$

$$= \vec{r_1} \cdot ((\vec{r_2} \cdot \vec{r_2})\, \vec{r_1} - (\vec{r_2} \cdot \vec{r_1})\, \vec{r_2})$$

$$= (\vec{r_1} \cdot \vec{r_1})(\vec{r_2} \cdot \vec{r_2}) - (r_1 \cdot \vec{r_2})^2$$

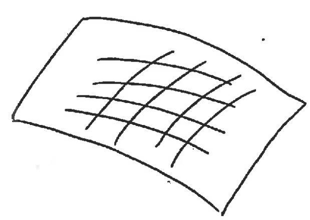
$$= g_{11}\, g_{12} - (g_{12})^2$$

$$= g$$

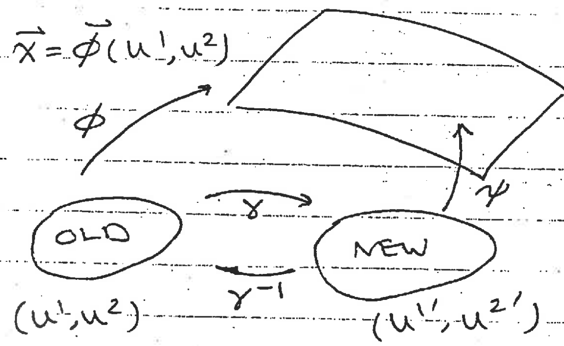
— THE POINTS AT WHICH $g > 0$ ON THE SURFACE ARE CALLED REGULAR POINTS OF THE SURFACE. THE ABOVE RESULT SHOWS THAT $g \geqslant 0$ ALWAYS. THE SINGULAR POINTS OF A SURFACE ARE THOSE AT WHICH $g = 0$.

SURFACE INTEGRAL

$$I = \int Q(u^1, u^2) \sqrt{g}\, du^1 du^2$$



CHANGE OF VARIABLES



$$u^{i'} = \gamma^{i'}(u^i)$$
$$u^i = \gamma(u^{i'})$$

$$\vec{r}_{i'} = \frac{\partial \vec{r}}{\partial u^i}\frac{\partial u^i}{\partial u^{i'}} = \frac{\partial u^i}{\partial u^{i'}}\vec{r}_i$$

$$g_{i'j'} = \vec{r}_{i'} \cdot \vec{r}_{j'} = \frac{\partial u^i}{\partial u^{i'}}\frac{\partial u^j}{\partial u^{j'}} g_{ij}$$

$$g_{ij} = \frac{\partial u^{i'}}{\partial u^i}\frac{\partial u^{j'}}{\partial u^j} g_{i'j'}$$

THESE RELATIONS MEAN THAT g_{ij} IS A COVARIANT TENSOR OF RANK 2. SIMILARLY, WE CAN SHOW THAT

$$g^{i'j'} = \frac{\partial u^{i'}}{\partial u^i}\frac{\partial u^{j'}}{\partial u^j} g^{ij}$$

$$g^{ij} = \frac{\partial u^i}{\partial u^{i'}}\frac{\partial u^j}{\partial u^{j'}} g^{i'j'}$$

g^{ij} IS A CONTRAVARIANT TENSOR OF RANK 2

LET $\vec{a} = a^1\vec{r}_1 + a^2\vec{r}_2 = a^{1'}\vec{r}_{1'} + a^{2'}\vec{r}_{2'}$

$$\Rightarrow \quad \vec{a} = a^1\frac{\partial u^{i'}}{\partial u^1}\vec{r}_{i'} + a^2\frac{\partial u^{i'}}{\partial u^2}\vec{r}_{i'}$$

$$\vec{a} = \left(a^1 \frac{\partial u^{1'}}{\partial u^1} + a^2 \frac{\partial u^{1'}}{\partial u^2}\right)\vec{r}_{1'}$$

$$+ \left(a^1 \frac{\partial u^{2'}}{\partial u^1} + a^2 \frac{\partial u^{2'}}{\partial u^2}\right)\vec{r}_{2'}$$

$$\therefore \quad a^{i'} = \frac{\partial u^{i'}}{\partial u^i} a^i$$

SIMILARLY

$$a^i = \frac{\partial u^i}{\partial u^{i'}} a^{i'} \quad \therefore \left\{ \begin{array}{l} a^i \text{ IS THE COMPONENT} \\ \text{OF A } \underline{\text{CONTRAVARIENT}} \\ \text{VECTOR} \end{array} \right.$$

WE CAN SHOW THAT, USING ABOVE RESULTS

$$\vec{a} \cdot \vec{b} = g_{ij}\, a^i a^j$$

$$= g_{i'j'}\, a^{i'} a^{j'}$$

LET US DEFINE THE <u>COVARIANT</u> COMPONENTS OF $\vec{a}$ BY a_i

$$a_i = g_{ij} \cdot a^j$$

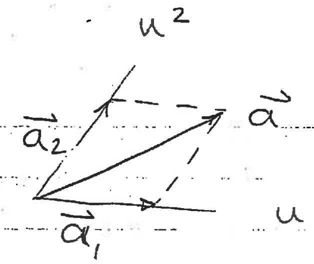
$$\Rightarrow \quad \vec{a} \cdot \vec{b} = g_{ij}\, a^i b^j$$

$$= a_j\, b^j$$

$$= a^j\, b_j.$$

WE HAVE $a^i = g^{ij} a_j$ ALSO.

9 RESSION 8



$$\vec{a} = \vec{a}_1 + \vec{a}_2$$
$$= a^1 \vec{r}_1 + a^2 \vec{r}_2$$

$$\vec{a}_1 = (a^1, 0)$$
$$\vec{a}_2 = (0, a^2)$$

$$|\vec{a}_1| = \sqrt{g_{11}}\,|a^1| \quad , \quad |\vec{a}_2| = \sqrt{g_{22}}\,|a^2|$$

— THE CONTRAVARIANT BASIS VECTORS $\vec{r}^{\,i}$

LET US FIND VECTORS $\vec{r}^{\,1}$ AND $\vec{r}^{\,2}$ ∋

$$\vec{r}^{\,i} \cdot \vec{r}_j = \delta^i_j$$

LET $\vec{r}^{\,1} = \alpha \vec{r}_1 + \beta \vec{r}_2$

$$\Longrightarrow \begin{cases} \vec{r}^{\,1} \cdot \vec{r}_1 = 1 = \alpha g_{11} + \beta g_{12} \\ \vec{r}^{\,1} \cdot \vec{r}_2 = 0 = \alpha g_{12} + \beta g_{22} \end{cases}$$

$$\Longrightarrow \quad \alpha = g^{11} \quad , \quad \beta = g^{12}$$

SIMILARLY IF $\vec{r}^{\,2} = \alpha' \vec{r}_1 + \beta' \vec{r}_2$

$$\Longrightarrow \quad \alpha' = g^{21} \quad , \quad \beta' = g^{22}$$

$$\therefore \quad \vec{r}^{\,i} = g^{ij}\, \vec{r}_j$$ {CONTRAVARIANT BASIS VECTORS

NOW LET

$$\vec{a} = a^1 \vec{r}_1 + a^2 \vec{r}_2$$
$$= a_1 \vec{r}^{\,2} + a_2 \vec{r}^{\,2}$$

HOW CAN WE FIND a_1 AND a_2? WE USE THE RELATION $\vec{r}_i \cdot \vec{r}^j = \delta_i^j$

$$a^i \vec{r}_i = a_i \vec{r}^i$$

$$a^i \vec{r}_i \cdot \vec{r}_j = a_i \vec{r}^i \cdot \vec{r}_j = a_i \delta^i_j = a_j$$

i.e.

$$a_j = g_{ji} a^i \quad \text{AS BEFORE}$$

NOTE THAT $\vec{r}^1 \perp \vec{r}_2$, $\vec{r}^2 \perp \vec{r}_1$

$$|\vec{r}^1|^2 = ?$$

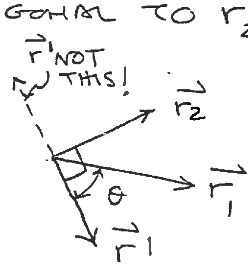
$$\vec{r}^1 = g^{1i} \vec{r}_i = (g^{11}, g^{12})$$

$$|\vec{r}^1|^2 = g_{ij} g^{1i} g^{1j} = \delta^1_j g^{1j} = g^{11}$$

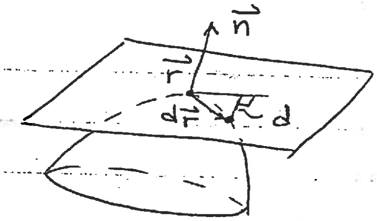
$$|\vec{r}^2|^2 = g^{22}$$

$$\therefore |\vec{r}^1| = \sqrt{g^{11}}, \quad |\vec{r}^2| = \sqrt{g^{22}}$$

— SINCE $\vec{r}_1 \cdot \vec{r}^1 = 1 = \sqrt{g_{11} g^{11}} \cos\theta$ WHERE θ IS THE ANGLE BETWEEN $\vec{r}^1$ AND $\vec{r}_1$ $\Longrightarrow$ $\cos\theta > 0$, i.e. $\vec{r}^1$ IS ORTHOGONAL TO $\vec{r}_2$ AND POINTING IN A DIRECTION WHICH MAKES AN ACUTE ANGLE WITH $\vec{r}_1$ AS SHOWN ON THE RIGHT



$d = \vec{n} \cdot d\vec{r}$ $\left\{ \text{A SIGNED QUANTITY} \right.$



$d\vec{r} = \vec{r}_i \, du^i$

$\quad + \frac{1}{2}\left(du^i \frac{\partial}{\partial u^i}\right)^2 \vec{r} + \cdots$

$\quad = \vec{r}_i \, du^i + \frac{1}{2} \vec{r}_{ij} \, du^i du^j + O(du^i)^3$

$\vec{n} \cdot d\vec{r} = \frac{1}{2} \vec{r}_{ij} \cdot \vec{n} \, du^i du^j + O(du^i)^3$

WE CALL $\text{II} = b_{ij} \, du^i du^j$, WHERE

$$b_{ij} = \vec{r}_{ij} \cdot \vec{n}$$

<u>THE SECOND FUNDAMENTAL FORM</u>

— ANOTHER USEFUL FORMULA FOR b_{ij}

$$\vec{n} \cdot \vec{r}_i = 0 \Rightarrow \vec{n}_j \cdot \vec{r}_i + \underbrace{\vec{n} \cdot \vec{r}_{ij}}_{b_{ij}} = 0$$

$$\therefore \; b_{ij} = - \vec{r}_i \cdot \vec{n}_j$$

— $b_{ij} = b_{ji}$ SINCE $\vec{r}_{ij} = \vec{r}_{ji}$.

— $b = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$ THE DISCRIMINANT OF THE 2ND FUNDAMENTAL FORM

— CLASSIFICATION OF POINTS ON A SURFACE :
LET b BE CALCULATED AT A POINT P

1) $b>0 \Rightarrow \Pi$ IS POSITIVE OR NEGATIVE DEFINITE, i.e. $d>0$ OR $d<0$ $\forall$ POINTS IN THE VICINITY OF POINT P, THE SURFACE LIES ON ONE SIDE OF THE TANGENT PLANE (SEE FIG. ON P. 10). THESE POINTS ARE KNOWN AS <u>ELLIPTIC POINTS</u>.

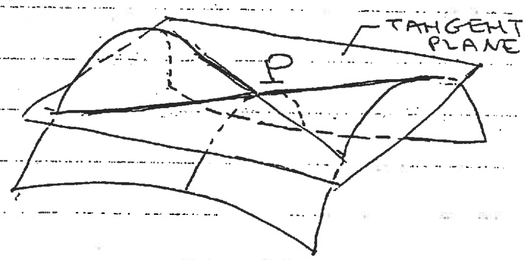
2) $b=0$, WE HAVE TWO CASES

i) $b_{ij} \neq 0$ FOR SOME i,j $\Rightarrow$ A DIRECTION (du^1, du^2) WHERE $\Pi = 0$, OTHERWISE d HAS THE SAME SIGN $\forall$ POINTS NEAR P. ONE HAS TO USE TAYLOR EXPANSION NEAR P TO STUDY THE BEHAVIOR OF THE SURFACE. (SEE FIG. 17.2 AND 17.3 GOETZ) THESE POINTS ARE KNOWN AS <u>PARABOLIC POINTS</u>.

ii) $b_{ij} = 0$ $\forall$ i,j. THE TANGENT PLANE HAS A CONTACT OF ORDER HIGHER THAN 2 WITH THE SURFACE. THIS IS CALLED A <u>FLAT POINT</u> OF THE SURFACE.

3) $b<0$. IN THIS CASE $\Pi = b_{ij}\,du^i du^j$ HAS TWO DISTINCT SOLUTIONS FOR THE RATIO OF $(du^1 : du^2)$. POINTS OF THE SURFACE

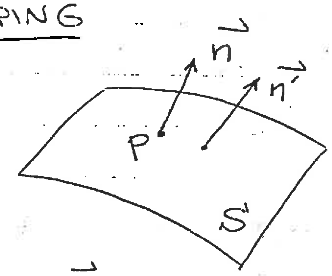
IN THE VICINITY OF POINT P ARE BOTH ABOVE AND BELOW THE TANGENT PLANE AS SHOWN. SUCH POINT CALLED A HYPERBOLIC POINT.

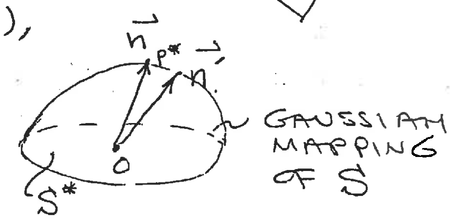


SPHERICAL OR GAUSSIAN MAPPING

$\vec{n} = \vec{n}(u^1, u^2)$ UNIT NORMAL

USING $\vec{n}$ AS THE PARAMETRIC REPRESENTATION OF A NEW SURFACE (A SPHERE), A SPHERICAL OR GAUSSIAN MAPPING OF THE ORIGINAL SURFACE S IS OBTAINED





THE VECTORS $\vec{n}_1 = \dfrac{\partial \vec{n}}{\partial u^1}$ AND $\vec{n}_2 = \dfrac{\partial \vec{n}}{\partial u^2}$ ARE TANGENT TO THE SPHERE. IF $\vec{n}_1 \times \vec{n}_2 \neq 0$ AT $P^*: (u^1, u^2)$, THEN P^* ON THE SPHERE IS A REGULAR POINT OF REPRESENTATION. WE HAVE $\vec{n}_1 \times \vec{n}_2 \parallel \vec{n}$. LET $\vec{n}_1 \times \vec{n}_2 = \varepsilon \vec{n}$. IF $\varepsilon > 0$, WE SAY THE PARAMETRIZATION INDUCES THE SAME ORIENTATION ON S AND S^*. IF $\varepsilon < 0$, THE PARAMETRIZATION INDUCES OPPOSIT ORIENTATION ON S AND S^*. IF $\varepsilon = 0$

THE ORIENTATION IS UNDETERMINED.

NOW TAKE A DOMAIN Ω ON S AND FIND ITS SPHERICAL IMAGE Ω^*. WE DEFINE THE GAUSSIAN CURVATURE OF S AT A POINT AS

$$K = \lim_{dia(\Omega)\to 0} \frac{\text{AREA OF } \Omega^*}{\text{AREA OF } \Omega}$$

WHERE WE CAN THINK OF $dia(\Omega)$ AS THE MAXIMUM EUCLIDEAN DISTANCE BETWEEN TWO POINTS OF Ω.

THM: $K = \dfrac{b}{g}$ AT REGULAR POINTS OF S.

PROOF: $\dfrac{\text{AREA } \Omega^*}{\text{AREA } \Omega} = \dfrac{\int_\Omega |\vec{n}_1 \times \vec{n}_2|\, du^1 du^2}{\int_\Omega \sqrt{g}\, du^1 du^2}$

$$\therefore \quad K = \varepsilon \frac{|\vec{n}_1 \times \vec{n}_2|}{\sqrt{g}}$$

WHERE $\varepsilon = 1$ IF ORIENTATION IS PRESERVED, $\varepsilon = -1$ IF IT IS REVERSED, i.e. $\varepsilon = \mathrm{SGN}(\vec{n}_1, \vec{n}_2 \vec{n})$

$$b = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} -\vec{r}_1\cdot\vec{n}_1 & -\vec{r}_1\cdot\vec{n}_2 \\ -\vec{r}_2\cdot\vec{n}_1 & -r_2\cdot\vec{n}_2 \end{vmatrix}$$

$$= (\vec{r}_1 \times \vec{r}_2)\cdot(\vec{n}_1 \times \vec{n}_2) = \sqrt{g}\,\vec{n}\cdot(\vec{n}_1\times\vec{n}_2) = \varepsilon\sqrt{g}\,|\vec{n}_1\times\vec{n}$$

$$\therefore \quad K = \frac{b}{g}$$

— IF $(u^1, u^2) \to (u^{1'}, u^{2'}) \Longrightarrow$

$$b' = \left[\frac{\partial(u^1, u^2)}{\partial(u^{1'}, u^{2'})}\right]^2 b$$

$$g' = \left[\frac{\partial(u^1, u^2)}{\partial(u^{1'}, u^{2'})}\right]^2 g$$

$$\therefore \quad \frac{b'}{g'} = \frac{b}{g} \equiv K$$, i.e. THE GAUSSIAN CURVATURE DOES NOT DEPEND ON PARAMETRIC REPRESENTATION OF THE SURFACE. IT IS THEREFORE A GEOMETRIC PROPERTY OF THE SURFACE

— COR: i) $K > 0$ AT ELLIPTIC POINTS
ii) $K < 0$ AT HYPERBOLIC POINTS
iii) $K = 0$ AT PARABOLIC POINTS

— A USEFUL RESULT: $\vec{n}_1 \times \vec{n}_2 = K \, \vec{r}_1 \times \vec{r}_2$

PROOF: $\vec{n}_1 \times \vec{n}_2 = \alpha \, \vec{r}_1 \times \vec{r}_2 = \alpha \sqrt{g} \, \vec{n}$

$$(\vec{n}_1 \times \vec{n}_2) \cdot \vec{n} = \varepsilon \, |\vec{n}_1 \times \vec{n}_2| = \alpha \sqrt{g}$$

$$\alpha = \varepsilon \frac{|\vec{n}_1 \times \vec{n}_2|}{\sqrt{g}} \equiv K$$

GAUSS AND WEINGARTEN FORMULAS

$$\vec{r}_{ij} = \Gamma^k_{ij}\,\vec{r}_k + b_{ij}\,\vec{n} \qquad \underline{\text{GAUSS FORMULA}}$$

WHERE Γ^k_{ij} IS THE CHRISTOFFEL SYMBOLS OF 2ND KIND

— $\vec{r}_{ij}$ HAS A COMPONENT IN TANGENT PLANE WHICH WE CAN WRITE AS $\Gamma^k_{ij}\,\vec{r}_k$ AND A COMPONENT ALONG $\vec{n}$ WHICH WE CAN WRITE AS $\beta\vec{n}$. TO FIND β, WE USE $\vec{r}_{ij}\cdot\vec{n} = b_{ij}$. THIS GIVES $\beta = b_{ij}$.

— $\vec{n}_i = \dfrac{\partial\vec{n}}{\partial u^i}$. SINCE $\vec{n}\cdot\vec{n}_i = \frac{1}{2}\frac{\partial}{\partial u^i}(\vec{n}\cdot\vec{n}) = 0$

$\Rightarrow \vec{n}_i \perp \vec{n}$. LET $\vec{n}_i = -\,b^k_i\,\vec{r}_k$.

$\Rightarrow \vec{n}_i\cdot\vec{r}_j = -\,b_{ij} = -\,b^k_i\,\vec{r}_k\cdot\vec{r}_j = -g_{jk}\,b^k_i$

OR $b_{ij} = g_{jk}\,b^k_i \quad \therefore \; b^j_i = g^{jk}\,b_{ki}$

i.e. b^j_i IS A MIXED TENSOR OF RANK 2.

THE FORMULA

$$\vec{n}_i = -\,b^j_i\,\vec{r}_j$$

IS CALLED THE <u>WEINGARTEN FORMULA</u>.

— BOTH GAUSS AN WEINGARTEN FORMULAS HAVE MANY USES.

THE CHRISTOFFEL SYMBOLS

FROM GAUSS FORMULA, WE HAVE

$$\vec{r}_{ij} \cdot \vec{r}_\ell = \Gamma_{ij}^{\;k} \, \vec{r}_k \cdot \vec{r}_\ell = \Gamma_{ij}^{\;k} g_{k\ell}$$

$$\equiv \Gamma_{ij\ell}$$ — CHRISTOFFEL SYMBOL OF FIRST KIND

$$\Gamma_{ij\ell} = \Gamma_{ij}^{\;k} g_{k\ell}$$

— USING $g_{ij} g^{jk} = \delta_i^{\;k}$, WE GET

$$\Gamma_{ij}^{\;k} = \Gamma_{ij\ell} \, g^{\ell k}$$

— BOTH Γ_{ijk} AND $\Gamma_{ij}^{\;k}$ ARE SYMMETRIC WRT i AND j INDICES.

— $$\Gamma_{ijk} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right]$$

$$\Gamma_{ij}^{\;k} = g^{k\ell} \, \Gamma_{ij\ell}$$ [THEN USE ABOVE RESULT FOR $\Gamma_{ij\ell}$]

PROOF: DIFFERENTIATE $\vec{r}_i \cdot \vec{r}_j = g_{ij}$ WRT $u^k \Rightarrow \Gamma_{ikj} + \Gamma_{jki} = \dfrac{\partial g_{ij}}{\partial u^k}$, PERMUTE INDICES:

$$\Gamma_{kij} + \Gamma_{jik} = \frac{\partial g_{jk}}{\partial u^i}$$

$$\Gamma_{kji} + \Gamma_{ijk} = \frac{\partial g_{ki}}{\partial u^j}$$

ADD THE LAST TWO SUBTRACT THE FIRST TO GET Γ_{ijk}.

— IF $(u^1, u^2) \to (u^{1'}, u^{2'}) \Rightarrow$

$$\Gamma_{i'j'k'} = \frac{\partial u^i}{\partial u^{i'}} \frac{\partial u^j}{\partial u^{j'}} \frac{\partial u^k}{\partial u^{k'}} \Gamma_{ijk} + \frac{\partial^2 u^i}{\partial u^{i'} \partial u^{j'}} \frac{\partial u^k}{\partial u^{k'}} g_{ik}$$

$\therefore$ Γ_{ijk} IS NOT A TENSOR, NEITHER IS Γ_{ij}^{k}

— A USEFUL RESULT :

$$\frac{\partial \sqrt{g}}{\partial u^i} = \Gamma_{ik}^{k} \sqrt{g}$$

— IN RIEMANNIAN GEOMETRY Γ_{ijk} IS CALLED A CONNECTION. THE MEANING OF THIS TERM COMES FROM THE FOLLOWING:

$$\begin{aligned}
\Gamma_{ijk} &= \vec{r}_{ij} \cdot \vec{r}_k \\
&\approx \frac{1}{\Delta u^j} \left[\vec{r}_i (u^j + \Delta u^j) - \vec{r}_i (u^j) \right] \cdot \vec{r}_k \\
&= \frac{1}{\Delta u^j} \left[\vec{r}_k \cdot \vec{r}_i (u^j + \Delta u^j) - \vec{r}_k \cdot \vec{r}_i (u^j) \right]
\end{aligned}$$

i.e. Γ_{ijk} GIVES AN IDEA OF HOW $\vec{r}_i (u^j + \Delta u^j)$ IS CONNECTED (RELATED) TO $\vec{r}_i (u^j)$; IN OTHER WORDS, HOW $\vec{r}_i$ VARIES IN THE VICINITY OF A POINT.

— FORMULAS OF GAUSS AND CODAZZI

THM: ON A SURFACE OF CLASS C^3, WE HAVE

$$b = \partial^2_{12} g_{12} - \frac{1}{2}\left(\partial^2_{22} g_{11} + \partial^2_{11} g_{22}\right) - \left(\Gamma^i_{11}\Gamma^j_{12} - \Gamma^i_{12}\Gamma^j_{12}\right) g_{ij} \qquad \text{GAUSS FORMULA}$$

AND

$$\left.\begin{aligned} \partial_2 b_{11} + \Gamma^i_{11} b_{i2} &= \partial_1 b_{12} + \Gamma^i_{12} b_{i1} \\ \partial_2 b_{21} + \Gamma^i_{21} b_{i2} &= \partial_1 b_{22} + \Gamma^i_{22} b_{i2} \end{aligned}\right\} \text{CODAZZI FORMULAS}$$

WHERE

$$\partial_i = \frac{\partial}{\partial u^i}, \quad \partial^2_{ij} = \frac{\partial^2}{\partial u^i \partial u^j}$$

— GAUSS FORMULA TELLS US THAT b DEPENDS ON g_{ij} AND THEIR FIRST AND SECOND DERIVATIVES. THIS LEADS TO THEOREMA EGREGIUM OF GAUSS:

THM: THE GAUSSIAN CURVATURE OF A CLASS C^3 SURFACE DEPENDS ONLY ON THE COEFFICIENTS OF 1ST FUNDAMENTAL FORM AND THEIR FIRST AND SECOND DERIVATIVES.

PROOF: $K = b/g$. USE GAUSS FORMULA FOR b.

— WE CAN ALSO WRITE GAUSS FORMULA AS

$$b = [\partial_2 \Gamma^i_{11} - \partial_1 \Gamma^i_{12} + (\Gamma^j_{11}\Gamma^i_{j2} - \Gamma^j_{12}\Gamma^i_{j1})]\, g_{i2}$$

— THE CODAZZI FORMULAS IMPLY THAT THE COEFFICIENTS OF 1ST AND 2ND FUNDAMENTAL FORMS ARE DEPENDENT.

— THERE ARE DEEP GEOMETRIC REASONS BEHIND GAUSS AND CODAZZI FORMULAS WHICH CAN BE SEEN USING THE MODERN GEOMETRIC METHODS BASED ON DIFFERENTIAL FORMS.

— THE FUNDAMENTAL THEOREM OF SURFACE THEORY: A SURFACE IS COMPLETELY DETERMINED TO WITHIN THE POSITION IN SPACE BY TWO QUADRATIC DIFFERENTIAL FORMS $g_{ij}\, du^i du^j$ ($g_{ij} \in C^2$) AND $b_{ij}\, du^i du^j$ ($b_{ij} \in C^1$) IF AND ONLY IF

i) $g = g_{11} g_{22} - (g_{12})^2 > 0, \quad g_{11} > 0$

ii) g_{ij} AND b_{ij} SATISFY GAUSS AND CODAZZI'S FORMULAS WHERE Γ^k_{ij} IS GIVEN BY THE RELATION

$$\Gamma^k_{ij} = g^{kl}\Gamma_{ijl} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l}\right)$$

— THE ABOVE THEOREM IS KNOWN AS BONNET'S THM.

— THM : ANY QUADRATIC DIFFERENTIAL $g_{ij}\, du^i du^j$, $g_{ij} \in C^1$, WHICH IS POSITIVE DEFINITE, IS LOCALLY THE 1ST. FUND. FORM OF SOME SURFACE OF CLASS C^2. THERE ARE INFINITELY MANY SUCH SURFACES OF DIFFERENT SHAPES.

— SOME USEFUL RESULTS

i) FOR ORTHOGONAL COORDINATES $g_{12} = 0$, THE GAUSSIAN CURVATURE IS:

$$K = \frac{1}{\sqrt{g_{11} g_{22}}} \left[\partial_1 \left(g_{11}^{-1/2} \partial_1 g_{22}^{1/2} \right) + \partial_2 \left(g_{22}^{-1/2} \partial_2 g_{11}^{1/2} \right) \right]$$

ii) FOR ISOTHERMIC COORDINATES $g_{11} = g_{22} = \rho^2(u^1, u^2)$, $g_{12} = 0$:

$$K = -\frac{1}{\rho^2} \left(\partial_{11}^2 + \partial_{22}^2 \right) \ln \rho$$

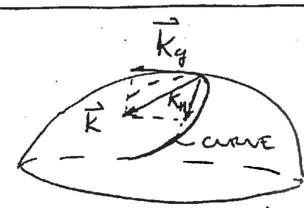
$$\equiv -\frac{1}{\rho^2} \nabla^2 \ln \rho$$

(READ ABOUT ISOTHERMIC COORDINATES IN GOETZ)

THE INTRINSIC GEOMETRY OF SURFACES

— INTRINSIC GEOMETRY OF SURFACES INVOLVE THE STUDY OF THOSE PROPERTIES WHICH DO NOT REQUIRE IMBEDDING OF THE SURFACE IN A HIGHER DIMENSIONAL SPACE. A CREATURE LIVING ON A SURFACE WILL BE ABLE TO DISCOVER THE INTRINSIC PROPERTIES OF THE SURFACE UNAWARE OF HOW THE SURFACE IS IMBEDDED IN 3D. THE STUDY OF INTRINSIC PROPERTIES OF A SURFACE LED TO THE DEVELOPMENT OF RIEMANNIAN GEOMETRY.

— THE CURVATURE OF A CURVE ON THE SURFACE



LET US CONSIDER A CURVE ON A SURFACE. THIS CURVE, VIEWED IN 3D, HAS A CURVATURE k. LET US CALL $\vec{k}$ THE CURVATURE VECTOR DEFINED BY $\vec{k} = k\vec{n}'$ WHERE $\vec{n}'$ IS THE NORMAL TO THE CURVE OBTAINED FROM FRENET-SERET FORMULA. WE DECOMPOSE THIS VECTOR INTO NORMAL AND TANGENTIAL COMPONENTS TO THE SURFACE $\vec{K}_n$ AND $\vec{K}_g$, RESPECTIVELY. THE FIRST VECTOR $\vec{K}_n$ IS CALLED THE VECTOR OF NORMAL CURVATURE AND $\vec{K}_g$ IS CALLED THE VECTOR OF GEODESIC CURVATURE

WE NOW OBTAIN THE FORMULAS FOR $\vec{K}_n$ AND $\vec{K}_g$
LET $\vec{r} = \vec{r}(u^1, u^2)$ AND $u^i(s)$ BE THE PARAMETRIC EQ. OF THE CURVE WITH s AS LENGTH PARAMETER : $\vec{K} = k\vec{n} = \vec{r}''$, $(\ )' = \frac{d}{ds}$

$$\vec{r}' = \vec{r}_i \frac{du^i}{ds}$$

$$\vec{r}'' = \vec{r}_{ij} \frac{du^i}{ds}\frac{du^j}{ds} + \vec{r}_i \frac{d^2u^i}{ds^2}$$

$$= (\Gamma_{ij}^k \vec{r}_k + b_{ij}\vec{n})\frac{du^i}{ds}\frac{du^j}{ds} + \vec{r}_k \frac{d^2u^k}{ds^2}$$

$$= \underbrace{\left(\frac{d^2u^k}{ds^2} + \Gamma_{ij}^k \frac{du^i}{ds}\frac{du^j}{ds}\right)\vec{r}_k}_{\vec{K}_g} + \underbrace{b_{ij}\frac{du^i}{ds}\frac{du^j}{ds}\vec{n}}_{\vec{K}_n}$$

$$\vec{K}_n = b_{ij}\frac{du^i}{ds}\frac{du^j}{ds}\vec{n}$$

$$k_n = b_{ij}\frac{du^i}{ds}\frac{du^j}{ds} \quad \text{(A SIGNED QUANTITY)}$$

WE NOTE THAT THE UNIT TANGENT TO THE CURVE $\vec{t} = t^i \vec{r}_i$ HAS COMPONENTS $t^i = \frac{du^i}{ds}$

$$\therefore \quad k_n = b_{ij} t^i t^j$$

$$\vec{K}_g = \left(\frac{d^2u^k}{ds^2} + \Gamma_{ij}^k \frac{du^i}{ds}\frac{du^j}{ds}\right)\vec{r}_k$$

SINCE THIS DEPENDS ON g_{ij} AND ITS DERI-

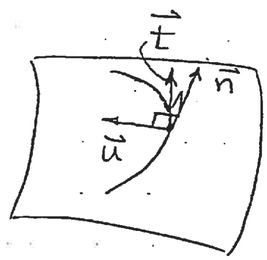
—VATIVES $\Longrightarrow \vec{k}_g$ IS AN INTRINSIC QUANTITY.

— LET US DEFINE GEODESIC NORMAL $\vec{u}$

$$\vec{u} = \vec{n} \times \vec{t}$$

WHERE $\vec{t} = \dfrac{d\vec{r}}{ds}$ IS UNIT TANGENT TO THE CURVE $\Longrightarrow \vec{u}$ IS A UNIT VECTOR. WE DEFINE GEODESIC CURVATURE k_g BY THE RELATION

$$\vec{k}_g = k_g \, \vec{u}$$



$\therefore$

$$k_g = \vec{k}_g \cdot \vec{u} \quad \left\{ \begin{array}{l} \text{A SIGNED} \\ \text{QUANTITY} \end{array} \right.$$

— WE CAN SHOW THAT

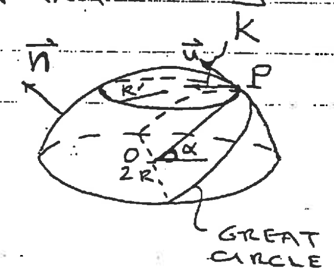
$$\left\{ \begin{array}{l} \vec{r}_1 \cdot \vec{u} = -\sqrt{g} \, \dfrac{du^2}{ds} \\ \vec{r}_2 \cdot \vec{u} = \sqrt{g} \, \dfrac{du^1}{ds} \end{array} \right.$$

$\therefore$

$$k_g = \sqrt{g} \begin{vmatrix} \dfrac{du^1}{ds} & \dfrac{du^2}{ds} \\ \dfrac{d^2u^1}{ds^2} + \Gamma^1_{ij} \dfrac{du^i}{ds} \dfrac{du^j}{ds} & \dfrac{d^2u^2}{ds^2} + \Gamma^2_{ij} \dfrac{du^i}{ds} \dfrac{du^j}{ds} \end{vmatrix}$$

— NOTE THAT EVEN THOUGH $\vec{k}_g$ DOES NOT DEPEND ON THE DIRECTION OF THE CURVE, k_g CHANGES SIGN IF THE DIRECTION OF THE CURVE CHANGES. NOTE ALSO THAT $\vec{u}$ ALSO CHANGES DIRECTION IN THIS CASE.

— THE VECTOR $\vec{k}_g$ IS NOT A PROPERTY OF THE SURFACE BUT A CURVE ON THE SURFACE. BY CHANGING THE CURVE, $\vec{k}_g$ ALSO CHANGES. THIS CAN BE SEEN EASILY BY TAKING A SPHERE AND CUTTING IT BY DIFFERENT PLANES PASSING THRU A POINT.



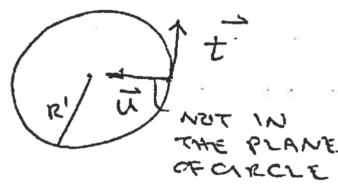
— FOR SMALL CIRCLE THRU P:

$$|\vec{k}| = \frac{1}{R'}$$

$$\vec{k}_n = -\frac{\cos\alpha}{R'}\,\vec{n}$$

$$\vec{k}_g = \frac{\sin\alpha}{R'}\,\vec{u}$$

WITH $\vec{u}$ AS SHOWN



$k_n = \frac{1}{R}$

$k_g = 0$

— IF WE USE t, AN ARBITRARY PARAMETER, INSTEAD OF s, THEN THE FORMULA FOR k_g ON LAST PAGE MUST BE CORRECTED BY DIVIDING $\sqrt{g}$ BY $\left(g_{ij}\,\frac{du^i}{dt}\,\frac{du^j}{dt}\right)^{3/2}$ AND CHANGING ALL $\frac{d}{ds}$ TO $\frac{d}{dt}$. NOTE THAT NOW $\vec{t}$ (THE UNIT TANGENT VECTOR) IS IN THE DIRECTION $\left(\frac{du^1}{dt}, \frac{du^2}{dt}\right)$.

— A <u>GEODESIC LINE</u> OR A <u>GEODESIC</u> IS A CURVE ON THE SURFACE WHOSE GEODESIC CURVATURE IS ZERO. TO FIND GEODESICS ON A SURFACE, WE MUST SOLVE

$$\frac{d^2u^k}{ds^2} + \Gamma_{ij}^{k}\frac{du^i}{ds}\frac{du^j}{ds} = 0 \qquad k=1,2$$

SUBJECT TO THE CONDITION $g_{ij}\frac{du^i}{ds}\frac{du^j}{ds} = 1$.
THIS LATTER CONDITION, WHEN USED IN THE FIRST EQ. GIVES A SINGLE EQ. FOR THE CURVE $u^2(u^1)$ OF THE GEODESIC:

$$\frac{d^2u^2}{(du^1)^2} = \Gamma_{22}^{1}\left(\frac{du^2}{du^1}\right)^3 + \left(2\Gamma_{12}^{1} - \Gamma_{22}^{2}\right)\left(\frac{du^2}{du^1}\right)^2 + \left(\Gamma_{11}^{1} - 2\Gamma_{12}^{2}\right)\frac{du^2}{du^1} - \Gamma_{11}^{2}$$

(SEE P 37)

WE HAVE THE FOLLOWING THMS:

THEOREM: THRU EVERY REGULAR POINT OF A SURFACE OF CLASS C^2 THERE IS AT LEAST ONE GEODESIC CURVE IN EVERY DIRECTION. IF THE SURFACE IS OF CLASS $C^3 \Rightarrow \exists$ EXACTLY ONE GEODESIC LINE IN EACH DIRECTION.

THEOREM: A CURVE OF CLASS C^2 ON A SURFACE OF CLASS C^2 IS A GEODESIC IFF THE OSCULATING PLANE OF THE CURVE INCLUDES THE NORMAL TO THE SURFACE AT EACH POINT OF THE CURVE OR THE CURVATURE IS ZERO AT THE POINT.

— AN IMPORTANT PROPERTY OF GEODESIC LINES: LET L BE THE LENGTH OF A CURVE ON THE SURFACE PASSING THRU POINTS

P AND Q PARAMETRIZED BY t, $t \in [0,1]$ THEN

$$L = \int_0^1 \left(g_{ij} \frac{du^i}{dt} \frac{du^j}{dt} \right)^{1/2} dt$$

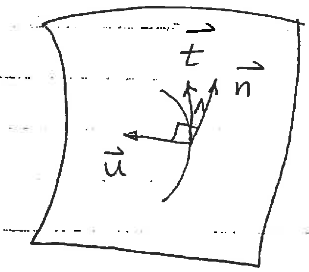
WE CAN SHOW THAT THE EULER EQ. FOR OBTAINING THE EXTREMAL CURVE THRU P AND Q, i.e., THE CURVE WHICH MAKES L EXTREMAL, IS EXACTLY THE EQUATION FOR GEODESICS DISCUSSED ABOVE. THIS TIME ONE HAS TO SOLVE A 2-POINT BOUNDARY VALUE PROBLEM TO GET THE GEODESIC BETWEEN P & Q

— THE BONNET-KOVALEVSKI FORMULAS FOR A CURVE ON A SURFACE

$$\frac{d\vec{t}}{ds} = k_g \vec{u} + k_n \vec{n}$$

$$\frac{d\vec{u}}{ds} = -k_g \vec{t} + \tau_g \vec{n}$$

$$\frac{d\vec{n}}{ds} = -k_n \vec{t} - \tau_g \vec{u}$$



τ_g IS GEODESIC TORSION GIVEN BY

$$\tau_g = \sqrt{g} \left(\delta_i^1 b_j^2 - \delta_i^2 b_j^1 \right) \frac{du^i}{ds} \frac{du^j}{ds}$$

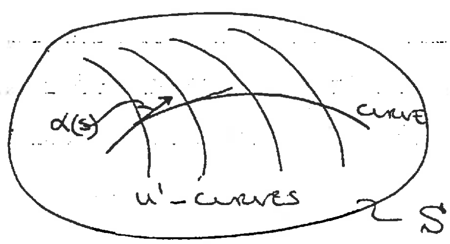
THM: τ_g AT A GIVEN POINT OF A CURVE ON A CLASS C^2 DEPENDS ON THE DIRECTION OF THE CURVE AT THAT POINT. IF THE SURFACE IS OF CLASS C^3, THEN

τ_g IS GIVEN BY THE TORSION OF GEODESIC CURVE IN THE DIRECTION OF THE CURVE AT THAT POINT.

— A USEFUL INTERPRETATION OF GEODESIC CURVATURE BY GAUSS

— LET A CURVE $(u^1(s), u^2(s))$ MAKE AN ANGLE $\alpha(s)$ WITH u^1-CURVE ON A SURFACE $\Rightarrow$

$$K_g = \frac{d\alpha}{ds} + \frac{\sqrt{g}}{g_{11}} \Gamma^2_{1i} \frac{du^i}{ds}$$



— THE GEODESIC CURVATURE OF A PLANE CURVE CONCIDES WITH THE ORDINARY CURVATURE (DEFINE SIGN OF CURVATURE).

— GEODESIC CURVATURE OF A CURVE GIVEN BY IMPLICIT EQ. $\phi(u^1, u^2) = 0$:

$$K_g = \frac{1}{\sqrt{g}} \left[\partial_1 \left(\frac{g_{12}\phi_2 - g_{22}\phi_1}{G} \right) + \partial_2 \left(\frac{g_{12}\phi_1 - g_{11}\phi_2}{G} \right) \right]$$

$$G = g_{11}(\phi_1)^2 - 2 g_{12} \phi_1 \phi_2 + g_{22}(\phi_2)^2$$

$$\phi_i = \partial_i \phi = \frac{\partial \phi}{\partial u^i}$$

HERE $\vec{u}$ IS ON THE SAME SIDE IS $\nabla\phi = (\phi_1, \phi_2)$.

— ABSOLUTE DIFFERENTIATION OF VECTORS

CONSIDER THE VECTOR $\vec{a} = \alpha^j \vec{r}_j$, THEN

$$\frac{\partial \vec{a}}{\partial u^i} = \partial_i \vec{a} = \frac{\partial \alpha^j}{\partial u^i} \vec{r}_j + \alpha^j \vec{r}_{ij}$$

$$= \frac{\partial \alpha^j}{\partial u^i} \vec{r}_j + \alpha^j \left(\Gamma^k_{ij} \vec{r}_k + b_{ij} \vec{n} \right)$$

$$= \left(\frac{\partial \alpha^k}{\partial u^i} + \Gamma^k_{ij} \alpha^j \right) \vec{r}_k + a^j b_{ij} \vec{n}$$

WE CALL $\left(\frac{\partial \alpha^k}{\partial u^i} + \Gamma^k_{ij} \alpha^j \right) \vec{r}_k = \frac{D\vec{a}}{\partial u^i}$ IS

THE ABSOLUTE DERIVATIVE OF $\vec{a}$ WRT u^i

WE NOTE THAT THIS CAN BE WRITTEN FOR A COMPONENT a^k OF $\vec{a}$ AS

$$\frac{Da^k}{\partial u^i} = \frac{\partial a^k}{\partial u^i} + \Gamma^k_{ij} a^j$$

THE ABSOLUTE DIFFERENTIAL OF $\vec{a}$ IS DEFINED AS

$$D\vec{a} = \left(da^k + \Gamma^k_{ij} a^i du^j \right) \vec{r}_k$$

$$\therefore \quad d\vec{a} = D\vec{a} + b_{ij} a^i du^j \vec{n}$$

NOTE THAT $D\vec{a}$ IS AN INTRINSIC QUANTITY. A CREATURE LIVING ON THE SURFACE CAN MEASURE $D\vec{a}$ BUT NOT $d\vec{a}$.

— SOME USEFUL RESULTS

i) LET $\vec{t} = \frac{d\vec{r}}{ds} = t^k \vec{r}_k$

$$\frac{D\vec{t}}{ds} = \frac{Dt^k}{ds}\vec{r}_k$$

$$= \left(\frac{dt^k}{ds} + \Gamma^k_{ij} t^i \frac{du^j}{ds}\right)\vec{r}_k$$

$$= \left(\frac{d^2u^k}{ds^2} + \Gamma^k_{ij}\frac{du^i}{ds}\frac{du^j}{ds}\right)\vec{r}_k$$

$$= \vec{k}_g = k_g \vec{u}$$

ii) $d(\vec{a}\cdot\vec{b}) = D(\vec{a}\cdot\vec{b}) = D\vec{a}\cdot\vec{b} + \vec{a}\cdot D\vec{b}$

iii) $$\begin{cases} \frac{D\vec{t}}{ds} = k_g \vec{u} \\ \frac{D\vec{u}}{ds} = -k_g \vec{t} \end{cases}$$

— PARALLEL TRANSPORT IN THE SENSE OF LEVI-CIVITA

CONSIDER A VECTOR $\vec{a}$ DEFINED ON A CURVE. WE SAY $\vec{a}$ IS TRANSPORTED PARALLEL ALONG THE CURVE IF $D\vec{a} = 0$ ON THE CURVE, i.e.

$$Da^k = da^k + \Gamma^k_{ij} a^i du^j = 0$$

THM: PARALLEL TRANSPORT OF VECTORS ALONG A CURVE PRESERVES SCALAR PRODUCT OF TWO VECTORS. IT FOLLOWS THAT THE LENGTH OF A VECTOR AND ANGLE BETWEEN TWO VECTORS ARE PRESERV

— LET $\vec{a}(s)$ BE A FIELD OF UNIT VECTOR PARALLEL ALONG A CURVE $\mathcal{C}: u^i = u^i(s)$. LET $\alpha(s)$ BE THE ANGLE BETWEEN $\vec{a}$ AND $\vec{t}$ THE TANGENT VECTOR ALONG $\mathcal{C}$ $\Rightarrow \frac{d\alpha}{ds} = k_g$
[$\alpha(s)$ IS THE ORIENTED ANGLE $\vec{t} \to \vec{a}$]

— A CURVE IS GEODESIC IFF THE FIELD OF UNIT TANGENT VECTORS IS PARALLEL ALONG THIS CURVE.

— A CURVE IS GEODESIC IFF THE FIELD OF GEODESIC NORMALS IS PARALLEL ALONG THIS CURVE.

FOR BOTH OF ABOVE RESULTS, USE (iii) OF PREVIOUS PAGE.

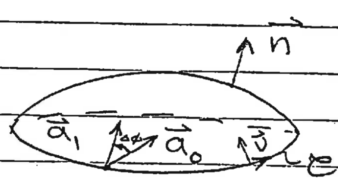
— ANOTHER RESULT ON ABSOLUTE DIFFERENTIALS IF TWO SURFACES ARE TANGENT ALONG A CURVE $\mathcal{C}$ AND $\vec{a}$ IS A TANGENT VECTOR FIELD ALONG $\mathcal{C}$ $\Rightarrow D\vec{a}$ DEFINED ON BOTH SURFACES COINCIDE. THIS FOLLOWS FROM THE FACT THAT $d\vec{a} = D\vec{a} + b_{ij} a^i du^j \vec{n}$. SINCE $D\vec{a}$ IS COMPONENT OF $d\vec{a}$ IN THE TANGENT PLANE AND THE TANGENT PLANES OF TWO SURFACES COINCIDE $\Rightarrow D\vec{a}$ IS THE SAME FOR BOTH SURFACES.

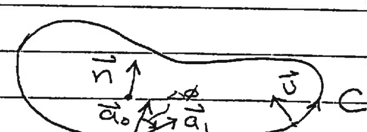
— THE ABSOLUTE DIFFERENTIAL OF A PLANE VECTOR FIELD CORRESPONDS TO THE ORDINARY DIFFERENTIAL (SINCE $b_{ij}=$

THM: $\mathcal{C}$ PIECEWISE SMOOTH BOUNDARY OF A SIMPLY CONNECTED DOMAIN Ω OF A C^3 SURFACE. $\bar{\Omega} \subset \Omega'$, Ω' AN OPEN SET OF THE SURFACE CONSISTING OF REGULAR POINTS AND ADMITTING A FIELD OF CLASS C^2 TANGENT VECTORS. LET $\Delta\phi$ BE THE ANGLE OF ROTATION OF A VECTOR TRANSPORTED PARALLEL AROUND $\mathcal{C}$ $\Rightarrow$ $\Delta\phi = \int_\Omega K\,dS = \int_\Omega K\sqrt{g}\,du^1du^2$ WHERE K IS THE GAUSSIAN CURVATURE.

NOTES: i) THE VECTOR WHICH IS TRANSPORTED PARALLEL ALONG THE CONTOUR IS ARBITRARY SINCE IF TWO VECTORS $\vec{a}$ AND $\vec{b}$ ARE TRANSPORTED PARALLEL ALONG $\mathcal{C}$, THEN $\angle(\vec{a},\vec{b})$, i.e. THE ANGLE BETWEEN $\vec{a}$ AND $\vec{b}$ REMAINS THE SAME SO THAT $\Delta\phi(\vec{a}) = \Delta\phi(\vec{b})$.

ii) $\Delta\phi$ IS ORIENTED ANGLE. LET $\mathcal{C}$ BE PARAMETRIZED BY $t \in [0,1]$ AND LET $\vec{a}_0 = \vec{a}(0)$ AND $\vec{a}_1 = \vec{a}(1)$ WHERE $\vec{a}$ IS TRANSPORTED PARALLEL ALONG $\mathcal{C}$. THEN $\mathcal{C}$ IS THE ORIENTED CONTOUR COMPATIBLE WITH $\vec{n}$ (i.e. THE GEODESIC NORMAL $\vec{U}$ IS INWARD) AND $\Delta\phi$ IS FROM $\vec{a}_0$ TO $\vec{a}_1$ AS SHOWN



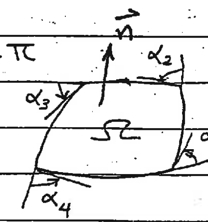


FOR LEFT FIGURE $\}$ $K > 0 \Rightarrow \Delta\phi > 0$ i.e. $\vec{a}_0 \times \vec{a}_1 \cdot \vec{n}(0) > 0$

FOR RIGHT FIGURE $\}$ $K < 0 \Rightarrow \Delta\phi < 0$ i.e. $\vec{a}_0 \times \vec{a}_1 \cdot \vec{n}(0) < 0$

GAUSS–BONNET THM: $\mathcal{C}$ PIECEWISE CLASS C^2 CONTOUR OF A SIMPLY CONNECTED DOMAIN Ω OF A CLASS C^3 SURFACE ADMITTING A FIELD OF C^2 UNIT TANGENT VECTORS $\Longrightarrow$

$$\int_\Omega K\, dS + \oint_{\mathcal{C}} K_g\, ds + \sum_i \alpha_i = 2\pi$$

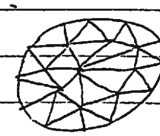


WHERE THE ANGLES α_i ARE SHOWN IN THE FIGURE.

NOTES: i) THIS THEOREM CAN BE GENERALIZED TO MULTIPLY CONNECTED REGIONS. (SEE GOETZ)
ii) MANY SURFACES DO NOT ADMIT A FIELD OF TANGENT VECTOR OVER THE ENTIRE SURFACE. THERE WILL BE SINGULARITIES ON THE SURFACE (SEE THE HAIRY BALL THM.). SUCH QUESTIONS ARE STUDIED IN TOPOLOGY. WE GET MULTIPLY CONNECTED REGIONS WHEN WE PUNCTURE THE SURFACE AT POINTS OF SINGULARITIES OF THE VECTOR FIELD TANGENT TO THE SURFACE. TOPOLOGY ENTERS OUR STUDY IN ANOTHER WAY:

THM: LET χ BE THE EULER–POINCARÉ CHARACTERISTIC OF A CLOSED ORIENTABLE SURFACE OF CLASS $C^3 \Longrightarrow \int K\, dS = 2\pi\chi$

THE EULER–POINCARÉ CHARACTERISTIC IS DEFINED BY TRIANGULATING THE SURFACE FIRST.
THEN $\chi = v - e + f$ WHERE



χ IS A TOPOLOGICAL INVARIANT OF A SURFACE
$\chi = 2$ SPHERE
$\chi = 0$ TORUS

v : THE NO. OF VERTICES
e : " " " EDGES
f : " " " TRIANGLES (CALLED FACE)

SOME USEFUL FACTS AND RESULTS

i) A SURFACE WHICH IS THE ENVELOPE OF FAMILY OF PLANES IS CALLED A DEVELOPABLE SURFACE. SUCH A SURFACE CAN BE ISOMETRICALLY MAPPED INTO A PLANE (ISOMETRIC ≡ DISTANCE PRESERVING).

EXAMPLES: (a) TAKE A CURVE IN 3D. THE SURFACE FORMED BY THE TANGENT LINES TO THE CURVE IS A DEVELOPABLE SURFACE. THE CURVE ITSELF IS THE EDGE OF REGRESSION OF THE TANGENT PLANES. THE SURFACE IS USUALLY IN TWO PIECES GLUED ALONG THE CURVE USED TO GENERATE THEM. THE EDGE OF REGRESSION MEANS THAT TWO NEIGHBORING CHARACTERISTIC LINES OF TWO NEIGHBORING PLANES OF FAMILY MEET ON THIS EDGE.

PARAMETRIZATION: LET THE CURVE BY SPECIFIED BY $\vec{r} = \vec{p}(u^1)$. LET $\vec{t} = \frac{d\vec{r}}{ds}$ BE THE UNIT TANGENT TO THE CURVE. TAKE $u^2 = s'$, THE DISTANCE ALONG THE TANGENT POSITIVE IN THE DIRECTION OF $\vec{t}$. THEN THE EQUATION OF THE DEVELOPABLE SURFACE IS

$$\vec{r}(u^1,u^2) = \vec{p}(u^1) + u^2\,\vec{t}(u^1)$$

$$\Rightarrow \quad \vec{r}_1 = \vec{t}(u^1) + u^2\frac{d\vec{t}}{du^1} \qquad (u^1 \equiv s)$$

$$= \vec{t}(u^1) + \kappa u^2 \vec{n}'$$

$\vec{n}'$ UNIT NORMAL TO THE CURVE

κ CURVATURE OF THE CURVE $\kappa = \kappa(u^1)$

$$\vec{r}_2 = \vec{t}$$

$$g_{11} = 1 + \kappa^2 (u^2)^2, \quad g_{12} = 1, \quad g_{22} = 1$$

TO MAP THIS SURFACE TO A PLANE, WE FIRST MAP THE CURVE $\mathcal{C}$ IN 3D TO A CURVE $\mathcal{C}'$ IN 2D BY USING $\kappa_{\mathcal{C}'}(u^1) = \kappa_{\mathcal{C}}(u^1)$, $\tau_{\mathcal{C}'}(u^1) = 0$ (OBVIOUS SINCE $\mathcal{C}'$ IS A PLANE CURVE), THIS CURVE $\mathcal{C}'$ IS UNIQUE UPTO THE POSITION IN SPACE. NOW DEFINE $\vec{t}$ AND u^2 AS BEFORE AND USE

$$\vec{r} = \vec{p}'(u^1) + u^2 \vec{t}(u^1)$$

AS PARAMETRIZATION OF THE 2D PLANE SURFACE HERE $\mathcal{C}'$ IS GIVEN BY $\vec{p}'(u^1)$. WE NOTE THAT THE TWO PIECES OF THE DEVELOPABLE SURFACE CORRESPONDING TO $u^2 > 0$ AND $u^2 < 0$ NOW ARE MAPPED TO THE SAME PART OF THE PLANE IN 2D. LET US CALL THE DEVELOPABLE SURFACE S AND ITS 2D MAP AS S'. CALL THE DISCRIMINANT OF THE 1ST FUNDAMENTAL FORMS AS g AND g' ON S AND S', RESPECTIVELY. THEN

$$g = \kappa(u^1)|u^2| = g' \quad (\neq 0 \text{ AWAY FROM EDGE OF REGRESSION})$$

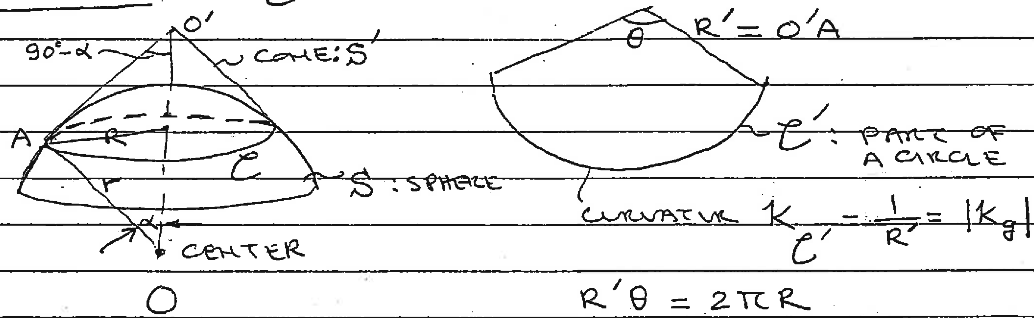
i.e S AND S' ARE ISOMETRIC.

IN THE ABOVE CASE, WE CAN SHOW THAT EACH PLANE OF THE FAMILY OF PLANES WHICH FORM S IS AN OSCULATING PLANE OF $\mathcal{C}$.

(b) TAKE A CURVE $\mathcal{C}$ ON A SURFACE S, CONSTRUCT THE TANGENT PLANES TO S ALONG $\mathcal{C}$. THE ENVELOPE OF THESE PLANES IS A DEVELOPABLE SURFACE S' WHICH IS DIFFERENT

FROM THE SURFACE CONSTRUCTED IN EX. (a), ABOVE. AGAIN S' CAN BE MAPPED INTO A PLANE ISOMETRICALLY AND $\mathcal{C}$ NOW BECOMES A CURVE IN 2D, DENOTED $\mathcal{C}'$, WITH CURVATURE $K_{\mathcal{C}'} = |K_g|_{\mathcal{C}}$. THIS IS AN IMPORTANT RESULT.

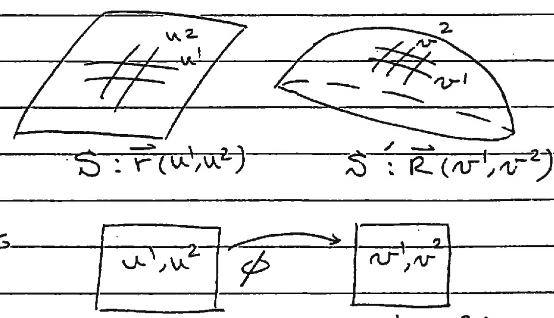
THE CURVE $\mathcal{C}'$ IS CALLED <u>THE DEVELOPMENT IN PLANE OF</u> $\mathcal{C}$.



ALONG $\mathcal{C}'$ PARALLEL DISPLACEMENT HAS A SPECIALLY SIMPLE MEANING WHICH WE WILL DISCUSS AFTER DEFINING COMPATIBLE MAPPING AND MAPPING OF TANGENT VECTORS.

ii)

LET S AND S' BE TWO SURFACES PARAMETRIZED BY (u^1, u^2) AND (v^1, v^2) RESPECTIVELY. A MAPPING FROM S TO S' IS DEFINED BY SPECIFYING $\phi : (u^1, u^2) \to (v^1, v^2)$, i.e. BY GIVING $v^1 = v^1(u^1, u^2)$, $v^2 = v^2(u^1, u^2)$.



THIS MAPPING INDUCES A MAPPING BETWEEN TANGENT VECTORS. LET $\vec{w} = \frac{d\psi}{dt}\Big|_{t=0} = \left(\frac{du^1}{dt}, \frac{du^2}{dt}\right)_0$ BE A TANGENT VECTOR ON S.

DEFINE $\psi'(t) = (v^1[u^1(t), u^2(t)], v^2[u^1(t), u^2(t)])$ ON S'

THEN WE DEFINE $\vec{w}'$, THE MAPPING OF $\vec{w}$ BY

$$\vec{w}' = \frac{d\psi'}{dt}\Big|_{t=0} = (w'^1, w'^2)$$

$$w'^1 = \frac{\partial v^1}{\partial u^i}\frac{du^i}{dt}\Big|_0 = \frac{\partial v^1}{\partial u^i} w^i$$

$$w'^2 = \frac{\partial v^2}{\partial u^i}\frac{du^i}{dt} = \frac{\partial v^1}{\partial u^i} w^2$$

i.e. $$w'^j = \frac{\partial v^j}{\partial u^i} w^i$$

WE CAN ALWAYS LET $\vec{R}^*(u^1, u^2) \equiv \vec{R}(v^1(u^1,u^2), v^2(u^1,u^2))$ SO THAT BOTH S AND S' ARE PARAMETRIZED WRT (u^1, u^2). WE THEN SAY THAT WE HAVE <u>COMPATIBLE MAPPING</u> AND IF $A \in S$, $A \overset{\Phi}{\to} A' \in S'$ THEN BOTH A AND A' HAVE THE SAME COORDINATES (u^1, u^2). WHAT IS MORE, BOTH $\vec{w}$ AND $\vec{w}'$ WILL HAVE THE SAME <u>COMPONENTS</u>. BELOW, WE WILL ASSUME THAT WE HAVE COMPATIBLE MAPPING.

LET $\mathcal{C}'$ BE THE DEVELOPMENT IN THE PLANE OF A CURVE $\mathcal{C}$ ON A SURFACE S. LET $\vec{a}'$ BE THE MAPPING OF A FIELD OF TANGENT VECTOR $\vec{a}$ ALONG $\mathcal{C} \Rightarrow D\vec{a}' = d\vec{a}'$. THIS IS BECAUSE $\vec{a}'$ IS IN A PLANE AND $b_{ij} = 0$. IF $\vec{a}$ IS A FIELD OF PARALLEL VECTORS $\Rightarrow D\vec{a}' = d\vec{a}' = 0$, i.e. $\vec{a}'$ ARE PARALLEL

IN THE PLANE. WE ALREADY KNOW THAT $|\vec{a}'| = |\vec{a}|$ = CONSTANT. THUS THE MAPPING NOT ONLY PRODUCES PARALLEL BUT ALSO EQUAL VECTORS IN THE PLANE.

iii) THE DIFF. EQUATION OF GEODESIC CURVE $u^2(u^1)$ GIVEN ON P25.

WE WANT TO GET A SINGLE EQ. IN u^2 FROM

$$\frac{d^2u^k}{ds^2} + \Gamma^k_{ij}\frac{du^i}{ds}\frac{du^j}{ds} = 0 \qquad k=1,2$$

LET $y = u^1$, $z = u^2$, $x = s$, $(\ )' = \frac{d}{ds}$.
THEN WE CAN WRITE THE ABOVE 2 EQS. AS

$$\begin{cases} y'' + \alpha y'^2 + \beta y'z' + \gamma z'^2 = 0 & (1-a) \\ z'' + \alpha' y'^2 + \beta' y'z' + \gamma' z'^2 = 0 & (1-b) \end{cases}$$

$$\frac{dz}{dy} = \frac{z'}{y'}$$

$$\frac{d^2z}{dy^2} = \frac{d}{dx}\left(\frac{z'}{y'}\right)\cdot\frac{dx}{dy} = \frac{z''y' - z'y''}{y'^3}$$

MULTIPLYING (1-a) BY z' AND SUBTRACTING THE RESULT FROM (1-b) $\times y'$ GIVES

$$z''y' - z'y'' = z'(\alpha y'^2 + \beta y'z' + \gamma z'^2) - y'(\alpha' y'^2 + \beta' y'z' + \gamma' z'^2)$$

DIVIDING BY y'^3 BOTH SIDES OF THIS EQ. GIVES

$$\frac{d^2z}{dy^2} = \alpha\frac{dz}{dy} + \beta\left(\frac{dz}{dy}\right)^2 + \gamma\left(\frac{dz}{dy}\right)^3 - \alpha' - \beta'\frac{dz}{dy} - \gamma'\left(\frac{dz}{dy}\right)^2$$

$$= \gamma\left(\frac{dz}{dy}\right)^3 + (\beta - \gamma')\left(\frac{dz}{dy}\right)^2 + (\alpha - \beta')\frac{dz}{dy} - \alpha'$$

THIS GIVES THE DIFFERENTIAL EQ. OF THE GEODESIC ON P 25 WHEN $y, z, \alpha, \beta, \gamma, \alpha', \beta', \gamma'$ ARE REPLACED APPROPRIATELY

iv) GEODESIC COORDINATES

WE CAN SHOW THAT

$$(K_g)_{u^2} = -\Gamma_{22}^{1} \frac{\sqrt{g}}{g_{22}^{3/2}} \quad \text{(i.e. ON } u^1 = \text{CONST. CURVES)}$$

$$(K_g)_{u^1} = \Gamma_{11}^{2} \frac{\sqrt{g}}{g_{11}^{3/2}} \quad \text{(ON } u^2 = \text{CONST. CURVES)}$$

WHEN WE HAVE ORTHOGONAL COORDINATES $\Longrightarrow$

$$\Gamma_{22}^{1} = -\frac{1}{2g_{11}} \frac{\partial g_{22}}{\partial u^1}, \quad \Gamma_{11}^{2} = -\frac{1}{2g_{22}} \frac{\partial g_{11}}{\partial u^2}$$

$\therefore$ IN THIS CASE SINCE $g_{12} = 0$

$$(K_g)_{u^2} = \frac{1}{2g_{22}\sqrt{g_{11}}} \frac{\partial g_{22}}{\partial u^1} = \frac{1}{\sqrt{g_{11}}} \frac{\partial}{\partial u^1}(\log \sqrt{g_{22}})$$

$$(K_g)_{u^1} = -\frac{1}{2g_{11}\sqrt{g_{22}}} \frac{\partial g_{11}}{\partial u^2} = -\frac{1}{\sqrt{g_{22}}} \frac{\partial}{\partial u^1}(\log \sqrt{g_{11}})$$

FROM THE ABOVE RESULTS, WE HAVE

THM: THE COORDINATE CURVES OF A SURFACE OF CLASS C^2 ARE GEODESIC IFF AT EVERY POINT OF THIS SURFACE $\Gamma_{22}^{1} = \Gamma_{11}^{2} = 0$. IN THE CASE OF ORTHOGONAL COORDINATES, THIS HOLDS IFF $g_{22} = g_{22}(u^2)$ AND $g_{11} = g_{11}(u^1)$.

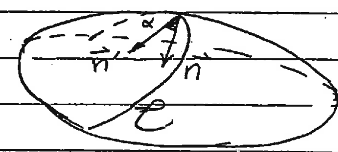
EXTRINSIC GEOMETRY OF SURFACES

- NORMAL CURVATURE WAS DEFINED AS

$$K_n = b_{ij} \frac{du^i}{ds} \frac{du^j}{ds}$$

$$= \frac{b_{ij}\, du^i du^j}{g_{ij}\, du^i du^j} \qquad \text{A SIGNED QUANTITY}$$

- MEUSNIER'S THEOREM



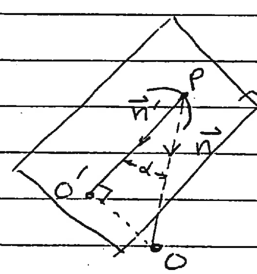
LET α BE THE ANGLE BETWEEN THE PRINCIPAL NORMAL VECTOR $\vec{n}'$ OF A CURVE $\mathcal{C}$ ON A SURFACE AND THE LOCAL NORMAL TO THE SURFACE $\vec{n}$. THEN THE CURVATURE K OF $\mathcal{C}$ IS

$$K = \frac{K_n}{\cos\alpha}$$

PROOF: $K_n \vec{n} = \vec{K}_n = K(\vec{n}' \cdot \vec{n})\vec{n} = K\cos\alpha\, \vec{n}$

MEUSNIER'S THM: THE CENTER OF CURVATURE OF A POINT P ON $\mathcal{C}$ IS THE PROJECTION OF THE CENTER OF NORMAL CURVATURE ONTO THE OSCULATING PLANE OF $\mathcal{C}$ AT P

THIS RESULT IS ILLUSTRATED ON THE RIGHT

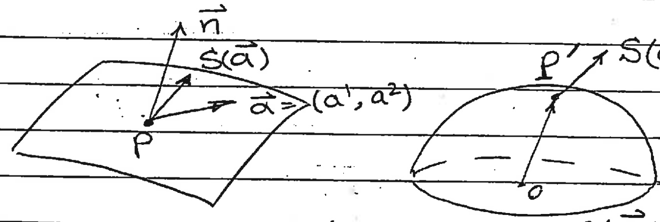


PROOF $K_n = \frac{1}{R}$, $K = \frac{1}{R'}$, SUBSTITUTE IN (*) ABOVE TO GET $R' = R\cos\alpha$

$PO = R$
$PO' = R'$

THE SPHERICAL IMAGE OF TANGENT VECTORS

LET $\vec{a} = a^i \vec{r}_i$ BE A TANGENT VECTOR, WE CALL $S(\vec{a}) = + b_i^j a^i \vec{r}_j$ AS THE SPHERICAL IMAGE OF $\vec{a}$. WE CAN VIEW THIS AS ANOTHER TANGENT VECTOR. THE ORIGIN OF THIS DEFINITION IS AS FOLLOWS. THE BASIS VECTORS ON THE SPHERICAL IMAGE OF A SURFACE ARE $\vec{n}_1$ AND $\vec{n}_2$. WE DEFINE THE SPHERICAL IMAGE OF $\vec{a}$ BY $S(\vec{a}) = -a^i \vec{n}_i$ AND THEN USE WEINGARTEN FORMULA.



$S(\vec{a}) = -a^1 \vec{n}_1 - a^2 \vec{n}_2$

$\vec{n}_1 = \frac{\partial \vec{n}}{\partial u^1}$

$\vec{n}_2 = \frac{\partial \vec{n}}{\partial u^2}$

$S(\vec{a}+\vec{b}) = S(\vec{a}) + S(\vec{b})$

$S(\lambda \vec{a}) = \lambda S(\vec{a})$

$$|S(\vec{a})|^2 = g_{ij}\, b_k^i\, b_l^j\, a^k a^l$$

$$= b_{kj}\, g^{jm}\, b_{lm}\, a^k a^l = \underbrace{g^{jm} b_{kj} b_{lm}}_{C_{kl}} a^k a^l$$

THUS $|S(\vec{a})| \neq |\vec{a}|$ IN GENERAL.

NOTE: $S(d\vec{r}) = -d\vec{n} = + b_i^j \vec{r}_j\, du^i$.

THM $\vec{a} \cdot S(\vec{a}) = + b_{ij} a^i a^j$. THEREFORE

$$K_n = + \frac{\vec{a} \cdot S(\vec{a})}{|\vec{a}|^2}$$ (NORMAL CURVATURE ALONG $\vec{a}$)

THM: i) $S(\vec{a}) \times S(\vec{b}) = K\, \vec{a} \times \vec{b}$, K: GAUSSIAN CURVATURE

$$S(\vec{a}) \times \vec{b} + \vec{a} \times S(\vec{b}) = + b_i^i\, \vec{a} \times \vec{b} = 2H\, \vec{a} \times \vec{b}$$

WHERE $H = \frac{1}{2} b_i^i$ IS THE MEAN CURVATURE.

THE MEAN CURVATURE CAN BE WRITTEN AS

$$H = \frac{1}{2g}\left[g_{11}b_{22} - 2g_{12}b_{12} + g_{22}b_{11}\right]$$

$$= \frac{1}{2g}\left[\begin{vmatrix} g_{11} & b_{12} \\ g_{21} & b_{22}\end{vmatrix} + \begin{vmatrix} b_{11} & g_{12} \\ b_{21} & g_{22}\end{vmatrix}\right]$$

THM: THE MEAN CURVATURE IS A SIGNED QUANTITY WHOSE ABSOLUTE VALUE IS INDEPENDENT OF CHOICE OF COORDINATES. A CHANGE OF ORIENTATION OF THE SURFACE CHANGES THE SIGN OF THE MEAN CURVATURE.

THM: $H^2 \geq K$. $H^2 = K$ IFF $b^1_1 = b^2_2$ AND $b^2_1 b^1_2 = 0$ WHICH HOLD IF THE TWO FUNDAMENTAL FORMS ARE PROPORTIONAL.

A POINT ON A SURFACE IS CALLED AN <u>UMBILICAL POINT</u> IFF THE NORMAL CURVATURE IS THE SAME IN EVERY DIRECTION AT THIS POINT, i.e.

$$b_{ij}\,du^i du^j = k\, g_{ij}\, du^i du^j$$

SINCE BY ASSUMPTION, THERE ARE MORE THAN TWO SOLUTIONS OF THIS $\Rightarrow b_{ij} = k\, g_{ij}$, i.e. THE TWO FUNDAMENTAL FORMS ARE PROPORTIONAL. WE HAVE PROVED THE FOLLOWING

THM: A POINT IS AN UMBILICAL POINT OF A SURFACE IFF $b_{ij} = \lambda g_{ij}$ $\forall i,j = 1,2$

— WE CAN SHOW THAT THE PLANE AND SPHERES ARE THE ONLY SURFACES CONSISTING OF UMBILICAL POINTS ONLY. OTHER SURFACES HAVE ISOLATED UMBILICAL POINTS

— THE THM ON PRECEDING PAGE ON $H^2 \geq K$ ALSO SAYS THAT $H^2 = K$ AT A POINT P <u>IFF</u> P IS AN UMBILICAL POINT.

<u>THM</u> A POINT P IS UMBILICAL IFF $\forall \vec{a}$ $S(\vec{a}) = k\,\vec{a}$ WHERE k IS THE NORMAL CURVATURE WHICH IS INDEPENDENT OF DIRECTION $\vec{a}$.

— AN UMBILICAL POINT IS EITHER AN ELLIPTIC OR PARABOLIC POINT: IF PARABOLIC, IT IS A FLAT POINT.

— PRINCIPAL DIRECTIONS AND PRINCIPAL CURVATURES

$\vec{a}$ IS IN THE PRINCIPAL DIRECTION IF $\vec{a} \parallel S(\vec{a})$, i.e. $\vec{a} \times S(\vec{a}) = 0$. LET $\vec{a} = du^i\,\vec{r}_i = d\vec{r}$

$$S(\vec{a}) \times \vec{a} = S(d\vec{r}) \times d\vec{r} = -\,d\vec{n} \times d\vec{r} = 0$$

<u>THM</u>: THE PRINCIPAL DIRECTIONS ARE GIVEN BY

$$b_2^1 (du^2)^2 + (b_1^1 - b_2^2)\,du^1 du^2 - b_1^2 (du^1)^2 = 0$$

IN GENERAL, UNLESS k_n IS THE SAME IN ALL DIRECTIONS (SUCH POINTS ARE KNOWN AS UMBILICAL POINTS), THERE ARE TWO DISTINCT PRINCIPAL DIRECTIONS.

THE ABOVE EQ. FOR PRINCIPAL DIRECTIONS IS FOUND BY WRITING

$$-d\vec{n} \times d\vec{r} = (b_i^j \vec{r}_j \, du^i) \times \vec{r}_K \, du^K$$

$$= (b_i^1 du^i \vec{r}_1 + b_i^2 du^2 \vec{r}_2) \times (\vec{r}_1 du^1 + \vec{r}_2 du^2)$$

$$= \underbrace{(b_i^1 du^i du^2 - b_i^2 du^1 du^i)}_{=0} \vec{r}_1 \times \vec{r}_2 = 0$$

— THE PRINCIPAL DIRECTIONS CAN ALSO BE FOUND FROM

$$\begin{vmatrix} (du^2)^2 & -du^1 du^2 & (du^1)^2 \\ b_{11} & b_{12} & b_{22} \\ g_{11} & g_{12} & g_{22} \end{vmatrix} = 0$$

THIS IS OBTAINED BY WRITING $b_i^j = g^{jK} b_{iK}$ AND THEN USING THE RELATION BETWEEN g_{ij} AND g^{ij} IN THE FORMULA IN THE ABOVE THM FOR PRINCIPAL DIRECTIONS.

THM: LET $\vec{a}$ BE ALONG A PRINCIPAL DIRECTION $\Rightarrow S(\vec{a}) = K\vec{a}$ WHERE K IS THE NORMAL CURVATURE ALONG $\vec{a}$ KNOWN AS THE PRINCIPAL CURVATURE.
THIS RESULT IS ALSO SUFFICIENT FOR K TO BE THE PRINCIPAL CURVATU
— THE ABOVE IS KNOWN AS RODRIGUES THM.

THE CONDITION $S(\vec{a}) = K\vec{a}$ CAN BE USED TO FIND THE PRINCIPAL CURVATURES

$$S(d\vec{r}) = -d\vec{n} = b_i^j \vec{r}_j du^i = K \vec{r}_j du^j$$

$$\therefore \; b_i^j du^i = K du^j$$

$$(b_i^j - \delta_j^i k)\, du^j = 0$$

$$\begin{vmatrix} b_1^1 - k & b_2^1 \\ b_1^2 & b_2^2 - k \end{vmatrix} = 0$$

$$k^2 - (b_1^1 + b_2^2) + b_1^1 b_2^2 - b_1^2 b_2^1 = 0$$

$$k^2 - 2Hk + K = 0$$

NOTE SINCE $H^2 \geq K$ ALWAYS, WE HAVE REAL SOLUTIONS.

$$\Rightarrow \begin{cases} k_1 + k_2 = 2H \\ k_1 k_2 = K = b_1^1 b_2^2 - b_1^2 b_2^1 \end{cases}$$

GAUSSIAN CURVATURE

THM: THE PRINCIPAL CURVATURES OF A SURFACE COINCIDE AT AN UMBILICAL POINT AND ARE DISTINCT AT OTHER POINTS.

THM: THE PRINCIPAL DIRECTIONS ARE ORTHOGONAL AT A NONUMBILICAL POINT.

THM (EULER) THE NORMAL CURVATURE IN THE DIRECTION MAKING AN ANGLE α WITH THE PRINCIPLE DIRECTION 1 IS GIVEN BY

$$k_n = k_1 \cos^2\alpha + k_2 \sin^2\alpha$$

PROOF: LET $\vec{e}_1$ AND $\vec{e}_2$ BE UNIT VECTORS IN THE PRINCIPAL DIRECTIONS 1 AND 2. THEN $S(\vec{e}_1) = k_1 \vec{e}_1$, $S(\vec{e}_2) = k_2(\vec{e}_2)$. LET $\vec{e} = \vec{e}_1 \cos\alpha + \vec{e}_2 \sin\alpha$

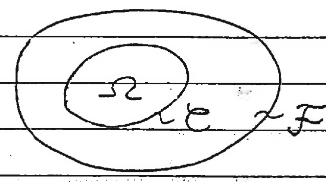
$$k_n = \vec{e} \cdot S(\vec{e}) = (\vec{e}_1 \cos\alpha + \vec{e}_2 \sin\alpha) \cdot (k_1 \vec{e}_1 \cos\alpha + k_2 \vec{e}_2 \sin\alpha)$$
$$= k_1 \cos^2\alpha + k_2 \sin^2\alpha$$

THM THE PRINCIPAL CURVATURES k_1 AND k_2 ARE THE MAXIMUM AND MINIMUM OF NORMAL CURVATURES FOR ALL DIRECTIONS. (USE EULER'S THM)

— SOME USEFUL RESULTS:

i) MINIMAL SURFACES: SURFACES OF ZERO MEAN CURVATURE ARE CALLED MINIMAL SURFACES.

THM: $\mathcal{C}$ PIECEWISE SMOOTH CLOSED CURVE ON A SURFACE $\mathcal{F}$, $\Omega \subset \mathcal{F}$ A DOMAIN $\ni \partial\Omega = \mathcal{C}$. LET Ω' BE ANY DOMAIN $\ni \partial\Omega' = \mathcal{C} \ni$ AREA $\Omega <$ AREA $\Omega' \Longrightarrow \mathcal{F}$ IS A MINIMAL SURFACE



SKETCH OF THE PROOF (GOETZ): LET $\vec{r}(u^1,u^2)$ DESCRIBE Ω AND $\alpha(u^1,u^2)$ BE A C^1 FUNCTION $\ni$ $\alpha = 0$ ON $\partial\Omega = \mathcal{C}$. FOR SMALL t, THE SURFACE $\vec{R} = \vec{r}(u^1,u^2) + t\,\alpha(u^1,u^2)\,\vec{n}$ REPRESENTS A SURFACE NEAR Ω (WE CALLED Ω' IN THE THM ABOVE) $\ni \partial\Omega' = \mathcal{C}$. WE CAN SHOW THAT

$$\sqrt{g'} = \sqrt{g} - 2t\alpha H\sqrt{g} - t^2\psi(u^1,u^2,t)$$

WHERE ψ IS A BOUNDED FN $\Longrightarrow$

S_t = AREA $\Omega' = \int_{\Omega'} \sqrt{g'}\, du^1 du^2$

IS MINIMUM IF $dS/dt = 0$. THIS LEADS TO

$$\int_{\Omega} \alpha H \sqrt{g}\, du^1 du^2 = 0 \quad \text{AT} \quad t = 0$$

SINCE α AND Ω ARE ARBITRARY $\Longrightarrow H = 0$, i.e. $\mathcal{F}$ MINIMAL

THE CONVERSE OF THIS THM IS NOT CORRECT UNLESS ADDITIONAL CONDITIONS ARE IMPOSED.

A SURFACE IS MINIMAL IF

$$2gH = g b^i_i = g(b^1_1 + b_2{}^2) = g_{11} b_{22} - 2 g_{12} b_{12} + g_{22} b_{11} = 0$$

ii) ANOTHER USEFUL DEFINITION OF K_n

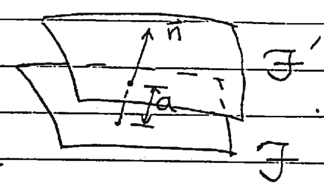
SINCE $S(d\vec{r}) = -d\vec{n}$

$$K_n = \frac{d\vec{r} \cdot S(d\vec{r})}{|dr|^2}$$

$$= -\frac{d\vec{r} \cdot d\vec{n}}{|d\vec{r}|^2}$$

iii) PARALLEL SURFACE : LET $\mathcal{F}$ BE A SURFAC DESCRIBED BY $\vec{r}(u^1, u^2)$. THE SURFACE $\mathcal{F}'$: $\vec{R}(u^1, u^2) = \vec{r}(u^1, u^2) + a\,\vec{n}(u^1, u^2)$, $a = \text{CONST}$ IS SAID TO BE PARALLEL TO $\mathcal{F}$. THE NATURAL BASIS VECTORS ON $\mathcal{F}'$ ARE

$$\vec{R}_1 = \vec{r}_1 + a\,\vec{n}_1$$
$$\vec{R}_2 = \vec{r}_2 + a\,\vec{n}_2$$



$\Longrightarrow$

$$\sqrt{g'} = (1 - 2Ha + K a^2)\sqrt{g}$$

$$H' = \frac{H - Ka}{1 - 2Ha + K^2 a^2}$$

$$K' = \frac{K}{1 - 2Ha + Ka^2}$$

SEE PROB. 3, PAGE 268, GOETZ

THESE RESULTS ARE QUITE USEFUL. FOR EXAMPLE,

BY VARYING a IN THE VICINITY OF $a=0$, WE HAVE AN EXTENSION OF THE VECTOR FIELD $\vec{n}(u^1, u^2)$ TO THE VICINITY OF $\mathcal{F}=0$. WE CAN THUS DEFINE $\nabla \cdot \vec{n}$ AS FOLLOWS USING THE INVARIANT DEFINITION OF DIVERGENCE:

$$\nabla \cdot \vec{Q} = \frac{1}{\sqrt{g_{(3)}}} \frac{\partial}{\partial u^i} \left(\sqrt{g_{(3)}}\, Q^i \right)$$

{ $g_{(3)}$ IS THE DET. OF COEF. OF 1ST FUND. FORM. HERE $g_{(3)} = g'$

WHERE $u^3 = a$ IN OUR CASE. ALSO NOTE THAT $\vec{n} = (0,0,1)$ SO THAT

$$\nabla \cdot \vec{n} = \frac{1}{\sqrt{g'}} \frac{\partial \sqrt{g'}}{\partial a}$$

$$= \frac{-2(+H - Ka)}{1 - 2Ha + Ka^2} = -2H'$$

IN PARTICULAR

$$\nabla \cdot \vec{n}(u^1, u^2, 0) = \nabla \cdot \vec{n}\Big|_{\mathcal{F}}$$

$$= -2H$$

WE NOW DERIVE ANOTHER IMPORTANT RESULT. LET US USE $(\vec{r}_1, \vec{r}_2, \vec{n})$ AS LOCAL BASIS VECTOR ON AND IN THE VICINITY OF A SURFACE $\mathcal{F}$. THEN, AGAIN USING THE FACT THAT $g_{(3)} = g'$, AND THE INVARIANT DEFINITION OF DIVERGENCE, WE CAN SHOW THAT

$$\nabla \cdot \vec{Q} = \nabla_2 \cdot \vec{Q}_2 - 2HQ_3 + \frac{\partial Q_3}{\partial n}, \quad \vec{Q}_2 = (Q^1, Q^2)$$

THIS RESULT IS USED IN LOADING AND QUADRUPOLE TERMS OF THE FW-H EQ.

iv) CONJUGATE DIRECTIONS : GIVEN A TANGENT VECTOR $\vec{a}$, THE DIRECTION $\vec{b}$ IS CONJUGATE TO $\vec{a}$ IF $\vec{b} \cdot S(\vec{a}) = 0$ i.e $\vec{b} \perp S(\vec{a})$.

— AT AN ELLIPTIC OR HYPERBOLIC POINT, CONJUGATE DIRECTION TO ANY VECTOR EXISTS. THE MATRIX OF SPHERICAL MAPPING IS $\begin{bmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{bmatrix}$. THE DETERMINANT OF THIS MAPPING IS THE GAUSSIAN CURVATURE K. THUS ONLY AT PARABOLIC POINTS, THE MAPPING IS SINGULAR. AT FLAT POINTS, THE SPHERICAL IMAGE OF ALL VECTORS IS UNDEFINED. FOR A NONFLAT PARABOLIC POINT, THE ABOVE MATRIX IS OF RANK 1 AND THERE IS ONE DIRECTION $\vec{a}'$ SUCH THAT $S(\vec{a}') = 0$. FOR $\vec{a} \not\parallel \vec{a}' \Rightarrow \vec{a}' \cdot S(\vec{a}) = 0 = \vec{a} \cdot S(\vec{a}')$, i.e. THE SPHERICAL IMAGE ALL VECTORS NOT PARALLEL TO $\vec{a}'$ IS PARALLEL TO $\vec{a}'$.

THM : $\vec{a} = (a^1, a^2)$ AND $\vec{b} = (b^1, b^2) \Rightarrow \vec{b}$ IS CONJUGATE TO DIRECTION $\vec{a}$ IFF $b_{ij} a^i b^j = 0$.

COR. : DIRECTION $\vec{b}$ IS CONJUGATE TO $\vec{a}$ IFF $\vec{a}$ IS CONJUGATE TO $\vec{b}$.

COR. : THE COORDINATE CURVES HAVE CONJUGATE DIRECTIONS AT A POINT P IFF $b_{12}(P) = 0$. $\left\{ \begin{array}{l} \text{TAKE } \vec{a} = (1,0) \\ \vec{b} = (0,1) \text{ ABOVE} \end{array} \right.$

— IF u^1 AND u^2 - CURVES HAVE CONJUGATE DIRECTIONS THEY ARE SAID TO FORM CONJUGATE NET. FROM ABOVE COR., WE MUST HAVE $b_{12}(u^1, u^2) = 0 \ \forall (u^1, u^2)$.

A SELF CONJUGATE DIRECTION IS CALLED AN ASYMPTOTIC DIRECTION. FROM ABOVE RESULTS, IT IS OBVIOUS THAT $\vec{a}$ IS IN ASYMPTOTIC DIRECTION IFF $b_{ij} a^i a^j = 0$. SINCE THIS EXPRESSION APPEARS IN NUMERATOR OF $K_n(\vec{a}) \Rightarrow \vec{a}$ IS AN ASYMPTOTIC DIRECTION IFF $K_n(\vec{a}) = 0$. THIS MEANS THAT THE OSCULATING PLANE OF ANY CURVE IN DIRECTION OF $\vec{a}$ LIES IN THE TANGENT PLANE.

THM: $\exists$ 2 DISTINCT ASYMPTOTIC DIRECTIONS AT A HYPERBOLIC POINT OF A SURFACE; AT A NONFLAT PARABOLIC POINT, $\exists$ ONE ASYMPTOTIC DIRECTION; AT AN ELLIPTIC POINT $\nexists$ ANY ASYMTOTIC DIRECTION, AT A FLAT POINT EVERY DIRECTION IS ASYMPTOTIC.

COR.: u^1-CURVE HAS ASYMPTOTIC DIRECTION IFF $b_{11} = 0$. SIMILARLY, u^2-CURVE HAS ASYMPTOTIC DIRECTION IFF $b_{22} = 0$.

A CURVE ON A SURFACE IS CALLED AN ASYMPTOTIC LINE IF IT HAS AN ASYMPTOTIC DIRECTION AT EVERY POINT. TO OBTAIN AN ASYMPTOTIC LINE PASSING THRU A GIVEN POINT, WE MUST SOLVE THE O.D.E. $b_{ij}(u^1,u^2)\, du^1 du^2 = 0$, WHERE $u^2 = u^2(u^1)$.

V) THE INDICATRIX OF DUPIN: THIS IS A CURVE CONSTRUCTED ON A TANGENT PLANE OF A POINT P ON A SURFACE $\mathcal{F}$. THIS CURVE IS LAID OUT ON TANGENT PLANE BY GOING A DISTANCE OF $1/\sqrt{|K_n|}$ ALONG EACH RAY FROM P. LET US USE

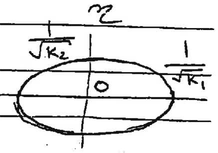
CARTESIAN VARIABLES ξ AND η IN TANGENT PLANE WITH AXIS IN PRINCIPAL DIRECTION

THM: THE INDICATRIX OF DUPIN IS AN ELLIPSE AT AN ELLIPTIC POINT P WITH EQUATION

$$|k_1|\xi^2 + |k_2|\eta^2 = 1$$

OR

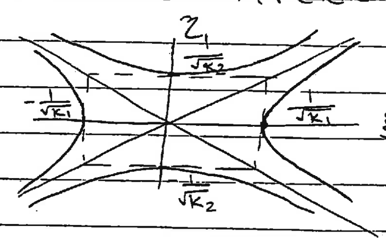
$$\frac{\xi^2}{|k_1|^{-1}} + \frac{\eta^2}{|k_2|^{-1}} = 1$$



— AT AN UMBILICAL POINT, THE ABOVE ELLIPSE DEGENERATES INTO A CIRCLE. AT A FLAT POINT, THIS CIRCLE HAS RADIUS ZERO AND WE SAY THAT THE INDICATRIX DOES NOT EXIST.

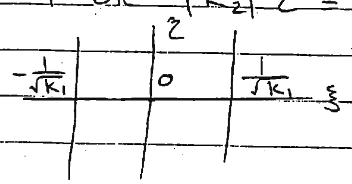
THM AT A HYPERBOLIC POINT, THE INDICATRIX OF DUPIN CONSISTS OF TWO CONJUGATE HYPERBOLAS

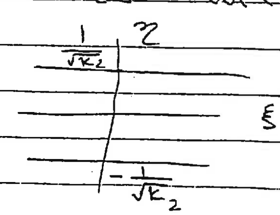
$$k_1\xi^2 + k_2\eta^2 = \pm 1$$



— AT A PARABOLIC POINT WHICH IS NOT FLAT, THE INDICATRIX OF DUPIN IS A PAIR OF LINES WITH EQ $|k_1|\xi^2 = 1$ OR $|k_2|\eta^2 = 1$

(THESE ARE DEGENERATE PARABOLAS)





— THE INDICATRIX OF DUPIN GIVES INFORMATION

ON HOW A PLANE PARALLEL TO THE TANGENT PLANE INTERSECTS THE SURFACE IN THE VICINITY OF THE TANGENT PLANE. LET T BE THE TANGENT PLANE AT POINT P AND LET T′ AND T″ BE PLANES PARALLEL TO T AND A SMALL DISTANCE h ON EITHER SIDES OF T. LET US PROJECT THE CURVES OF INTERSECTION OF T′ AND T″ ON T AND DILATE THESE CURVES ALONG RAYS FROM P WITH RATIO $1/\sqrt{h}$. AS $h \to 0$, THE ABOVE CONSTRUCTION GIVES THE INDICATRIX OF DUPIN.

V(i) LINES OF CURVATURE: THESE ARE CURVES THAT ARE IN PRINCIPAL DIRECTIONS AT EVERY POINT ALONG THE CURVE.

THM: ON A SURFACE OF CLASS C^3, THRU ANY NON-UMBILICAL POINT TWO LINES OF CURVATURE CAN BE DRAWN. WE CAN LAY AN ORTHOGONAL NET CONSISTING OF LINES OF CURVATURE ON ANY DOMAIN WITHOUT UMBILICAL OR SINGULAR POINTS.

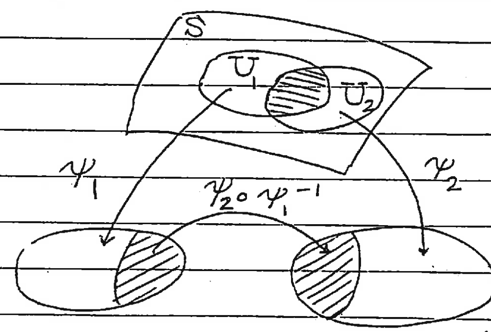
THE NORMALS ALONG A CURVE ON A SURFACE FORM A RULED SURFACE. WE HAVE THE FOLLOWING

THM: A CURVE ON A SURFACE IS A LINE OF CURVATURE IFF THE RULED SURFACE NORMAL TO THE SURFACE ALONG THIS CURVE IS DEVELOPABLE.

DIFFERENTIABLE MANIFOLDS (D.M.)

D.M. IS A GENERALIZATION OF THE IDEA OF PARAMETRIC DESCRIPTION OF SURFACES INTRODUC BY GAUSS. IT IS STUDIED PRIMARILY IN DIFFERENTIAL TOPOLOGY BUT HAS MANY APPLICATIONS

IN GENERAL A SINGLE PARAMETRIZATION DOES NOT APPLY TO A GIVEN SURFACE. ONE IS FORCED TO USE SEVERAL COORDINATE PATCHES DIFFERENTIABLE MANIFOLD (D.M.) SPECIFIES THE RELATION BETWEEN PATCHES IN ITS MOST GENERAL FORM. THE DEFINITIONS OF VECTORS AND TENSORS ON MANIFOLDS ARE POSSIBLE AND THIS GIVES NEW FREEDOM OF THOUGHT FROM IRRELEVANT DETAILS OF TENSOR ANALYSIS. THE USE OF D.M. CONCEPT CLARIFIES THE GEOMETRIC MEANING OF SOME RESULTS SUCH AS THE MEANING OF GAUSS AND CODAZZI'S FORMULAS.



LET US CONSIDER A SURFACE S WHICH TWO REGIONS (PATCHES) U_1 AND $U_2 \ni U_1 \cap U_2$ IS NONEMPTY. LET

$P \in U_1$, $Q \in U_2$

$\ni \psi_1(P) = (u^1, u^2)$ AND $\psi_2(Q) = (u^{1'}, u^{2'})$, i.e. U_1 AND U_2 ARE PARAMETRIZED BY (u^1, u^2) AND $(u^{1'}, u^{2'})$, RESPECTIVELY. IT IS SEEN

THAT OVER $U_1 \cap U_2$, THE RELATION BETWEEN (u^1, u^2) AND $(u^{1'}, u^{2'})$ IS GIVEN BY THE FUNCTION

$$(u^{1'}, u^{2'}) = \psi_2 \circ \psi_1^{-1} (u^1, u^2) \quad \left\{ \begin{array}{l} \text{COMPOSITIO} \\ \text{OF FNS:} \end{array} \right.$$

$$\equiv \phi (u^1, u^2)$$

WHERE $\phi = \psi_2 \circ \psi_1^{-1}$. FOR THIS TRANSFORMATION TO BE INVERTIBLE, WE REQUIRE

$$\frac{\partial (u^{1'}, u^{2'})}{\partial (u^1, u^2)} \neq 0 \quad \text{ON } U_1 \cap U_2$$

WE CALL (u^1, u^2) AND $(u^{1'}, u^{2'})$ THE LOCAL COORDINATES IN U_1 AND U_2, RESPECTIVELY. WE CAN REQUIRE CLASS C^r DIFFERENTIABILITY FOR FUNCTION ϕ.

IN THE DEFINITION OF DIFFERENTIABLE MANIFOLD, THE ABOVE REQUIREMENTS OF DIFFERENTIABILITY AND NONVANISHING OF JACOBIAN OF TRANSFORMATION ARE EXTENDED TO ALL POSSIBLE COORDINATE PATCHES ON S. ONE MUST ASK, HOWEVER, WHAT IS A SURFACE ANYWAY? WE CONSIDER A CONNECTED HAUSDORFF TOPOLOGICAL SPACE IN n DIMENSIONS COVERED BY A SYSTEM OF OPEN NEIGHBORHOODS HOMEOMORPHIC TO R^m, $1 < m < n$, AS A SURFACE IN n DIMENSIONS. WE WILL FIRST DEFINE SOME OF THESE TERMS.

A TOPOLOGICAL SPACE IS THE MOST PRIMITIVE OF ALL SPACES WHERE NONTRIVIAL MATHEMATICS

CAN BE WORKED IN. TO DEFINE THIS SPACE, WE RECALL SOME RESULTS FROM THE METRIC SPACE R^1 WITH $d(x,y) = |x-y|$ WHERE x AND y ARE REAL NUMBERS. THE OPEN AND CLOSED SETS ARE DEFINED AS USUAL. WE HAVE THE FOLLOWING PROPERTIES OF OPEN SETS

i) THE EMPTY SET AND R^1 ARE OPEN

ii) THE UNION OF ANY NUMBER OF OPEN SETS IS OPEN

iii) THE INTERSECTION OF A FINITE NUMBER OF OPEN SETS IS OPEN

iv) CLOSED SETS ARE COMPLEMENTS WRT R^1 OF OPEN SETS.

IN A TOPOLOGICAL SPACE X, A COLLECTION OF SETS g IS GIVEN CALLED THE OPEN SETS OF X SUCH AT

i) X AND THE EMPTY SET $\in g$

ii) IF $G_\alpha \in g$, $\alpha \in A \Rightarrow \bigcup_{\alpha \in A} G_\alpha \in g$

iii) IF G_i, $i = 1 \ldots, n$ (n FINITE), $\in g \Rightarrow \bigcap_{i=1}^{n} G_i \in g$

THE TOPOLOGICAL SPACE IS DENOTED BY (X, g)

IF $G \in g$, THEN $X \setminus G$ IS CALLED A CLOSED SET.

LET $Y \subset X$ AND $h = \{ Y \cap G, G \in g \}$, THEN (Y, h) IS A TOPOLOGICAL SPACE. WE SAY THAT THE TOPOLOGY IN Y IS THE TOPOLOGY INDUCED BY THE TOPOLOGY IN X.

FOR A GIVEN SET X, THERE MAY BE MANY COLLECTION OF SETS $g, h, k, \ldots$ WHICH I CAN

X INTO A TOPOLOGICAL SPACE. TWO PARTICULAR WAYS ANY SET CAN BE MADE INTO A TOPOLOGICAL SPACE ARE

i) $g = \{\phi, X\}$, ϕ: EMPTY SET

ii) $g = \{U \mid U \subseteq X\}$

THESE TWO ARE CALLED TRIVIAL TOPOLOGIES. THE T.S. (ii) IS CALLED THE DISCRETE TOPOLOGY.

— THE FAMILIAR TOPOLOGY OF REAL LINE IS FORMED BY COLLECTION OF OPEN SETS WHICH ARE UNIONS OF OPEN INTERVALS.

— SOME USEFUL DEFINITIONS:

i) (X, g), (X, h) T.S. ON THE SAME SET. THEN THE TOPOLOGY OF g IS FINER THAN THAT BY h IF $g \supset h$. IN THIS CASE THE TOPOLOGY OF h IS SAID TO BE COARSER THAN THAT OF g.

IN THE ABOVE EXAMPLES, (i) IS THE COARSEST TOPOLOGY ON X AND (ii) IS THE FINEST TOPOLOGY ON X POSSIBLE.

ii) (X, g) A T.S., $b \subset g$ IS A BASE OF TOPOLOGY IF $\forall G \in g$ $\exists B_\alpha \in b$, $\alpha \in A \ni G = \bigcup_{\alpha \in A} B_\alpha$

WE CAN SHOW THAT b IS A BASE OF (X, g) IFF $\forall x \in X$ AND $G \in g \ni x \in G$ $\exists H \in b \ni x \in H$ AND $H \subset G$.

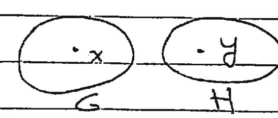
<u>EXAMPLE</u>: IN THE TOPOLOGY OF R^1, OPEN INTERVALS WITH RATIONAL ENDPOINTS FORM A BASE OF THE TOPOLOGY. IF THE INTERVALS DO NOT HAVE A RATIONAL END POINTS, WE GET ANOTHER BASE FOR THE TOPOLOGY.

iii) A METRIC SPACE CAN BE MADE INTO A TOPOLOGICAL SPACE IN A UNIQUE WAY. LET (X, d) BE A METRIC SPACE WITH THE METRIC d. THE BASE b OF T.S. CONSISTS OF ALL THE OPEN SPHERES IN THE METRIC SPACE. THE SET g (OF SETS) IS THE SET OF ALL UNIONS OF SUBSETS OF b. THEN (X, g) IS A T.S. THIS TOPOLOGY IS SAID TO BE <u>INDUCED</u> BY THE METRIC.

THIS IS THE WAY WE MADE R^1 INTO A T.S.

$\therefore$ A T.S. (X, g) IS SAID TO BE <u>METRIZABLE</u> IF $\exists\, d$, A METRIC, ON $X \ni (X, g)$ IS A T.S. WITH THE TOPOLOGY INDUCED BY THE METRIC d.

iv) FOR OUR PURPOSE A HAUSDORFF SPACE IS THE MOST IMPORTANT T.S. A TOPOLOGICAL SPACE (X, g) IS SAID TO BE A <u>HAUSDORFF SPACE</u> IF GIVEN $x, y \in X$, $x \neq y$ $\exists\, G, H \in g$
$\ni x \in G$, $y \in H$ AND $G \cap H = \phi$



OF THE TWO TRIVIAL T.SPACES MENTIONED ABOVE, (i) IS NOT HAUSDORFF BUT (ii) IS.

— EVERY METRIC SPACE IS A HAUSDORFF SPACE.

V) CONTINUITY CAN BE DEFINED ON A T.S. LET (X, g) AND (Y, h) BE TOPOLOGICAL SPACES AND $f: X \to Y$ A FUNCTION. f IS CONTINUOUS AT x IF $\forall H \in h \ni f(x) \in H \ \exists G \in g \ni f(G) \subset H$. f IS CONTINUOUS ON X IF IT IS CONTINUOUS AT ALL POINTS $x \in X$.

— f IS CONTINUOUS IF $\forall H \in h \Rightarrow f^{-1}(H) = G \in g$

VI) $A \subset X$, (X, g) A T.S., A IS DISCONNECTED IF $\exists G \,\&\, H \in g \ni A \cap G$ AND $A \cap H$ ARE DISJOINT NONEMPTY SETS AND $A = (A \cap G) \cup (A \cap H)$. A SET IS CONNECTED IF IT IS NOT DISCONNECTED

VII) TWO T.S (X, g) AND (Y, h) ARE HOMEOMORPHIC OR TOPOLOGICALLY EQUIVALENT IF $\exists f: X \to Y \ni f$ AND f^{-1} ARE CONTINUOUS.

VIII) A T.S. (X, g) IS SEPARABLE IF $\exists$ A COUNTABLE DENSE SUBSET V IN X, i.e. $\exists V \ni X = \overline{V}$, V COUNTABLE.

— WE WILL BE CONCERNED WITH METRIC SPACES WHERE THE TOPOLOGY IS INDUCED BY A METRIC AND THE RESULTING T.S. IS HAUSDORFF. IN ANY CASE, RECOGNITION OF A DIFFERENTIABLE MANIFOLDS IN PRACTICAL CASES IS EASY. FIRST WE MUST COMPLETE THE DEFINITION OF A DIFFERENTIABLE MANIFOLD.

WE DEFINE A D.M. IN SEVERAL STEPS.

DEFINITION: LET M BE A T.S. A CHART (V, ϕ) IS A HOMEOMORPHISM Φ OF AN OPEN SET V OF M TO AN OPEN SET IN R^m. TWO CHARTS ARE COMPATIBLE IF EITHER $V_1 \cap V_2$ IS EMPTY OR ON $V_1 \cap V_2$ $\phi_1 \circ \phi_2^{-1}$ AND $\phi_2 \circ \phi_1^{-1}$ ARE C^r MAPPINGS OF OPEN SETS OF R^m. AN ATLAS IS A SET OF COMPATIBLE CHARTS THAT COVER M. TWO ATLASES ARE COMPATIBLE IF ALL THEIR CHARTS ARE COMPATIBLE. (COMPATIBILITY OF ATLASES IS AN EQUIVALENCE RELATION)

— IF ALL CHARTS MAP M INTO R^m WITH THE SAME m, THE DIMENSION OF M IS SAID TO BE m.

— WE ASSUME THAT V'S IN CHARTS OF M ARE CONNECTED.

DEFN: A DIFFERENTIABLE MANIFOLD IS A SEPARABLE, METRIZABLE SPACE M WITH AN EQUIVALENCE CLASS OF ATLASES.

— IN PRACTICE ONE ONLY NEEDS C^r MAPPING FOR $\phi_1 \circ \phi_2^{-1}$ AND $\phi_2 \circ \phi_1^{-1}$. HOWEVER, MOST BOOKS ASSUME C^∞ MAPPING.

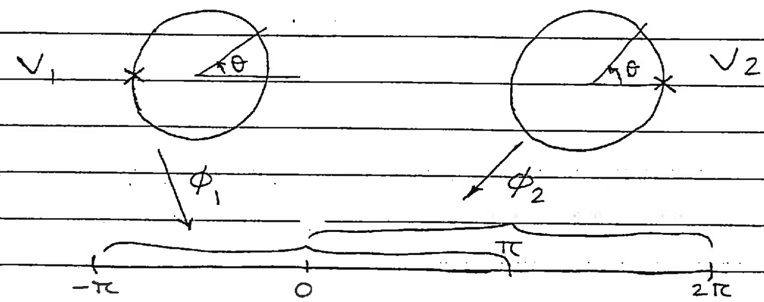
WE WILL NEXT GIVE SOME EXAMPLES.

1. $M = R^n = V$, $\phi = 1$ (FUNCTION 1). IN THIS

CASE, WE ONLY NEED ONE CHART.

2. ONE DIMENSIONAL SPHERE S^1: LET $M = \{(x_1, x_2) \in R^2 : x_1^2 + x_2^2 = 1\}$. WE NEED TWO CHARTS OF THIS:

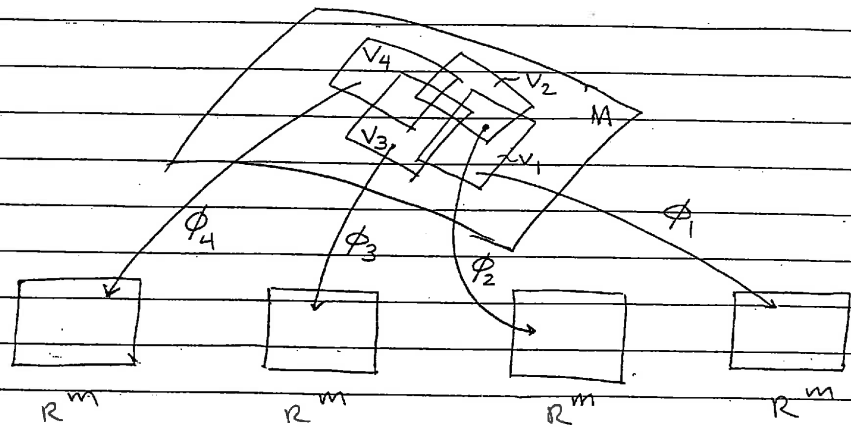
$V_1 = S^1 \setminus \{(-1,0)\}$, $\phi_1^{-1} : \theta \to (\cos\theta, \sin\theta)$, $\theta \in (-\pi,$
$V_2 = S^1 \setminus \{(1,0)\}$, $\phi_2^{-1} : \theta \to (\cos\theta, \sin\theta)$, $\theta \in (0, 2\pi)$



3. LET $f \in C^1$, $M = \{x \in R^n : f(x) = 0$ & $\forall x\ \exists j \ni \frac{\partial f}{\partial x_j} \neq 0\}$. THIS IS THE MOST COMMON WAY WE FORM A MANIFOLD. THE IMPLICIT FUNCTION THM GUARANTEES THE EXISTENCE OF SUITABLE CHARTS SO THAT M BECOMES AN $(n-1)$-DIMENSIONAL D.M.

4. LET f AND $g \in C^1$, $M = \{x \in R^4 : f(x) = g(x) = C$ & $\forall x$, $\frac{\partial(f,g)}{\partial(x_i, x_j)} \neq 0$ FOR SOME $i, j = 1-4\}$. M IS THEN A MANIFOLD WHICH IS TWO DIMENSIONAL.

THE FOLLOWING FIGURE IS THEN THE PICTURE OF A D.M WE HAVE IN MIND



SINCE $V_1 \cap V_2$ IS NONEMPTY, OVER $V_1 \cap V_2$, WE MUST HAVE $\phi_2 \circ \phi_1^{-1}$ AND $\phi_1 \circ \phi_2^{-1}$ CONTINUOUS, ETC

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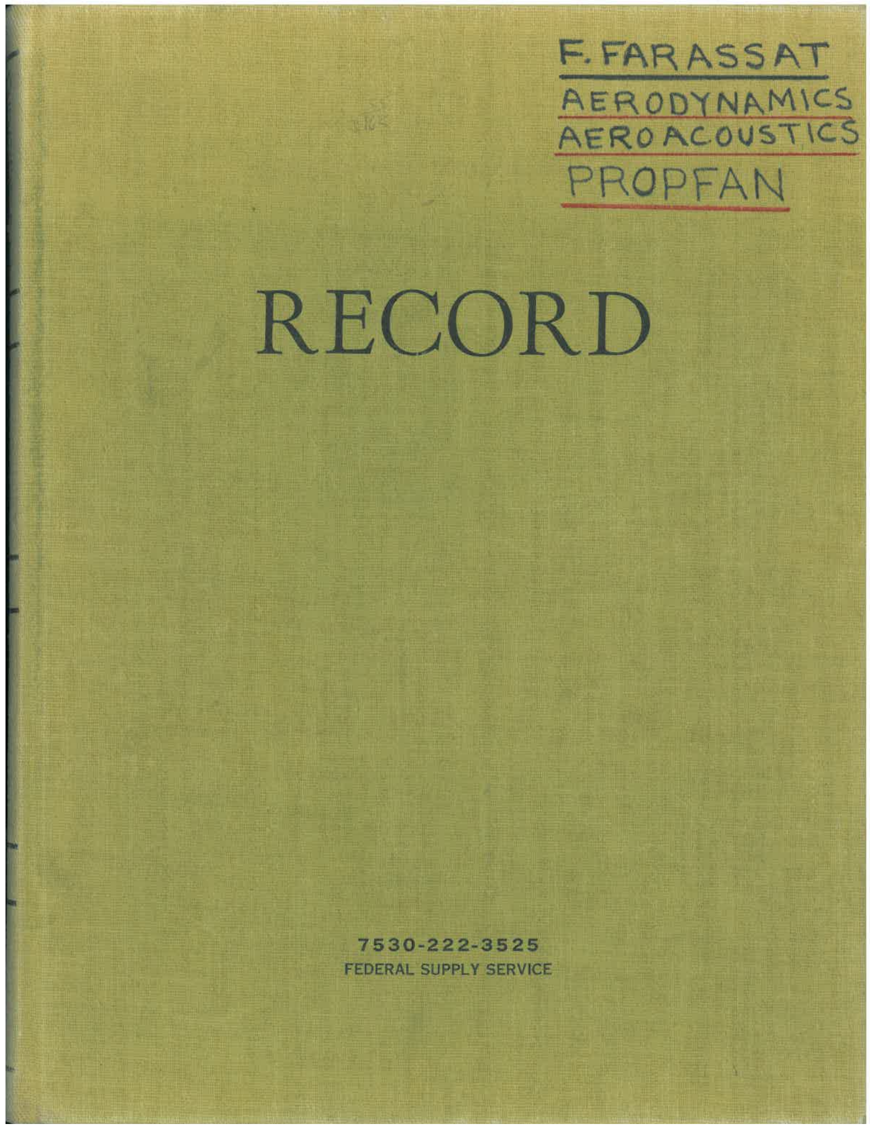
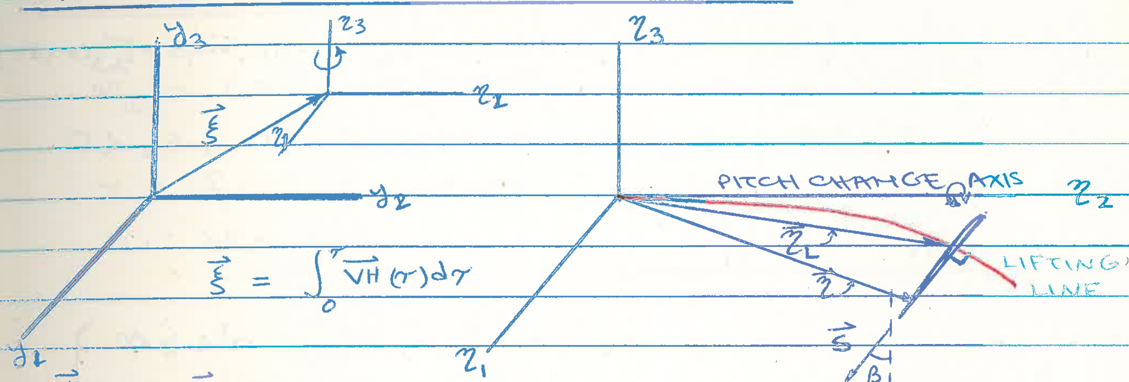


ACOUSTIC PROGRAM – SUBSONIC TIP (OR $M_r < 1$)

DUE TO SPECIAL GEOMETRY OF PROPFAN BLADES, THE FOLLOWING PROGRAM WILL BE DEVELOPED TO IMPROVE ACCURACY AND REDUCE COMPUTER TIME.

SPECIFICATION OF BLADE GEOMETRY



$$\vec{\xi} = \int_0^{\tau} \vec{VH}(\tau)\,d\tau$$

$\vec{y}$ AND $\vec{\eta}$-FRAMES COINCIDE AT $\tau = 0$

$$\vec{\eta}_L = (\eta_{L1}(\eta_{L2}),\ \eta_{L2},\ \eta_{L3}(\eta_{L2}))$$

$\alpha = 90 - \beta^\circ$ GEO. ANGLE OF ATTACK, DEGREES

$\alpha = \alpha(\eta_{L2})$

$CH = CH(\eta_{L2})$ CHORD

$|\vec{s}| = 1$

DETERMINATION OF $\vec{s}$

$$\begin{cases} \vec{s} \cdot \vec{e}_3 = -\cos\beta = s_3 \\ \vec{s} \cdot \dfrac{d\vec{\eta}_L}{d\eta_2} = s_1 \eta'_{L1} + s_2 + s_3 \eta'_{L3} \\ |\vec{s}| = s_1^2 + s_2^2 + s_3^2 = 1 \end{cases}$$

, $\vec{e}_3$ UNIT VECTOR ALONG η_3

$$\begin{cases} s_1 \eta'_{L1} + s_2 = \eta'_{L3} \cos\beta \\ s_1^2 + s_2^2 = \sin^2\beta \end{cases}$$

$$\Rightarrow \quad s_1^2 + (\eta'_{L3}\cos\beta - s_1 \eta'_{L1})^2 = \sin^2\beta$$

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$$(1+\eta_{L1}'^2)S_1^2 - 2\eta_{L1}'\eta_{L3}'\cos\beta S_1 + \eta_{L3}'^2\cos^2\beta - \sin^2\beta = 0$$

$$S_1 = \frac{1}{1+\eta_{L1}'^2}\left[\eta_{L1}'\eta_{L3}' + \sqrt{(1+\eta_{L1}'^2)\sin^2\beta - \eta_{L3}'^2\cos^2\beta}\right]$$

$$\varepsilon_1 = \sqrt{1+\eta_{L1}'^2}$$

$$S_1 = \frac{1}{1+\eta_{L1}'^2}\left[\eta_{L1}'\eta_{L3}' + \sqrt{(\varepsilon_1\sin\beta+\eta_{L3}'\cos\beta)(\varepsilon_1\sin\beta-\eta_{L3}'\cos\beta)}\right]$$

WE HAVE $\eta_{L1}' > 0$, $\eta_{L3}' < 0$, $\cos\beta > 0$, $S_1 > 0$, $S_2 < 0$.

$$S_2 = \eta_{L3}'\cos\beta - S_1\eta_{L1}'$$

THE CONDITION THAT $(1+\eta_{L1}'^2)\sin^2\beta - \eta_{L3}'^2\cos^2\beta \geq 0$ IS EQUIVALENT TO THE FACT THAT A CONE WITH VERTEX ANGLE 2β AND AXIS PARALLEL TO η_3-AXIS AT ANY POINT ON LIFTING LINE, SHOULD INTERSECT THE PLANE NORMAL TO THE LIFTING LINE. THIS GEOMETRIC CONDITION IS ALWAYS SATISFIED.

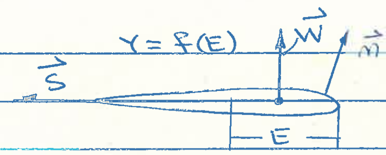
$$\vec{\eta} = \vec{\eta}_L + K\,CH(\eta_{L2})\,\vec{S} \qquad (T.E.)\,3/4 \leq K \leq -1/4\,(L.E.)$$

$$\vec{V}(\vec{\eta}) = \vec{\Omega}\times\vec{\eta} + \vec{VH} \qquad \text{LOCAL BLADE VELOCITY}$$

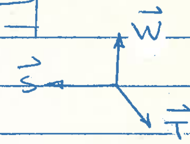
$$\dot{\vec{V}}(\vec{\eta}) = \vec{\Omega}\times(\vec{\Omega}\times\vec{\eta}) + \dot{\vec{VH}} \qquad \text{LOCAL BLADE ACCELERATION}$$
$$= (\vec{\Omega}\cdot\vec{\eta})\vec{\Omega} - \Omega^2\vec{\eta} + \dot{\vec{VH}}$$

$\vec{\Omega}$ WILL ALWAYS BE ALONG η_3-AXIS

$$\dot{\vec{V}}(\vec{\eta}) = \dot{\vec{VH}} - \Omega^2(\eta_1, \eta_2, 0)$$



$\vec{T}$: UNIT TANGENT TO LIFTING LINE



$$\vec{T} = \frac{1}{\sqrt{1+\eta_{L1}'^{2}+\eta_{L3}'^{2}}}\left(\eta_{L1}',\ 1,\ \eta_{L3}'\right)$$

$\vec{m}_{U,L} = (m_1',\ m_2',\ \pm m_3')$ IN $\vec{S}\,\vec{T}\,\vec{W}$ SYSTEM

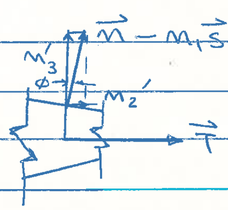
$Y = f(E) = \text{THKRAT}\left(C_1 E^{1/2} + C_2 E + C_3 E^2 + C_4 E^3 + C_5 E^4\right)$

Y : NON-DIM'L.

$$\begin{cases} m_1' = -\dfrac{f'(E)}{\sqrt{1+f'^{2}(E)}} \\[2ex] m_2' = \dfrac{-\dfrac{1}{2}\dfrac{d(\text{THK})_{MAX}}{d\eta_2}A}{\sqrt{1+f'^{2}(E)}} \\[2ex] m_3' = \dfrac{A}{\sqrt{1+f'^{2}(E)}} \\[2ex] A = \left(1+\eta_{L1}'^{2}+\eta_{L3}'^{2}\right)^{-1/2}\left[1+\dfrac{1}{4}\left(\dfrac{d(\text{THK})_{MAX}}{d\eta_2}\right)^2\right]^{-1/2} \end{cases}$$

$(\text{THK})_{MAX}$: MAXIMUM THICKNESS

THESE EQUATIONS ACCOUNTS FOR BLADE THICKNESS VARIATION



$$\begin{cases} \cos\phi = A \\[1ex] \sin\phi = -\dfrac{1}{2}\dfrac{d(\text{THK})_{MAX}}{d\eta_2}A \\[2ex] \tan\phi = -\dfrac{1}{2}\dfrac{d(\text{THK})_{MAX}}{d\eta_2}\dfrac{d\eta_2}{dS} \end{cases}$$

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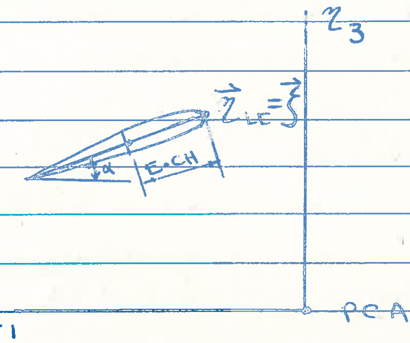
$$\frac{\partial}{\partial t}\left[\frac{v_m}{r(1-M_r)}\right]_{ret} = \left[\frac{\partial}{\partial \tau}\ \frac{v_m}{r-\vec{r}\cdot\vec{M}}\right]_{ret}$$

WHEN THE RIGHT SIDE IS WRITTEN IN TERMS OF THE COORDINATE SYSTEM SHOWN ON P1, THE RESULT IS TOO COMPLICATED TO BE OF USE IN NUMERICAL WORK. A NEW AND MUCH SIMPLER METHOD NEEDS TO BE DEVISED.



P.C.A. : PITCH CHANGE AXIS
ASSUMED TO LIE ON η_2-AXIS

SUBSCRIPTS :
U : UPPER
L : LOWER

E = DISTANCE FROM L.E. / CHORD

y = CH(η_2) TR(η_2) f(E) THICKNESS FUNCTION GIVING AIRFOIL THICKNESS DISTRIBUTION ABOVE MEAN SURFACE OR THE CAMBER.

TR = THICKNESS RATIO

ALL THE VECTORS BELOW ARE IN ROTATING $\vec{\eta}$-FRAME

$\vec{\xi} = (\xi_1(\eta_2), \eta_2, \xi_3(\eta_2))$: LEADING EDGE CURVE

$\alpha = \alpha(\eta_2)$ ANGLE OF ATTACK (GEOMETRIC)

$\vec{\eta}$ = POSITION VECTOR ON BLADE SURFACE

$\vec{VH} = (0, 0, VH(3))$ AIRCRAFT FORWARD SPEED

$\vec{V} = \vec{VH} + \vec{\Omega}\times\vec{\eta}$ LOCAL BLADE SPEED

$\vec{\Omega} = (0, 0, \Omega)$ ANGULAR VELOCITY VECTOR

$v_M = \vec{V}\cdot\vec{M}$

$\vec{M}$ = LOCAL NORMAL VECTOR (UNIT)

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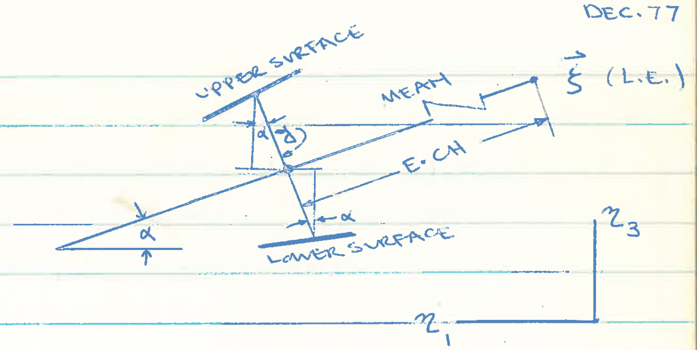

$$\begin{cases} \eta_{1U} = \xi_1 + ECH\cos\alpha + y\sin\alpha \\ \eta_{3U} = \xi_3 - ECH\sin\alpha + y\cos\alpha \end{cases}$$

$$\begin{cases} \eta_{1L} = \xi_1 + ECH\cos\alpha - y\sin\alpha \\ \eta_{3L} = \xi_3 - ECH\sin\alpha - y\cos\alpha \end{cases}$$

$$\vec{n}_{U,L} = \vec{N}_{U,L} / |\vec{N}_{U,L}|$$

$$\vec{N}_{U,L} = \frac{\partial \vec{\eta}_{U,L}}{\partial E} \times \frac{\partial \vec{\eta}_{U,L}}{\partial \eta_2}$$

ELEMENT OF AREA $dS_{U,L} = |\vec{N}_{U,L}|\, dE\, d\eta_2$

FOR THE MEAN SURFACE

$$\begin{cases} \eta_1 = \xi_1 + ECH\cos\alpha \\ \eta_3 = \xi_3 - ECH\sin\alpha \end{cases}$$

$$\vec{n}_m = \vec{N}_m / |\vec{N}_m|$$

$$\vec{N}_m = \frac{\partial \vec{\eta}}{\partial E} \times \frac{\partial \vec{\eta}}{\partial \eta_2}, \quad dS_m = |\vec{N}_m|\, dE\, d\eta_2$$

$$\cos\theta = \frac{\vec{r}}{r} \cdot \vec{n}_m$$

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Note: There has been a change of symbols, the distance along the chord as measured from the leading edge is now represented by "Q" rather than "E".

Computer Program - TEST

The program will be described in six sections:

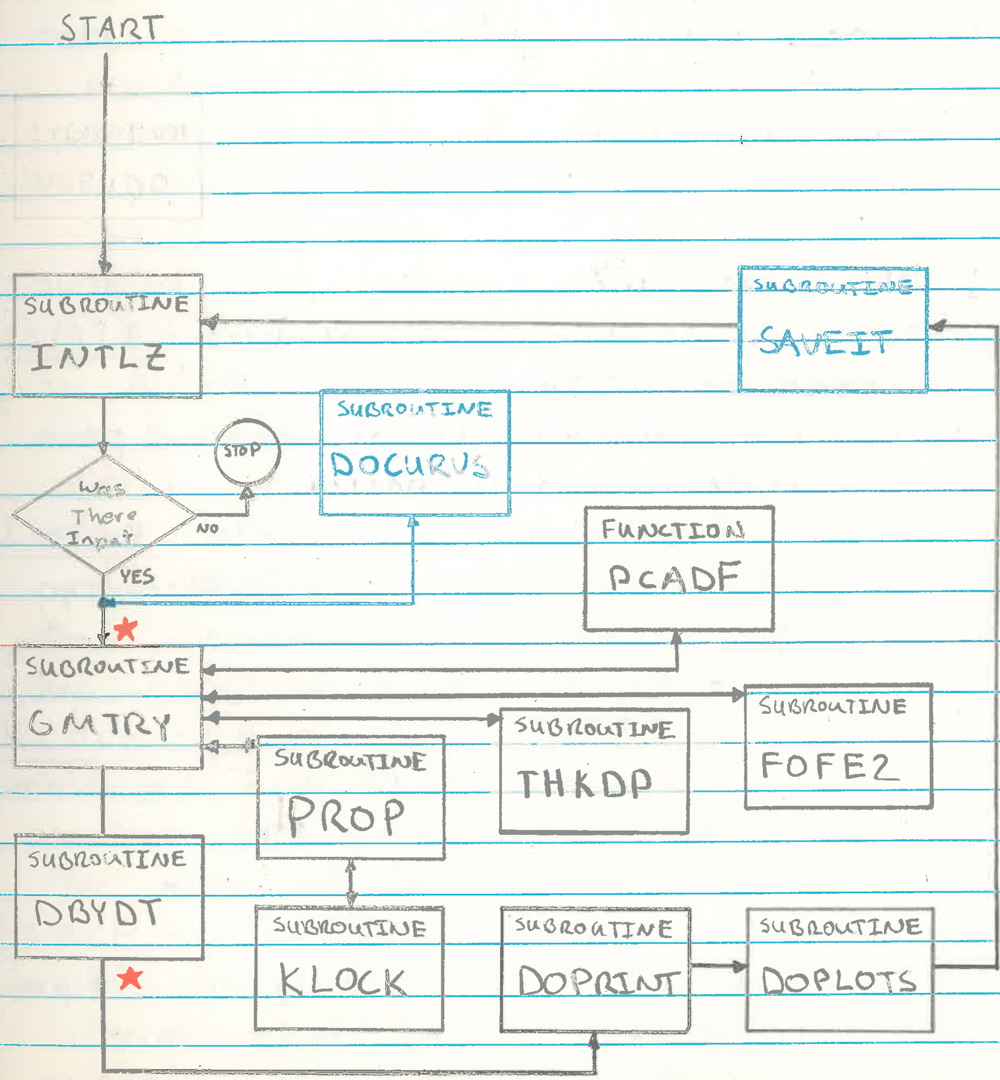
Note: In the previous sections, the blade half-thickness is represented by the symbol "y"; in this section, the blade half thickness is broken into factors like this,

$$y = CH \cdot Thkrat \cdot Thk \quad \text{where,}$$

CH: Chord
Thkrat: Thickness ratio
Thk: Normalized Half-Thickness

Flowchart

Below is an overall flowchart of PROGRAM TEST, blocks in blue have not yet been included.



Brief Description of Subprograms

The subprograms used in PROGRAM TEST and PROGRAM TEST are briefly described below. All of them, except SUBROUTINE PSEUDO and those not yet included (DOCURUS and SAVEIT), are more fully explained in the last section. The programs are listed in alphabetical order.

DBYDT - Sums the positive and negative components, performs the required differentiation (either 2 or 3 point), shifts with wraparound to center peak and calculates the spectra.

DOCURUS - This routine will plot curves defined by the input functions. They are tenatively planned to be

1.) η_2 vs. PCA
2.) ζ_1 vs. η_2
3.) ζ_3 vs. η_2
4.) α vs. η_2
5.) CH vs. η_2
6.) ΔP vs. Q $\exists$ η_2 division

DOPLOTS - This routine plots the pressure signatures and spectra of the noise components. The plots are

1.) Overall Pressure
2.) Overall Spectrum
3.) Thickness Noise
4.) Thickness Spectrum
5.) Loading Noise Components
6.) Loading Noise
7.) Loading Spectrum

DOPRINT - Prints a data sheet which lists input parameters and some output variables and (optionally) list the pressure signatures and spectra.

FOFEZ - Input subroutine in which properties of the blade are calculated. These quantities can be functions of η_2 only.

1.) α 2.) $\partial\alpha/\partial\eta_2$
3.) CH 4.) $\partial CH/\partial\eta_2$
5.) Thkrat 6.) ∂ Thkrat $/\partial\eta_2$
7.) ξ_1 8.) $\partial\xi_1/\partial\eta_2$
9.) ξ_3 10.) $\partial\xi_3/\partial\eta_2$

GMTRY - segments the blade into elements as directed by the various division schemes and for each element (mean, upper and lower) calculates the position, unit normal and surface area.

INTLZ - Intializes all of the input parameters and allows any changes that need to be made. Some constants are generated and arrays are zeroed.

KLOCK - Given the coordinates of the element and of the observer, this routine will calculate the retarded time for any observer time.

PCADF - This input function defines a curve which locates the η_2 coordinates of the chordwise sections.

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PROP - Using the quantities calculated in SUBROUTINE GMTRY, this routine steps the elements through time and adds their contributions. The components calculated are thickness and near and far field loading. For better accuracy, negative and positive values are summed separately.

SAVEIT - This routine will store on a file all the input parameters and results. A method of storing the input routines (FOFE2, PCADF and THKDP) needs to be determined.

TEST - This is the actual program routine where the decisions as to which routines are to be called are made. The CPU timing is also done in this routine.

THKDP - This input subroutine is used to calculate properties which can also be chord dependent, they are:

1.) Thk
2.) ΔP
3.) $\partial Thk/\partial Q$

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Input and Output

The input is one Fortran Namelist (/Input/), one function routine (PCADF) and two subroutines (FOFEZ and THKDP). Each element will be described separately. The output consists of a "PROPFAN DATA SHEET" (p.19) and (optionally) plots of curves representing the input functions, listings, storage and/or plotting of the pressure signatures and spectra.

NAMELIST /INPUT/ - Below are the definitions of the input parameters and their default values.

C	Speed of sound - 345 m/sec
CHECK	Logical variable that when true enables the printing of intermediate values - .FALSE.
CURVES	Logical variable that when true enables the plotting of the input functions - .FALSE.
EPSILON	Allowable error in the calculation of the retarded time - 1. x 10^{-8} sec
IDENT	Ten character string used to further identify output - 10HTWIN OTTER
NBLADES	Number of blades - 2

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NDIF — Number of points to be used in differentiation. If positive, observer is assumed to be moving parallel with the blade; if negative, observer is assumed to be stationary. IABS(NDIF) = 2 or 3. - 2

NLE — Number of points taken along the chord from the leading edge to THKMAXL. See Chordwise Division Scheme, p. 42 - 5

NPCA — Number of points taken along the pitch change axis from RINNER to R. - See FUNCTION PCADF, p 13 - 15

NPTS — Number of points per period, rounded down if necessary to make even. - 200

NSPEC — Number of spectral lines to calculate. - 50

NTE — Number of points taken along the chord from THKMAXL to trailing edge. See Chordwise Division Scheme, p. 42 - 10

PLOTS — Logical variable that when true enables the plotting of the pressure signatures and spectra - .TRUE.

PRINT — Logical variable that when true enables the printing of the pressure signatures and spectra - .TRUE.

R — Blade radius, outer. - 1.3m

REV — Rotational blade speed. - 2145. RPM

RHO — Ambient air density - 1.2029 kg/m^3

RINNER	Blade radius, inner. - 0.3m
STORE	Logical variable that when true enables the storage of all input parameters and output onto TAPE3. - .FALSE.
SYMBOLS	Logical variable that when true enables the plotting of symbols on the plots of the input functions at blade division points. - .TRUE.
THKMAXL	Location of "leading edge - trailing edge" transition. See Chordwise Division Scheme. p 42 - 0.05
V3	Translational blade speed in η_3 direction, also observer speed if NDIF>0. - 40 m/sec.
X0(3)	Initial observer position. - (7.28, 0., 0.) M

FUNCTION PCADF - This function is used to divide the blade in the spanwise direction. The domain and range of this function is the interval [RINNER, R]. The function is evaluated at the end points of NPCA equally spaced segments to obtain the η_2 coordinates of the slices (See figure, next page). By dividing the blade this way, it is possible to force a region of smaller blade elements anywhere on the blade. Note:

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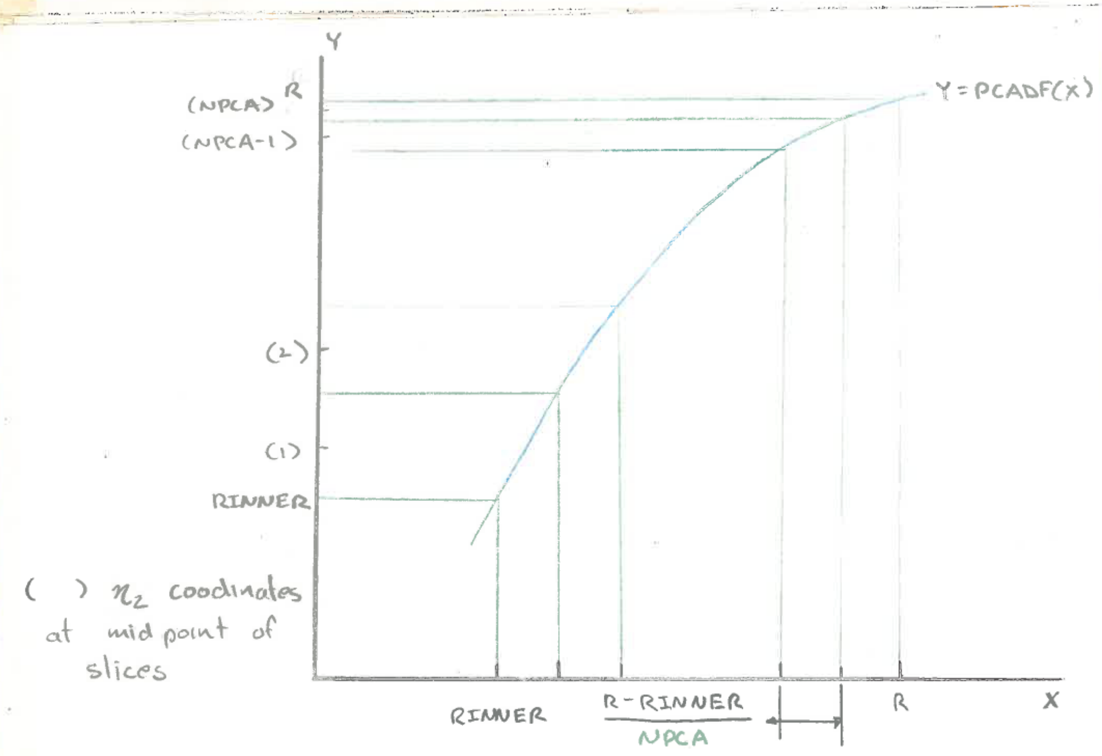
the derivative of PCADF must always be positive. A sample listing is shown below, (Twin Otter calculations)



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SUBROUTINE FOFEZ - This routine is used to calculate properties of the blade which can be functions of η_2 only. The input to this routine is through COMMON /GMTRY/ (pp 24 & 27) which passes the η_2 coordinate (E2) and the normalized η_2 coordinate, η_2/R (ER). The output of this routine are the following quantities, which are passed through COMMON /EVAL/ (pp. 24 & 27)

AA	Angle of attack. (rad)
AAP	Derivative of the above with respect to η_2. (rad/m).
CH	Chord (m)
CHP	Derivative of the above with respect to η_2.
THKRAT	Thickness ratio.
THKRATP	Derivative of the above with respect to η_2. (m^{-1})
Z1	η_1 coordinate of the leading edge. (m)
Z1P	Derivative of the above with respect to η_2.
Z3	η_3 coordinate of the leading edge. (m)
Z3P	Derivative of the above with respect to η_2.

By using a subroutine to obtain these quantities, it is possible to have either functions or tables to define the above quantities. A sample list

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is shown below, (Twin Otter calculations).

SUBROUTINE THKDP - This routine is used to calculate quantities which can be functions of both η_2 and Q. The input to this routine is through COMMON /GMTRY/ (pp 24 & 27) which passes the η_2 coordinate (E2), the normalized η_2 coordinate, η_2/R (ER) and the normalized chord as measured from the leading edge Q (Q). The output of this routine are the following quantities which are passed through COMMON /EVAL/ (pp 24 & 27):

DP	Pressure difference (PA).
THK	Normalized half-thickness.
THKP	Derivative of the above with respect to Q.

By using a subroutine to obtain these quantities, it is possible to have either functions or tables to define the above quantities. A sample listing is shown on the next page, (Twin Otter calculations).

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<u>Output</u> - Every run generates a "PROPFAN DATA SHEET" as shown on the opposite page. In the upper right hand corner of all the output (including the plots) is the run identifier block which consist of the date (YY/MM/DD) and time at which the calculations started followed by the ten character blade identifier (input variable IDENT). Additionally, the CPU time for execution is given below the identifier block on the "PROPFAN DATA SHEET", it represents the CPU elasped time

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between the two red stars on the Flowchart (p. 7). With the exception of the below terms, the output consist of a repetition of the input variables (See NAMELIST /INPUT/ p.11).

BPF	Blade passage frequency. (Hz)
DT	Time between points. (sec)
M-TIP	Tip Mach Number
OASPL	Overall sound pressure level, divided also into thickness and loading contributions (dB re. 20 μPA).
PER	Period of blade passage. (sec)
POWER	Power per blade (kW/blade and Hp/blade).
THRUST	Thrust per blade (N/blade)
TORQUE	Torque per blade (N-m/blade)
V3	Translational blade speed in η_3 direction, also observer speed in NDIF>0. (mph)

Note: "M-TIP" is now labelled "HELICAL M-TIP" to clarify that it does take into account the forward flight.

Additional Output, See p. 33

Provided that PRINT = .TRUE. the following listings are also produced, which should be self explanatory.

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PRESSURE SIGNATURES OF NOISE COMPONENTS — 78/02/08. 09.35.32. TWIN OTTER

POINT NUMBER	OBSERVER TIME (SEC)	THICKNESS NOISE (PA)	LOADING NOISE FAR FIELD (PA)	LOADING NOISE NEAR FIELD (PA)	LOADING NOISE COMBINED (PA)	OVERALL NOISE (PA)
1	0.0000	1.2308	1.0228	-.9949	.0279	1.2587
2	.0000	1.2350	1.0674	-.9945	.0729	1.3079
3	.0001	1.2394	1.1121	-.9941	.1180	1.3574
4	.0001	1.2439	1.1567	-.9935	.1632	1.4071
5	.0001	1.2486	1.2014	-.9930	.2085	1.4571
6	.0002	1.2535	1.2461	-.9923	.2538	1.5074
7	.0002	1.2587	1.2909	-.9916	.2993	1.558[illegible]
[illegible]	.0002	1.2639	1.3357	-.9908	.3449	1.6[illegible]
	.0002	[illegible]	1.3806	-.9899		
	.0003		[illegible]4255	-.9890		
	[illegible]			-.9880		

PRESSURE SPECTRA OF NOISE COMPONENTS — 78/02/08. 09.35.32. TWIN OTTER

HARMONIC NUMBER	HARMONIC FREQUENCY (HZ)	THICKNESS NOISE (DB)	LOADING NOISE (DB)	OVERALL NOISE (DB)
1	107.25	100.80	105.70	107.55
2	214.50	102.42	101.35	105.59
3	321.75	102.19	97.32	104.02
4	429.00	101.30	93.64	102.53
5	536.25	100.08	90.23	101.00
6	643.50	98.68	87.03	99.41
7	750.75	97.17	83.99	97.78
8	858.00	95.59	81.07	96.[illegible]
9	965.2[illegible]	[illegible]3.96	78.25	
10	107[illegible]	[illegible]	75.5[illegible]	

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Provided that PLOTS = .TRUE. the seven plots from SUBROUTINE DOPLOTS, p. 8 will be generated. They are also self explanatory.

The storage of output (SAVE = .TRUE.) and the generation of curves defined by the input functions (CURVES = .TRUE.) have not yet been finalized and so are not included in this discussion.

Intermediate values can be printed by setting CHECK = .TRUE. If done, the output on the following page is generated, where:

I - Spanwise strip number, starting at RINNER.
J - Chordwise element number, starting from leading edge
E2OLD - Outer spanwise slice

The other variables have been previously defined or will be define in SUBROUTINE GMTRY (p. 35).

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I	J	E2OLD	Q	E2	DP
Z1	Z3	CH	AA	THKRAT	THK
Z1P	Z3P	CHP	AAP	THKRATP	THKP
EM1	EM3	NM1	NM2	NM3	DSM
EU1	EU3	NU1	NU2	NU3	DSU
EL1	EL3	NL1	NL2	NL3	DSL
1	15	5.5820E-01	9.4350E-01	4.2910E-01	2.2963E+02
-7.1533E-02	4.6947E-02	1.7113E-01	5.8079E-01	1.4898E-01	1.8021E-02
-8.5953E-02	-6.1845E-02	7.5851E-02	-1.1555E+00	-6.8905E-01	-4.6909E-01
6.3450E-02	-4.1642E-02	5.4659E-01	-8.7358E-02	8.3283E-01	4.2136E-03
6.3702E-02	-4.1258E-02	6.0331E-01	-8.7534E-02	7.9269E-01	4.2240E-03
6.3198E-02	-4.2026E-02	-4.8722E-01	8.6687E-02	-8.6896E-01	4.2237E-03
2	14	6.6515E-01	8.4850E-01	6.1167E-01	1.5133E+03
-8.3143E-02	3.6535E-02	1.8163E-01	4.1403E-01	8.5592E-02	8.2226E-02
-4.2523E-02	-5.1518E-02	3.6408E-02	-7.0772E-01	-1.4294E-01	-8.5393E-01
5.7950E-02	-2.5465E-02	4.0190E-01	-4.4753E-02	9.1459E-01	1.8473E-03
5.8465E-02	-2.4295E-02	4.6752E-01	-4.3622E-02	8.8290E-01	1.8521E-03
5.7436E-02	-2.6636E-02	-3.3415E-01	4.5514E-02	-9.4142E-01	1.8523E-03
3	[illegible]	[illegible]E-01	7.5350E-01	7.0618E-01	[illegible]
[illegible]6E-02	3.[illegible]	[illegible]1	3.5496E-01	7.6350E[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	-5.4911E-01	-6.33[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	-2.5575E-02	9.[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	-2.3630E-02	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible].7132E-02	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]0E-01	[illegible]	[illegible]

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Common Block Definitions.

Below is a listing of the statements between the "C/ LIST, NONE" and "C/ LIST, ALL" directives.

The variables in each common block will be defined and the program section which first defines it will be noted.

Some naming conventions - Many variable are similar except for a prefix or a suffix. In these cases, the options will be shown in braces, "{ }.

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Prefixes

D	Differential.
S	Spectral.

Suffixes

N	Near field loading noise.
F	Far field loading noise.
DP	Loading noise.
GT	Components greater than zero.
L	Lower surface.
LT	Components less than zero.
M	Mean surface.
P	First derivative ($\partial/\partial\eta_{2}$ or $\partial/\partial Q$).
TH	Thickness noise.
U	Upper surface. VS Viscous Shear
1	Component in η_1 direction.
2	Component in η_2 direction.
3	Component in η_3 direction.
(No suffix)	Overall quantity

COMMON /ANSWER/

$PP\left\{\begin{matrix} DP\left\{\begin{matrix}A\\B\end{matrix}\right\} \\ TH \end{matrix}\right\}\left\{\begin{matrix}GT\\LT\end{matrix}\right\}$	Acoustic pressure signature. (PROP).
IOPL	Location of shifted endpoint, one period later. (DBYDT)

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OPL	Values of PPTH, PPDPB, PPDPA, PPDP and PP at IOPL (DBYDT)

COMMON /CALC/

TORQUE	Torque per blade. (GMTRY)
THRUST	Thrust per blade. (GMTRY)
POWER	Power per blade. (GMTRY)
OASPL$\{{}^{DP}_{TH}\}$	Overall sound pressure level, dB re 20 μPA. (DBYDT)

COMMON /DIVIDE/ - If not listed below, see "NAMELIST /INPUT/", p. 11.

NTIME	Number of points per period, NTIME = NPTS - MOD(NPTS, 2)+1. (INTLZ)
DT	Differential time between points. (INTLZ)
DX3	Differential observer motion in η_3 direction between time points. (INTLZ)
DPHI	Angle between blades. (INTLZ)

COMMON /ETC/ - If not listed below, see "NAMELIST /INPUT/", p 11.

PI	π (INTLZ)

DTR	Conversion factor between degrees to radians. (INTLZ)
DAY	Hollerith string containing the date when the calculations were started. In the form "ΔYY/MM/DD". (TEST)
TIME	Hollerith string containing the time of day when the calculations were started. (TEST)
CPUTIME	Elapsed CPU time for calculations (corresponds to time between red stars on "Flowchart" p. 7. (TEST)
TWICE	Logical variable that when .TRUE. caused the printed output to be duplicated. The default value is .FALSE. and is input through NAMELIST /INPUT/. (INTLZ)

COMMON /EVAL/ - See SUBROUTINE FOFE2, p. 15 and SUBROUTINE THKDP, p. 17 for definition of variables.

COMMON /GMTRY/

Q	Fraction of chord as measured from the leading edge. (GMTRY)

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$E\begin{Bmatrix}L\\M\\U\end{Bmatrix}\begin{Bmatrix}1\\2\\3\end{Bmatrix}$ — η coordinates of surface elements. (GMTRY)

ER — Normalized η_2 coordinate, η_2/R. (GMTRY)

$N\begin{Bmatrix}L\\M\\U\end{Bmatrix}\begin{Bmatrix}1\\2\\3\end{Bmatrix}$ — η coordinates of surface unit normals. (GMTRY)

$DS\begin{Bmatrix}L\\M\\U\end{Bmatrix}$ — Differential surface area of elements. (GMTRY)

COMMON /KLOCK/

T — Observer time. (PROP)

TAU — Retarded time. (KLOCK)

PHI — Original blade angle (PROP)

$X\begin{Bmatrix}1\\2\\3\end{Bmatrix}$ — Observer position, y-frame (PROP)

$M\begin{Bmatrix}1\\2\\3\end{Bmatrix}$ — η frame components of Mach number. (PROP)

$SR\begin{Bmatrix}1\\2\\3\end{Bmatrix}$ — η frame components of radiation direction (KLOCK)

SRM — $\vec{SR}\cdot\vec{M}$ (KLOCK)

COMMON /PARAM/ - If not listed below, see "NAMELIST /INPUT/", p11.

CINV — $1./c$ (INTLZ)

RINV — $1./R$ (INTLZ)

OMEGA — Rotational blade speed in $\sec^{-1}$. (INTLZ)

DIMENSION

$PP\left\{\begin{matrix} DP\left\{\begin{matrix}A\\B\end{matrix}\right\} \\ TH \end{matrix}\right\}$	Acoustic pressure signature (DBYDT)
$S\left\{\begin{matrix} DP \\ TH \end{matrix}\right\}$	Pressure spectra. (DBYDT
SPP	Overall pressure spectrum. (DBYDT)

EQUIVALENCE - To save room, the listed equivalences were made.

Full Descriptions and Method

SUBROUTINE DBYDT - This subroutine adds together the positive and negative parts of the acoustic pressure components and stores them in the "-LT" arrays. Then, either two or three point differentiation is performed and the results are stored in temporary location in the "-GT" arrays. The differentiation is performed as follows.

Two Point - The derivative is the slope of the line joining two adjacent points. See the figure on the next page.

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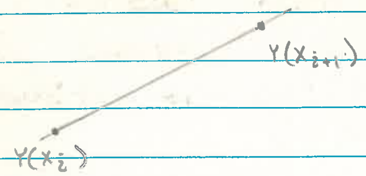
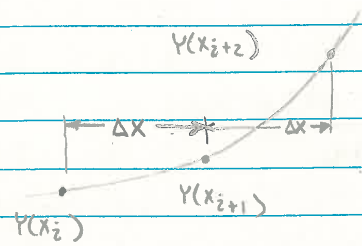


$$dY/dx\Big|_{x_i} = \frac{Y(x_{i+1}) - Y(x_i)}{x_{i+1} - x_i}$$

<u>Three Point</u> - A parabola is fitted to three adjacent points and its derivative taken and evaluated at the midpoint. See the figure below.



$$A = \frac{Y(x_{i+2}) - 2\cdot Y(x_{i+1}) + Y(x_i)}{2\cdot \Delta X^2}$$

$$B = \frac{Y(x_{i+1}) - Y(x_i)}{\Delta X} - A\cdot\Delta X$$

$$dY/dx\Big|_{x_{i+1}} = 2\cdot A\cdot\Delta X + B$$

The two loading components are then summed to obtain the loading signature and that is then summed with the thickness noise to get the overall signature.

The location of the maximum absolute overall pressure is found and the signatures are shifted with wrap-around to center the peak. The last point (NTIME) is set equal

to the first point and the location and values of the old endpoint is stored in IOPL and OPL, respectively.

The spectra for thickness noise, loading noise and the overall noise are calculated as follows (Simpson's Rule) (Overall used in example)

$$\omega_j = j \cdot \frac{2\pi}{T} \quad , \Delta t = NDIF \cdot DT \quad , T = NPTS \cdot \Delta t$$

$$A_j = \frac{2}{3} \frac{\Delta t}{T} \sum_{i=1}^{NPTS} f_i \cdot PP(i) \cdot \cos(i\, \omega_j\, \Delta t)$$

$$B_j = \frac{2}{3} \frac{\Delta t}{T} \sum_{i=1}^{NPTS} f_i \cdot PP(i) \cdot \sin(i\, \omega_j\, \Delta t)$$

$$C_j = 1/\sqrt{2}\ \sqrt{A_j^2 + B_j^2} \qquad c = a + ib$$

$$S_j = 20 \cdot \log_{10}\left[\frac{C_j}{20 \times 10^{-6}}\right] \quad (\text{dB re } 20\ \mu PA)$$

And the overall levels are computed as follows,

$$OASPL = 20 \cdot \log_{10}\left[\frac{\sum_1^{NSPEC} A_j^2 + B_j^2}{20 \times 10^{-6}}\right] \quad (\text{dB re } 20\ \mu PA)$$

$$f_i = \begin{cases} 1 & , i = 1 \\ 3 + (-1)^i & , i = 2, 3, \ldots, NPTS-1 \\ 1 & , i = NPTS \end{cases}$$

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SUBROUTINE DOPLOTS - This routine draws a common frame around each plot it produces. The plots are scaled to fit the page and the following plots are forced to the same scaling.

PPTH vs. T, PPDPA & PPDPB vs. T and PPDP vs. T
PPTH vs. W and PPDP vs. W.

In the plots of the spectrum, the minimum value can be forced no lower than Ø. SUBROUTINE MYAXES fills COMMON /PSTN/ with the correct values to properly plot the data. SUBROUTINE SCALE will generate the proper parameters for a call to MYAXES.

SUBROUTINE DOPRINT - This routine generates the "PROPFAN DATA SHEET" and the "INPUT FUNCTIONS" (see next page) output. If PRINT = .TRUE. the listings of the pressure signatures and spectra are also generated. If TWICE = .TRUE., the above is repeated.

INPUT FUNCTIONS - To be able to record what the input function are for the case that is being run, this addition was made to the output. A file was created which consisted of FUNCTION PCADF(X) - p 35, SUBROUTINE FOFEZ - p , and SUBROUTINE THKDP - p which was the edited. The edit information is then able to be listed as shown on the next page. The following section of procedure and edit command files was used.

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SUBROUTINE FOFEZ - See p. 15 for a description of the routine. The following lines have to be inserted into the program to bracket the essential code.

START△OF△FOFEZ and
END△OF△FOFEZ

For example, at the location of the blue stars on the listing, p. 16.

SUBROUTINE GMTRY - This routine segments the blade into elements as directed by the two division schemes. The blade is first "sliced" according to FUNCTION PCADF, p. 34 and the midpoint between adjacent slices is taken as the η_2 coordinate, E2. The outer slice is at E2OLD. The η_2 coordinate is then normalized, η_2/R = E2 and a call to SUBROUTINE FOFEZ, p. 35 is made. Several non-chord dependent factors are calculated and then the strip of blade is divided chordwise according to the "Chordwise Division Scheme, p. 42. For each surface (mean, upper and lower) the η coordinates, unit normal and

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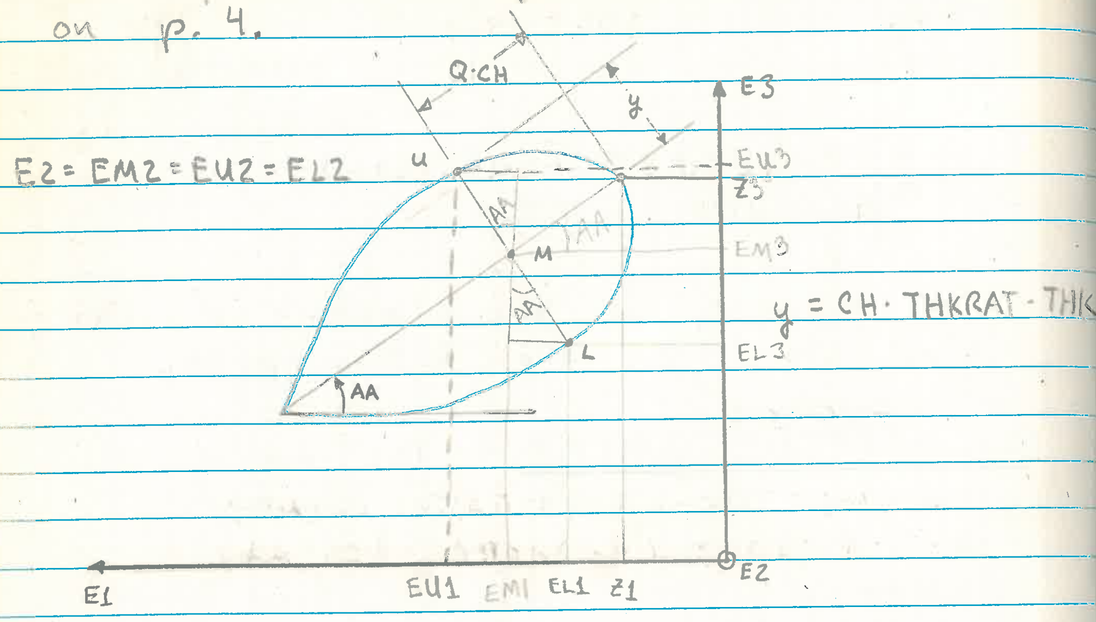
differential surface area is calculated. See figure and equations below and on p. 4.



Mean Surface

$$EM1 = Z1 + Q \cdot \boxed{CH \cdot COS(AA)}^{F1}$$

$$EM3 = Z3 - Q \cdot \boxed{CH \cdot SIN(AA)}^{F2}$$

$$\partial EM1/\partial Q = F1$$

$$\partial EM1/\partial E2 = Z1P + Q\left(\boxed{CHP \cdot COS(AA) - CH \cdot SIN(AA) \cdot AAP}^{F3}\right)$$

$$\partial EM2/\partial Q = 0 \qquad \partial EM2/\partial E2 = 1$$

$$\partial EM3/\partial Q = -F2$$

$$\partial EM3/\partial E2 = Z3P - Q\left(\boxed{CHP \cdot SIN(AA) + CH \cdot COS(AA) \cdot AAP}^{F4}\right)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F1 & 0 & -F2 \\ Z1P+Q\cdot F3 & 1 & Z3P-Q\cdot F4 \end{vmatrix}$$

$$NM1 = F2$$
$$NM2 = -F2\cdot(Z1P+Q\cdot F3) - F1\cdot(Z3P-Q\cdot F4)$$
$$NM3 = F1$$

Normal is "unitized" and forced upwards pointing, i.e. $NM3 \geq 0$

UPPER
LOWER Surfaces

$$E^{U}_{L}1 = EM1 \pm THK\cdot \underbrace{\boxed{CH\cdot THKRAT\cdot SIN(AA)}}_{F5}$$
$$E^{U}_{L}3 = EM3 \pm THK\cdot \underbrace{\boxed{CH\cdot THKRAT\cdot COS(AA)}}_{F6}$$

$$\partial E^{U}_{L}1/\partial Q = F1 \pm THKP\cdot F5$$
$$\partial E^{U}_{L}1/\partial E2 = Z1P + Q\cdot F3 \pm THK\cdot \underbrace{\boxed{[CHP\cdot THKRAT\cdot SIN(AA) + CH\cdot(THKRATP\cdot SIN(AA) + THKRAT\cdot COS(AA)\cdot AAP)]}}_{F7}$$

$$\partial E^{U}_{L}2/\partial Q = 0 \qquad \partial E^{U}_{L}2/\partial E2 = 1$$

$$\partial E^{U}_{L}3/\partial Q = -F2 \pm THKP\cdot F6$$
$$\partial E^{U}_{L}3/\partial E2 = Z3P - Q\cdot F4 \pm THK\cdot \underbrace{\boxed{[CHP\cdot THKRAT\cdot COS(AA) + CH\cdot(THKRATP\cdot COS(AA) - THKRAT\cdot SIN(AA)\cdot AAP)]}}_{F8}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F1 \pm THKP\cdot F5 & 0 & -F2 \pm THKP\cdot F6 \\ Z1P+Q\cdot F3 \pm THK\cdot F7 & 1 & Z3P-Q\cdot F4 \pm THK\cdot F8 \end{vmatrix}$$

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$NU_L1 = F2 \mp THKP \cdot F6$

$NU_L2 = (-F2 \pm THKP \cdot F6)(Z1P + Q \cdot F3 \pm THK \cdot F7) - (F1 \pm THKP \cdot F5)(Z3P - Q \cdot F4 \pm THK \cdot F8)$

$NU_L3 = F1 \pm THKP \cdot F5$

Upper surface normal "unitized" and is forced upwards, i.e. $NU3 \geq 0$ and the lower surface normal "unitized" and is forced downwards, i.e. $NL3 < 0$.

After this the differential surface areas are calculated, these values are printted if CHECK = .TRUE. and the element lies on one of the diagonals i.e. MOD(NPCA-I, NLE+NTE) = J-1

The contribution of that element to the thrust per blade (THRUST) and torque per blade (TORQUE) is added and a call to SUBROUTINE PROP, p. 40 is made to step the elements through time.

$THRUST = \sum NM3 \cdot DSM \cdot DP$ (N)

$TORQUE = \sum EM2 \cdot NM1 \cdot DSM \cdot DP$ (N·m)

$POWER = TORQUE \cdot OMEGA \cdot 0.001$ (kW)

SUBROUTINE INTLZ - This is the routine that accepts NAMELIST /INPUT/, p. 11 and sets up all the default values. Besides zeroing the storage arrays, this routine also generates constants that are needed elsewhere, see COMMON /DIVIDE/, p. 24 & 26 and COMMON /PARAM/, p. 24 & 28.

SUBROUTINE KLOCK - This routine calculates the retarded time, TAU. The total amount of blade rotation is calculated (OMEGA·TAU + PHI) and the observer position in the η frame is calculated. The radiation vector, SR, is then computed. Then, if the retarded time plus the radiation time (SR·CINV) is within EPSILON of the observer time, the retarted time is assumed to be correct otherwise the retarded time is corrected and the process is repeated until convergence.

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FUNCTION PCADF(X) - See p. 13 for a description of this routine. The following lines have to be inserted into the program to bracket the essential code.

START△OF△ PCADF and
*END△OF△ PCADF.

For example, at the location of the blue stars on the listing, p. 14.

SUBROUTINE PROP - This routine uses the output from SUBROUTINE GMTRY, p. 35 to add the contributions of the element to the pressure signatures.

The following summations are performed, the operations blocked in blue are done is SUBROUTINE DBYDT, p. 29.

$$PPTH = \boxed{RHO/4\pi \ \partial/\partial t} \boxed{\sum \left[\frac{VN}{SR - SRM} \right] \Big|_{TAU} DS} \Big|^{UPPER}_{LOWER}$$

$$PPDPA = \boxed{-1/4\pi} \sum \left[\frac{\vec{NM} \cdot \vec{SR}}{(SR)^2} \frac{DP}{SR - SRM} \right] \Big|_{TAU} DSM$$

$$PPDPB = \boxed{-1/4\pi C \ \partial/\partial t} \sum \left[\frac{\vec{NM} \cdot \vec{SR} \cdot DP}{SR \ (SR - SRM)} \right] \Big|_{TAU} DSM$$

Time is stepped through by increments of DT; after every NDIF th step, the observer position is incremented by DX3. After all the time points for one blade passage is completed, the program steps to the next blade. The positive and negative components are summed separately for better accuracy. They are summed together in SUBROUTINE DBYDT, p. 29.

PROGRAM TEST - This is sort of the base of operations; not much is done here except the definition of the identifier block and subroutine calls. PROGRAM TEST is pretty well defined by the Flowchart, p. 7.

SUBROUTINE THKDP - See p. 17 for a description of this routine. The following lines have to be inserted into the program to bracket the essential code.

"*START△OF△THKDP*" and
"*END△OF△THKDP*".

For example, at the location of the blue stars on the listing on p. 18.

In the second region, the midpoints of NTE equally spaced divisions from THKMAXL to the trailing edge are used. If either NLE ≤ 0 or THKMAXL ≤ 0. then, THKMAXL is set to 0. and the entire blade is considered to be in the second region.

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SR-2 CRUISE

E	E2	V_{TIP} HELICAL	M	P	L	CH
	m	m/sec		N/m^2	N/m	m
.3	.732	248.22	.84	3386	300.4	.727
.4	.976	256.46	.86	4811	853.5	.727
.5	1.220	266.68	.90	5499	975.6	.727
.6	1.464	278.67	.94	6225	1097	.722
.7	1.708	292.20	.99	6634	1145	.708
.8	1.952	307.07	1.04	6951	1167	.688
.9	2.196	323.10	1.09	6785	1034	.625
1.0	2.440	340.12	1.15	0.	0	.268

$E = r_2/R$, $E = r_2$ m, V_{TIP} HELICAL

$P = \frac{1}{2} \rho_0 V_{TIP}^2 C_\ell$ N/m^2 $L = P \times CH \times .1R$

EXCEPT AT ENDS WHERE $L = \frac{1}{2} P \times CH \times .1R$

SR-1 CRUISE

E	P	L
.3	3129	277.6
.4	4499	798.3
.5	5768	1023
.6	6916	1219
.7	7540	1302
.8	7112	1194
.9	5381	820.1
1.0	0	0.0

SR2 - TUNNEL

E	E2	V_{TIP} HELICAL	M	P	L	CH
	m	m/sec		N/m²	N/m	m
.3	.093	143.60	.44	1396	2.013	.093
.4	.124	167.52	.51	5355	15.44	.093
.5	.156	193.99	.59	9497	27.39	.093
.6	.187	222.10	.68	12750	36.53	.092
.7	.218	251.30	.77	14580	40.91	.090
.8	.249	281.25	.86	13880	37.87	.088
.9	.280	311.74	.95	8972	22.23	.080
1.0	.311	342.61	1.04	0	0.0	.034

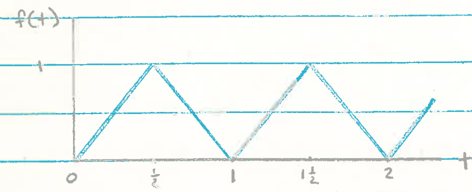
SR1 - TUNNEL

E	V_{TIP} HELICAL	M	P	L
.3	144.67	.44	3314	4.779
.4	168.58	.51	7167	20.67
.5	195.06	.59	11490	33.15
.6	223.20	.67	14320	41.01
.7	252.44	.76	14950	41.96
.8	282.45	.85	16230	30.65
.9	312.99	.94	6895	17.08
1.0	343.94	1.04	0	0.0

Check of Fourier Transform Coding

The code used to generate the spectra was extracted and placed into another program. A series of known waveforms were then input and the outputs were compared with the theoretical calculations. Comparisons are done on page 49.

Triangular



$$f(t) = \begin{cases} 2t & 0 \le t \le \frac{1}{2} \\ 1-2t & \frac{1}{2} \le t \le 1 \end{cases}$$

$2(1-t)$

From symmetry,

$$a_n = \frac{2}{\frac{1}{2}} \int_0^{\frac{1}{2}} 2t \cos(2n\pi t)\, dt$$

If we let $\alpha = 2n\pi$, then

$$I(\alpha) = \int_0^{\frac{1}{2}} \sin(\alpha t)\, dt = \left. \frac{-\cos(\alpha)}{\alpha} \right|_0^{\frac{1}{2}}$$

$$I'(\alpha) = \int_0^{\frac{1}{2}} t \cos(\alpha t)\, dt = \left[\frac{1}{\alpha} - \frac{\cos(\alpha/2)}{\alpha} \right]'$$

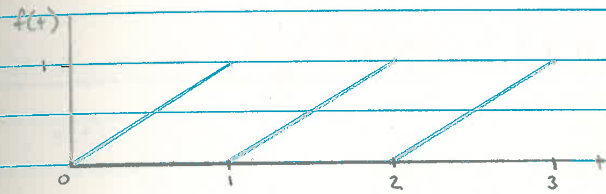
$$a_n = 8 \left\{ \frac{\sin(\alpha/2)}{2\alpha} + \frac{1}{\alpha^2} \left[\cos(\alpha/2) - 1 \right] \right\}$$

Since $\alpha = 2n\pi$ then $\sin(n\pi) = 0$ & $\cos(n\pi) = \pm 1$

$a_n = \frac{-4}{n^2\pi^2}$, n = odd $\qquad a_n = 0$, n = even

$SPL_n = -20 \log_{10} \left(0.25 \sqrt{2}\ n^2\pi^2\ 20 \times 10^{-6} \right)$, n = odd

Ramp



$f(t) = t \,,\ 0 \le t < 1$

$$a_n = \frac{1}{\frac{1}{2}} \int_0^1 t \cos(2n\pi t)\, dt$$

$$b_n = \frac{1}{\frac{1}{2}} \int_0^1 t \sin(2n\pi t)\, dt$$

If we let $\alpha = 2n\pi$, then

$$I(\alpha) = \int_0^1 \sin(\alpha t)\, dt = -\frac{\cos(\alpha t)}{\alpha}\Big|_0^1$$

$$I'(\alpha) = \int_0^1 t \cos(\alpha t)\, dt = \left[\frac{1}{\alpha} - \frac{\cos(\alpha)}{\alpha} \right]'$$

$$a_n = \frac{2}{\alpha}\left[\sin(\alpha) + \frac{1}{\alpha}\left\{ \cos(\alpha) - 1 \right\} \right]$$

$$J(\alpha) = -\int_0^1 \cos(\alpha t)\, dt = -\frac{\sin(\alpha t)}{\alpha}\Big|_0^1$$

$$J'(\alpha) = \int_0^1 t \sin(\alpha t)\, dt = \left[-\frac{\sin(\alpha)}{\alpha} \right]'$$

$$b_n = \frac{2}{\alpha}\left[\frac{\sin(\alpha)}{\alpha} - \cos(\alpha) \right]$$

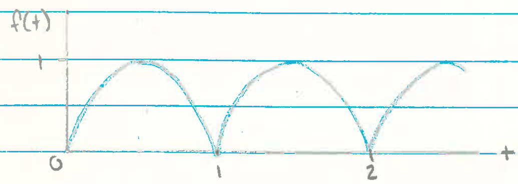
Since $\alpha = 2n\pi$ then $\sin(2n\pi) = 0$ & $\cos(2n\pi) = 1$

$$c_n = \sqrt{a_n^2 + b_n^2}$$

$$c_n = \frac{2}{\alpha}$$

$$SPL_n = -20 \log_{10}\left(n\pi\sqrt{2} \cdot 20\times10^{-6}\right)$$

Sine wave, half cycle



$$f(t) = \sin(\pi t),\ 0 \le t \le 1$$

$$a_n = \frac{1}{2}\int_0^1 \sin(\pi t)\cos(2n\pi t)\,dt$$

$$b_n = \frac{1}{2}\int_0^1 \sin(\pi t)\sin(2n\pi t)\,dt, \quad \alpha = 2n\pi$$

$$a_n = \int_0^1 \{\sin([\pi+\alpha]t) + \sin([\pi-\alpha]t)\}\,dt$$

$$a_n = -\frac{\cos([\pi+\alpha]t)}{\pi+\alpha} - \frac{\cos([\pi-\alpha]t)}{\pi-\alpha}\Bigg|_0^1$$

$$a_n = \frac{2\pi}{\pi^2-\alpha^2}\left[1 + \cos(\alpha)\right]$$

$$b_n = \int_0^1 \{\cos([\pi-\alpha]t) - \cos([\pi+\alpha]t)\}\,dt$$

$$b_n = -\frac{\sin([\pi-\alpha]t)}{\pi-\alpha} + \frac{\sin([\pi+\alpha]t)}{\pi+\alpha}\Bigg|_0^1$$

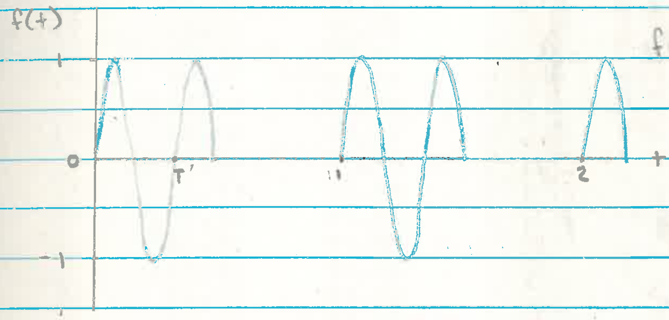
$$b_n = \frac{-2\pi}{\pi^2-\alpha^2}\sin(\alpha)$$

Since $\alpha = 2n\pi$ then $\sin(2n\pi) = 0$ & $\cos(2n\pi) = 1$

$$c_n = \frac{4}{\pi(1-4n^2)}$$

$$SPL_n = 20 \log_{10}\left(0.25\,\pi\sqrt{2} \,/\, \left|[4n^2-1]\right| 20\times10^{-6}\right)$$

Sine wave, $1\frac{1}{2}$ cycles



$$f(t) = \begin{cases} \sin\left(\frac{2\pi t}{T'}\right), & 0 \le t \le \frac{3T'}{2} \\ 0, & \frac{3T'}{2} \le t \le 1 \end{cases}$$

$$a_n = \frac{1}{\frac{1}{2}} \int_0^{\frac{3T'}{2}} \sin\left(\frac{2\pi t}{T'}\right) \cos(2n\pi t)\,dt$$

$$b_n = \frac{1}{\frac{1}{2}} \int_0^{\frac{3T'}{2}} \sin\left(\frac{2\pi t}{T'}\right) \sin(2n\pi t)\,dt$$

Let $\alpha = 2n\pi$ and $\beta = \frac{2\pi}{T'}$

$$a_n = -\frac{\cos([\beta+\alpha]t)}{\beta+\alpha} - \frac{\cos([\beta-\alpha]t)}{\beta-\alpha}\Bigg|_0^{\frac{3T'}{2}}$$

$$a_n = \frac{T'[1+\cos(3n\pi T')]}{\pi(1-n^2T'^2)}$$

$$b_n = -\frac{\sin([\beta-\alpha]t)}{\beta-\alpha} + \frac{\sin([\beta+\alpha]t)}{\beta+\alpha}\Bigg|_0^{\frac{3T'}{2}}$$

$$b_n = \frac{-T'\sin(3n\pi T')}{\pi(1-n^2T'^2)}$$

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$$c_n = \frac{T'}{\pi(1-n^2T'^2)} \sqrt{2\left[1+\cos(3n\pi T')\right]}$$

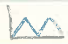
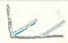
$$SPL_n = 20 \log_{10}\left[\left|\frac{T'\sqrt{1+\cos(3n\pi T')}}{\pi(1-n^2T'^2)\, 20.\times10^{-6}}\right|\right]$$

Harmonic Number				T=0.1
1	83.12	81.03	83.52	66.13
2	0.00	75.01	69.55	62.79
3	64.04	71.48	62.19	51.75
4	0.00	68.98	57.08	58.37
5	55.17	67.05	53.15	66.54
6	0.00	65.46	49.96	70.48
7	49.32	64.12	47.27	78.79
8	0.00	62.96	44.94	74.08
9	44.95	61.94	42.88	74.61
10	0.00	61.03	41.05	73.56
11	41.47	60.20	39.39	73.74
12	0.00	59.44	37.87	72.34
13	38.57	58.75	36.48	70.16
14	0.00	58.10	35.19	66.96
15	36.08	57.50	33.99	62.08
16	0.00	56.94	32.87	52.96
17	33.91	56.42	31.82	46.46
18	0.00	55.92	30.82	55.43
19	31.97	55.45	29.88	57.71
20	0.00	55.01	28.99	57.50
21	30.24	54.58	28.14	55.38
22	0.00	54.18	27.33	50.73
23	28.66	53.79	26.56	38.22
24	0.00	53.42	25.82	43.38
25	27.21	53.07	25.11	49.64
26	0.00	52.73	24.43	51.40
27	25.87	52.40	23.77	50.98
28	0.00	52.08	23.14	48.50
29	24.63	51.78	22.53	42.77
30	0.00	51.48	21.94	-8.21
31	23.47	51.20	21.37	41.51
32	0.00	50.92	20.82	46.99
33	22.38	50.66	20.29	47.04
34	0.00	50.40	19.77	46.13
35	21.36	50.14	19.27	43.00
36	0.00	49.90	18.78	35.25
37	20.40	49.66	18.30	28.96
38	0.00	49.43	17.84	39.88
39	19.48	49.20	17.38	43.00
40	0.00	48.98	16.95	43.52
41	18.61	48.77	16.52	42.06
42	0.00	48.56	16.10	37.98
43	17.79	48.36	15.69	25.96
44	0.00	48.16	15.29	31.62
45	17.00	47.96	14.90	38.37
46	0.00	47.77	14.52	40.53
47	16.24	47.58	14.14	40.46
48	0.00	47.40	13.78	38.33
49	15.52	47.22	13.42	32.91
50	0.00	47.05	13.07	-13.14

Theoretical Calculations
(dB re 20×10^{-6} Pa)

Red underline
is where the numerical calculations differ from the theoretical calculations by more than 0.5 dB.

T'

Harmonic Number				 T'=0.1
1	83.12	81.03	83.52	66.13
2	-198.79	75.01	69.55	62.79
3	64.04	71.48	62.19	51.75
4	-208.59	68.98	57.08	58.36
5	55.17	67.05	53.15	66.54
6	-198.85	65.46	49.96	70.49
7	49.32	64.12	47.27	72.79
8	-195.37	62.96	44.94	74.08
9	44.95	61.94	42.88	74.61
10	-190.88	61.03	41.05	74.49
11	41.47	60.20	39.39	73.74
12	-189.77	59.44	37.87	72.34
13	38.56	58.75	36.48	70.16
14	-187.34	58.11	35.19	66.96
15	36.07	57.51	33.98	62.09
16	-187.79	56.95	32.86	52.98
17	33.89	56.42	31.80	45.39
18	-186.94	55.93	30.81	55.41
19	31.95	55.46	29.86	57.70
20	-184.93	55.01	28.97	57.48
21	30.21	54.59	28.11	55.36
22	-187.54	54.19	27.30	50.71
23	28.61	53.81	26.52	38.24
24	-187.09	53.44	25.77	43.25
25	27.14	53.09	25.05	49.58
26	-191.75	52.75	24.36	51.33
27	25.78	52.43	23.69	50.88
28	-184.68	52.11	23.04	48.41
29	24.51	51.81	22.41	42.68
30	-187.13	51.53	21.81	-87.59
31	23.31	51.25	21.21	41.33
32	-191.79	50.98	20.64	45.71
33	22.17	50.72	20.08	46.83
34	-184.82	50.47	19.53	45.90
35	21.08	50.23	18.99	42.74
36	-188.86	49.99	18.46	34.98
37	20.04	49.77	17.94	28.51
38	-188.43	49.55	17.43	39.46
39	19.02	49.34	16.92	42.53
40	-179.72	49.13	16.42	43.00
41	18.02	48.93	15.93	41.47
42	-199.72	48.74	15.43	37.34
43	17.04	48.56	14.94	25.32
44	-191.05	48.38	14.45	30.71
45	16.05	48.21	13.95	37.38
46	-188.76	48.04	13.45	39.43
47	15.04	47.89	12.95	39.23
48	-192.39	47.73	12.43	36.96
49	14.01	47.59	11.91	31.39
50	-177.55	47.45	11.37	-86.72

Numerical Calculations
200 points per period
(dB re 20×10^{-6} Pa)

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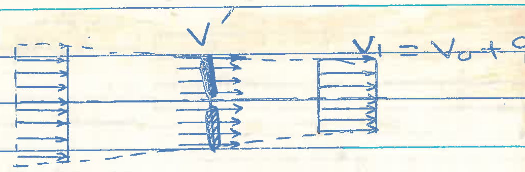
ACTUATOR DISK ANALYSIS OF SCALE MODEL OF PROPFAN



$V_0 = 105.1$ m/sec

$V' = V_0 + q/2$

$\rho = 1.231$ kg/m^3

THRUST $T = A\Delta p = A\rho V' q = A\rho (V_0 + q/2) q$

THRUST $= 2 \times 163.00 = 326.00$ N

$A = \pi \times .311^2 = .304$ m^2

$A\rho q^2 + 2A\rho V_0 q - 2T = 0$

$$q = \frac{1}{A\rho}\left[-A\rho V_0 + \sqrt{(A\rho V_0)^2 + 2A\rho T}\right]$$

$$= V_0\left[-1 + \sqrt{1 + \frac{2T}{A\rho V_0^2}}\right]$$

$$\approx \frac{T}{A\rho V_0}$$

$q = 8.29$ m/sec

$V' = 109.24$ m/sec

EFFICIENCY $\eta = \dfrac{TV_0}{TV'} = .96$

$A\rho V' = 40.88$ kg/sec

CHANGE OF SPEED OF SOUND IN THE VICINITY OF PROPFAN, $C_0 = 328.5$ m/sec, $\gamma = 1.4$

$$\frac{V_0^2}{2} + \frac{\gamma R T_0}{\gamma - 1} = \frac{V'^2}{2} + \frac{\gamma R T'}{\gamma - 1}$$

$$C'^2 = C_0^2 + \frac{\gamma - 1}{2}(V_0^2 - V'^2)$$

$$C' \approx C_0 + \frac{\gamma - 1}{4C_0}(V_0^2 - V'^2) = 328.23 \text{ m/sec}$$

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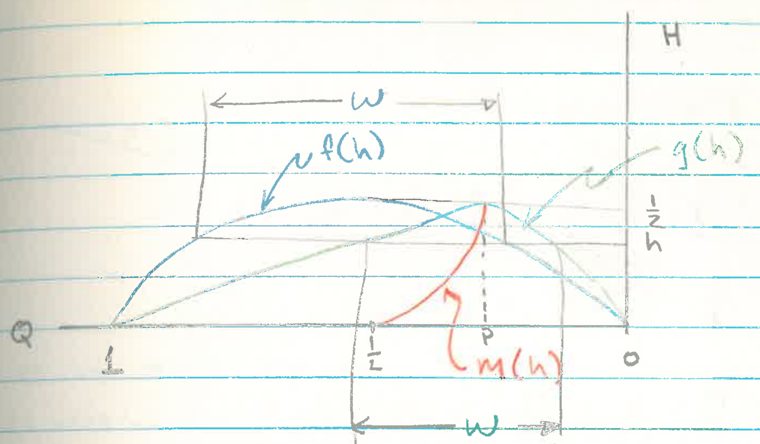
Deformed Biconvex Parabolic



$f(h) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-2h}$ (convex parabolic)

For and h between 0 and $\frac{1}{2}$ we have two values for $f(h)$ and $g(h)$. We will define $g(h)$ such that the difference between those values are the same as for $f(h)$, i.e at any h, $w = w$

$g(h) = m(h) \pm \frac{1}{2}\sqrt{1-2h}$

(For $f(h)$, $m(h) = \frac{1}{2}$)

For a blunt leading edge, $g'(h) = 0$

$0 = g'(h)\big|_{h=0} = m'(h)\big|_{h=0} - \frac{1}{4}\frac{-2}{\sqrt{1-2\cdot 0}}$

$m'(h)\big|_{h=0} = -\frac{1}{2}$

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$m(0) = \frac{1}{2}$ $\qquad$ $m'(0) = -\frac{1}{2}$

$m(\frac{1}{2}) = P$ $\qquad$ $m'(\frac{1}{2}) = 0$

$$m(h) = Ah^3 + Bh^2 + Ch + D$$

$m(0) = \boxed{D = \frac{1}{2}}$ $\qquad$ $m'(0) = \boxed{C = -\frac{1}{2}}$

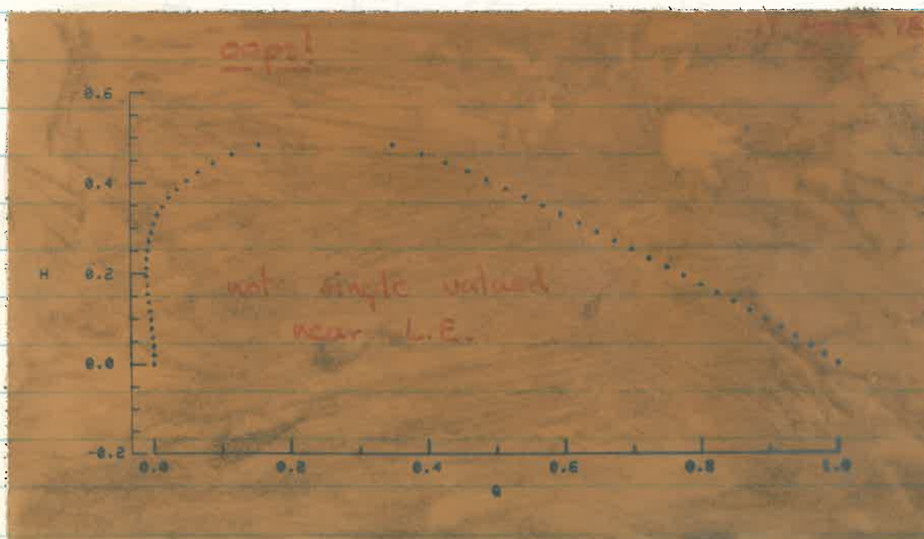
$$m(\tfrac{1}{2}) = \frac{A}{8} + \frac{B}{4} + \frac{1}{4} + \cancel{\frac{1}{2}} = P$$

$$m'(\tfrac{1}{2}) = \frac{3A}{4} + B - \frac{1}{2} = 0$$

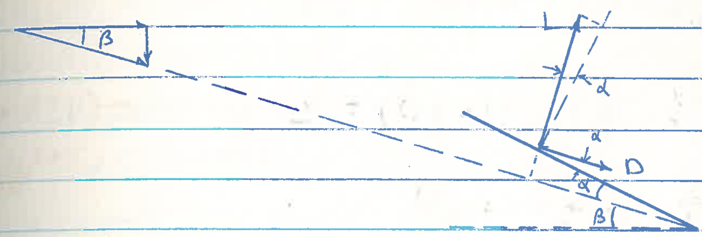
$$\frac{4A}{8} - \frac{6A}{8} + \frac{3}{2} = 4P \Rightarrow \boxed{A = 6 - 16P}$$

$$\boxed{B = 12P - 4}$$

For $P = \frac{1}{4}$, $m(h) = 2h^3 - h^2 - \frac{h}{2} + \frac{1}{2}$



THE INCLUSION OF FRICTION AND WAVE DRAG IN THE ACOUSTIC CALCULATIONS



GEOMETRIC ANGLE OF ATTACK $= \alpha + \beta$

α = AERODYNAMIC ANGLE OF ATTACK

D IS OBTAINED FROM GARABEDIAN-KORN PROGRAM AND INCLUDES BOTH SKIN FRICTION AND WAVE DRAGS. IN OUR PROGRAM, WE NEED THE NORMAL STRESS (ΔP) AND TANGENTIAL STRESS (σ) OVER THE CHORD. WE SEE THAT

$$\Delta P = \Delta P_{GK} \cos\alpha + \sigma_{GK} \sin\alpha$$

$$\sigma = \sigma_{GK} \cos\alpha - \Delta P_{GK} \sin\alpha$$

(GK: GARABEDIAN-KORN), $\sigma_{GK} = D_{GK}/CH$

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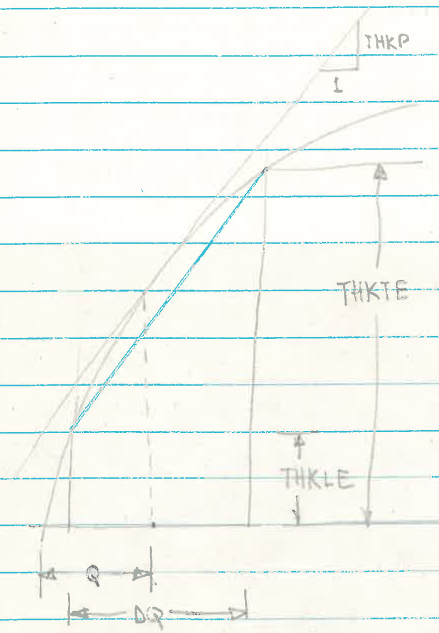
Possible problem in the calculation of ΔS when close to a blunt leading edge.

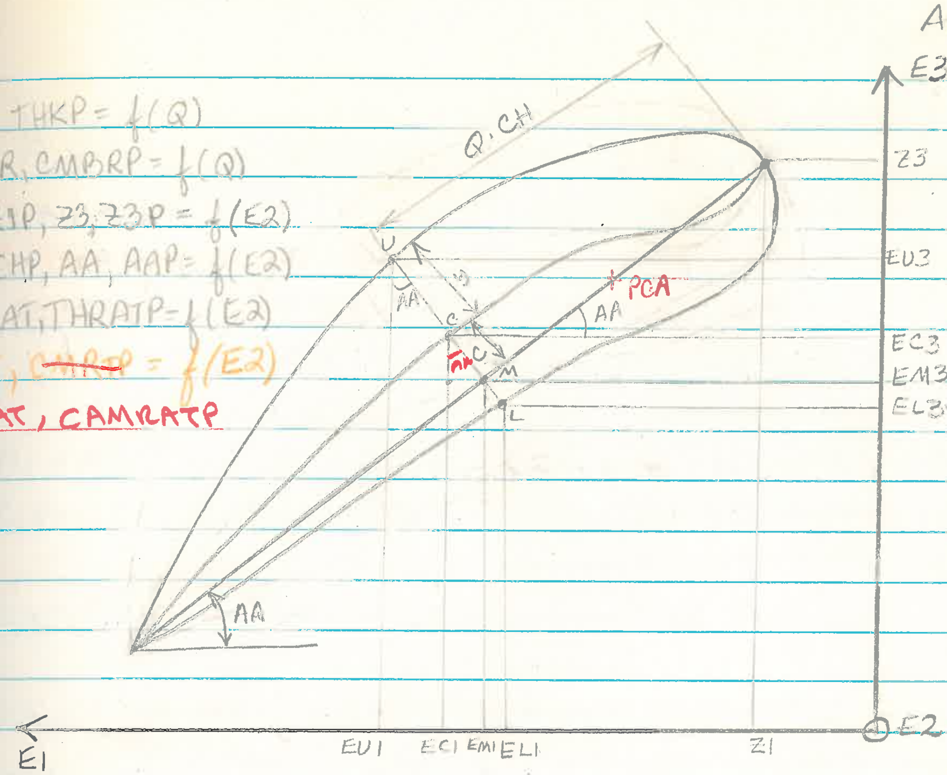
If $\partial Y/\partial \eta_1 = \partial/\partial \eta_1 \left[THK \cdot THKRAT \cdot CH \right] > \tan(45°)$,

(i.e. THKP · THKRAT > 1.) we then approximate THKP by using the slope of the below blue line.



$$THKP_{new} = \frac{(THKTE - THKLE)}{DQ}$$

THK, THKP = f(Q)
CMBR, CMBRP = f(Q)
Z1, Z1P, Z3, Z3P = f(E2)
CH, CHP, AA, AAP = f(E2)
THKRAT, THRATP = f(E2)
~~CAMRT~~, ~~CAMRP~~ = f(E2)
CAMRAT, CAMRATP



$E2 = EU2 = EL2 = EC2$ $\quad$ $COSAA = \cos(AA)$

$y = CH \cdot THKRAT \cdot THK$ ✓ $\quad$ $SINAA = \sin(AA)$

$C = CMBR \cdot CH \cdot \cancel{THKRAT}\ AMRAT$

Mean Surface:

$EM1 = Z1 + (Q \cdot CH) \cdot COSAA$ ✓

$EM3 = Z3 - (Q \cdot CH) \cdot SINAA$ ✓

Camber Surface:

note if $CMBR = 0$, $EC1 = EM1$ and $EC3 = EM3$

$EC1 = EM1 + C \cdot SINAA = Z1 + Q \cdot CH \cdot COSAA$ F9 ✓

$+ CMBR \cdot CH \cdot \cancel{THKRAT}\ AMRAT \cdot SINAA = Z1 + Q \cdot F1 + CMBR \cdot \cancel{F5}$

$EC3 = EM3 + C \cdot COSAA = Z3 - Q \cdot CH \cdot SINAA$ F10 ✓

$+ CMBR \cdot CH \cdot \cancel{THKRAT}\ AMRAT \cdot COSAA = Z3 - Q \cdot F2 + CMBR \cdot \cancel{F6}$

consistent w/ derivation on pp 36-38, define these non-chord dependent constants:

$F1 = CH \cdot COSAA$ $\qquad F9 = CH \cdot CAMRAT \cdot SINAA$

$F2 = CH \cdot SINAA$ $\qquad F10 = CH \cdot CAMRAT \cdot COSAA$

$F3 = CHP \cdot COSAA - CH \cdot SINAA \cdot AAP$

$F4 = CHP \cdot SINAA + CH \cdot COSAA \cdot AAP$

$F5 = CH \cdot THKRAT \cdot SINAA$

$F6 = CH \cdot THKRAT \cdot COSAA$

$$\overset{F11}{\cancel{F7}} = CHP \cdot \overset{CAMRAT}{\cancel{THKRAT}} \cdot SINAA + CH\left(\overset{CAMRATP}{\cancel{THKRATP}} \cdot SINAA + \overset{CAMRAT}{\cancel{THKRAT}} \cdot COSAA \cdot AAP\right)$$

$$\overset{F12}{\cancel{F8}} = CHP \cdot \overset{CAMRAT}{\cancel{THKRAT}} \cdot COSAA + CH\left(\underset{CAMRATP}{\cancel{THKRATP}} \cdot COSAA - \underset{CAMRAT}{\cancel{THKRAT}} \cdot SINAA \cdot AAP\right)$$

$$\frac{\partial EC1}{\partial Q} = CH \cdot COSAA + CH \cdot \overset{CAMRAT}{\cancel{THKRAT}} \cdot SINAA \cdot CMBRP$$
$$= F1 + \underset{F9}{\cancel{F5}} \cdot CMBRP$$

$$\frac{\partial EC1}{\partial E2} = ZIP + Q\,(CHP \cdot COSAA - CH \cdot SINAA \cdot AAP) + CMBR\left[CHP \cdot \overset{CAMRAT}{\cancel{THKRAT}} \cdot SINAA + CH\left\{\overset{CAMRATP}{\cancel{THKRATP}} \cdot SINAA + \overset{CAMRAT}{\cancel{THKRAT}} \cdot COSAA \cdot AAP\right\}\right]$$
$$= ZIP + Q \cdot F3 + CMBR \cdot \cancel{F7}\ F11$$

$$\frac{\partial EC2}{\partial Q} = 0 \quad \text{and} \quad \frac{\partial EC2}{\partial E2} = 1$$

$$\frac{\partial EC3}{\partial Q} = -CH \cdot SINAA + CH \cdot \overset{CAMRAT}{\cancel{THKRAT}} \cdot COSAA \cdot CMBRP$$
$$= -F2 + \underset{F10}{\cancel{F6}} \cdot CMBRP$$

$$\frac{\partial EC3}{\partial E2} = Z3P - Q\,(CHP \cdot SINAA + CH \cdot COSAA \cdot AAP) + CMBR\left[CHP \cdot \cancel{THKRAT}\,CAMRAT \cdot COSAA + CH\left\{\cancel{THKRATP}\,CAMRATP \cdot COSAA - \cancel{THKRAT}\,CAMRAT \cdot SINAA \cdot AAP\right\}\right]$$

$$= Z3P - Q \cdot F4 + CMBR \cdot \cancel{F8}\,F12$$

$$\vec{NC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F1 + \cancel{F5}\,F9 \cdot CMBRP & 0 & -F2 + \cancel{F6}\,F10 \cdot CMBRP \\ Z1P + Q \cdot F3 + CMBR \cdot \cancel{F7}\,F11 & 1 & Z3P - Q \cdot F4 + CMBR \cdot \cancel{F8}\,F12 \end{vmatrix}$$

$$NC1 = F2 - \cancel{F6}\,F10 \cdot CMBRP$$

$$NC2 = (-F2 + \cancel{F6}\,F10 \cdot CMBRP)(Z1P + Q \cdot F3 + CMBR \cdot \cancel{F7}\,F11) - (F1 + \cancel{F5}\,F9 \cdot CMBRP)(Z3P - Q \cdot F4 + CMBR \cdot \cancel{F8}\,F12)$$

$$NC3 = F1 + \cancel{F5}\,F9 \cdot CMBRP$$

Upper & Lower Surfaces:

$$E^{U}_{L}1 = EC1 \pm y \cdot SINAA$$
$$= EC1 \pm CH \cdot THKRAT \cdot THK \cdot SINAA$$
$$= EC1 \pm F5 \cdot THK$$

$$E^{U}_{L}3 = EC3 \pm y \cdot COSAA = EC3 \pm CH \cdot THKRAT \cdot THK \cdot COSAA$$
$$= EC3 \pm F6 \cdot THK$$

$$\frac{\partial E^{U}_{L}1}{\partial Q} = \frac{\partial EC1}{\partial Q} \pm F5 \cdot THKP$$
$$= F1 + \cancel{F5}\,F9 \cdot CMBRP \pm F5 \cdot THKP$$
$$\cancel{= F1 + F5\,(CMBRP \pm THKP)}$$

F1

$$\frac{\partial E^u_L 1}{\partial E2} = \frac{\partial EC1}{\partial E2} \pm THK\Big[CHP\cdot THKRAT\cdot SINAA + CH\,(THKRATP\cdot SINAA + THKRAT\cdot COSAA\cdot AAP)\Big]$$

$$= Z1P + Q\cdot F3 + CMBR\cdot \cancel{F4}\,F11 \pm THK\cdot F7 \checkmark \checkmark$$

$$\cancel{= Z1P + Q\cdot F3 + F7\,(CMBR \pm THK)}$$

$$\frac{\partial E^u_L 2}{\partial Q} = 0 \quad \text{and} \quad \frac{\partial E^u_L 2}{\partial E2} = 1 \checkmark$$

$$\frac{\partial E^u_L 3}{\partial Q} = \frac{\partial EC3}{\partial Q} \pm F6\cdot THKP = -F2 + \cancel{F6}\,F10\cdot CMBRP \pm F6\cdot THKP \checkmark$$

$$\cancel{= -F2 + F6\,(CMBRP \pm THKP)}$$

$$\frac{\partial E^u_L 3}{\partial E2} = \frac{\partial EC3}{\partial E2} \pm THK\Big[CHP\cdot THKRAT\cdot COSAA + CH\,(THKRATP\cdot COSAA - THKRAT\cdot SINAA\cdot AAP)\Big]$$

$$= Z3P - Q\cdot F4 + CMBR\cdot \cancel{F8}\,F12 \pm THK\cdot F8 \checkmark \checkmark$$

$$\cancel{= Z3P - Q\cdot F4 + F8\,(CMBR \pm THK)}$$

~~let~~ $\cancel{A = CMBR \pm THK}$

$\cancel{B = CMBRP \pm THKP}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F1 + F5\cdot B & 0 & -F2 + F6\cdot B \\ Z1P + Q\cdot F3 + F7\cdot A & 1 & Z3P - Q\cdot F4 + F8\cdot A \end{vmatrix}$$

$$N^u_L 1 = F2 - F6\cdot B = F2 - F6\,(CMBRP \pm THKP)$$

$$N^u_L 2 = [-F2 + F6\cdot B][Z1P + Q\cdot F3 + F7\cdot A] - [F1 + F5\cdot B][Z3P - Q\cdot F4 + F8\cdot A]$$

$$= [-F2 + F6\,(CMBRP \pm THKP)] * [Z1P + Q\cdot F3 + F7\,(CMBR \pm THK)] - [F1 + F5\,(CMBRP \pm THKP)] * [Z3P - Q\cdot F4 - F8\,(CMBR \pm THK)]$$

$$\cancel{N^U_L 3 = F1 + F5 \cdot B = F1 + F5\,(CMBRP \pm THKP)}$$

Note: For the case of CMBR = CMBRP = 0, all these equations reduce to those derived on pp. 36–38 (symmetric airfoil).

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ F1 + F9 \cdot CMBRP \pm F5 \cdot THKP & 0 & -F2 + F10 \cdot CMBRP \pm F6 \cdot THKP \\ Z1P + Q \cdot F3 + F11 \cdot CMBR \pm F7 \cdot THK & 1 & Z3P - Q \cdot F4 + F12 \cdot CMBR \pm F8 \cdot THK \end{vmatrix}$$

$$N^U_L 1 = +F2 - F10 \cdot CMBRP \mp F6 \cdot THKP$$

$$N^U_L 2 = [-F2 + F10 \cdot CMBRP \pm F6 \cdot THKP][Z1P + Q \cdot F3 + F11 \cdot CMBR \pm F7 \cdot THK] - [F1 + F9 \cdot CMBRP \pm F5 \cdot THKP][Z3P - Q \cdot F4 + F12 \cdot CMBR \pm F8 \cdot THK]$$

$$N^U_L 3 = F1 + F9 \cdot CMBRP \pm F5 \cdot THKP$$

NOTE IN THE PROGRAM, THE CAMBER SURFACE COORDINATES (EC1, EC3) ARE LABLED (EM1, EM3).

Feb - May 1982

PROPFAN program as described in Martin & Farassat uses regula falsi method to find emission time(s) τ, whenever $M_H > .95$.

This root finder is reliable but much slower than Newton's method. What we need is a better decision criterion for when regula falsi is necessary.

Write:

$$\textcircled{1}\quad r^2 = (x_3 - v_3 t)^2 + d^2 + \eta^2 - 2d\eta\cos\big(\omega(\tau - t) + \psi - (\beta - \omega t)\big)$$
$$+ v_3^2(\tau - t)^2 - 2v_3(x_3 - v_3 t)(\tau - t) = c^2(\tau - t)^2$$

First, show that eq ① is equivalent to original emission time equation [see eq (10) & (11) TP 1662]

$$r^2 = x_3^2 - \cancel{2x_3 v_3 t} + \cancel{v_3^2 t^2} + d^2 + \eta^2 - 2d\eta\cos(\omega\tau + \psi - \beta)$$
$$+ v_3^2\tau^2 - 2v_3^2\tau t + \cancel{v_3^2 t^2} - 2v_3 x_3 \tau + \cancel{2v_3^2 t\tau} + \cancel{2v_3 x_3 t}$$
$$- \cancel{2v_3^2 t^2} = c^2(\tau - t)^2$$

Now collect terms and divid by $2d\eta$

$$\frac{c^2-v_3^2(\tau-t)^2}{2d\eta} + \frac{2v_3(x_3-v_3t)(\tau-t)}{2d\eta}$$

$$-\frac{(x_3-v_3t)^2}{2d\eta} - \frac{d^2+\eta^2}{2d\eta} + \cos\left(\omega(\tau-t)+\psi-\beta+\omega t\right)$$

$$= 0$$

Let $\phi = \omega(\tau - t)$ physically this is the angle in radians between the present position (of the source pt) and the retarded time position. Always a negative quantity.

Write $A\phi^2 + B\phi + C + \cos\left[\phi + D\right] = 0$ (2)

where $A = \{(c^2-v_3^2)/(2\omega^2 d\eta)\}$ $A > 0$

$B = \{2v_3(x_3-v_3t)/(2\omega d\eta)\}$

$C = \{-[(x_3-v_3t)^2+d^2+\eta^2]/(2d\eta)\}$ $C < 0$

$D = \psi - \beta + \omega t$

Problem to find all negative real roots.

The roots of eq (2) are the intersection of a cosine curve [of amplitude = 1 and Shift = D] and a parabola. The parabola is always upright and always intersects the ϕ axis.

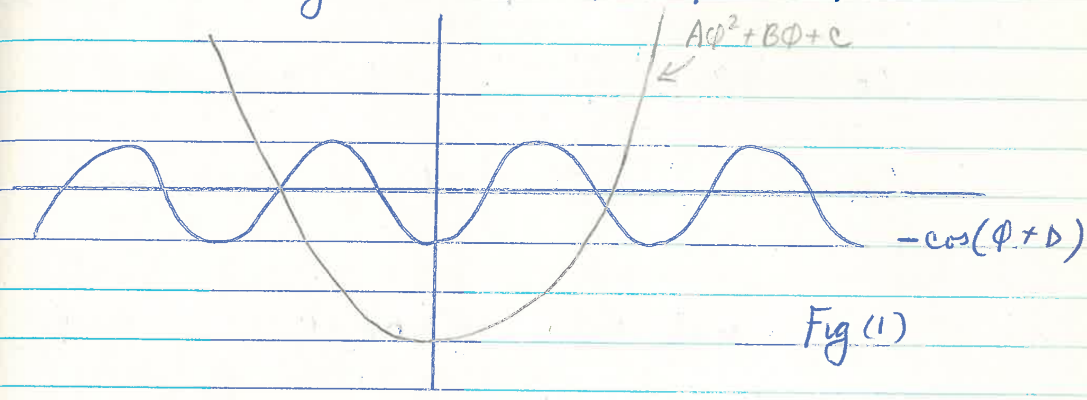


Fig (1)

A, B, C, D and ϕ are non-dimensional and generally in the range [0, 10]

When the observer is moving at the same speed as the source then

$$X_3 = X_3(t=0) + v_3 t$$

Thus the term $X_3 - v_3 t = X_3(t=0)$ regardless of the value of t. In this case A, B, C are functions of source position only but $D = \gamma - \beta + \omega t$ as before.

Refer to figure (1): Thus, the parabola doesn't change with t but the cosine gradually shifts left.

To find the roots of the equation ② first estimate their location.

Define: $\bar{X}_{low}$ such that $A\phi^2 + B\phi + C = 0$ if $\phi = \bar{X}_{low}$

$$LB = \text{fix}\left[\frac{D + X_{low}}{\pi}\right] \quad \text{if } D + X_{low} \geq 0$$

$$LB = \text{fix}\left[\frac{D + X_{low}}{\pi}\right] - 1 \quad \text{if } D + X_{low} < 0$$

X_{lim} such that $A\phi^2 + B\phi + C = 1$ if $\phi = X_{lim}$

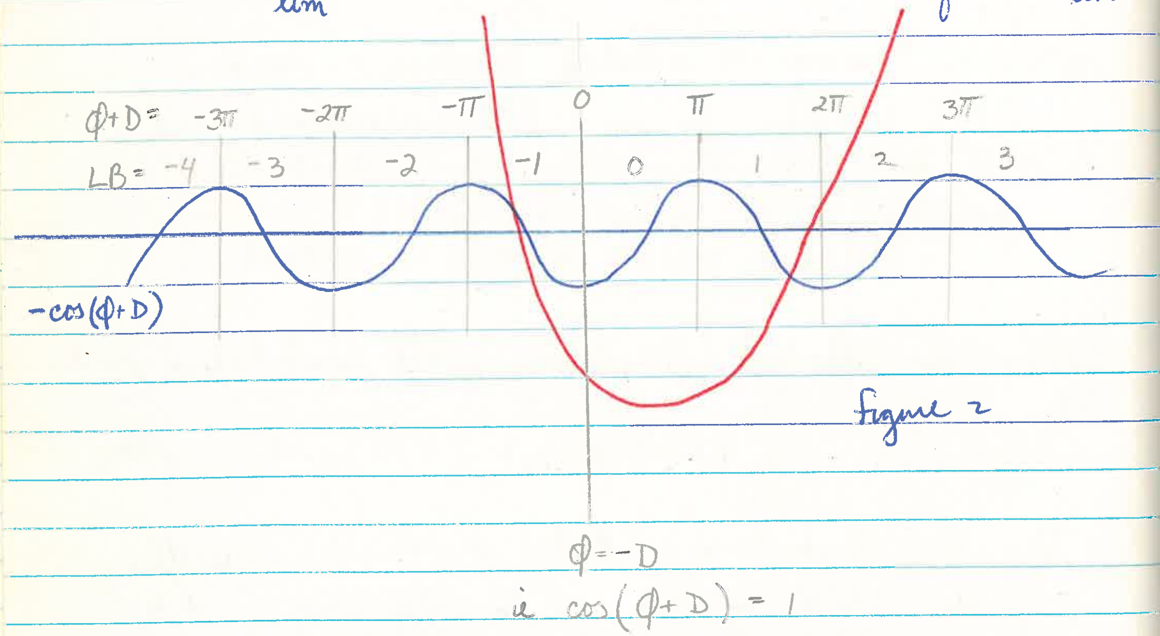
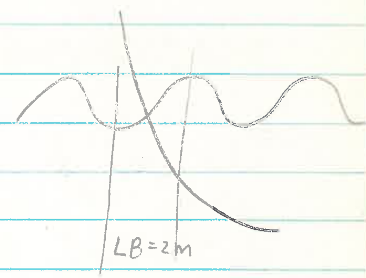


figure 2

Figure 2 shows the value of LB depending on which half cycle of the $-\cos(\phi + D)$ curve the parabola intersects.

If X_{low} falls between $\phi + D = 0$ and $\phi + D = -\pi$ then $LB = -1$. That is the case pictured.

CASE #1 LB is even



Only one root is possible.

$f(\phi) = A\phi^2 + B\phi + C + \cos(\phi + D)$
$f'(\phi) = 2A\phi + B - \sin(\phi + D)$

Note $-\cos(\phi + D)$ increases from -1 to 1 in this region thus $-\sin(\phi + D)$ goes from 0 to -1 to 0

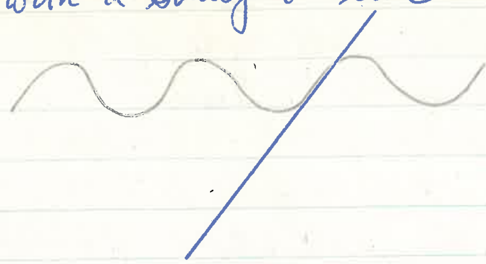
Therefore $f'(\phi) < 0$ since $(2A\phi + B) < 0$ and $-\sin(\phi + D) < 0$

Thus $f(\phi)$ is monotonic on the interval and Newton's method works well with good initial guess.

Replace $\cos(\phi + D) \cong -\phi - D + a$

where $a = (2LB + 1)\pi/2$

This replaces cosine with a straight line with slope -1

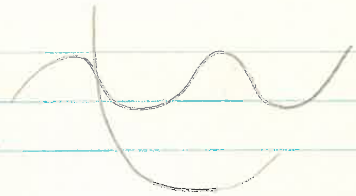


Solve the new quadratic eq. for ϕ

$$A\phi^2 + (B-1)\phi + C - D + a = 0$$

This will be a very good initial guess of the actual root.

CASE #2 $\qquad$ LB is odd



We can show that Newton's method works $(f'(\phi)<0)$ if the initial guess is less than the actual smallest root and if the actual root is less than X_{low}.

PROOF

Say ϕ_1 is initial guess and ϕ^* is actual root with $A\phi^{*2} + B\phi^* + C > 0$.

Then $2A\phi_1 + B < 2A\phi^* + B$ $\quad$ since $A>0$ $\phi<0$, $\phi_1 < \phi^*$

and $2A\phi^* + B - \sin(\phi^* + D) \leq 0$ or this is not the smallest root

and $-\sin(\phi_1 + D) < -\sin(\phi^* + D)$

#

To find initial guess(es):

Replace cosine curve

$$\cos(\phi + D) \approx 1 - \frac{4}{\pi^2}(\phi + D + a)^2$$

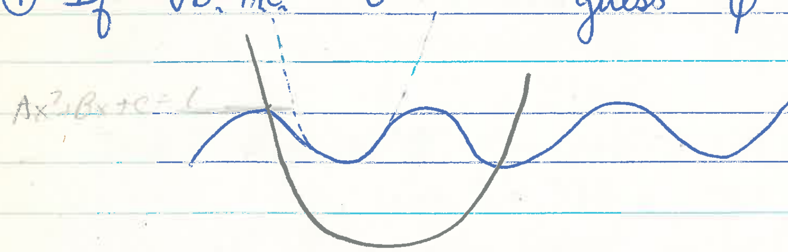
where $a = (LB+1)\pi$

Solve resulting quadratic eq for ϕ

$A_n \phi^2 + B_n \phi + C = 0$ Solve for ϕ^+ ϕ^-

$$\phi^+ = \frac{-B_n + \sqrt{B_n^2 - 4A_nC_n}}{2A_n} \qquad \phi^- = \frac{-B_n - \sqrt{}}{2A_n}$$

① If $\sqrt{B_n^2 - 4AC_n} < 0$ guess $\phi = X_{lim}$

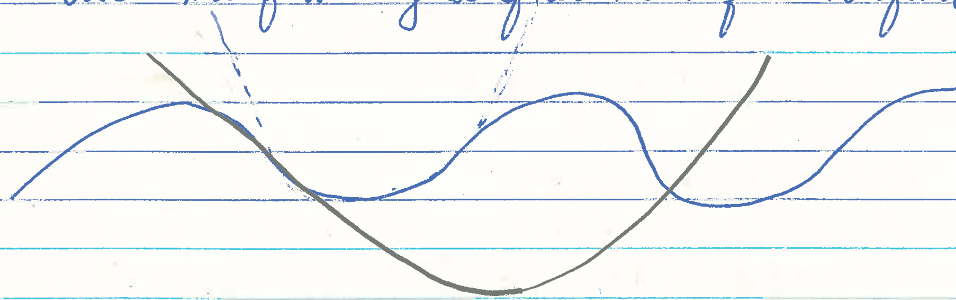


Here the - - - - approx. cosine curve does not intersect the parabola

②

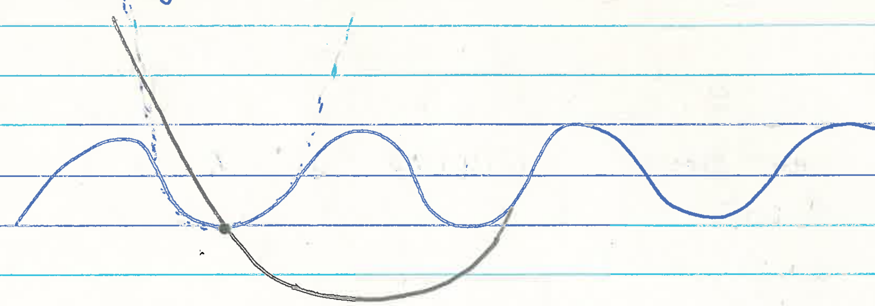
If $X_{lim} < \phi^+ < 0$
$X_{lim} < \phi^- < 0$ then there may be three roots to the equation

Use modified regula falsi technique to find



Here the - - - approx cosine curve intersects the parabola twice below the axis.

③ If either x^+ or x^- is out-of-range then guess the other one



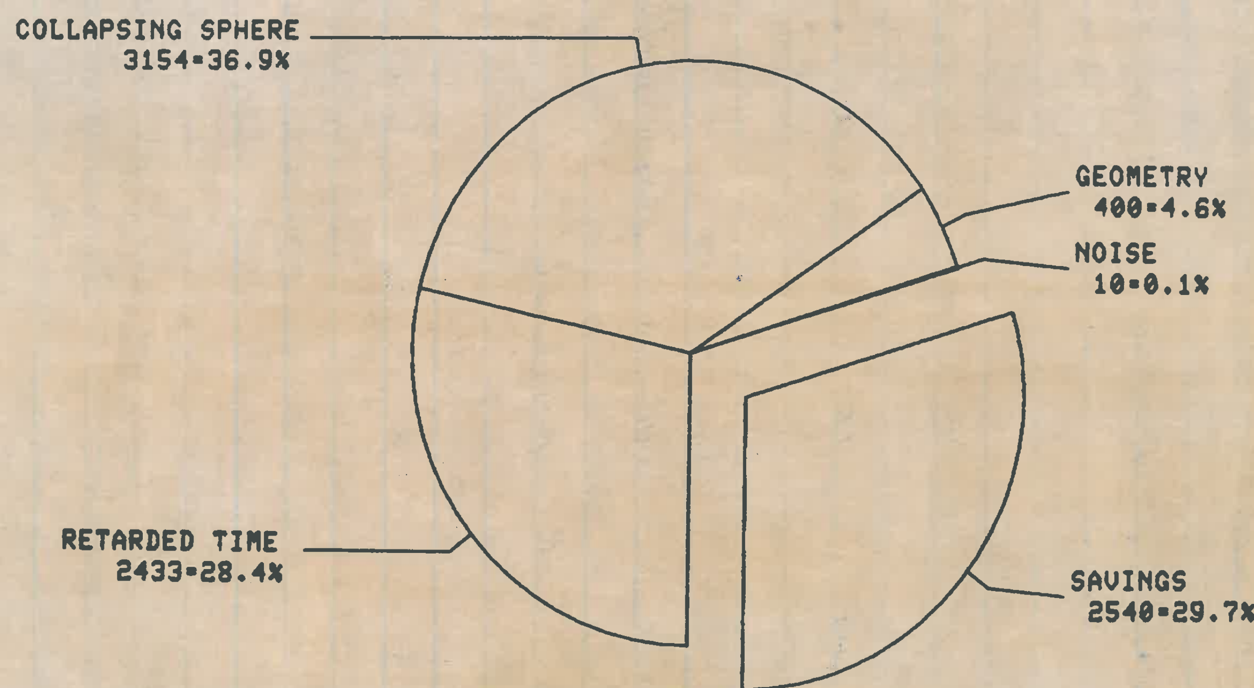
ATKYGIT - ex 3 run the original way
answers very close to those published

ATKYCJV - ex 3 run with initial guessing scheme described here. If three roots were possible QULOCK executes normally otherwise Newton's method finds the single root

ID	Cost $	CP sec	Remarks
ATKYGIT	$159	8535	Matches published results well
ATKYCJV	$112	5995	Identical results to GIT
ATKY06M	$22	1140	NTAU = 30, NPCAF = NCHF = 1
ATKYSCE	$51	2734	like CJV except calls to collapsing sphere are supressed
ATKYRKO	$11	550	ex 4 original
4ABQSZV	?	408	ex 4 modified code

4-29-82

TOTAL EXECUTION TIME =8537 CP SEC.
EXAMPLE 3. TM83135



May 6, 1982

Current research items

1) Should motion = 0, 1 cases be handled separately? In other words, should the problem be programed differently if you know that the observer is fixed on the ground. Why?

2) Design a series of tests to determine what the best set of parameters is for solving Jet Star problem. How does refinement of η_1, η_2 or t grid help?

3) Must we use collapsing sphere method whenever there are 3 roots to emission time eq or only near $1-M_r$ singularity

4) Can I reduce the number of times at which $\bar{p}(\bar{x},t)$ is evaluated? Perhaps put a lot of time points near the peak and few elsewhere?

5) In PROPFAN the time steps for differentiation & Fourier analysis and also the convergence criteria for root solvers is based on ~~DT~~ NPTS and & FRACDT. Is there a better way?

JET STAR much like ex 3 TM 83135

$C = 305.6$, $\rho = .457$, $\vec{x} = (.81, 0., -.25)$, $v_3 = .246.1$

$r = .31$, $r_{inner} = .073$, rev = 7599, 8 blades, FRACDT = 2.5

NPTS = 32 (NLE + NTE) NPCA panels NPCAF = NCAF = 1

ID	Cost $	CPsec	Remarks
ATKY OHD	4	200	(5+10) 15 panels NTAU=20 not smooth, misses peak value a bit
EPM	4	215	as above but NTAU = 40 not much different
DDD	7	340	(10+10) 20 panels NTAU=20 peaks match better but not smooth
DDQ	16	810	(15+15) 30 panels NTAU=20 acceptable NLE NTE NPCA
ENX	15	782	as above but FRACDT = 4.0 poor results
EPJ	8 18	924	nice results like DDQ with NTAU = 40 – all others compared to this –
	$50		

Note: 32 time points is about optimum 64 pts costs more and gives worse results unless FRACDT is adjusted carefully.

I suspect that there is an optimum value of dt which determines what NPTS should be set at.

Pressure Spectra Overall Noise (dB) Compared

	EPJ	DDQ	DDD	OHD
1	139.65	139.63	139.38	139.49
2	135.08	135.12	133.51	132.28
3	129.91	130.15	130.25	128.47
4	122.40	123.03	122.72	123.13
5	118.06	117.95	117.63	116.96
6	111.93	111.16	110.71	106.50
7	101.52	98.85	107.18	107.73
8	106.05	103.64	110.11	111.89

Fourier Transforms

The standard FFT routine from the Math Library can replace the inline coding as follows.

Let NT be the estimated number of times $p(\vec{x}, t)$ should be evaluated to characterize the time history.
Find M such that $2^{M-1} < NT \leq 2^M$

$$\text{ie} \quad M = \text{ifix}\left(\frac{\ln NT}{\ln 2}\right) + 1$$

Calculate or interpolate to get

$$p_j = p(\vec{x}, t_j) \qquad j = 1, 2 \ldots NT$$

$$t_j = (j-1)\left(\frac{2\pi}{NT}\right)$$

$$z_j = (p_j, 0.) \qquad \text{complex } z_j$$

CALL FFT(Z, -M, IWK)

$$|c_n| = \text{CABS}(Z(N)) \qquad N = 2, 3, \ldots NSPEC+1$$

where $NSPEC = 2^{M-1}$

$$SPL = 10 \log\left[2|c_n|^2\right] - 20 \log p_{ref}$$

$$p = \sum_{-\infty}^{\infty} c_0 + c_n$$

output from FFT with M=8

m n

N strip (printed / handwritten):
N
1 0
2 1
3 2
4 3
5 4
6 5
7 6
8 7
9 8
10 9
11 10
12 11
13 12
14 13
15 14
16 15
17 16
18 17
19 18
20 19
21 20
26 25
27 26
28 27
29 28
30 29
31 30
32 31
33 32
34 33
35 34
36 35
37 36
38 37
39 38
40 39
41 40
42 41
43 42
44 43
45 44
46 45
47 46
48 47
49 48
50 49

SPL	SPL	SPL	SPL
.90969100E+02	.90935104E+02	.93067194E+02	.67034479E+02
.83124741E+02	.81026321E+02	.83525204E+02	.66119504E+02
-.10602060E+03	.75006375E+02	.69547112E+02	.62773881E+02
.64043379E+02	.71485640E+02	.62189757E+02	.51742395E+02
-.10602060E+03	.68988391E+02	.57087359E+02	.58351410E+02
.55176407E+02	.67052154E+02	.53165392E+02	.66526234E+02
-.10602060E+03	.65470928E+02	.49976173E+02	.70479472E+02
.49341754E+02	.64134827E+02	.47287872E+02	.72781433E+02
-.10602060E+03	.62978260E+02	.44964305E+02	.74075299E+02
.44989937E+02	.61958919E+02	.42918476E+02	.74610077E+02
-.10602060E+03	.61047915E+02	.41091361E+02	.74490716E+02
.41521389E+02	.60224645E+02	.39441043E+02	.73745596E+02
-.10602060E+03	.59473894E+02	.37936671E+02	.72341993E+02
.38640322E+02	.58784111E+02	.36554870E+02	.70170491E+02
-.10602060E+03	.58146315E+02	.35277505E+02	.66977388E+02
.36178872E+02	.57553386E+02	.34090219E+02	.62114573E+02
-.10602060E+03	.56999586E+02	.32981450E+02	.53008296E+02
.34032543E+02	.56480222E+02	.31941751E+02	.45424901E+02
-.10602060E+03	.55991404E+02	.30963304E+02	.55451208E+02
.32131853E+02	.55529877E+02	.30039562E+02	.57742278E+02
-.10602060E+03	.55092884E+02	.29164990E+02	.57541061E+02
.30428248E+02	.54678074E+02	.28334866E+02	.55433155E+02
-.10602060E+03	.54283425E+02	.27545130E+02	.50799334E+02
.28886465E+02	.53907183E+02	.26792264E+02	.38412116E+02
-.10602060E+03	.53547819E+02	.26073201E+02	.43346546E+02
.27480081E+02	.53203991E+02	.25385250E+02	.49692323E+02
-.10602060E+03	.52874516E+02	.24726039E+02	.51470762E+02
.26188789E+02	.52558345E+02	.24093463E+02	.51044911E+02
-.10602060E+03	.52254542E+02	.23485650E+02	.48597790E+02
.24996644E+02	.51962272E+02	.22900922E+02	.42917610E+02
-.10602060E+03	.51680784E+02	.22337777E+02	.18513617E+02
.23890899E+02	.51409400E+02	.21794856E+02	.41582184E+02
-.10602060E+03	.51147508E+02	.21270933E+02	.45997249E+02
.22861203E+02	.50894552E+02	.20764896E+02	.47154424E+02
-.10602060E+03	.50650027E+02	.20275730E+02	.46274926E+02
.21899039E+02	.50413470E+02	.19802511E+02	.43181476E+02
-.10602060E+03	.50184459E+02	.19344393E+02	.35592277E+02
.20997313E+02	.49962607E+02	.18900600E+02	.29241070E+02
-.10602060E+03	.49747556E+02	.18470416E+02	.40010869E+02
.20150056E+02	.49538978E+02	.18053185E+02	.43155607E+02
-.10602060E+03	.49336569E+02	.17648298E+02	.43710592E+02
.19352198E+02	.49140049E+02	.17255193E+02	.42284215E+02
-.10602060E+03	.48949156E+02	.16873347E+02	.38300724E+02
.18599398E+02	.48763649E+02	.16502277E+02	.27219637E+02
-.10602060E+03	.48583302E+02	.16141531E+02	.31853283E+02
.17887912E+02	.48407906E+02	.15790690E+02	.38560133E+02
-.10602060E+03	.48237264E+02	.15449360E+02	.40756904E+02
.17214485E+02	.48071193E+02	.15117176E+02	.40729914E+02
-.10602060E+03	.47909521E+02	.14793793E+02	.38677180E+02
.16576277E+02	.47752089E+02	.14478891E+02	.33489719E+02

Calculate

$$\nabla_4 \cdot \vec{A}_4 = \nabla \cdot \left\{ \alpha \left[(\cos\theta + M_n)\,\vec{n} - \alpha_n^2\,\vec{\hat{r}} \right] \right\}$$

$$+ \frac{1}{c} \frac{\partial}{\partial \tau} \left[\alpha (1 - M_n \cos\theta) \right]$$

where

$$\alpha_n^2 = 1 + M_n^2 \qquad \vec{\hat{r}} = \vec{r}/|r|$$

$$\cos\theta = \vec{n} \cdot \vec{r}$$

$$\alpha = \frac{c}{\alpha_n^2 - 2M_n\cos\theta + \sin^2\theta}$$

$$M_n = \frac{\vec{v} \cdot \vec{n}}{c}$$

MACSYMA

EW LRCI, $$
EE SLP, FFDEL $$

↑Z

A↑K
(C1) BATCH(SLP, FFDEL, DSK, LRCI);

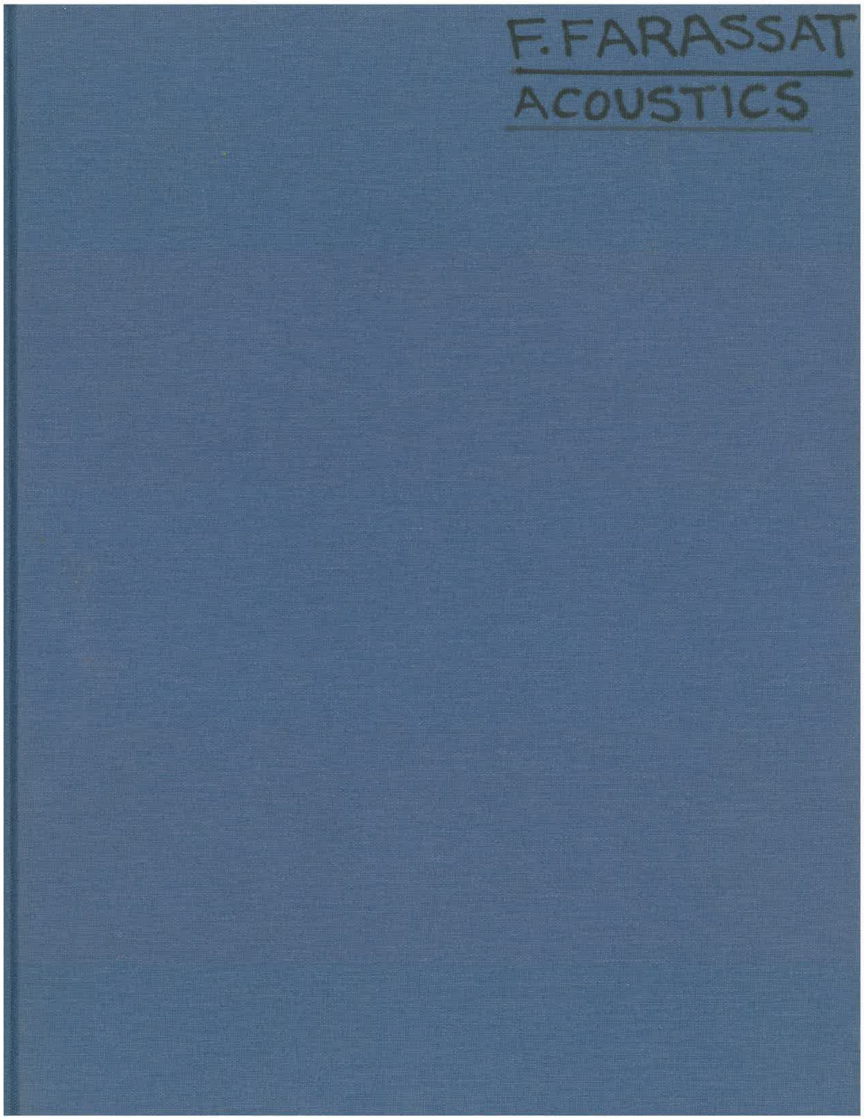
↑Z
$J
$P
J HT $$
ER SLP FFDEL $Y HT $$

↑Z
;LOGOUT

Basic TECO commands

(1) Insert string I string $
(2) Search for string S string $
(3) Move from line to line 2L, -L, 0L, ;L, J
(4) Move by character 2C, -C
(5) Delete 2D, -D, K
(6) Type T, HT, V

18 Acoustics



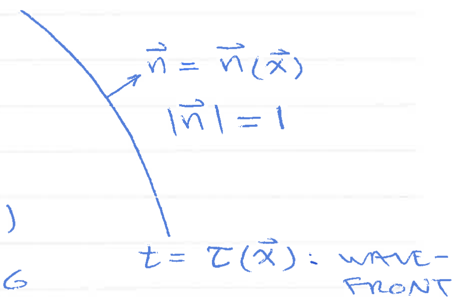
SOME RESULTS FROM RAY ACOUSTICS

1. STATIONARY MEDIUM – $c = c(\vec{x})$ SPEED OF SOUND

THE WAVEFRONT IS DESCRIBED BY THE EQUATION

$t = \tau(\vec{x})$, t : TIME

I WILL BE USING NOTATIONS OF PIERCE'S BOOK.



HUYGENS PRINCIPLE

$$\frac{d\vec{x}}{dt} = c\vec{n} \qquad (1)$$

WE ARE INTERESTED IN DERIVING THE D.E. FOR THE EVOLUTION OF THE SLOWNESS VECTOR $\vec{s} = \nabla\tau \parallel \vec{n}$. WE FIRST DERIVE A RELATION BETWEEN $\vec{n}$ AND $\vec{s}$. TAKE d/dt OF BOTH SIDES OF $t = \tau(\vec{x})$ REMEMBERING THAT $\vec{x} = \vec{x}(t)$:

$$1 = \frac{d\vec{x}}{dt} \cdot \nabla\tau = c\vec{n} \cdot \vec{s} \qquad (2)$$

SINCE $\vec{s} \parallel \vec{n}$, LET $\vec{s} = \alpha\vec{n}$ $\Rightarrow$ FROM (2) $\alpha = \frac{1}{c}$

i.e.

$$\vec{s} = \frac{\vec{n}}{c} \quad , \quad c\vec{s} = \vec{n} \qquad (3a,b)$$

WE NOTE THAT $|\vec{s}| = 1/c$, HENCE THE TERMINOLOGY SLOWNESS VECTOR FOR $\vec{s}$, i.e. THE LARGER THE SPEED OF SOUND, THE SMALLER $|\vec{s}|$.

DIFFERENTIAL EQUATION FOR EVOLUTION OF $\vec{s}$

$\vec{s} = \vec{s}(\vec{x})$, TAKE d/dt OF THIS EQUATION

$$\frac{d\vec{s}}{dt} = \frac{d\vec{x}}{dt} \cdot \nabla\vec{s} = c\vec{n} \cdot \nabla\left(\frac{\vec{n}}{c}\right) = \underbrace{\vec{n} \cdot \nabla\vec{n}}_{\nabla(\vec{n}\cdot\vec{n}) = 0} - \frac{\nabla c}{c}$$

$$\therefore \quad \frac{d\vec{s}}{dt} = -\frac{\nabla c}{c} \qquad (4)$$

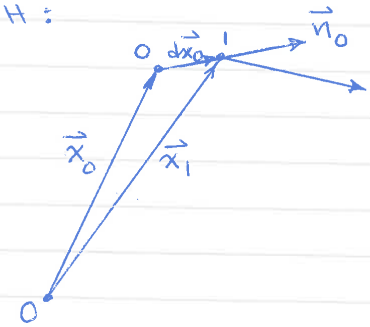
TO FIND THE RAYS WE MUST SIMULTANEOUSLY SOLVE THE TWO EQUATIONS :

$$\begin{cases} \dfrac{d\vec{x}}{dt} = c\vec{n} = c^2\vec{s}(\vec{x}) & (1) \\ \dfrac{d\vec{s}}{dt} = -\dfrac{\nabla c}{c}(\vec{x}) & (4) \end{cases}$$

AN INFINITESIMAL VIEW OF RAY CONSTRUCTION IS INSTRUCTIVE. WE HAVE

$$(5) \quad \begin{cases} d\vec{x}_0 = c_0\vec{n}_0\, dt \equiv c(\vec{x}_0)\,\vec{n}(\vec{x}_0)\, dt \\ d\vec{s}_0 = -(\nabla c/c)_0\, dt \equiv -(\nabla c/c)(\vec{x}_0)\, dt \\ \vec{s}_1 = \vec{s}_0 + d\vec{s}_0 \equiv \vec{n}_1/c_1 \approx \vec{n}_1/c_0 \\ \vec{x}_1 = \vec{x}_0 + d\vec{x}_0 \end{cases}$$

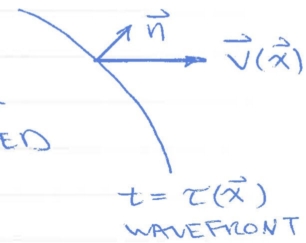
BELOW IS THE EXAGERATED CONSTRUCTION OF A RAY PATH:



$\vec{n}_1 = c_1\vec{s}_1 = c(\vec{x}_1)\,\vec{s}_1$ KNOWN FROM $\vec{n}_0$ AND $c = c(\vec{x})$. THIS STEP CAN BE REPEATED TO GET THE RAY PATH.

TO CONSTRUCT THE RAY, WE NEED $c = c(\vec{x})$ AND $\vec{n}_0$, THE INITIAL UNIT NORMAL TO THE WAVEFRONT. WE SEE THAT THE FUNCTION OF THE SLOWNESS VECTOR IS TO DETERMINE THE LOCAL TANGENT TO THE RAY: $\vec{n} \perp$ WAVEFRONT $\tau(\vec{x}) = t$. THIS CONSTRUNCTION IS SOMEWHAT SIMILAR TO THAT OF BICHARACTERISTICS OF A HYPERBOLIC EQUATION.

2 - <u>MOVING MEDIUM</u> – $c = c(\vec{x})$ SPECIFIED, INDEPENDENT OF TIME, FLUID VELOCITY $\vec{V} = \vec{V}(\vec{x})$,
WAVEFRONT SURFACE GIVEN BY
$t = \tau(\vec{x})$, $\vec{s} = \nabla\tau$ SLOWNESS VECTOR
$\vec{s} \parallel \vec{n}$, $\vec{s} = \alpha\vec{n}$, α TO BE DETERMINED



<u>HUYGENS PRINCIPLE</u>

$$\frac{d\vec{x}}{dt} = c\vec{n} + \vec{V} \qquad (6)$$

BY TAKING DERIVATIVE WRT t OF BOTH SIDES OF $t = \tau(\vec{x})$ AND USING EQ. (6), WE GET

$$1 = \frac{d\vec{x}}{dt} \cdot \nabla\tau = (c\vec{n} + \vec{V}) \cdot \vec{s} = \alpha(c + \vec{V}\cdot\vec{n}) \qquad (7)$$

$$\alpha = \frac{1}{c(1+M_n)} \equiv \frac{\Omega}{c} \; , \; \Omega = \frac{1}{1+M_n} \qquad (8)$$

$$M_n = \vec{V}\cdot\vec{n}/c \qquad (9)$$

$$\vec{s} = \frac{\Omega}{c}\vec{n} \quad , \quad c\vec{s} = \Omega\vec{n} \qquad (10a,b)$$

$$\alpha c = \Omega \Rightarrow \text{FROM (7) THAT } \Omega = 1 - \vec{V}\cdot\vec{s} \qquad (11)$$

Ω IS DIMENSIONLESS, $c|\vec{s}| = \Omega$ IS, THEREFORE, DIMENSIONLESS.

<u>DIFFERENTIAL EQUATION FOR EVOLUTION OF $\vec{s}$</u>

$$\frac{d\vec{s}}{dt} = (c\vec{n} + \vec{V})\cdot\nabla\vec{s} = \left(\frac{c^2}{\Omega}\vec{s} + \vec{V}\right)\cdot\nabla\vec{s}$$

$$= \frac{c^2}{\Omega}\vec{s}\cdot\nabla\vec{s} + \vec{V}\cdot\nabla\vec{s} \qquad (12)$$

$$\vec{s}\cdot\nabla\vec{s} = \nabla\left(\frac{s^2}{2}\right) = \frac{1}{2}\nabla\left(\frac{\Omega^2}{c^2}\right) \qquad \text{FROM (10a)}$$

$$= \frac{\Omega}{c^2}\nabla\Omega - \frac{\Omega^2}{c^3}\nabla c$$

$$= -\frac{\Omega}{c^2}\nabla(\vec{V}\cdot\vec{s}) - \frac{\Omega^2}{c^3}\nabla c \qquad (13)$$

WE HAVE USED $\nabla\Omega = -\nabla(\vec{V}\cdot\vec{s})$ FROM EQ. (11).

EQUATION (12) BECOMES

$$\frac{d\vec{S}}{dt} = -\frac{\Omega}{c}\nabla c - \nabla(\vec{V}\cdot\vec{S}) + \vec{V}\cdot\nabla\vec{S} \qquad (14)$$

WE NOW USE SOME VECTOR IDENTITIES TO REMOVE THE SPACE DERIVATIVES FROM $\vec{S}$.

$$\begin{aligned}\nabla(\vec{V}\cdot\vec{S}) &= \nabla_V(\vec{V}\cdot\vec{S}) + \nabla_S(\vec{V}\cdot\vec{S}) \\ &= \vec{S}\times(\nabla\times\vec{V}) + \vec{S}\cdot\nabla\vec{V} \qquad (= \nabla_V(\vec{V}\cdot\vec{S})) \\ &\quad + \vec{V}\times\underbrace{(\nabla\times\vec{S})}_{\nabla\times\nabla\tau = 0} + \vec{V}\cdot\nabla\vec{S} \qquad (= \nabla_S(\vec{V}\cdot\vec{S})) \\ &= \vec{S}\times(\nabla\times\vec{V}) + \vec{S}\cdot\nabla\vec{V} + \vec{V}\cdot\nabla S \qquad (15)\end{aligned}$$

∇_V MEANS THAT THE GRADIENT OPERATOR ACTS ON $\vec{V}$ ONLY, i.e. $[\nabla_V(\vec{V}\cdot\vec{S})]_i = S_j \partial_i v_j$. SUBSTITUTING (15) IN (14) GIVES

$$\frac{d\vec{S}}{dt} = -\frac{\Omega}{c}\nabla c - \vec{S}\times\underbrace{(\nabla\times\vec{V})}_{\text{VORTICITY}} - \vec{S}\cdot\nabla\vec{V} \qquad (16)$$

WE CAN AGAIN USE AN INFINITESIMAL DESCRIPTION OF HOW WE FIND THE RAY PATH USING EQS. (6) AND (16). EQUATION (6) GIVES $d\vec{x}$ AND EQ. (16) GIVES $d\vec{S}$ FROM WHICH THE LOCAL RAY TANGENT $\vec{n}$ CAN BE CONSTRUCTED : $\vec{n} = (c/\Omega)\vec{S}$.

SEE SECTIONS 8-2 TO 8-4 OF PIERCE'S BOOK.

AMPLITUDE VARIATION ALONG RAYS (SEC. 8-5, PIERCE)

SINCE $t = \tau(\vec{x})$ DESCRIBES THE WAVEFRONT, FOR $\vec{V}=0$, THE VARIATION OF A PERIODIC WAVE ALONG THE RAY SATISFIES

$$\hat{p}(\vec{x}) = P(\vec{x},\omega)\, e^{i\omega\tau(\vec{x})} \tag{18}$$

WHERE $P(\vec{x})$ IS THE AMPLITUDE OF THE WAVE. NOTE THAT WE ARE LOOKING AT A PARTICULAR WAVEFRONT. THE GENERALIZATION OF THIS WILL BE GIVEN LATER. THE FUNCTION $\hat{p}(\vec{x})$ MUST SATISFY THE HELMHOLTZ EQUATION $\nabla^2\hat{p} + k^2\hat{p} = 0$, $k = \omega/c(\vec{x}) \equiv k(\vec{x})$. USING EQ. (18) IN THIS EQUATION GIVES

$$\nabla\hat{p} = (\nabla P + i\omega P\nabla\tau)\, e^{i\omega\tau(\vec{x})} \tag{19a}$$

$$\nabla^2\hat{p} = (\nabla^2 P + i\omega P\nabla^2\tau + i\omega\nabla P\cdot\nabla\tau)\, e^{i\omega\tau(\vec{x})}$$
$$+ i\omega(\nabla P\cdot\nabla\tau + i\omega P|\nabla\tau|^2)\, e^{i\omega\tau(\vec{x})} \tag{19b}$$

THE COEFF. OF $e^{i\omega\tau}$ IN HELMHOLTZ EQ. IS

$$\nabla^2 P + i\omega(2\nabla P\cdot\nabla\tau + P\nabla^2\tau) - \omega^2 P\left[|\nabla\tau|^2 - \frac{1}{c^2}\right] = 0 \tag{20}$$

IF WE NOW USE AN ASYMPTOTIC EXPANSION FOR P:

$$P(\vec{x},\omega) = P_0(\vec{x}) + \frac{1}{\omega}P_1(\vec{x}) + \frac{1}{\omega^2}P_2(\vec{x}) + \cdots \tag{21}$$

WE REQUIRE THAT THE HELMHOLTZ EQ. BE SATISFIED IDENTICALLY, i.e. THE COEFFICIENTS OF ALL POWERS OF ω VANISH $\Rightarrow$ THE COEFF. OF ω^2 IS

$$\left(|\nabla\tau|^2 - \frac{1}{c^2}\right)P_0 = 0 \qquad \text{(SEE NEXT PAGE)*}$$

$$|\nabla\tau|^2 = \frac{1}{c^2} \qquad (\text{SINCE } P_0 \neq 0) \tag{22}$$

THE COEFF. ω IS

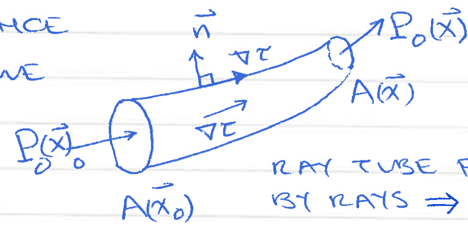
$$i(2\nabla P_0\cdot\nabla\tau + P_0\nabla^2\tau) - P_1\underbrace{\left[|\nabla\tau|^2 - \frac{1}{c^2}\right]}_{=0} = 0$$

$$2\nabla P_0\cdot\nabla\tau + P_0\nabla^2\tau = 0 \tag{23}$$

BY MULTIPLYING THIS EQ. BY P_0, WE GET

$$\nabla\cdot(P_0^2\,\nabla\tau)=0 \qquad (24)$$

APPLYING THE DIVERGENCE THM TO A RAY TUBE, WE GET:



RAY TUBE FORMED BY RAYS $\Rightarrow \vec{n}\cdot\nabla\tau=0$ ON TUBE WALL

$$0=\int_V \nabla\cdot(P_0^2\,\nabla\tau)\,d\vec{x}$$

$$=\int_{\partial V} P_0^2\,\nabla\tau\cdot\vec{n}\,d\vec{S}$$

$$=P_0^2(\vec{x})\,A(\vec{x})-P_0^2(\vec{x}_0)\,A(\vec{x}_0) \qquad (25)$$

OR $P_0^2(\vec{x})\,A(\vec{x})=$ CONST. ALONG A RAY TUBE, i.e. A SMALLER $A(\vec{x})$ IS ASSOCIATED WITH HIGHER P_0, THE WAVE AMPLITUDE.

COEFF. INDEPENDENT OF ω IN EQ.(20)

$$\nabla^2P_0+i\,(2\nabla P_1\cdot\nabla\tau+P_1\nabla^2\tau)-P_2\underbrace{\left[|\nabla\tau|^2-\frac{1}{c^2}\right]}_{=0}=0$$

$$2\nabla P_1\cdot\nabla\tau+P_1\nabla^2\tau=+i\,\nabla^2P_0 \qquad (26)$$

IF P_0 IS ASSUMED REAL $\Rightarrow$ P_1 IS PURELY IMAGINARY.

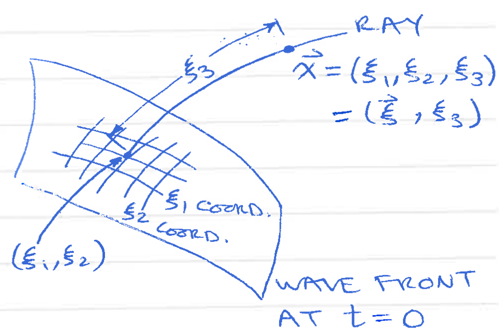
(*) COMMENT ON EXPANSION: WHEN WE SUBSTITUTE EQ.(21) IN EQ.(20), THE HIGHEST POWER OF ω IS ω^2 AND THIS IS THE POWER WE START WITH.

ENERGY CONSERVATION ALONG RAYS (NO FLOW) - SEC. 8.5 PIERCE

ACCORDING TO PIERCE, HERE IS A HEURISTIC DERIVATION OF CONSERVATION OF ACOUSTIC ENERGY ALONG RAYS. WE ASSUME $\vec{V}=0$, $\rho=\rho(\vec{x})$, $c=c(\vec{x})$ AND THE ACOUSTIC WAVE LENGTH (TYPICAL) $\ll$ LENGTH SCALE OF VARIATION OF ρ AND c IN SPACE. THE ACOUSTIC PRESSURE CAN BE WRITTEN AS

$$p(\vec{x},t) = B(\vec{x})\, f(t-\tau(\vec{x}), \vec{\xi}) \qquad (27)$$

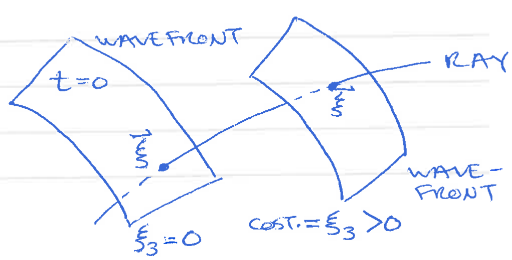
WHERE $\vec{\xi}=(\xi_1, \xi_2)$ IS THE VARIABLE SPECIFYING THE PARTICULAR RAY WE ARE INTERESTED IN (SEE FIG. BELOW). THIS EQUATION IS DEDUCED FROM THE STUDY OF RAY ACOUSTICS PROBLEMS THAT CAN BE SOLVED ANALYTICALLY (e.g., CAUSTICS, P 398-399, PIERCE).



AGAIN $t=\tau(\vec{x})$ OR $\tau(\vec{x})=$ CONST. DESCRIBES THE WAVEFRONT AT TIME t. THEREFORE, $B(\vec{x})f(t-\tau,\vec{\xi})$ HAS A SIMPLE STRUCTURE ON A RAY IF WE INTRODUCE A THIRD COORDINATE ξ_3 ALONG THE RAY, SAY DISTANCE ALONG THE RAY FROM $t=0$ AS SHOWN. THEN ALONG A GIVEN RAY, WE HAVE

$$p(\xi_3,t) = B(\xi_3)\, f[t-\tau(\xi_3)] \qquad (28)$$

THE FUNCTION $p(\vec{x},t)/B(\vec{x}) = f[t-\tau(\vec{x}),\vec{\xi}]$ WHICH MEANS THAT ON A GIVEN WAVEFRONT, WE HAVE $\xi_3=$ CONST. THEREFORE, THE FUNCTION f DESCRIBES THE SHAPE OF THE WAVEFORM ON A RAY BUT IT IS NOT UNIQUE! TO FIND f WITHIN A SCALE FACTOR ON A RAY,



FIX A POINT $\xi_3 =$ CONST. ON THE RAY, THEN $f[t-\tau(\xi_3)]$ $\propto p(\xi_3, t)$ BECAUSE $B(\xi_3) =$ CONST.

NOW, ASSUMING $p(\vec{x},t)$ DESCRIBED BY EQ. (27), THEN THE ACOUSTIC WAVE ACTS LIKE A PLANE WAVE LOCALLY SO THAT

$$v = \text{ACOUSTIC VELOCITY} = \frac{p\,\vec{n}}{\rho c} = \frac{Bf}{\rho}\nabla\tau \qquad (29)$$

$$\text{AC. ENERGY DENSITY} = w = \frac{1}{2}\rho v^2 + \frac{1}{2}\frac{p^2}{\rho_0 c^2} \equiv \rho v^2 \quad \text{PLANE WAVE}$$

$$= \frac{B^2}{\rho c^2} f^2 \quad \left(\text{USING } |\nabla\tau|^2 = \frac{1}{c^2}\right) \qquad (30)$$

$$\text{AC. INTENSITY} = I = \vec{n}\, c\, w = c^2 w \nabla\tau \qquad (31)$$

THE ACOUSTIC CONSERVATION LAW IS $\partial w/\partial t + \nabla\cdot I = 0$. USING EQS. (30) AND (31), WE GET

$$2\frac{B^2 f}{\rho c^2} f \frac{\partial f}{\partial t} + f^2 \nabla\cdot\left(\frac{B^2}{\rho}\nabla\tau\right) + 2\frac{B^2 f}{\rho}\nabla\tau\cdot\nabla f = 0 \qquad (32)$$

WE HAVE $\nabla f \approx -\frac{\partial f}{\partial \tau}\nabla\tau$, NEGLECTING DEPENDENCE OF f ON $\vec{\xi}$. THEN THE FIRST AND THIRD TERMS OF EQ. (32) CANCELS AND WE GET

$$\nabla\cdot\left(\frac{B^2}{\rho}\nabla\tau\right) = \nabla\cdot\left(\frac{B^2\vec{n}}{\rho c}\right) = 0 \qquad (33)$$

FOR A RAY TUBE, USING THE FIGURE AND THE ARGUMENT OF PAGE 6, WE GET

$$\frac{A(\vec{x})\, B^2(\vec{x})}{\rho(\vec{x})\, c(\vec{x})} = \text{CONST. ALONG A RAY TUBE} \qquad (34)$$

NOTE THAT ALONG A RAY TUBE, TIME AND SPACE COORDINATES ARE CONNECTED BY THE RELATION $t = \tau(\vec{x})$ SO THAT, $\vec{x} = x(t)$ ON A WAVEFRONT. THEN RELATION OF EQ. (34) HOLDS FOR TWO POINTS ON A RAY AT THE TWO TIMES THAT THE WAVEFRONTS AT THOSE POINTS.

ENERGY CONSERVATION ALONG RAYS (WITH FLOW)

WE WILL DERIVE SEVERAL IMPORTANT RESULTS HERE. THE SUBJECT IS DIFFICULT WITHOUT SOME SIMPLIFYING ASSUMPTIONS. THE ASSUMPTION WE MAKE IS THAT ANY DERIVATIVE OF THE BACKGROUND FLOW PARAMETERS IS A FIRST ORDER QUANTITY, i.e., SMALL COMPARED TO ZEROTH ORDER QUANTITIES OF BACKGROUND FLOW. WE ALSO ASSUME THAT THE BACKGROUND FLOW IS TIME INDEPENDENT.

MASS CONTINUITY: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$ (35)

$$\partial_t (\rho_0 + \rho') + \nabla \cdot [(\rho_0 + \rho')(\vec{u}_0 + \vec{u}')] = 0 \qquad (36)$$

$$\nabla \cdot (\rho_0 \vec{u}_0) = 0 \qquad (37)$$

$$\partial_t \rho' + \vec{u}_0 \cdot \nabla \rho' + \rho_0 \nabla \cdot \vec{u}' + \underbrace{\vec{u}' \cdot \nabla \rho_0 + \rho' \nabla \cdot \vec{u}_0}_{\text{2ND ORDER QUANTITIES}} = 0 \qquad (38)$$

$$D_t = \partial_t + \vec{u}_0 \cdot \nabla \qquad (39)$$

$$\boxed{D_t \rho' + \rho_0 \nabla \cdot \vec{u}' = 0} \qquad (40)$$

MOMENTUM EQ.: $\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \frac{\nabla P}{\rho} = 0$ (41)

$$\partial_t (\vec{u}_0 + \vec{u}') + (\vec{u}_0 + \vec{u}') \cdot \nabla (\vec{u}_0 + \vec{u}') + \frac{\nabla (P_0 + P')}{\rho_0 + \rho'} = 0 \qquad (42)$$

$$u_0 \cdot \nabla \vec{u}_0 + \frac{\nabla P_0}{\rho_0} = 0 \qquad (43)$$

$$\partial_t \vec{u}' + \vec{u}_0 \cdot \nabla \vec{u}' + \frac{\nabla P'}{\rho_0} + \underbrace{\vec{u}' \cdot \nabla \vec{u}_0 - \frac{\rho' \nabla P_0}{\rho_0^2}}_{\text{2ND ORDER QUANTITIES}} = 0 \qquad (44)$$

$$\boxed{D_t \vec{u}' + \frac{\nabla P'}{\rho_0} = 0} \qquad (45)$$

ENTROPY EQUATION

$$\frac{\partial S}{\partial t} + \vec{u} \cdot \nabla S = 0 \qquad (46)$$

$$\partial_t (S_0 + S') + (\vec{u}_0 + \vec{u}') \cdot \nabla (S_0 + S') = 0 \qquad (47)$$

$$\vec{u}_0 \cdot \nabla S_0 = 0 \Rightarrow \vec{u}_0 \perp \nabla S_0 \text{ OR } S_0 = \text{CONST. EVERYWHERE} \qquad (48)$$

$$\partial_t S' + \vec{u}_0 \cdot \nabla S' + \underbrace{\vec{u}' \cdot \nabla S_0}_{\text{2ND ORDER QUANT.}} = 0 \qquad (49)$$

$$\boxed{D_t S' = 0} \qquad (50)$$

THIS MEANS THAT THE FIRST ORDER QUANTITY S' DOES NOT CHANGE FOR A MOVING PARTICLE OF FLUID. THIS MAKES SENSE BECAUSE THERE IS NO PHYSICAL MECHANISM OF AFFECTING ENTROPY CHANGE SUCH AS HEAT CONDUCTIVITY AND VISCOSITY.

CONSTITUTIVE RELATION $\quad p = p(\rho, S) \qquad (51)$

$$dp = \left(\frac{\partial p}{\partial \rho}\right)_S d\rho + \left(\frac{\partial p}{\partial S}\right)_\rho dS = c^2 d\rho + \left(\frac{\partial p}{\partial S}\right)_\rho dS \qquad (52)$$

$$p' = c^2 \rho' + \left(\frac{\partial p}{\partial S}\right)_0 S' \qquad (53)$$

IN AIR, WE CAN SHOW THAT

$$\left(\frac{\partial p}{\partial S}\right)_0 = \frac{\rho_0}{C_p} c^2 = \frac{(\gamma - 1)\rho_0}{\gamma R_{AIR}} c^2 \ll c^2 \qquad (54)$$

WHERE C_p IS THE SPECIFIC HEAT AT CONSTANT PRESSURE, γ IS THE RATIO OF SPECIFIC HEATS AND $R_{AIR} = 8314/28 = 297$.

WE ARE GOING TO ASSUME THAT THE MAIN CHANGE IN p' IS CAUSED BY THE CHANGE IN ρ', i.e. LOCALLY, WE

HAVE $$\boxed{p' = c^2(\vec{x})\,\rho'} \qquad (55)$$

ENERGY CONSERVATION

WE HAVE

$$\partial_t \rho' + \vec{u}_0\cdot\nabla\rho' + \rho_0\nabla\cdot\vec{u}' = 0 \qquad (40)$$

MULTIPLY BY $c^2(\vec{x})$, USE EQ. 55

$$\partial_t p' + \vec{u}_0\cdot\nabla(\rho' c^2) - \underbrace{\rho'\vec{u}_0\cdot\nabla c^2}_{\text{SMALL}} + \rho_0 c^2\nabla\cdot\vec{u}' = 0 \qquad (56)$$

$$\partial_t p' + \vec{u}_0\cdot\nabla p' + \rho_0 c^2\nabla\cdot\vec{u}' = 0 \qquad (57)$$

$$\rho_0\partial_t\vec{u}' + \rho_0\vec{u}_0\cdot\nabla\vec{u}' + \nabla p' = 0 \qquad (45)$$

$$\partial_t(\rho_0\vec{u}') + \vec{u}_0\cdot\nabla(\rho_0\vec{u}') - \underbrace{(\vec{u}_0\cdot\nabla\rho_0)}_{\text{SMALL}}u' + \nabla p' = 0 \qquad (58)$$

$$\partial_t(\rho_0\vec{u}') + \vec{u}_0\cdot\nabla(\rho_0\vec{u}') + \nabla p' = 0 \qquad (59)$$

DOT WITH $\vec{u}'$

$$\frac{\partial}{\partial t}\left(\frac{\rho_0 u'^2}{2}\right) + \vec{u}_0\cdot\nabla\left(\frac{\rho_0 u'^2}{2}\right) + \vec{u}'\cdot\nabla p' = 0 \qquad (60)$$

$$\partial_t\left(\frac{\rho_0 u'^2}{2}\right) + \nabla\cdot\left(\frac{\rho_0 u'^2}{2}\vec{u}_0\right) - \underbrace{\frac{\rho_0 u'^2}{2}\nabla\cdot\vec{u}_0}_{O(u'^3)\text{:SMALL}} + \vec{u}'\cdot\nabla p' = 0 \qquad (61)$$

$$\partial_t\left(\frac{\rho_0 u'^2}{2}\right) + \nabla\cdot\left(\frac{\rho_0 u'^2}{2}\vec{u}_0 + p'\vec{u}'\right) - p'\nabla\cdot\vec{u}' = 0 \qquad (62)$$

FROM EQ. (57), WE HAVE

$$p'\nabla\cdot\vec{u}' = -\partial_t\left(\frac{p'^2}{2\rho_0 c^2}\right) - \nabla\cdot\left(\frac{p'^2\vec{u}_0}{2\rho_0 c^2}\right) + \underbrace{p'^2\nabla\cdot\left(\frac{\vec{u}_0}{\rho_0 c^2}\right)}_{O(p'^3)\text{:SMALL}} = 0 \qquad (63)$$

SUBSTITUTING THIS RESULT IN EQ. (62), WE GET

$$\partial_t\left(\frac{\rho_0 u'^2}{2} + \frac{p'^2}{2\rho_0 c^2}\right) + \nabla\cdot\left[p'\vec{u}' + \left(\frac{\rho_0 u'^2}{2} + \frac{p'^2}{2\rho_0 c^2}\right)\vec{u}_0\right] = 0 \qquad (64)$$

OR FOR $$\boxed{w = \frac{\rho_0 u'^2}{2} + \frac{p'^2}{2\rho_0 c^2}\,,\quad \vec{I} = p'\vec{u}' + w\,\vec{u}_0(\vec{x})} \qquad (65a,b)$$

$$\boxed{\partial_t w + \nabla\cdot\vec{I} = 0}^{(*)} \qquad (66)$$

THIS RESULT MAKES SENSE PHYSICALLY. THE ACOUSTIC ENERGY

(*) PIERCE DEFINES $\vec{I} = p'\vec{u}'$ WHICH IS NOT OUR $\vec{I}$!

DENSITY w IS LIKE THAT OF A PLANE WAVE WITH LOCAL $\rho_0(\vec{x})$ AND $c(\vec{x})$. SINCE THE ACOUSTIC WAVE RIDES ON THE PARTICLE WHICH MOVES AT $\vec{u}_0(\vec{x})$, THE ACOUSTIC INTENSITY $\vec{I}$ IS THE SUM OF $p'\vec{u}'$ (FROM THE ACOUSTIC WAVE) AND THE ENERGY/UNIT AREA $w\vec{u}_0$ CARRIED AWAY BY THE FLUID.

NOTE: PIERCE DERIVES THE FOLLOWING RESULT, BASED ON OUR NOTATION, FOR ENERGY CONSERVATION LAW:

$$\partial_t\left(\frac{w}{\Omega}\right) + \nabla\cdot\left(\frac{\vec{I}}{\Omega}\right) = 0 \;(*) \qquad (67)$$

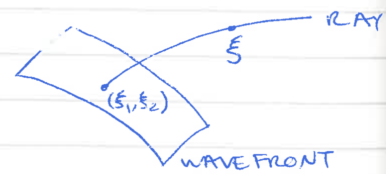
HOWEVER, BY OUR ASSUMPTION $\nabla\Omega(\vec{x})$ IS SMALL SO THAT EQ.(67) IS EQUIVALENT TO OUR EQ.(66).

(*) THIS IS EQ.(8-6-8) OF PIERCE, P403

WE MENTION HERE THAT $\boxed{\vec{u}' = \frac{p'}{\rho_0 c}\vec{n}}$ (68)
WHERE $\vec{n}$ IS THE NORMAL TO THE WAVEFRONT.

THE BLOKHINTZEV INVARIANT

$$p' = P(\vec{x})\, f[t - \tau(\vec{x}), \xi]$$



WHERE ξ IS A VARIABLE ALONG THE RAY.

$$w = \frac{1}{2}\rho_0 u'^2 + \frac{1}{2}\frac{p'^2}{\rho_0 c^2} = \frac{p'^2}{\rho_0 c^2} \qquad \text{(USING EQ.(68))}$$

$$= \frac{P^2 f^2}{\rho_0 c^2} \qquad (69)$$

$$\vec{I} = p'\vec{u}' + w\vec{u}_0 = \frac{p'^2}{\rho_0 c^2}(c\vec{n} + \vec{u}_0) = w\vec{V}_{ray} \qquad (70)$$

$$\partial_t w = \frac{2P^2}{\rho_0 c^2} f f' \qquad (71)$$

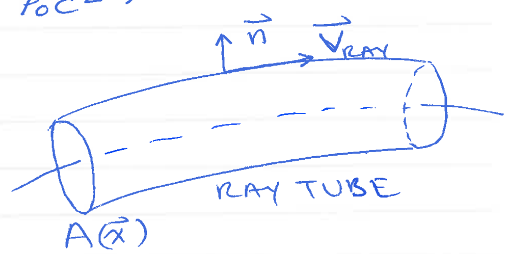
$$\nabla\cdot\vec{I} = \nabla\cdot\left(\frac{P^2\vec{V}_{ray}}{\rho_0 c^2} f^2\right) = f^2\nabla\cdot\left(\frac{P^2\vec{V}_{ray}}{\rho_0 c^2}\right) - \frac{2fP^2}{\rho_0 c^2}\vec{V}_{ray}\cdot\nabla f \qquad (72)$$

$$\nabla f = -f' \nabla \tau = -f' \vec{s} \tag{73}$$

$$\nabla f \cdot \vec{V}_{ray} = -f' \vec{s} \cdot (c\vec{n} + \vec{u}_0) = $$
$$= -f' \frac{\Omega}{c} \vec{n} \cdot (c\vec{n} + \vec{u}_0) = -f' \tag{74}$$

$$\therefore \quad \partial_t w + \nabla \cdot \vec{I} = f^2 \nabla \cdot \left(\frac{P^2 \vec{V}_{ray}}{\rho_0 c^2} \right) = 0 \tag{75}$$

$$\nabla \cdot \left(\frac{P^2 \vec{V}_{ray}}{\rho_0 c^2} \right) = 0 \tag{76}$$



OR APPLYING DIVERGENCE THM

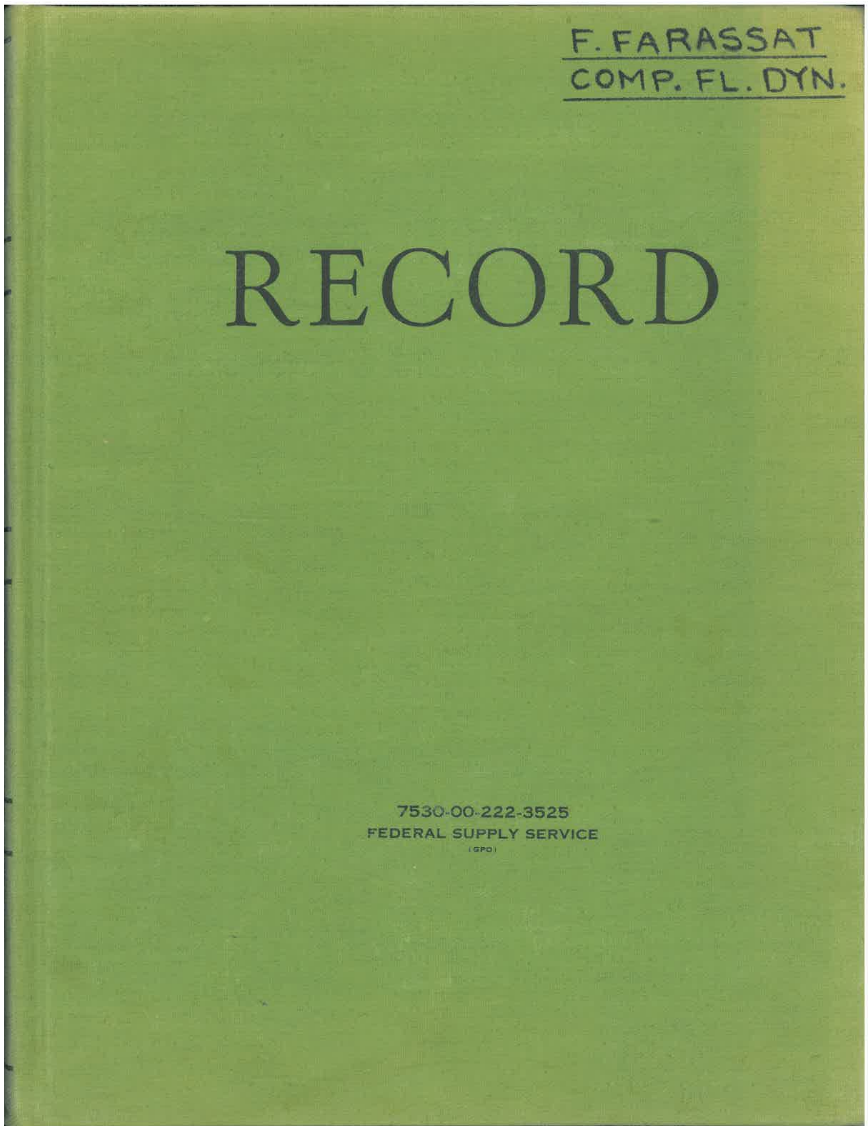
$$\frac{P^2 |\vec{V}_{ray}| A(\vec{x})}{\rho_0 c^2} = \text{CONST.} \tag{77}$$
(ALONG A RAY TUBE)

BLOKHINTZEV USED EQ. (67) AND SHOWED

$$\frac{P^2 |\vec{V}_{ray}| A(\vec{x})}{\Omega \rho_0 c^2} = \text{CONST.} \tag{78}$$
(ALONG A RAY TUBE)

THIS LAST QUANTITY IS CALLED THE <u>BLOKHINTZEV INVARIANT</u>. HERE $\Omega = 1/(1+M_n)$, $M_n = \vec{u}_0 \cdot \vec{n}/c$.
BECAUSE $|\nabla \Omega| \ll 1$, EQS. (77) AND (78) ARE CONSISTENT.

19 Computational Fluid Dynamics



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* NOTES ON NUMERICAL SOLUTION OF P.D.E.'S

IN THE SOLUTION OF A P.D.E. (WITH SOME GIVEN B.C. & I.C.), THE FOLLOWING FOUR STEPS MUST BE CONSIDERED:

i) CONSTRUCTION OF FINITE DIFFERENCE SYSTEM OF EQS.,

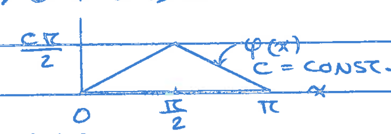
ii) PROVIDING METHODS OF SOLVING F.D. SYSTEM OF EQS.,

iii) STUDY OF STABILITY OF THE METHODS,

iv) STUDY OF PRECISION OF THE METHODS AND RATE OF CONVERGENCE.

THE FOLLOWING EXAMPLE ILLUSTRATES THESE STEPS. THE PROBLEM OF HEAT FLOW IN A BAR IS STUDIED ANALYTICALLY AND NUMERICALLY USING F.D. APPROACH

$$\begin{cases} u_t - \sigma u_{xx} = 0 & x \in [0,\pi],\ \sigma > 0,\ t > 0 \\ u(x,0) = \varphi(x) & x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0 & \forall\, t > 0 \end{cases}$$



THE ANALYTIC SOLUTION OF THIS PROBLEM IS

$$U(x,t) = \sum_{m=-\infty}^{\infty} A_m \exp(imx - m^2 \sigma t)$$

$$A_m = \begin{cases} 0 & m \text{ EVEN} \\ (-1)^{(m+1)/2}\, 2ic/(\pi m^2) & m \text{ ODD} \end{cases}$$

i) FINITE DIFFERENCE EQ. — $U_{i,j} = U(x_i, t_j)$. LET $[0,\pi]$ BE DIVIDED INTO N SEGMENTS WITH DIVIDING POINTS $x_1, x_2, x_3, \ldots x_{N-1}$ AND END POINTS x_0 AND x_N. LET $t \in [0,T]$ AND $\Delta t = T/M$, $t_j = j\Delta t$ $j = 0,1,2,\ldots,M$. THE P.D.E. IN F.D. FORM CAN BE WRITTEN AS

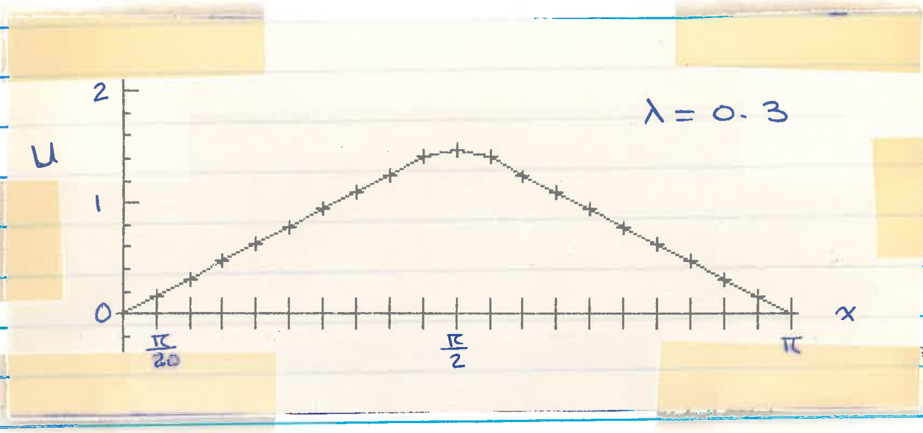
$$\frac{1}{\Delta t}\left[U_{i,j+1} - U_{i,j}\right] - \frac{\sigma}{(\Delta x)^2}\left[U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right] = 0$$

LET $\lambda = \sigma \Delta t/(\Delta x)^2 \Longrightarrow$

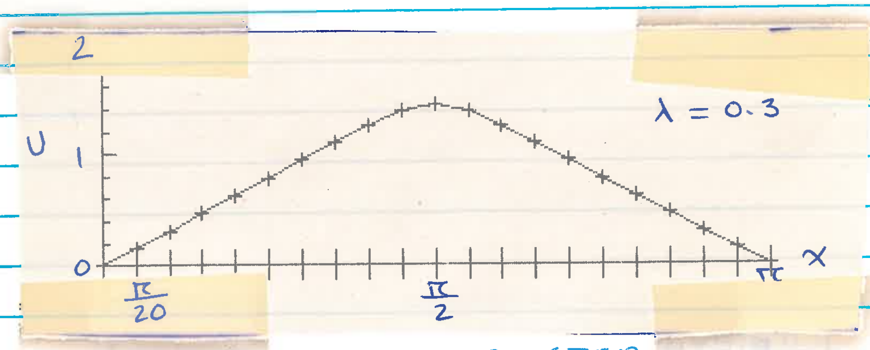
$$\begin{cases} U_{i,j+1} = \lambda U_{i-1,j} + (1-2\lambda) U_{i,j} + \lambda U_{i+1,j} \\ U_{0,j} = 0,\ U_{N,j} = 0 \quad \text{B.C.} \end{cases}$$

ii) THIS EQ. WAS PROGRAMMED ON HP SYSTEM 45 COMPUTER AS SHOWN BELOW : U = Temp

FOR $\lambda = 0.3$, FOR ALL STEPS IN TIME, THE RESULT IS A SMOOTH CURVE AS SHOWN.

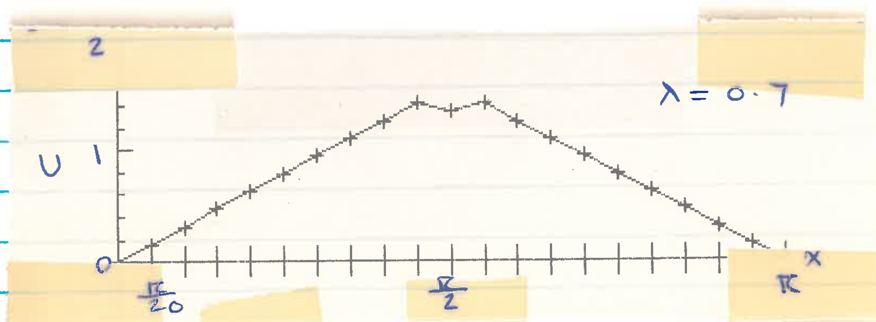


FIRST TIME STEP



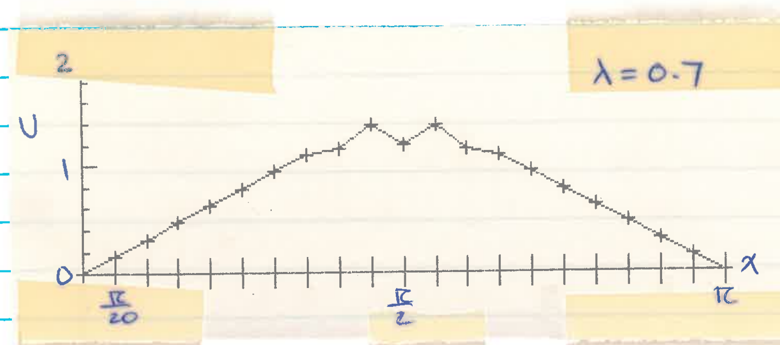
SECOND TIME STEP

FOR $\lambda = 0.7$, OSCILLATIONS APPEAR IN U.

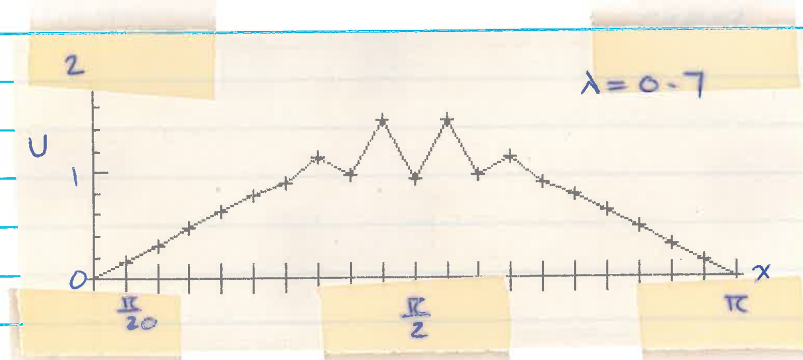


FIRST TIME STEP

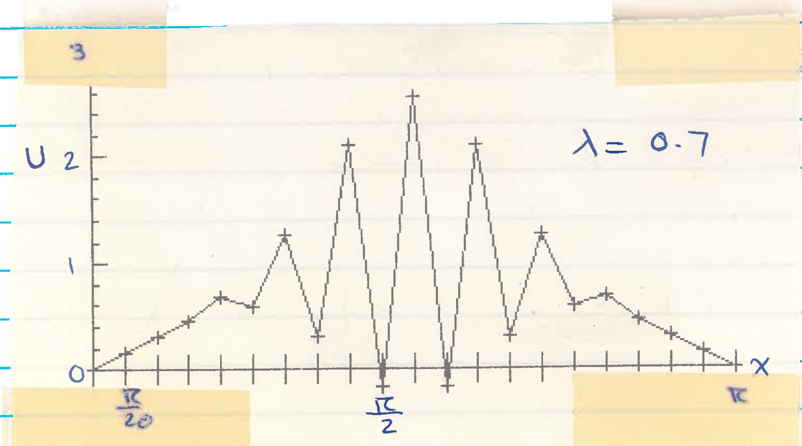
4 APR. 79



THIRD TIME STEP



FIFTH TIME STEP



EIGHTH TIME STEP

IT IS SEEN THAT THE SOLUTION TO F.D. EQ. CAN BE CONSTRUCTED BY MARCHING IN TIME. HOWEVER DEPENDING ON THE VALUE OF λ, THE F.D. SOLUTION SHOWS WAVINESS INDICATING INSTABILITY OF THE METHOD. THIS CAN BE STUDIED AS FOLLOWS. WE USE $U_{n,j}$ RESERVING THE SYMBOL i FOR $\sqrt{-1}$ BELOW.

iii) THE FD EQ. CAN BE SOLVED EXPLICITLY BY ASSUMING A SOLUTION OF THE TYPE $A_m \xi^j \exp\left(i\frac{m\pi n}{N}\right) = U_{n,j}$ WHERE $\xi = \xi(m)$. THE REASON FOR SELECTION OF THIS FN IS THAT THE FD. EQ. IS WITH CONSTANT COEFFICIENT WITH PERIODIC B.C. USING THIS SOLUTION, WE FIND

$$\xi^{j+1}\exp\left(\frac{m\pi i}{N}\right) = \lambda\xi^j \exp\left[\frac{m\pi(i-1)}{N}\right] + (1-2\lambda)\xi^j \exp\left[i\frac{m\pi n}{N}\right] + \lambda\xi^j \exp\left[i\frac{m\pi(i+1)}{N}\right]$$

$$\Rightarrow \quad \xi = 1 - 2\lambda\left(1-\cos\frac{m\pi}{N}\right)$$

$$\therefore U_{n,j} = \sum_{m=-\infty}^{\infty} A_m \xi^j \exp\left[\frac{i m\pi n}{N}\right]$$

WE NOTE THAT FOR $j=0$, USING THE I.C., WE FIND THE SAME SEQ. $\{A_m\}$ AS IN THE ANALYTIC SOLUTION ON P 1. NOTE THAT $\frac{\pi}{N} = \Delta x$.

WE SEE THAT THE TERM $e^{-\sigma m^2 t}$ HAS BEEN REPLACED BY $[\xi(m)]^j$. SINCE $t = j\Delta t$, WE SHOULD COMPARE $\xi(m)$ WITH $e^{-\sigma m^2 \Delta t}$. WE HAVE

$$\xi(m) = 1 - m^2\sigma\Delta t + \frac{1}{12}m^4\sigma\Delta t(\Delta x)^2 \ldots$$
$$e^{-m^2\sigma\Delta t} = 1 - m^2\sigma\Delta t + \frac{1}{2}m^4\sigma^2(\Delta t)^2 \ldots$$

FOR ANY HARMONIC, THESE TWO GROWTH FACTORS CAN

BE MADE TO AGREE BY TAKING Δx AND Δt SUFFICIENTLY SMALL. HOWEVER, IF $\max_m |\xi(m)|$ IS GREATER THAN 1, THEN LARGE HARMONICS WILL AMPLIFY AND CAUSE INSTABILITY. THE STABILITY CONDITION IS THEREFORE

$$\max_m |\xi(m)| \leq 1 .$$

WE HAVE, IN THIS CASE

$$|u_{m,j}| \leq \sum_{-\infty}^{\infty} |A_m| \, |\xi(m)|^j$$

$$\leq \sum_{-\infty}^{\infty} |A_m|$$

THE LAST SUM CONVERGES IF THE FOURIER SERIES FOR $\varphi(x)$ IS ABSOLUTELY CONVERGENT WHICH IS IN THIS EXAMPLE. WE NOTE THAT $\xi(m)$ IS REAL:

$$\xi(m) = 1 - 2\lambda\left[1 - \cos(m\Delta x)\right]$$

$$= 1 - 2\lambda + 2\lambda\cos(m\Delta x)$$

THE MAX. POSITIVE VALUE OF ξ IS 1, THE MINIMUM OF ξ IS $1-4\lambda$ WHEN $\cos(m\Delta x) = -1$. WE SHOULD HAVE $1-4\lambda \geq -1$ OR $\lambda \leq 1/2$

THIS IS THE STABILITY CONDITION FOR THE F.D. METHOD USED FOR THIS PROBLEM. THIS CONDITION CAN BE PROVED TO BE SUFFICIENT AS FOLLOWS:

$$|U_{m,j+1}| \leq (\lambda + (1-2\lambda) + \lambda) \max_n |U_{m,j}|$$

$$= \max_m |U_{m,j}|$$

$$\therefore \quad \max_m |U_{m,j+1}| \leq \max_m |U_{m,j}|$$

$$\leq \max_m |U_{m,j-1}|$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\leq \max_m |U_{m,0}|$$

$\Rightarrow$ THE SOLUTION IS BOUNDED.

iv) LET $\tilde{U}_{i,j} = u(i\Delta x, j\Delta t)$ WHERE $u(x,t)$ IS THE ANALYTIC SOLUTION OF THE PRESENT PROBLEM. THEN USING TAYLOR'S EXPANSION, WE CAN SHOW THAT

$$\frac{\tilde{U}_{i,j+1} - \tilde{U}_{i,j}}{\Delta t} - 6\,\frac{\tilde{U}_{i+1,j} - 2\tilde{U}_{i,j} + \tilde{U}_{i-1,j}}{(\Delta x)^2}$$
$$= O(\Delta t) + O[(\Delta x)^2] \text{ AS } \Delta t, \Delta x \to 0$$

LET $\varepsilon_{i,j} = u_{i,j} - \tilde{U}_{i,j} \Rightarrow$

$$\varepsilon_{i,j+1} = \lambda\,\varepsilon_{i-1,j} + (1-2\lambda)\,\varepsilon_{i,j} + \lambda\,\varepsilon_{i+1,j} + O(\Delta t)^2 + O[\Delta t(\Delta x)^2]$$

IF $\lambda \leq 1/2$

$$|\varepsilon_{i,j+1}| \leq \max_i |\varepsilon_{i,j}| + K_1(\Delta t)^2 + K_2\,\Delta t(\Delta x)^2$$

OR

$$\max_i |\varepsilon_{i,j+1}| \leq \max_i |\varepsilon_{i,j}| + K_1(\Delta t)^2 + K_2\,\Delta t(\Delta x)^2$$

$$\max_i |\varepsilon_{i,j+1}| - \max_i |\varepsilon_{i,j}| \leq K_1(\Delta t)^2 + K_2\,\Delta t\,(\Delta x)^2$$
$$\max_i |\varepsilon_{i,j}| - \max_i |\varepsilon_{i,j-1}| \leq K_1(\Delta t)^2 + K_2\,\Delta t\,(\Delta x)^2$$
$$\vdots$$
$$\max_i |\varepsilon_{i,1}| - \max_i |\varepsilon_{i,0}| \leq K_1(\Delta t)^2 + K_2\,\Delta t\,(\Delta x)^2$$

ADD BOTH SIDES AND USE THE FACT THAT $\max_i |\varepsilon_{i,0}| = 0$ TO GET

$$\max_i |\varepsilon_{i,j+1}| \leq (j+1)\,\Delta t\,[K_1\Delta t + K_2(\Delta x)^2]$$

OR REPLACE $j+1$ BY j

$$\max_i |\varepsilon_{i,j}| \leq t\,[K_1\Delta t + K_2(\Delta x)^2]$$

i.e. $|\varepsilon_{i,j}|$ IS BOUNDED BY t TIMES THE LOCAL

TRUNCATION ERROR. WE SEE THAT $|E_{i,j}| \to 0$ AS $\Delta x \to 0$ WITH λ KEPT FIXED.

TO STUDY THE RATE OF CONVERGENCE WE WRITE

$$|U_{i,j} - U| = \Big|\Big(\sum_{|m|\le m_0} + \sum_{|m|>m_0}\Big) A_m e^{imx}\Big[\xi^{t/\Delta t} - e^{-m^2 \delta t}\Big]\Big|$$

$$= |\Sigma^1 + \Sigma^2|$$

WE HAVE THE FOLLOWING INEQUALITY

$$|\Sigma^2| \le 2 \sum_{|m|>m_0} |A_m|$$

NOW WE USE THE FOLLOWING THM: $\phi(x) \in C^{p-1}$, $\frac{d^p\phi}{dx^p}$ OF BOUNDED VARIATION $\Rightarrow A_m = O\left(\frac{1}{m^{p+1}}\right)$ AS $p \to \infty$

$$\therefore \quad |\Sigma^2| \le 2 \sum_{|m|>m_0} |A_m| = O\left(\frac{1}{m_0^p}\right)$$

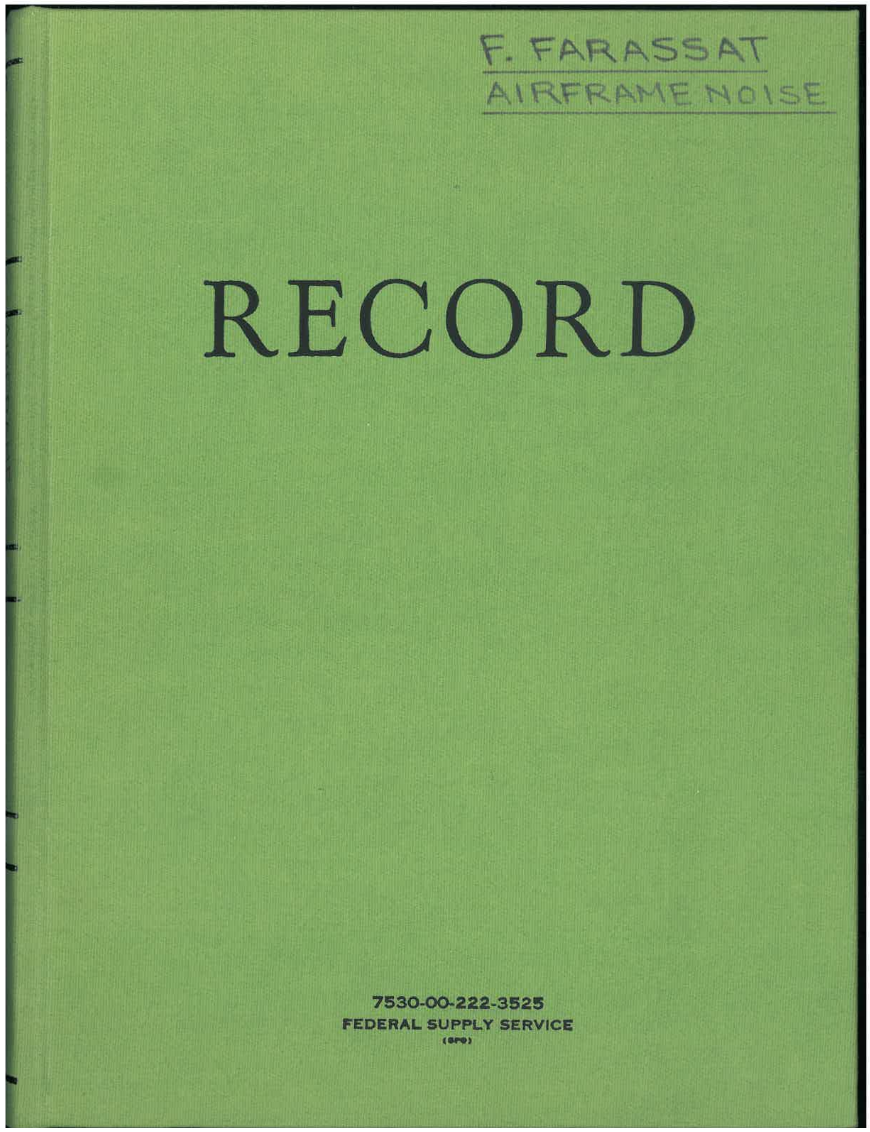
WE CAN ALSO SHOW THAT

$$|\Sigma^1| \le \begin{cases} C_1 m_0^{4-p} \Delta t & p < 3 \\ C_2 (\ln m_0) \Delta t & p = 4 \\ C_3 \Delta t & p > 4 \end{cases}$$

WHERE C_1, C_2, C_3 DEPEND ON x, t, p, δ BUT NOT ON Δt OR m. THESE CONSTANTS CAN BE CALCULATED. NOW m_0 IS CHOSEN IN SUCH A WAY AS TO MINIMIZE $|\Sigma^1| + |\Sigma^2|$. FOR $p \le 4$, THE MINIMIZING m_0 IS PROPORTIONAL TO $(\Delta t)^{-1/4}$. WE GET THE FINAL ESTIMATE THAT

$$|U_{i,j} - U| = \begin{cases} O[(\Delta t)^{p/4}] = O[(\Delta x)^{p/2}] & p < 3 \\ O[\Delta t |\ln \Delta t|] = O[(\Delta x)^2 \ln |\Delta x|] & p = 4 \\ O[\Delta t] = O[(\Delta x)^2] & p > 4 \end{cases}$$

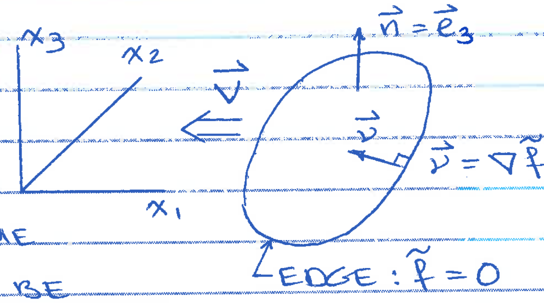
20 Airframe Noise



NEW FORMULATION FOR NOISE FROM UNSTEADY SURFACE PRESSURE – FORMULATION 1B

WE WILL FIRST CONSIDER A FLAT SURFACE MOVING IN $x_1 x_2$-PLANE. THE EDGE OF THE FLAT SURFACE IS GIVEN $\tilde{f}(x_1, x_2, t) = 0$ SUCH THAT $\tilde{f} > 0$ ON THE SURFACE.

ASSUME THAT THIS SURFACE IS MOVING IN THE PLANE $x_3 = 0$.



NOTE THAT THE $\vec{x}$-FRAME AND THE $\vec{y}$-FRAME TO BE USED SOON ARE FIXED TO THE MEDIUM AT REST. THE FUNCTION $\tilde{f}$ IS SO DEFINED THAT $\nabla \tilde{f} = \vec{\nu}$, THE UNIT INWARD POINTING GEODESIC NORMAL. THE GEODESIC NORMAL IS TANGENT TO THE SURFACE (i.e., IN OUR CASE LIES IN $x_1 x_2$-PLANE) AND IS NORMAL TO THE EDGE CURVE.

WE WANT TO FIND THE SOLUTION OF

$$\begin{aligned}
\Box^2 p' &= -\nabla \cdot [p \vec{n} H(\tilde{f}) \delta(x_3)] \\
&= -\nabla \cdot [p \vec{e}_3 H(\tilde{f}) \delta(x_3)] \\
&= -p(x_1, x_2, t) H(\tilde{f}) \delta'(x_3) \qquad (1)
\end{aligned}$$

WHERE $\vec{e}_3$ IS THE UNIT VECTOR ALONG x_3-AXIS AND $H(\tilde{f})$ IS THE HEAVISIDE FUNCTION.

SOLUTION

WE WILL SOLVE THIS PROBLEM IN A FRAME FIXED TO THE MEDIUM.

$$4\pi p'(\vec{x},t) = -\int \frac{\delta(g)}{r}\, p(y_1,y_2,\tau)\, H(f)\, \delta'(y_3)\, d\vec{y}\, d\tau \tag{2}$$

LET $\tau \rightarrow g$, INTEGRATE WRT g

$$4\pi p'(\vec{x},t) = -\int \frac{1}{r}\,[p]_{ret}\, H(\tilde{F})\, \delta'(y_3)\, d\vec{y} \tag{3}$$

WHERE $\tilde{F}(y_1,y_2;\vec{x},t) = \tilde{f}(y_1,y_2,t-r/c) = [\tilde{f}]_{ret}$

NEXT INTEGRATE WRT y_3:

$$4\pi p'(\vec{x},t) = \int \frac{\partial}{\partial y_3}\left\{\frac{[p]_{ret}}{r} H(\tilde{F})\right\}_{y_3=0} dy_1 dy_2 \tag{4}$$

$$\frac{\partial}{\partial y_3}\left\{\frac{[p]_{ret}}{r} H(\tilde{F})\right\} = \frac{1}{cr}\left[\hat{r}_3\, \dot{p}\right]_{ret} H(\tilde{F}) + \frac{1}{cr}\left[f_\tau\, \hat{r}_3\, p\right]_{ret} \delta(\tilde{F}) + \frac{[\hat{r}_3\, p]_{ret}}{r^2} H(\tilde{F}) \tag{5}$$

HERE $\hat{r}_3$ IS THE THIRD COMPONENT OF THE UNIT RADIATION VECTOR $\hat{r} = (\vec{x}-\vec{y})/r$:

$$\hat{r}_3 = \vec{n}\cdot\hat{r} = \vec{e}_3\cdot\hat{r} = \cos\theta \tag{6}$$

WHERE θ IS THE ANGLE BETWEEN $\hat{r}$ AND $\vec{e}_3$.

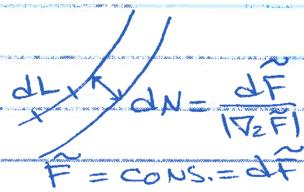
WE CAN SHOW THAT

$$\int Q\, H(\tilde{F})\, dy_1 dy_2 = \int_{\tilde{F}>0} Q\, d\Sigma = \int_{\tilde{f}>0} \frac{Q}{[|1-M_r|]_{ret}}\, dS \tag{7}$$

WHERE $d\Sigma$ IS THE ELEMENT OF THE SURFACE AREA OF THE ACOUSTIC PLANFORM OF $\tilde{f}>0$. ALSO

$M_r = \vec{M} \cdot \hat{\vec{r}}$, WHERE $\vec{M}$ IS THE LOCAL MACH NUMBER OF THE SURFACE. WE NEXT NEED TO FIND $I = \int q(y_1, y_2)\, \delta(\tilde{F})\, dy_1\, dy_2$

FROM THE FIGURE ON THE RIGHT, WE HAVE



$$dy_1\, dy_2 = dL\, dN$$
$$= \frac{dL\, d\tilde{F}}{|\nabla_2 \tilde{F}|} \qquad (8)$$

∇_2: SURFACE GRADIENT IN $y_1 y_2$-PLANE

$$\Rightarrow \quad I = \int q(y_1, y_2)\, \delta(\tilde{F}) \frac{dL\, d\tilde{F}}{|\nabla_2 \tilde{F}|}$$
$$= \int_{\tilde{F}=0} \frac{q(y_1, y_2)}{|\nabla_2 \tilde{F}|}\, dL \qquad (9)$$

$$\nabla_2 \tilde{F} = \nabla_2 \left[\tilde{f}(y_1, y_2, t - r/c) \right]$$
$$= \left(\tilde{f}_1 + \frac{\hat{r}_1}{c} \tilde{f}_\tau\, ,\, \tilde{f}_2 + \frac{\hat{r}_2}{c} \tilde{f}_c\, ,\, 0 \right) \qquad (10)$$

WE HAVE

$$\nu_i = \tilde{f}_i = \frac{\partial \tilde{f}}{\partial y_i}\, , \quad \tilde{f}_\tau = -v_\nu = -\vec{v} \cdot \vec{\nu} \qquad (11\text{-a,b})$$

WHERE $\vec{v}$ IS THE LOCAL VELOCITY OF THE SURFACE AND v_ν IS THE LOCAL VELOCITY IN THE DIRECTION OF THE GEODESIC NORMAL $\vec{\nu}$.

$\therefore$ WE HAVE FOUND

$$\nabla_2 \tilde{F} = (\nu_1 - M_\nu \hat{r}_1, \nu_2 - M_\nu \hat{r}_2, 0)$$
$$= \vec{\nu} - M_\nu \vec{\hat{r}}_t \qquad (12)$$

WHERE $M_\nu = \vec{M} \cdot \vec{\nu}$ AND $\vec{\hat{r}}_t = (\hat{r}_1, \hat{r}_2, 0)$.

NOTE THAT $\vec{\hat{r}}_t$ IS THE PROJECTION OF $\vec{\hat{r}}$ ON x_1x_2-PLANE. WE HAVE, THEREFORE,

$$\boxed{|\nabla_2 \tilde{F}|^2 = 1 + M_\nu^2 \sin^2\theta - 2 M_\nu \cos\theta'} \tag{13}$$

$$\sin^2\theta = \vec{\hat{r}}_t \cdot \vec{\hat{r}}_t = 1 - (\vec{\hat{r}} \cdot \vec{e}_3)^2 \tag{14-a}$$

$$\cos\theta' = \vec{\nu} \cdot \vec{\hat{r}}_t = \vec{\nu} \cdot \vec{\hat{r}} \tag{14-b}$$

WE HAVE SHOWN PREVIOUSLY, IN OTHER WORKS, THAT

$$\frac{dL}{|\nabla_2 \tilde{F}|} = \frac{d\ell}{[|1 - M_r|]_{ret}} \tag{15}$$

WHERE dL IS THE ELEMENT OF THE LENGTH OF THE ACOUSTIC PLANFORM AND $d\ell$ IS THE ELEMENT OF THE LENGTH OF THE EDGE OF THE ACTUAL SURFACE DESCRIBED BY $\tilde{f} = 0$.

WE HAVE THUS SHOWN THAT

$$I = \int_{\tilde{f}=0} \frac{q(y_1, y_2)}{[|1 - M_r|]_{ret}} \, d\ell \tag{16}$$

BEFORE WRITING THE SOLUTION, WE MUST INTERPRET THE MEANING OF $\dot{p}$. THE DERIVATION SHOWS THAT THIS IS THE RATE OF CHANGE OF THE PRESSURE MEASURED BY A TRANSDUCER FIXED TO THE $\vec{x}$-FRAME. LET US NOW DENOTE IT BY $\dot{p}_F$, F FOR FIXED TO $\vec{x}$-FRAME. IT IS MORE CONVINIENT TO WRITE THIS IN TERMS OF PRESSURE FLUCTIONS ON THE SURFACE ITSELF (i.e., THE TRANSDUCER MEASURING p IS FIXED TO THE SURFACE).

IF WE NOW USE $\dot{p}$ TO DESIGNATE THE RATE OF CHANGE OF p AS MEASURED BY THIS TRANSDUCER, THEN

$$\dot{p}_F = \dot{p} - v\frac{\partial p}{\partial s} \qquad (17)$$

WHERE $\partial p/\partial s$ IS DIRECTIONAL DERIVATIVE OF p IN THE DIRECTION OF LOCAL VELOCITY $\vec{v}$ WITH $v = |\vec{v}|$.

WE HAVE DERIVED THE <u>FORMULATION 1B</u>:

$$\boxed{\begin{aligned} 4\pi p'(\vec{x},t) = &\int_{\tilde{f}>0} \left[\frac{(\dot{p} - v\,\partial p/\partial s)\cos\theta}{c\,r\,|1-M_r|}\right]_{ret} dS \\ &+ \int_{\tilde{f}>0} \left[\frac{p\cos\theta}{r^2|1-M_r|}\right]_{ret} dS \\ &- \int_{\tilde{f}=0} \left[\frac{M_\nu\, p\cos\theta}{r|1-M_r|}\right]_{ret} dl \end{aligned}} \qquad (18)$$

NOTE THAT FOR AN INFINITELY THIN AIRFOIL, p IS IN FACT THE PRESSURE JUMP $-\Delta p$.(*) THIS MEANS THAT ON THE TE, THE KUTTA CONDITION REQUIRES THAT $\Delta p = 0$ AND THE LINE INTEGRAL VANISHES.

(*) $\Delta p = p_l - p_u$. TO GET THE RESULT INVOLVING Δp, REPLACE p IN EQ (18) BY $-\Delta p = p_u - p_l$ AND USE $\cos\theta = \vec{n}_u \cdot \hat{r}$. (10/3/02)

STATISTICAL FORMULATION

THE FOLLOWING ARE FROM MY PAPER WITH JAY CASPER THAT WILL BE PUBLISHED IN THE INTERNATIONAL JOURNAL OF AEROACOUSTICS. THE TITLE OF THE PAPER IS "A NEW TIME DOMAIN FORMULATION FOR BROADBAND NOISE PREDICTIONS"

5. Statistical Formulation

Often, when aeroacoustic experiments are performed, surface pressure correlations are extremely useful in characterizing noise due to an airflow over a model. Under certain conditions, Formulation 1B is readily transformed into an expression that aids in the statistical analysis of broadband noise. Such an expression for the autocorrelation of the far field acoustic pressure is developed in this section.

A far field autocorrelation formulation is derived for the case of a thin airfoil moving rectilinearly at constant velocity. The assumption of uniform velocity simplifies the algebra considerably, and the extension to general motion is straightforward. The following derivation follows the statistical analysis for jet noise presented by Morris and Farassat [28]. In the case of an observer in the acoustic far field, $r \gg \lambda$, the near field term (the second integral in Eq. (6)) can be ignored because of its $1/r^2$ proportionality. Also, for notational convenience, the spatial and temporal derivatives of pressure in the first integral are written as a single time derivative of pressure, $\dot{p}$, when evaluated in a reference frame that is fixed relative to the medium at rest, as in Eq. (5b). In this fixed reference frame, the far field acoustic pressure can be written

$$4\pi p'(\vec{x},t) = \int_{\tilde{f}>0} \left[\frac{\dot{p}\cos\theta}{c_0\, r\,(1-M_r)}\right]_{\text{ret}} dS - \int_{\tilde{f}=0} \left[\frac{M_\nu\, p\cos\theta}{r\,(1-M_r)}\right]_{\text{ret}} d\ell \qquad (22)$$

Eq. (22) is now transformed to the reference frame fixed to the airfoil, i.e. the wind tunnel reference frame. In this frame of reference, the emmision distance r_{ret} has the following form:

$$r_{\text{ret}} = \frac{1}{\beta^2}[M(x_1 - y_1) + \tilde{R}] \qquad (23a)$$

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where

$$\tilde{R} = [((x_1 - y_1)^2 + \beta^2 (x_2 - y_2)^2 + \beta^2 x_3^2]^{1/2} \tag{23b}$$

Furthermore,

$$[r\,(1 - M_r)]_{\rm ret} = \tilde{R} \tag{23c}$$

and

$$[\cos\theta]_{\rm ret} = \frac{x_3}{r_{\rm ret}} = \frac{\beta^2 x_3}{M(x_1 - y_1) + \tilde{R}} \tag{23d}$$

Therefore, in the wind tunnel reference frame, a time independent function $\mathcal{K}(\vec{x},\vec{y})$ is defined by

$$\left[\frac{\cos\theta}{r\,(1 - M_r)}\right]_{\rm ret} = \frac{\beta^2 x_3}{\tilde{R}[M(x_1 - y_1) + \tilde{R}]} \equiv \mathcal{K}(\vec{x},\vec{y}) \tag{24}$$

Eq. (24) is an important result because it shows that, in the laboratory reference frame, the only time dependent quantities in Eq. (22) are p and $\dot{p}$. Thus, Eq. (22) can be written in a form that shows the dependence of $p'(\vec{x},t)$ on $\vec{x}$, $\vec{y}$, and t explicitly:

$$4\,\pi\, p'(\vec{x},t) = \frac{1}{c_0}\int_{\bar{f}>0} \mathcal{K}(\vec{x},\vec{y})\,\dot{p}(\vec{y},t - r_{\rm ret}/c_0)\,d\vec{y}_S - \int_{\bar{f}=0} M_\nu(\vec{y})\,\mathcal{K}(\vec{x},\vec{y})\,p(\vec{y},t - r_{\rm ret}/c_0)\,d\ell \tag{25}$$

where $d\vec{y}_S$ denotes an integral surface element $dy_1 dy_2$. Using the notation $\langle\cdot\,,\cdot\rangle$ to represent an ensemble average, as in Eq. (19g), assuming ergodicity, the autocorrelation of the acoustic pressure is evaluated as follows:

$$16\,\pi^2\,\langle\, p'(\vec{x},t)\,,p'(\vec{x},t+\bar{\tau})\,\rangle =$$

$$\begin{aligned}
&\frac{1}{c_0^2}\int_{\bar{f}>0}\int_{\bar{f}>0} \mathcal{K}\,\mathcal{K}'\,\langle\,\dot{p}(\vec{y},t - r_{\rm ret}/c_0)\,,\dot{p}(\vec{y}',t+\bar{\tau} - r'_{\rm ret}/c_0)\,\rangle\,d\vec{y}_S\,d\vec{y}'_S \\
-\;&\frac{1}{c_0}\int_{\bar{f}>0}\int_{\bar{f}=0} M'_\nu\,\mathcal{K}\,\mathcal{K}'\,\langle\,\dot{p}(\vec{y},t - r_{\rm ret}/c_0)\,,p(\vec{y}',t+\bar{\tau} - r'_{\rm ret}/c_0)\,\rangle\,d\vec{y}_S\,d\ell' \\
-\;&\frac{1}{c_0}\int_{\bar{f}=0}\int_{\bar{f}>0} M_\nu\,\mathcal{K}\,\mathcal{K}'\,\langle\,p(\vec{y},t - r_{\rm ret}/c_0)\,,\dot{p}(\vec{y}',t+\bar{\tau} - r'_{\rm ret}/c_0)\,\rangle\,d\vec{y}'_S\,d\ell \\
+\;&\int_{\bar{f}=0}\int_{\bar{f}=0} M_\nu\,M'_\nu\,\mathcal{K}\,\mathcal{K}'\,\langle\,p(\vec{y},t - r_{\rm ret}/c_0)\,,p(\vec{y}',t+\bar{\tau} - r'_{\rm ret}/c_0)\,\rangle\,d\ell\,d\ell'
\end{aligned} \tag{26}$$

All primed quantities on the right-hand side of Eq. (26) indicate replacement of the variable $\vec{y}$ with $\vec{y}'$, e.g., $\mathcal{K}' = \mathcal{K}(\vec{x}, \vec{y}')$.

Now, let $\vec{y}$ and $\vec{y}'$ be related by $\vec{y}' = \vec{y} + \vec{\eta}$, where $\vec{\eta} = [\eta_1, \eta_2, 0]^T$ is a vector in the plane of the airfoil. Furthermore, it is assumed that $|\vec{\eta}| \ll r_{\text{ret}}$, in which case r'_{ret} can be approximated by

$$r'_{\text{ret}} \approx r_{\text{ret}} + \vec{\eta} \cdot \nabla_y r_{\text{ret}} \tag{27a}$$

where

$$\nabla_y r_{\text{ret}} = -\frac{1}{\beta^2} \left[M + \frac{x_1 - y_1}{\tilde{R}} \,,\, \beta^2 \frac{x_2 - y_2}{\tilde{R}} \,,\, 0 \right]^T \tag{27b}$$

Two autocorrelation functions, $F_{\dot{p}\dot{p}}$ and F_{pp}, and a cross-correlation function $F_{p\dot{p}}$ are therefore defined by

$$\begin{aligned} F_{\dot{p}\dot{p}} &= \langle \dot{p}(\vec{y}, \tau), \dot{p}(\vec{y} + \vec{\eta}, \tau + \bar{\tau}) \rangle \\ F_{pp} &= \langle p(\vec{y}, \tau), p(\vec{y} + \vec{\eta}, \tau + \bar{\tau}) \rangle \\ F_{p\dot{p}} &= \langle p(\vec{y}, \tau), \dot{p}(\vec{y} + \vec{\eta}, \tau + \bar{\tau}) \rangle \end{aligned} \tag{28}$$

Using Eqs. (27) and (28), the ensemble-averaged quantities on the right-hand side of Eq. (26) have the following meaning:

$$\begin{aligned} \langle \dot{p}(\vec{y}, t - r_{\text{ret}}/c_0) \,, \dot{p}(\vec{y}', t + \bar{\tau} - r_{\text{ret}}/c_0 - \vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0) \rangle &= F_{\dot{p}\dot{p}}(\vec{y};\, \vec{\eta}, \bar{\tau} - \vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0) \\ \langle p(\vec{y}, t - r_{\text{ret}}/c_0) \,, p(\vec{y}', t + \bar{\tau} - r'_{\text{ret}}/c_0 - \vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0) \rangle &= F_{pp}(\vec{y};\, \vec{\eta}, \bar{\tau} - \vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0) \\ \langle p(\vec{y}, t - r_{\text{ret}}/c_0) \,, \dot{p}(\vec{y}', t + \bar{\tau} - r'_{\text{ret}}/c_0 - \vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0) \rangle &= F_{p\dot{p}}(\vec{y};\, \vec{\eta}, \bar{\tau} - \vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0) \\ \langle \dot{p}(\vec{y}, t - r_{\text{ret}}/c_0) \,, p(\vec{y}', t + \bar{\tau} - r'_{\text{ret}}/c_0 - \vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0) \rangle &= F_{p\dot{p}}(\vec{y};\, -\vec{\eta}, -\bar{\tau} + \vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0) \end{aligned} \tag{29}$$

Eq. (26) can thus be rewritten using the above correlation functions explicitly:

9

$$16\,\pi^2\,\langle\, p'(\vec{x},t)\,,p'(\vec{x},t+\bar{\tau})\,\rangle =$$

$$\begin{aligned}
&\frac{1}{c_0^2}\int_{\tilde{f}>0} d\vec{y}_S\;\mathcal{K}(\vec{x},\vec{y})\int_{A_c}\mathcal{K}(\vec{x},\vec{y}+\vec{\eta})\,F_{\dot{p}\dot{p}}(\vec{y};\,\vec{\eta},\bar{\tau}-\vec{\eta}\cdot\nabla_y r_{\text{ret}}/c_0)\,d\vec{\eta} \\
-&\frac{1}{c_0}\int_{\tilde{f}>0} d\vec{y}_S\;\mathcal{K}(\vec{x},\vec{y})\int_{L_c} M_\nu(\vec{y}+\vec{\eta})\,\mathcal{K}(\vec{x},\vec{y}+\vec{\eta})\,F_{p\dot{p}}(\vec{y};\,-\vec{\eta},-\bar{\tau}+\vec{\eta}\cdot\nabla_y r_{\text{ret}}/c_0)\,d\ell \\
-&\frac{1}{c_0}\int_{\tilde{f}=0} d\ell\;M_\nu(\vec{y})\,\mathcal{K}(\vec{x},\vec{y})\int_{A_c}\mathcal{K}(\vec{x},\vec{y}+\vec{\eta})\,F_{p\dot{p}}(\vec{y};\,\vec{\eta},\bar{\tau}-\vec{\eta}\cdot\nabla_y r_{\text{ret}}/c_0)\,d\vec{\eta} \\
+&\int_{\tilde{f}=0} d\ell\;M_\nu(\vec{y})\,\mathcal{K}(\vec{x},\vec{y})\int_{\tilde{f}=0} M_\nu(\vec{y}+\vec{\eta})\,\mathcal{K}(\vec{x},\vec{y}+\vec{\eta})\,F_{pp}(\vec{y};\,\vec{\eta},\bar{\tau}-\vec{\eta}\cdot\nabla_y r_{\text{ret}}/c_0)\,d\ell
\end{aligned} \tag{30}$$

Here, A_c and L_c are correlation area and correlation length, respectively. Clearly, performing the integration in the four inner integrals on the right-hand side of Eq. (30) will result in four functions of $\vec{x}$, $\vec{y}$, and $\bar{\tau}$. Calling these function F_1 to F_4 in order of their appearance, Eq. (30) can be rewritten

$$\begin{aligned}
16\,\pi^2\,\langle\, p'(\vec{x},t)\,,p'(\vec{x},t+\bar{\tau})\,\rangle \;=\; &\frac{1}{c_0^2}\int_{\tilde{f}>0}\mathcal{K}(\vec{x},\vec{y})\,[\,F_1(\vec{x},\vec{y},\bar{\tau})-c_0F_2(\vec{x},\vec{y},\bar{\tau})\,]\,d\vec{y}_S \\
&-\frac{1}{c_0}\int_{\tilde{f}=0} M_\nu(\vec{y})\,\mathcal{K}(\vec{x},\vec{y})\,[\,F_3(\vec{x},\vec{y},\bar{\tau})-c_0F_4(\vec{x},\vec{y},\bar{\tau})\,]\,d\ell
\end{aligned} \tag{31}$$

The usefulness of the analysis presented above depends on the knowledge of two autocorrelation functions and one cross-correlation function, all of which are expected to be obtained from experimental measurement. If the surface pressure fluctuations are assumed to be stationary in time, then only the correlation function $F_{pp}(\vec{y};\,\vec{\eta},\bar{\tau})$ needs to be measured, as the other two correlation functions can be derived from it [29]. It is expected that, of the three correlation functions, $F_{\dot{p}\dot{p}}(\vec{y};\,\vec{\eta},\bar{\tau})$ will dominate the acoustic pressure, and therefore, $F_1(\vec{y};\,\vec{\eta},\bar{\tau})$ will be the only significant contributor to the right-hand side of Eq. (31). Eq. (30) indicates that, at a point $\vec{y}$ on the airfoil, only the turbulent pressure fluctuations over a correlation area near that point will contribute to the sound at the observer location $\vec{x}$. Furthermore, the time dependence of the correlation functions is

displaced by $\vec{\eta} \cdot \nabla_y r_{\text{ret}}/c_0$ because of the retarded time effect. This displacement is dependent upon airfoil Mach number and the radiation direction $\vec{x} - \vec{y}$ (the visual direction). Perhaps as important as its direct application to the prediction of far field noise from surface pressure measurements, Eq. (31) can be used for qualitative analysis of broadband and trailing edge noise, which is the subject of ongoing research.

FULL SURFACE STATISTICAL FORMULATION

FOR A LONG TIME, I THOUGHT THAT THE DERIVATION OF A FULL SURFACE FORMULATION FOR FINDING THE ACOUSTIC PRESSURE p' BY A METHOD SIMILAR TO THAT I USED FOR FORMULATION 1B WAS TRIVIAL. BUT WHEN I FINALLY SAT DOWN TO DERIVE IT, I WAS SHOCKED THAT IT WASN'T! AFTER WORKING A FEW DAYS ON IT, I FOUND THAT FORMULATION 1A WAS STILL THE SIMPLEST FORMULATION. I MUST SAY THAT FORMULATION 1B IS BY FAR THE SIMPLEST RESULT FOR A SUPERSONIC FLAT SURFACE ALTHOUGH THIS SITUATION DOES NOT APPLY TO PROPFANS, STILL THIS DISCOVERY MAKES ME HAPPY BECAUSE SINGULARITY FREE SOLUTIONS OF THIS SIMPLICITY ARE RARE.

WE WILL CONSIDER THE OBSERVER IN THE FAR FIELD. THE ACOUSTIC PRESSURE IS GIVEN BY

$$4\pi p'(\vec{x}, t) = \frac{1}{c} \int_{f=0} \left[\frac{\dot{p} \cos\theta}{r(1-M_r)^2} \right]_{ret} dS \qquad (1)$$

IN THE SITUATION THAT WE ARE INTERESTED, $\dot{M}_i = 0$ AND WE HAVE NEGLECTED HERE THE FAR FIELD INTEGRAL INVOLVING THIS TERM. HERE $\dot{p}$ IS THE RATE OF PRESSURE FLUCTUATIONS AS MEASURED BY A TRANSDUCER FIXED TO THE AIRFOIL. DEFINE

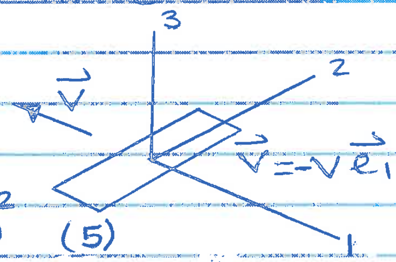
$$K_1(\vec{x},\vec{y}) = \left[\frac{\cos\theta}{r(1-M_r)^2}\right]_{ret}$$

$$= \left[\frac{r\cos\theta}{(r(1-M_r))^2}\right]_{ret} \qquad (2)$$

NOTE THAT $\vec{x}$- AND $\vec{y}$-FRAME ARE FIXED TO THE AIRFOIL SURFACE. WE HAVE

$$[r\cos\theta]_{ret} = x_3 \qquad (3)$$

$$[r(1-M_r)]_{ret} = \tilde{R} \qquad (4)$$



$$\tilde{R}^2 = (x_1-y_1)^2+\beta^2(x_2-y_2)^2+\beta^2(x_3-y_3)^2 \qquad (5)$$

$$\Rightarrow K_1(\vec{x},\vec{y}) = \frac{x_3}{\tilde{R}^2} \qquad (6)$$

WE NOTE THAT $y_3 \neq 0$ IN $\tilde{R}^2$ FOR THIS FULL SURFACE PROBLEM. FROM RESULTS IN OUR IJA PAPER (PAGES 6-10), WE HAVE

$$16\pi^2\langle p(\vec{x},t)p'(\vec{x},t+\tau)\rangle = \frac{1}{c^2}\int_{f>0} dS\, K_1(\vec{x},\vec{y})$$

$$\int_{A_c} K_1(\vec{x},\vec{y}+\vec{\eta})F_{\dot{p}\dot{p}}(\vec{y};\vec{\eta},\tau-\vec{\eta}\cdot\nabla r_{ret}/c)\,d\vec{\eta} \qquad (7)$$

A_c: CORRELATION AREA

WE NOW DERIVE ∇r_{ret}. FROM

$$r_{ret} = \frac{1}{\beta^2}\left[M(x_1-y_1)+\tilde{R}\right] \qquad (8)$$

$$\Rightarrow \quad \nabla r_{ret} = \frac{1}{\beta^2}\left[-M\vec{e}_1 + \nabla\tilde{R}\right] \qquad (9)$$

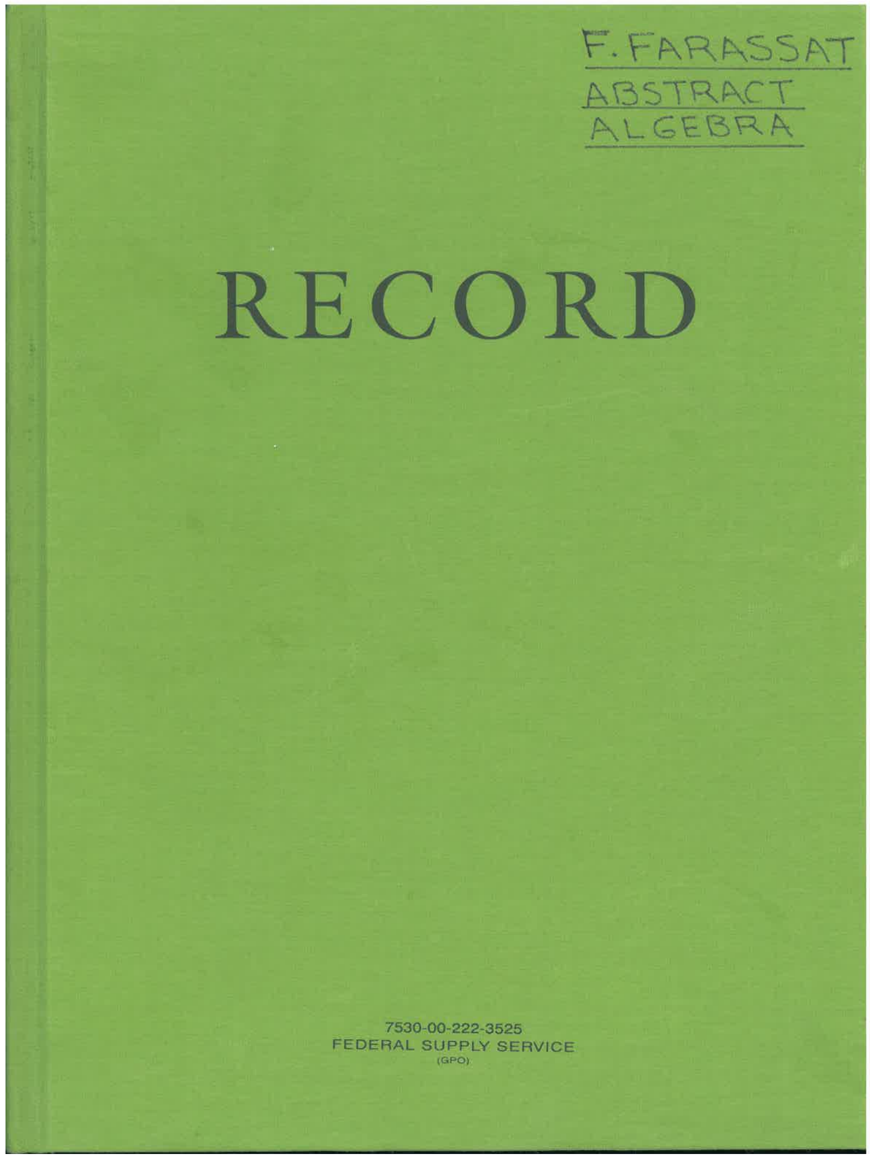
$$\beta^2 = 1-M^2, \quad M = V/c$$

$$\nabla\tilde{R} = -\left(\frac{x_1-y_1}{\tilde{R}}, \frac{\beta^2(x_2-y_2)}{\tilde{R}}, \frac{\beta^2(x_3-y_3)}{\tilde{R}}\right) \quad (10)$$

$$\Rightarrow \quad \nabla r_{ret} = -\left(\frac{1}{\beta^2}\left(M+\frac{x_1-y_1}{\tilde{R}}\right), \frac{x_2-y_2}{\tilde{R}}, \frac{x_3-y_3}{\tilde{R}}\right) \quad (11)$$

NOTE THAT $\nabla \equiv \partial/\partial y_i$

21 Abstract Algebra



SEP. 04 1

ABSTRACT ALGEBRA, LIKE GEOMETRY, HAS IMPORTANT APPLICATIONS IN WIDE RANGING SUBJECTS IN SCIENCE AND ENGINEERING. IT IS USED IN OTHER AREAS OF MATHEMATICS AND HELPS IN DEEPER UNDERSTANDING OF MANY SUBJECTS IN MATHMATICS. I HAVE TRIED TO LEARN ABS. ALG. BY MYSELF FOR INTELLECTUAL STIMULATION. THE REASONING IN PROOFS OF ABS. ALG. IS OFTEN SUBTLE AND BASED ON LONG CHAIN OF RESULTS. LEARNING ABS. ALG., THEREFORE, IS CHALLENGING AND EXCITING. IN THIS NOTEBOOK, I WILL WRITE ISOLATED RESULTS THAT I LEARN. MY MAIN AIM IS LEARNING ALGEBRAIC STRUCTURES, GROUP THEORY AND GALOIS THEORY. I HAVE MANY BOOKS, ARTICLES AND LECTURE NOTES. I FOUND THE INTERNET TO BE VERY USEFUL IN FINDING ARTICLES AND LECTURE NOTES. IN FACT, I REALLY HAVE TOO MUCH MATERIAL TO READ AND HAVE TO BE SELECTIVE. I WILL MENTION REFERENCES I HAVE USED FOR ANYTHING I WRITE HERE. ALTHOUGH I APPRECIATE AND USE ABSTRACTION, I AVOID EXTREME ABSTRACTION OF THE BOURBAKI GROUP. I CAN SAY A LOT HERE BUT I DON'T HAVE TO! VLADIMIR ARNOLD HAS ALREADY SEVERLY CRITICIZED THE SUPER ABSTRACTION OF THE FRENCH. FOR MYSELF, I NEED A CONNECTION OF WHATEVER I LEARN IN MATH TO CONCRETE EXAMPLES. MY PURPOSE HERE IS WORKING OUT A LOT OF EXAMPLES.

2 SEP 04

EUCLID'S ALGORITHM

THIS ALGORITHM IS USED TO FIND GCD OF TWO INTEGERS. IT HAS IMPORTANT IMPLICATIONS IN OTHER AREAS SUCH AS THEORY OF EQS. AND GENERATION OF NEW FIELDS (RESIDUE FIELDS). WE START WITH AN EXAMPLE: FIND THE GCD OF 126 AND 36. HERE IS EUCLID'S ALGORITHM

$$126 = 3 \times 36 + 18$$

$$36 = 2 \times 18 \Rightarrow 18 \text{ IS THE GCD OF } 126 \,\&\, 36$$

NO REMAINDER

FOR 623 AND 119, WE HAVE

$$623 = 5 \times 119 + 28$$

$$119 = 4 \times 28 + 7$$

$$28 = 4 \times 7 \Rightarrow 7 \text{ IS THE GCD OF } 623 \,\&\, 119$$

— HERE IS A VERY IMPORTANT RESULT:

THM: GIVEN TWO INTEGERS a AND b WITH C AS THEIR GCD $\Rightarrow \exists$ INTEGERS A AND B $\ni$

$$Aa + Bb = C \qquad (a \,\&\, b \text{ NOT BOTH ZERO!}) \quad (1)$$

WE WILL GIVE AN ELEGANT PROOF OF THIS BELOW BUT WE FIRST GIVE A FEW FACTS.

1 – THE NUMBERS A AND B ARE NOT UNIQUE.

PROOF: REPLACE A BY $A' = A + \alpha b$, $B' = B - \alpha a$

$$\Rightarrow A'a + B'b = (A + \alpha b)a + (B - \alpha a)b$$

$$= Aa + Bb = C$$

2 – THE RESULT IN EQ. (1) IS USED IN MANY PROOFS IN ABS. ALGEBRA. IT IS NOT VERY INTUITIVE AND TO GIVE A PROOF WITHOUT THE KNOWLEDGE OF EUCLID'S

ALGORITHM IS NOT EASY. OUR SECOND PROOF FROM HUNGERFORD IS INGENIOUS AND PURE ABS. ALGEBRA!

FIRST PROOF: FROM JEAN-PIERRE TIGNOL "GALOIS' THEORY OF ALGEBRAIC EQUATIONS", 2001, 45-47:
WE ASSUME THE FOLLOWING LISTING FOR FINDING THE GCD OF INTEGERS P_1 AND P_2 WHICH IS P_n

$$\begin{cases} P_1 = Q_1 P_2 + P_3 \\ P_2 = Q_2 P_3 + P_4 \\ \vdots \\ P_{n-1} = Q_{n-1} P_n \end{cases}$$

(MY NOTATION)

USING MATRIX NOTATION, WE CAN WRITE THE FIRST LINE AS

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} Q_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \equiv \mathbb{A}_1 \begin{bmatrix} P_2 \\ P_3 \end{bmatrix} \qquad (2)$$

SIMILARLY, THE iTH LINE CAN BE WRITTEN AS

$$\begin{bmatrix} P_i \\ P_{i+1} \end{bmatrix} = \begin{bmatrix} Q_i & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_{i+1} \\ P_{i+2} \end{bmatrix} \equiv \mathbb{A}_i \begin{bmatrix} P_{i+1} \\ P_{i+2} \end{bmatrix}$$

AND THE LAST LINE IS

$$\begin{bmatrix} P_{n-1} \\ P_n \end{bmatrix} = \begin{bmatrix} Q_{n-1} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_n \\ 0 \end{bmatrix} = \mathbb{A}_{n-1} \begin{bmatrix} P_n \\ 0 \end{bmatrix}$$

IN (2), LET US REPLACE $\begin{bmatrix} P_2 \\ P_3 \end{bmatrix}$ BY $\mathbb{A}_2 \begin{bmatrix} P_3 \\ P_4 \end{bmatrix}$ AND $\begin{bmatrix} P_3 \\ P_4 \end{bmatrix}$ BY $\mathbb{A}_3 \begin{bmatrix} P_4 \\ P_5 \end{bmatrix}$, ETC., WE GET

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \mathbb{A}_1 \mathbb{A}_2 \cdots \mathbb{A}_{n-1} \begin{bmatrix} P_n \\ 0 \end{bmatrix} \equiv \mathbb{A} \begin{bmatrix} P_n \\ 0 \end{bmatrix}$$

$$\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbb{A}^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

ASSUMING FOR THE MOMENT THAT $\mathbb{A}$ IS NOT

SINGULAR $\Rightarrow$

$$\begin{bmatrix} P_n \\ 0 \end{bmatrix} = \mathbb{A}^{-1} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

THE FIRST ROW GIVES

$$P_n = \alpha_{11} P_1 + \alpha_{12} P_2 \qquad (3)$$

WHICH IS WHAT WE INTENDED TO PROVE! THE SECOND ROW GIVES

$$\alpha_{21} P_1 + \alpha_{22} P_2 = 0 \qquad (4)$$

WHICH IS A TRIVIAL RESULT BECAUSE TAKING $\alpha_{21} = \alpha P_2$, $\alpha_{22} = -\alpha P_1$, α A CONST. INTEGER $\Rightarrow$ THAT THE ABOVE RESULT HOLDS.

WE NOW SHOW THAT $\mathbb{A}$ IS NOT SINGULAR. WE MUST SHOW THAT $\mathbb{A}_i$, $i=1,2,\ldots,n-1$, ARE NOT SINGULAR. BUT THIS IS SIMPLE!

$$\det \mathbb{A}_i = -1 \neq 0$$

$$\mathbb{A}_i^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -Q_i \end{bmatrix}$$

THE ABOVE METHOD GIVES α_{11} AND α_{12} UNIQUELY BUT WE KNOW THAT THESE INTEGERS ARE NOT UNIQUE. IF WE ARE GIVEN P_1, P_2 AND P_n, THE GCD OF P_1 & P_2, FINDING α_{11} AND α_{12} IS EQUIVALENT TO SOLVING THE DIOPHANTINE EQ. (3). BUT EQ. (4) TELLS US THAT THE HOMOGENEOUS EQ. $\alpha P_1 + \beta P_2 = 0$ HAS NONTRIVIAL SOLUTION. WHY DON'T WE SEE NONUNIQUENESS OF α_{11} AND α_{12} IN THIS METHOD? IS THERE A SIGNIFICANCE TO α_{11} AND α_{12} OBTAINED BY THIS METHOD?

I WILL THINK ABOUT THIS FURTHER.

SECOND PROOF: FROM THOMAS W. HUNGERFORD "ABSTRACT ALGEBRA – AN INTRODUCTION", 1990, THM 1.3, P7
ASSUME a & b, NOT BOTH ZERO, HAVE GCD c. WE WANT TO PROVE THAT $\exists$ INTEGERS A AND B $\ni$

$c = Aa + Bb$ (MY NOTATION)

PROOF DETAILS: CONSIDER THE SET OF INTEGERS $S = \{ua + vb \mid u \,\&\, v \in \mathbb{Z}\}$. THE INTEGERS IN THIS SET TAKE POSITIVE & NEGATIVE VALUES. IN PARTICULAR FOR $u = a$, $v = b$, WE HAVE $ua + vb = a^2 + b^2 > 0$. $\Rightarrow \exists\, u = A, v = B \ni t = Aa + Bb$ IS THE SMALLEST POSITIVE INTEGER IN S. WE WILL SHOW THAT t IS THE GCD OF a AND b AND THUS $t = c$. WE SHOW THAT IN TWO STEPS

i) $t \mid a$ AND $t \mid b$, i.e., t IS A DIVISOR OF a & b

ii) IF $d \mid a$ AND $d \mid b \Rightarrow d \mid t \Rightarrow t$ IS THE GCD OF a AND b

PROOF OF i: USING DIVISION ALGORITHM FOR a & t, WE HAVE

$$a = qt + r \quad , 0 \leqslant r < t$$

$$0 \leqslant r = a - qt = a - q(Aa + Bb)$$

$$= \underbrace{(1 - qA)}_{u} a + \underbrace{(-qB)}_{v} b$$

$\therefore$ WE HAVE FOUND A NEW MINIMUM POS. INTEGER IN S IF $r > 0$! THIS IS IMPOSSIBLE BY CONSTRUCTION.
$\Rightarrow r = 0$, i.e. $t \mid a$. SIMILARLY $t \mid b$.

PROOF OF ii: LET $d|a$ AND $d|b$, i.e., LET d BE ANOTHER COMMON DIVISOR OF a & b. $\Rightarrow$

$$t = Aa + Bb \qquad a = \alpha d,\ b = \beta d$$
$$= A\alpha d + B\beta d$$
$$= d(A\alpha + B\beta) \Rightarrow d|t \Rightarrow d \leq |t|$$

BUT $t > 0 \Rightarrow d \leq t$, i.e. t IS THE GCD OF a AND B, i.e., $t = c$.

REMARKS – 1. THIS PROOF IS NONCONSTRUCTIVE. IT IS A PURE EXISTENCE PROOF DISAVOWED BY CONSTRUCTIVISTS. HOW CAN WE FIND THE SMALLEST POSITIVE INTEGER IN S? EVEN ANCIENT GREEK MATHEMATICIANS GAVE EXISTENCE PROOFS OF THIS KIND. AN EXISTENCE PROOF CAN LEAD TO A SEARCH FOR AN ALGORITHM TO GET WHAT WE HAVE SHOWN TO EXIST.

2. THE CONVERSE OF "GIVEN INTEGERS a & b, NOT BOTH ZERO, $\exists$ INTEGERS A & B $\ni$ THE GCD c OF a & b CAN BE WRITTEN AS $c = Aa + Bb$" IS NOT TRUE! WE SHOW THAT IF c CAN BE WRITTEN AS $c = Aa + Bb$, THEN c MAY NOT BE THE GCD OF a AND b. TAKE $a = 9$, $b = 5$, $c = 4$ THEN : $4 = 9 - 5$, $A = B = 1$ BUT $4 \nmid 9$, $4 \nmid 5$ AND CANNOT POSSIBLY BE THE GCD OF 9 & 5 WHICH IS 1. WE HAVE USED THE COMMON NOTATION THAT $a|b$ MEANS THAT a IS A DIVISOR OF b.

LATELY I HAVE BEEN HAVING HEATED ARGUMENTS ABOUT COUNTABILITY OF REAL NUMBERS WITH PROF. WOLFGANG MÜCKENHEIM OF GERMANY. OUR ARGUMENTS HAVE BEEN BY EMAIL AND I WAS ABLE TO ATTRACT THE ATTENTION OF PROF. BOB & ELLEN KAPLAN (OF HARVARD U.) AND ERIC SCHECHTER OF VANDERBILT UNIV. I ORDERED A LOT OF BOOKS ABOUT COUNTABILITY, INFINITE SETS AND TRANSFINITE ARGUMENT OF CANTOR. SLOWLY, I BEGAN TO ACCEPT THAT THE INFINITE DIGITS OF AN IRRATIONAL NUMBER AND THE INFINITY OF INFINITE SERIES AND PRODUCTS MUST BE CONSIDERED COMPLETED INFINITY. BOB KAPLAN FULLY AGREES WITH ME AND I GIVE HIM THE CREDIT OF ARGUING ABOUT ACCEPTANCE OF THE CONCEPT. I THINK THIS CONCEPT HAS IMPORTANT IMPLICATIONS IN MATHEMATICS, PARTICULARLY NSA.

THE GIST OF MY REASONING WITH MÜCKENHEIM IS THAT HIS CONSTRUCTION OF THE "REALS" LEAVES OUT ALL OF THE IRRATIONAL NUMBERS FROM THE BEGINNING AND, THEREFORE, HE IS ONLY COUNTING THE RATIONAL NUMBERS!

ONE OF THE BOOKS THAT I RECENTLY BOUGHT IS: D.E. LITTLEWOOD "THE SKELETON KEY OF MATHEMATICS – A SIMPLE ACCOUNT OF COMPLEX ALGEBRAIC THEORIES", 1949. I WAS PLEASANTLY SURPRISED ABOUT THIS SMALL BOOK (138 PAGES). THE NEXT FEW PAGES ARE FROM THIS BOOK.

COGRUENCE IN $\mathbb{Z}$ AND MODULAR ARITHMETIC

IF p IS A PRIME, TO FIND $a/b = ab^{-1} \pmod{p}$, WE CAN SOLVE $a = bx \pmod{p}$ FOR x. AGAIN WE CAN USE THE RESULT FROM EUCLID'S ALGORITHM THAT $\exists$ B AND P $\ni$ $Bb + Pp = 1 \Rightarrow Bb = 1 \pmod{p}$ $\Rightarrow B = b^{-1} \pmod{p}$, AND $a/b = aB \pmod{p}$. WE NOTE THAT $p = 0 \pmod{p}$, SO THAT b CAN NOT BE EQUAL p. THIS MEANS THAT IF $xy = 0$ $\pmod{p} \Rightarrow$ EITHER $x = 0 \pmod{p}$ OR $y = 0 \pmod{p}$, OR BOTH.

IF $f(x)$ IS A POLYNOMIAL OF DEGREE n AND $f(\alpha) = 0 \pmod{p} \Rightarrow f(x) = (x-\alpha)\, g(x)$ WHERE $g(x)$ IS OF DEGREE $n-1$. THIS CAN BE PROVEN BY DIVISION ALGORITHM: $f(x) = (x-\alpha)\, g(x) + R$, WHERE R IS A CONSTANT. SINCE $f(\alpha) = 0 \pmod{p}$ $\Rightarrow R = 0 \pmod{p}$.

IF $f(x)$ HAS ROOTS $\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow f(x) = A(x-\alpha_1)$ $\cdots (x-\alpha_n)$, A IS OF DEGREE 0 & $A \neq 0 \pmod{p}$. $\Rightarrow f(\beta) = 0 \pmod{p}$, THEN $\beta = \alpha_i$ FOR SOME $i \pmod{p}$. $\therefore$ $f(x)$ CAN <u>NOT</u> HAVE MORE THAN n ROOTS $\pmod{p}$. (SEE ALSO PAGE 8)

THERE ARE $p-1$ NONZERO RESIDUES MOD p. THESE ARE $1, 2, 3, \ldots\ p-2, p-1$ (AN EVEN NUMBER OF INTEGERS). NOTE THAT 1 AND $p-1$ SATISFY $x^2 = 1$. ALSO FOR EACH $a = 2, 3, \ldots, p-2$, $\exists\ a^{-1} \in \{2, \ldots$ $p-2\} \ni a a^{-1} = a^{-1} a = 1 \pmod{p}$ AND $a^{-1} \neq a$! WE NOTE

THE ONLY SOLUTION OF $x^2 = 1 \pmod{p}$ IS $x = \pm 1$ WITH $p-1 = -1 \pmod{p}$. THIS MEANS THAT $2, 3, \ldots, p-2$ ARE PAIRED IN SUCH A WAY THAT THE PRODUCTS OF EACH TWO OF THEM IS 1. (*) (SEE BELOW)

EXAMPLE:

$p = 7$:

$$\begin{cases} 2 & 3 & 4 & 5 \\ 2 & 3 & 2^{-1} & 3^{-1} \end{cases} \quad : \quad \begin{array}{l} 2 \times 4 = 1 \\ 3 \times 5 = 1 \end{array} \qquad (\mathrm{MOD}\ 7)$$

$p = 11$:

$$\begin{cases} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 3^{-1} & 5 & 2^{-1} & 7 & 7^{-1} & 5^{-1} \end{cases} \qquad (\mathrm{MOD}\ 11)$$

$$2 \times 6 = 1, \quad 3 \times 4 = 1, \quad 5 \times 9 = 1, \quad 7 \times 8 = 1 \qquad (\mathrm{MOD}\ 11)$$

$$\Rightarrow 1 \times 2 \times 3 \times \cdots \times \overbrace{p-1}^{-1\ (\text{PAIRED WITH } 1)} \quad 1 \pmod{p}$$

OR $\boxed{(p-1)! = -1} \pmod{p}$, p PRIME

THIS IS AN IMPORTANT RESULT.

NOW LET $r \neq 0$ AND CONSIDER (p PRIME)

$$r, 2r, 3r, \ldots, (p-1)r \qquad (\mathrm{MOD}\ p)$$

THESE NUMBERS MUST BE A PERMUTATION OF NUMBERS $1, 2, \ldots, p-1$ BECAUSE, TAKING $1 \leq r \leq p-1$, WE HAVE IF $r\alpha = r\beta \pmod{p}$, α & $\beta \in \{1, 2, \ldots, p-1\} \Rightarrow \alpha = \beta$

$$\Rightarrow r^{p-1}(p-1)! = -r^{p-1} = -1 \qquad (\mathrm{MOD}\ p)$$

$$\boxed{r^{p-1} = 1} \quad (\mathrm{MOD}\ p), \ p \text{ PRIME}$$

I ONCE SPENT A RIDICULOUS AMOUNT OF TIME TO PROVE THIS BUT I FAILED!

(*) IN THE TABLE OF MULTIPLICATION (MOD p), WE HAVE

ONLY $1^2 = 1$, $(p-1)^2 = (-1)^2 = 1$. ALL THE OTHER NUMBERS ON THE DIAGONAL ARE NOT 1 BECAUSE THE ONLY SOLUTIONS OF $x^2 = 1$ ARE 1 AND $p-1 \pmod{p}$. THEREFORE, NO OTHER NUMBER BUT THESE IS ITS OWN INVERSE. SINCE $\{2^{-1}, 3^{-1}, \ldots (p-2)^{-1}\}$ IS A PERMUTATION OF $\{2, 3, \ldots p-2\}$ $\Rightarrow$ EACH NUMBER IN $2, 3, \ldots, p-2$ IS PAIRED UNIQUELY WITH ITS INVERSE IN THIS SET.

EXAMPLE $p = 7$

$1 \times 2 \times 3 \times 4 \times 5 \times 6 = -1$

PRODUCT OF $1 \times 6 = -1 \pmod{p}$
" $2 \times 4 = 1$
" $3 \times 5 = 1$

NOTE: LET US USE $\mathbb{Z}_n$ FOR RESIDUE CLASS OF INTEGERS (MOD n). IF n IS NOT A PRIME, THEN $f(x) = 0$ A POLYNOMIAL OF DEGREE r MAY HAVE MORE THAN r SOLUTIONS IN $\mathbb{Z}_n$.

EXAMPLE: IN $\mathbb{Z}_8$, $x^2 - 1 = 0$ HAS SOUTIONS 1, 3, 5, 7.

THIS SITUATION DOES NOT HAPPEN IF n IS A PRIME.

THE RESULT $r^{p-1} = 1 \pmod{p}$ IS SOMETIME WRITTEN AS $\boxed{r^p = r} \pmod{p}$. IT IS A REMARKABLE RESULT.

22 Ffowcs Williams Response

N.

<u>NOTE</u>: WE REPLACE SOME NOTATIONS IN FW(63) PAPE

OUR $\vec{M}_F = \frac{\vec{V}_F}{C}$ IS $\vec{N}$ IN FW, $\vec{V}_F$ IS THE FLIGHT VEl

OUR C IS a_0 IN FW, OUR ρ' IS $C^2(\rho-\rho_0)$ IN F

OUR SUBSCRIPTS : E : EMISSION , V : VISUAL

RELATION BETWEEN FIXED $\vec{y}$-FRAME AND MOVING $\vec{z}$-FRAME :

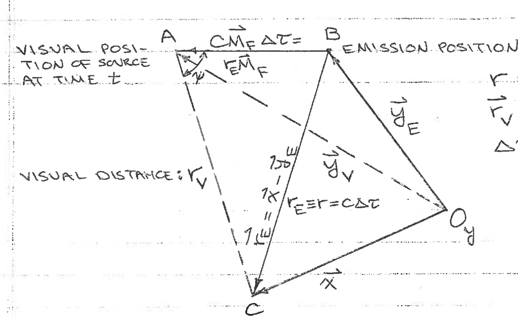
$$\vec{y} = \vec{z} + C\vec{M}_F \tau = \vec{y}(\vec{z},\tau)$$

$$\Rightarrow \quad \vec{y} = \vec{z} \quad \text{AT } \tau = 0$$

$$\vec{y}_V(\vec{z},t) = \vec{y}(\vec{z},t) = \vec{z} + C\vec{M}_F t$$

THIS IS THE <u>VISUAL POSITION</u> OF THE SOURCE AT OBSERVER TIME t.

WE FIRST DERIVE SOME RELATIONS BETWEEN VARIABLES.



$$r \equiv r_E = C\Delta\tau$$

$$\vec{r}_V = \vec{x} - \vec{y}_V(\vec{z},t)$$

$$\Delta\tau \geqslant 0$$

N

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FROM THE GARRICK TRIANGLE ABC, WE HAVE

$$r_E^2 = c^2 \Delta\tau^2$$

$$= r_V^2 + r_E^2 M_F^2 - 2 r_V r M_F \cos\psi$$

$$\cos\psi = -\frac{\vec{r}_V}{r_V} \cdot \frac{\vec{M}_F}{M_F} = -\frac{\vec{r}_V \cdot \vec{M}_F}{r_V M_F}$$

$$r_E^2 = r_V^2 + r_E^2 M_F^2 + 2\, \vec{r}_V \cdot \vec{M}_F\, r_E$$

$$\Rightarrow (1 - M_F^2)\, r_E^2 - 2\, \vec{r}_V \cdot \vec{M}_F\, r_E - r_V^2 = 0$$

$$r_E = \frac{1}{1 - M_F^2}\left(\vec{r}_V \cdot \vec{M}_F + \sqrt{(\vec{r}_V \cdot \vec{M}_F)^2 + (1 - M_F^2)\, r_V^2}\right) \quad ($$

$$\vec{r}_E = \vec{x} - (\vec{y}_V(\vec{\eta}, t) - r_E \vec{M}_F) \qquad (2)$$

$$= \vec{x} - \vec{\eta} - c\vec{M}_F t + r_E \vec{M}_F$$

$$= \vec{x} - \vec{\eta} + (r_E - ct)\, \vec{M}_F = \vec{r}_E(\vec{\eta}, t)$$

WE NOTE THAT, EXCLUDING VARIABLE $\vec{x}$, WE HAVE

$$\vec{r}_V = \vec{x} - \vec{y}_V(\vec{\eta}, t) = \vec{r}_V(\vec{\eta}, t) \qquad (3)$$

$$\Rightarrow r_E = r_E(\vec{\eta}, t) \qquad (4)$$

OTHER RELATIONS

$$\tau_E = t - r_E/c \quad , \bar{M}_r = \vec{M}_F \cdot \hat{\vec{r}}_E = \bar{M}_r(\vec{\eta}, t)$$

$$\Rightarrow \tau_E = \tau_E(\vec{\eta}, t)$$

$$1 - \bar{M}_r = 1 - \vec{M}_F \cdot \hat{\vec{r}}_E$$

N.

WE WILL NOW REDERIVE THE RESULTS OF FW(1963) PAPER IN DETAIL.

AS ALWAYS WE USE $p' = (\rho - \rho_0)c^2$. EQ.(1.2) OF FW (1963) IS, IN OUR NOTATION:

$$4\pi p'(\vec{x},t) = \int \left[\frac{\hat{r}_i \hat{r}_j}{r(1-\bar{M}_r)^3} \frac{\partial T_{ij}}{\partial \tau^2}(\vec{\eta},\tau) \right]_{ret} d\vec{\eta}$$

<u>NOTE</u>: IN THE INTEGRATION (SOLUTION) OF $\Box^2 p' = \partial^2 T_{ij}/\partial x_i \partial x_j$, WE USE THE TRANSFORMATION OF $\vec{y} \rightarrow \vec{\eta}$ AS THE FIRST STEP. THIS TRANSFORMATION DOES <u>NOT</u> CHANGE THE INTERPRETATION OF THE FLUID VELOCITY $\vec{u}$ IN T_{ij} WHICH IS <u>ALWAYS</u> WRT THE FRAME FIXED TO UNDISTURBED MEDIUM WHICH IS STATIONARY.

$$16\pi^2 \overline{p'(\vec{x},t)\, p'(\vec{x},t+\tau^*)} \equiv 16\pi^2 D(\vec{x},t,\tau^*)$$

$$= \int \left[\frac{\hat{r}_i \hat{r}_j}{r(1-\bar{M}_r)^3} \right]_{ret} \left[\frac{\hat{r}_k \hat{r}_\ell}{r(1-\bar{M}_r)^3} \right]_{ret^*} \overline{\left[\frac{\partial^2 T_{ij}(\vec{\eta},\tau_1)}{\partial \tau_1^2} \right]_{ret} \left[\frac{\partial^2 T_{k\ell}(\vec{\xi},\tau_2)}{\partial \tau_2^2} \right]_{ret^*}} \times d\vec{\eta}\, d\vec{\xi}$$

WHERE ret AND ret* REFER TO EVALUATION AT EMISSION TIMES ASSOCIATED WITH OBSERVER TIME t AND $t+\tau^*$, RESPECTIVELY. THE FIRST TWO SQ. BRACKETS ONLY APPLY TO THE ORIGIN OF THE $\vec{\eta}$-FRAME WHICH IS FIXED TO THE JET. FOR THE OTHER TERMS INVOLVING T_{ij}, WE MUST BE CAREFUL.

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WE CAN SHOW THAT, BY WRITING $\tau_2 = \tau_1 + \tau$

$$\overline{\frac{\partial^2}{\partial \tau_1^2} T_{ij}(\vec{\eta}, \tau_1) \frac{\partial^2}{\partial \tau_2^2} T_{kl}(\vec{\xi}, \tau_2)}$$

$$= \frac{\partial^4}{\partial \tau^4} \underbrace{\overline{T_{ij}(\vec{\eta}, \tau_1) T_{kl}(\vec{\xi}, \tau_1 + \tau)}}_{R^*_{ijkl}(\vec{\eta}, \vec{\xi}, \tau)} \qquad (5)$$

EQ. (1.9) OF FW(1963)

$$\tau = \tau_2 - \tau_1 = \tau^* + \frac{|\vec{x} - \vec{y}(\vec{\eta}, \tau_E(t))|}{c} - \frac{|\vec{x} - \vec{y}(\vec{\xi}, \tau_E(t+\tau^*))|}{c} \qquad (6$$

WHERE $\tau_E(\cdot)$ IS EMISSION TIME. WE HAVE TO BE VERY CAREFUL IN INTERPRETING THE MEANING OF EACH TERM IN τ.

LET US TAKE $\vec{\xi} = \vec{\eta} + \vec{\Delta}$, REFINE

$$R_{ijkl}(\vec{\eta}, \vec{\Delta}, \tau) \equiv R^*_{ijkl}(\vec{\eta}, \vec{\xi}, \tau)$$

WE NOTE THAT $\vec{y}(\vec{\xi}, \tau_E(t+\tau^*)) \equiv \vec{y}(\vec{\eta} + \vec{\Delta}, \tau_E(\vec{\eta} + \vec{\Delta}, t + \tau^*))$. WE MUST BE CAREFUL IN FINDING THE TERM $|\vec{x} - \vec{y}(\vec{\xi}, \tau_E(t+\tau^*))|$ IN TERMS OF $|\vec{x} - \vec{y}(\vec{\eta}, \tau_E(t))|$ + INCREMENT !

N

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r_E IS A FUNCTION OF $\vec{r}_V = |\vec{x} - \vec{y}(\vec{\eta}, t)|$. SO WE CAN WRITE, SUPPRESSING $\vec{x}$, $r_E = r_E(\vec{\eta}, t)$

~~WE CAN WRITE~~

$$\tau = \tau^* + \frac{r_E(\vec{\eta}, t)}{c} - \frac{r_E(\vec{\xi}, t+\tau^*)}{c} \checkmark$$

$$r_E(\vec{\xi}, t+\tau^*) = r_E(\vec{\eta}+\Delta, t+\tau^*)$$

$$= r_E(\vec{\eta}, t) + \nabla_\eta r_E \cdot \vec{\Delta}$$

$$+ \frac{\partial r_E}{\partial t}(\vec{\eta}, t)\, \tau^* + \cdots \checkmark$$

$$\tau = \tau^* - \frac{1}{c}\nabla_\eta r_E(\vec{\eta}, t) \cdot \vec{\Delta} - \frac{1}{c}\frac{\partial r_E(\vec{\eta}, t)}{\partial t}\tau^*$$

$$r_E = \frac{1}{1-M_F^2}\left(\vec{r}_V \cdot \vec{M}_F + \sqrt{(\vec{r}_V \cdot \vec{M}_F)^2 + (1-M_F^2)\, r_V^2}\right)$$

$$\vec{r}_V(\vec{\xi}, t+\tau^*) = \vec{r}_V = \vec{x} - \vec{y}_V(\vec{\xi}, t+\tau^*)$$

$$= \vec{x} - \vec{\xi} - c\vec{M}_F(t+\tau^*)$$

$$= \vec{x} - \vec{\eta} - c\vec{M}_F t - \vec{\Delta} - c\vec{M}_F \tau^*$$

$$= \vec{r}_V(\vec{\eta}, t) - \vec{\Delta} - c\vec{M}_F \tau^*$$

$$\begin{cases} \vec{r}_V \equiv R_i & \text{(TEMPORARILY)} \\ M_F = M_i & (\quad " \quad) \end{cases}, \quad 1 - M_F^2 \equiv \beta^2$$

$$r_E = \beta^{-2}\left[R_i M_i + \sqrt{R_i M_i R_K M_K + \beta^2 R_i R_i}\right]$$

$$R_i = x_i - \eta_i - cM_i t - \Delta_i - cM_i \tau^*$$

$$\frac{\partial R_i}{\partial \eta_j} = -\delta_{ij} \qquad \frac{\partial R_i}{\partial t} = -cM_i$$

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$$\frac{\partial r_E}{\partial \eta_j} = \beta^{-2}\Big[M_i \frac{\partial R_i}{\partial \eta_j} + \frac{1}{2}\left(R_i M_i R_k M_k + \beta^2 R_i R_i \right)^{-1/2}$$

$$\times \left(M_i R_k M_k \frac{\partial R_i}{\partial \eta_j} + R_i M_i M_k \frac{\partial R_k}{\partial \eta_j} + 2\beta^2 R_i \frac{\partial R_i}{\partial \eta_j} \right) \Big]$$

$$= -\beta^{-2}\Big[M_i \delta_{ij} + \frac{1}{2}\left(R_i M_i R_k M_k + \beta^2 R_i R_i \right)^{-1/2}$$

$$\times \left(M_i R_k M_k \delta_{ij} + R_i M_i M_k \delta_{kj} + 2\beta^2 R_i \delta_{ij} \right) \Big]$$

$$= -\beta^{-2}\Big[M_j + \left(R_i M_i R_k M_k + \beta^2 R_i R_i \right)^{-1/2} \times \left(M_i R_i M_j + \beta^2 R_j \right) \Big]$$

$$\frac{\partial r_E}{\partial t} = \beta^{-2}\Big[M_i \frac{\partial R_i}{\partial t} + \frac{1}{2}\left(M_i R_i M_k R_k + \beta^2 R_i R_i \right)^{-1/2}$$

$$\times \left(M_i M_k R_k \frac{\partial R_i}{\partial t} + M_i R_i M_k \frac{\partial R_i}{\partial t} + 2\beta^2 R_i \frac{\partial R_i}{\partial t} \right) \Big]$$

$$= -C\beta^{-2}\Big[M_i M_i + \left(M_i R_i M_k R_k + \beta^2 R_i R_i \right)^{-1/2} \underbrace{\left(M_i M_i M_k R_k + \beta^2 R_i M_i \right)}_{M_i R_i} \Big]$$

$$= -C\beta^{-2}\left[M_F^2 + \frac{\vec{M}_F \cdot \vec{r}_V}{\sqrt{(\vec{M}_F \cdot \vec{r}_V)^2 + \beta^2 r_V^2}} \right]$$

$$\frac{\partial r_E}{\partial \eta_j} = \nabla_\eta r_E = -\beta^{-2}\left[\vec{M}_F + \frac{(\vec{M}_F \cdot \vec{r}_V)\vec{M}_F + \beta^2 \vec{r}_V}{\sqrt{(\vec{M}_F \cdot \vec{r}_V)^2 + \beta^2 r_V^2}} \right]$$

2.

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$$\vec{r}_V = \vec{x} - \vec{y}_V$$

$$= \vec{x} - \vec{y}_E - r_E \vec{M}_F$$

$$= \vec{r}_E - r_E \vec{M}_F$$

$$= r_E\, (\hat{r}_E - \vec{M}_F)$$

$$\boxed{\vec{r}_V \cdot \vec{M}_F = (\bar{M}_r - M_F^2)\, r_E}$$

$$\boxed{E_1 = (\vec{M}_F \cdot \vec{r}_V)^2 + \beta^2 r_V^2}$$

$$r_V^2 = r_E^2\,(1 + M_F^2 - 2\bar{M}_r)$$

$$\frac{E_1}{r_E^2} = (\bar{M}_r - M_F^2)^2 + \beta^2 (1 + M_F^2 - 2\bar{M}_r)$$

$$= \bar{M}_r^2 + \cancel{M_F^4} - 2 \cancel{M_r} M_F^2 + 1 + \cancel{M_F^2} - 2\bar{M}_r$$

$$- \cancel{M_F^2} - \cancel{M_F^4} + 2 \cancel{\bar{M}_r M_F^2}$$

$$= (1 - M_r)^2 \quad \checkmark$$

$$\boxed{E_1 = (1 - \bar{M}_r)^2 r_E^2} \quad \boxed{\sqrt{E_1} = (1 - \bar{M}_r)\, r_E}$$

$$\frac{1}{c}\frac{\partial r_E}{\partial t} = -\bar{\beta}^2 \left[M_F^2 + \frac{(\bar{M}_r - M_F^2)\cancel{r_E}}{(1 - \bar{M}_r)\cancel{r_E}} \right]$$

$$= -\bar{\beta}^2 \, \frac{\bar{M}_r (1 - M_F^2)}{1 - \bar{M}_r}$$

$$= - \frac{\bar{M}_r}{1 - \bar{M}_r} \quad \checkmark$$

N

$$\nabla_\eta r_E = -\beta^{-2}\left[\vec{M}_F + \frac{(\bar{M}_r - M_F^2)\cancel{r_E}\,\vec{M}_F + \beta^2 \cancel{r_E}(\hat{\vec{r}}_E - \vec{M}_F)}{\cancel{r_E}(1-\bar{M}_r)}\right]$$

$$= -\beta^{-2}\left[\vec{M}_F + \frac{(\bar{M}_r - \cancel{M_F^2} - 1 + \cancel{M_F^2})\vec{M}_F + \beta^2\hat{\vec{r}}_E}{1-\bar{M}_r}\right]$$

$$= -\beta^{-2}\left[\cancel{\vec{M}_F} - \cancel{\vec{M}_F} + \frac{\beta^2\hat{\vec{r}}_E}{1-\bar{M}_r}\right]$$

$$= -\frac{\hat{\vec{r}}_E}{1-\bar{M}_r}$$

$$\boxed{\nabla_\eta r_E = \frac{-\hat{\vec{r}}_E}{1-\bar{M}_r}}$$

GOING BACK TO P5, WE HAVE

$$\tau = \tau^* - \frac{1}{c}\nabla_\eta r_E \cdot \vec{\Delta} - \frac{1}{c}\frac{\partial r_E}{\partial t}\tau^* + \dots$$

$$= \tau^* + \frac{1}{c}\frac{\vec{\Delta}\cdot\hat{\vec{r}}_E}{1-\bar{M}_r} + \frac{\bar{M}_r}{1-\bar{M}_r}\tau^*$$

$$= \left(1 + \frac{\bar{M}_r}{1-\bar{M}_r}\right)\tau^* + \frac{1}{c}\frac{\vec{\Delta}\cdot\hat{\vec{r}}_E}{1-\bar{M}_r}$$

$$= \frac{1}{1-\bar{M}_r}\tau^* + \frac{1}{c}\frac{\vec{\Delta}\cdot\hat{\vec{r}}_E}{1-\bar{M}_r}$$

$$= \frac{1}{c(1-\bar{M}_r)}\left(c\tau^* + \vec{\Delta}\cdot\hat{\vec{r}}_E\right) \checkmark$$

THIS IS EXACTLY EQ. 1.13 OF FW (1963)!

SO FAR WE HAVE SHOWN THAT:

$$16\pi^2 D(\vec{x}, t, \tau^*) = \iint \left[\frac{\hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l}{r^2(1-\bar{M}_r)^6}\right]_{ret} \frac{\partial^4}{\partial \tau^4} R_{ijkl}(\vec{z}, \vec{\Delta}, \tau(\tau^*, \vec{\Delta}, \vec{z}, t)) \, d\vec{z} \, d\vec{\Delta}$$

WHERE

$$D(\vec{x}, t, \tau^*) = \overline{p'(\vec{x}, t) \, p'(\vec{x}, t+\tau^*)}$$

AND WE HAVE NEGLECTED THE EFFECT OF $\vec{\Delta}$ IN THE FIRST SQ. BRACKETS IN THE INTEGRAND. WE NOW NEED TO STUDY THE EFFECT OF EDDY CONVECTION ON $D(\vec{x}, t, \tau^*)$.

NOTE THAT $\vec{\Delta}$-FRAME IS ACTUALLY THE SAME AS THE $\vec{z}$-FRAME. NOW LET US TAKE A FRAME CALLED $\vec{\lambda}$-FRAME, FIXED TO AN EDDY. LET US NOW DEFINE

$$\vec{\Delta}(\vec{\lambda}, \tau) \equiv \vec{\Delta} = \vec{\lambda} + (\vec{M} - \vec{M}_F)\tau$$

WHERE $\vec{M}$ IS THE EDDY CONVECTION MACH NUMBER WRT THE GROUND-FIXED FRAME.

WE HAVE

$$d\vec{\Delta} = \frac{1-\bar{M}_r}{1-M_r} \, d\vec{\lambda}$$

WHERE $M_r = \vec{M} \cdot \hat{r}$ (AT EMISSION TIME.

WE NEXT DEFINE $\mathcal{R}_{ijkl}(\vec{z}, \vec{\lambda}, \tau) \equiv R_{ijkl}(\vec{z}, \Delta, \tau)$ THAT IS THIS IS THE CROSS CORRELATION AS MEASURED IN THE $\vec{\lambda}$-FRAME.

ESSENTIALLY, WE CAN SHOW THAT

$$\left.\frac{\partial}{\partial \tau}\right|_{(\vec{z}, \vec{\Delta})} = \left[\frac{1-\bar{M}_r}{1-M_r} \left.\frac{\partial}{\partial \tau}\right|_{(\vec{z}, \vec{\lambda})}\right]_{ret}$$

THEREFORE, THE EQUATION FOR THE AUTO-CORRELATION OF THE ACOUSTIC PRESSURE IS

$$16\pi^2 D(\vec{x},t,\tau^*) = \iint \left[\frac{\hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l \,(1-\bar{M}_r)^5}{r^2(1-\bar{M}_r)^6(1-M_r)^5}\right]_{ret} \frac{\partial^4}{\partial\tau^4} P_{ijkl}(\vec{\xi},\vec{\lambda},\tau)\, d\vec{\xi}\, d$$

$$= \iint \left[\frac{\hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l}{r^2(1-\bar{M}_r)(1-M_r)^5}\right]_{ret} \frac{\partial^4}{\partial\tau^4} P_{ijkl}(\vec{\xi},\vec{\lambda},\tau)\, d\vec{\xi}\, d$$

NOTE THE FOLLOWING:

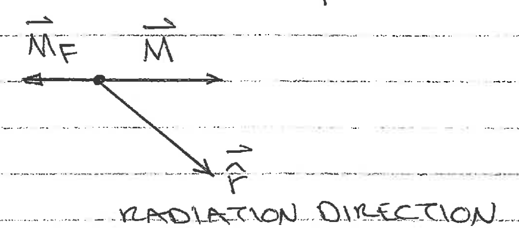
1- $\bar{M}_r = \vec{M}_F \cdot \vec{\hat{r}}$ (BASSED ON FLIGHT VELOCITY

2- $M_r = \vec{M} \cdot \vec{\hat{r}}$

3- $\vec{M}$ IS THE EDDY CONVECTION MACH NUMBER WITH RESPECT TO THE GROUND-FIXED FRAME.

4- THIS IS THE DIRECTIONS OF $\vec{M}_F$ AND $\vec{M}$ FOR AN EDDY:

$$\begin{cases} \bar{M}_r < 0 \\ M_r > 0 \end{cases}$$



5- $P_{ijkl}(\vec{\xi},\vec{\lambda},\tau)$ IS MEASURED IN THE FRAME MOVING WITH THE EDDY

6- RETARDED TIME OF THE EDDY IS BASED ON $\vec{M}_F$ BECAUSE OF THE SHORT LIFETIME OF THE EDDY

23 Final Derivations

STAPLE HERE

NOTE: WE LEAVE US THICKNESS NOISE HERE!

1

$$4\pi\,\Pi(\vec{x},t) = \frac{1}{c}\int_{f=0}\left[\frac{p\cos\theta}{r(1-M_r)}\right]_{ret} dS + \int_0^t dt' \int \left[\frac{p\cos\theta}{r^2(1-M_r)}\right]_{ret} dS$$

$$\phi(\vec{x},t) = -\Pi(\vec{x},t)/\rho_0$$

$$\vec{V} = \nabla\phi \quad \begin{cases}\text{AC. VELOCITY WRT}\\ \text{STATIONARY MEDIUM}\end{cases}$$

WE NOTE THAT

$$4\pi\,\Pi(\vec{x},t) = \frac{1}{c}\int \frac{p\cos\theta\,\delta(f)\,\delta(g)}{r}\,d\vec{y}\,d\tau + \int_0^t dt'\int \frac{p\cos\theta\,\delta(f)\,\delta(g)}{r^2}\,d\vec{y}\,d\tau$$

$$-4\pi\rho_0\vec{V}(\vec{x},t) = \frac{1}{c}\int p\,\delta(f)\,\nabla_x\left[\frac{\cos\theta\,\delta(g)}{r}\right]d\vec{y}\,d\tau + \int_0^t dt'\int p\,\delta(f)\,\nabla_x\left[\frac{\cos\theta\,\delta(g)}{r^2}\right]d\vec{y}\,d\tau$$

$$\cos\theta = \vec{n}\cdot\hat{r} = \frac{\vec{n}\cdot\vec{r}}{r}$$

$$\nabla_x\left[\frac{\cos\theta\,\delta(g)}{r}\right] = \nabla_x\left[\frac{\vec{n}\cdot\vec{r}\,\delta(g)}{r^2}\right]$$

$$= \left[\frac{\vec{n}}{r^2} - \frac{2\,\vec{n}\cdot\vec{r}}{r^2}\right]\delta(g) + \frac{\cos\theta\,\hat{r}}{cr}\,\delta'(g)$$

$$= \frac{\vec{n} - 2\cos\theta\,\hat{r}}{r^2}\,\delta(g) - \frac{\cos\theta\,\hat{r}}{cr}\,\frac{\partial\delta(g)}{\partial t}$$

SIMILARLY

$$\nabla_x\left[\frac{\cos\theta\,\delta(g)}{r^2}\right] = \frac{\vec{n} - 3\cos\theta\,\hat{r}}{r^3}\,\delta(g) - \frac{\cos\theta\,\hat{r}}{cr^2}\,\frac{\partial\delta(g)}{\partial t'}$$

2

$$
\begin{aligned}
-4\pi\rho_0\vec{V}(\vec{x}) = {} & \frac{1}{c}\int_{f=0}\left[\frac{p(\vec{n}-2\cos\theta\,\hat{r})}{r^2(1-M_r)}\right]_{ret} dS \\
& -\frac{1}{c^2}\frac{\partial}{\partial t}\int\left[\frac{p\cos\theta\,\hat{r}}{r(1-M_r)}\right]_{ret} dS \\
& +\int_0^t dt'\int_{f=0}\left[\frac{p(\vec{n}-3\cos\theta\,\hat{r})}{r^3(1-M_r)}\right]_{ret} dS \\
& \underbrace{-\frac{1}{c}\int_0^t dt'\,\frac{\partial}{\partial t'}\int_{f=0}\left[\frac{p\cos\theta\,\hat{r}}{r^2(1-M_r)}\right]_{ret} dS}_{\frac{1}{c}\int_{f=0}\left[\frac{p\cos\theta\,\hat{r}}{r^2(1-M_r)}\right]_{ret} dS} \\
= {} & \frac{1}{c^2}\frac{\partial}{\partial t}\int_{f=0}\left[\frac{p\cos\theta\,\hat{r}}{r(1-M_r)}\right]_{ret} dS \\
& +\frac{1}{c}\int_{f=0}\left[\frac{p(\vec{n}-3\cos\theta)}{r^2(1-M_r)}\right]_{ret} dS \\
& +\int_0^t dt'\int_{f=0}\left[\frac{p(\vec{n}-3\cos\theta\,\hat{r})}{r^3(1-M_r)}\right]_{ret} dS
\end{aligned}
$$

THE ONLY FAR FIELD TERMS ARE IN THE FIRST INTEGRAL AND IT IS EASY TO GET THEM:

$$
\begin{aligned}
-4\pi\rho_0\vec{V}(\vec{x},t) \overset{FF}{=} {} & \frac{1}{c^2}\int_{f=0}\left[\frac{\dot{p}\cos\theta\,\hat{r}}{r(1-M_r)^2}\right]_{ret} dS \\
& -\frac{1}{c^2}\int\left[\frac{p\,\dot{M}_r\cos\theta\,\hat{r}}{r(1-M_r)^3}\right]_{ret} dS
\end{aligned}
$$

THIS IS AN AMAZINGLY SIMPLE RESULT!

NOTE: $\dfrac{\partial}{\partial t}\displaystyle\int[\ldots]\,dS = \int\left[\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\cdots\right] dS$

MARK DUNN!

1

$$\square^2 p' = -\frac{\partial}{\partial x_i}\left[p n_i \delta(f)\right]$$

$$4\pi p'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int_{f=0}\left[\frac{p\cos\theta}{r(1-M_r)}\right]_{ret} dS + \int_{f=0}\left[\frac{p\cos\theta}{r^2(1-M_r)}\right] dS$$

$$= \frac{\partial}{\partial t}\left\{\int_{f=0}\left[\frac{p\cos\theta}{cr(1-M_r)}\right] dS + \int_0^t dt \int_{f=0}\left[\frac{p\cos\theta}{r^2(1-M_r)}\right] dS\right\}$$

$$p(\vec{x},t) = \frac{\partial}{\partial t}\left\{\Pi(\vec{x},t)\right\}$$

$$\Pi(\vec{x},t) = \frac{1}{4\pi}\int_{f=0}\left[\frac{p\cos\theta}{cr(1-M_r)}\right]_{ret} dS + \frac{1}{4\pi}\int_0^t\int_{f=0}\left[\frac{p\cos\theta}{r^2(1-M_r)}\right]_{ret} dS\, dt'$$

NOW $R_p(\vec{x},\tau) = \langle p(\vec{x},t)\, p(\vec{x},t+\tau)\rangle$

$$= \left\langle \frac{\partial \Pi(\vec{x},t)}{\partial t}\frac{\partial}{\partial t}\Pi(\vec{x},t+\tau)\right\rangle$$

$$= \left\langle \frac{\partial \Pi(\vec{x},t)}{\partial t}\frac{\partial}{\partial \tau}\Pi(\vec{x},t+\tau)\right\rangle$$

$$= \frac{\partial}{\partial t}\left\langle \frac{\partial \Pi(\vec{x},t)}{\partial t}\Pi(\vec{x},t+\tau)\right\rangle$$

$$= -\frac{\partial}{\partial \tau}\left\langle \Pi(\vec{x},t)\frac{\partial}{\partial t}\Pi(\vec{x},t+\tau)\right\rangle$$

$$= -\frac{\partial}{\partial \tau}\left\langle \Pi(\vec{x},t)\frac{\partial}{\partial \tau}\Pi(\vec{x},t+\tau)\right\rangle$$

$$= -\frac{\partial^2}{\partial \tau^2}\underbrace{\left\langle \Pi(\vec{x},t)\,\Pi(\vec{x},t+\tau)\right\rangle}$$

$$= -\frac{\partial^2}{\partial \tau^2} R_\Pi(\vec{x},\tau)$$

$R_\Pi(\vec{x},t)$ IS GENERALLY A SMOOTH FUNCTION OF TIME!

2

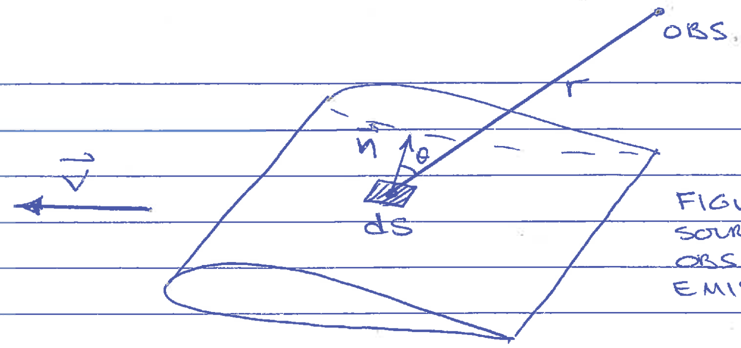
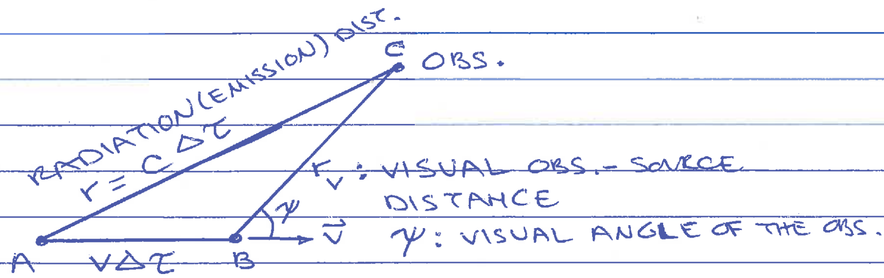


FIGURE SHOWS SOURCE AND OBS. AT THE EMISSION TIME

FORMULATION 1 — LOADING TERM ONLY

$$4\pi p'(\vec{x},t) = \frac{\partial}{c\,\partial t}\int_{f=0}\left[\frac{p\cos\theta}{r(1-M_r)}\right]dS + \int_{f=0}\left[\frac{p\cos\theta}{r^2(1-M_r)}\right]dS$$



A : SOURCE AT EMISSION TIME τ

B : " " RECEPTION (OBS.) TIME t

FIGURE OF RADIATION PROCESS AS SEEN FROM A FRAME FIXED TO THE MEDIUM

APPLYING THE COSINE RULE TO ΔABC, WE GET

$$r^2 = (V\Delta\tau)^2 + r_V^2 + 2(V\Delta\tau)\, r_V \cos\psi$$
$$= M^2 r^2 + 2(M r_V \cos\psi)\, r + r_V^2$$

$$M = \frac{V}{c}, \quad \beta^2 = 1 - M^2$$

$$\beta^2 r^2 - 2M r_V \cos\psi\, r - r_V^2 = 0$$

$$r = \frac{1}{\beta^2}\left[M r_V \cos\psi + \sqrt{M^2 r_V^2 \cos^2\psi + \beta^2 r_V^2}\,\right]$$

③

$$r = \frac{1}{\beta^2}\left[M r_v \cos\psi + r_v\sqrt{1 - M^2\sin^2\psi}\right]$$

$$= \frac{r_v}{\beta^2}\left[M\cos\psi + \sqrt{1 - M\sin^2\psi}\right]$$

THIS IS THE EMISSION OR RETARDED DISTANCE

$$\Pi(\vec{x}, t) = \frac{1}{4\pi}\int_{f=0}\left[\frac{p\cos\theta}{cr(1-M_r)}\right]dS + \frac{1}{4\pi}\int_0^t dt\int_{f=0}\left[\frac{p\cos\theta}{r^2(1-M_r)}\right]dS$$

(4)

WE NOTE THAT, SINCE

$$\Pi(\vec{x},t) = -\rho_0 \frac{\partial \phi}{\partial t}$$

$$\Longrightarrow \phi(\vec{x},t) = -\frac{1}{\rho_0}\Pi(\vec{x},t)$$

$$\Longrightarrow \vec{v}(\vec{x},t) = -\frac{1}{\rho_0}\nabla\Pi(\vec{x},t)$$

VELOCITY POTENTIAL!

24 Publications of Feridoun Farassat

This section contains references to publications and papers of Dr. Farassat. An effort was made to be comprehensive, though it is difficult to be certain a publication was not included given the wide range of research Feri performed. The NASA Technical Reports Server (NTRS), multiple journals, multiple databases, and search engines were used to construct the list. These publications are freely accessible through the journals, societies, and NASA NTRS as cited, and are not republished within this document. References are presented in chronological order.

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14. ABSTRACT

Dr. Fereidoun 'Feri' Farassat was a theoretical aeroacoustician at the National Aeronautics and Space Administration (NASA) Langley Research Center. This book contains technical derivations, notes, and classes that Dr. Farassat produced during his professional career. The layout of the book has been carefully crafted so that foundational ideas through advanced theories, which altered the technical discipline of aeroacoustics, build upon one another. The book can be used to understand the theories of acoustics and learn one contemporary aeroacoustic prediction approach made popular by Dr. Farassat. Most importantly, this book gives the general reader insight into how one of NASA's best aeroacoustics theoreticians thought, constructed, and solved problems throughout his career.

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